Counting Induced Subgraphs: A Topological Approach to #W[1]-hardness

Marc Roth
Saarland University and Cluster of Excellence (MMCI), Saarbrücken, Germany
mroth@mmci.uni-saarland.de

Johannes Schmitt[1]
ETH Zürich, Zürich, Switzerland
johannes.schmitt@math.ethz.ch

Abstract

We investigate the problem #IndSub(Φ) of counting all induced subgraphs of size k in a graph G that satisfy a given property Φ. This continues the work of Jerrum and Meeks who proved the problem to be #W[1]-hard for some families of properties which include, among others, (dis)connectedness [JCSS 15] and even- or oddness of the number of edges [Combinatorica 17]. Using the recent framework of graph motif parameters due to Curticapean, Dell and Marx [STOC 17], we discover that for monotone properties Φ, the problem #IndSub(Φ) is hard for #W[1] if the reduced Euler characteristic of the associated simplicial (graph) complex of Φ is non-zero. This observation links #IndSub(Φ) to Karp's famous Evasiveness Conjecture, as every graph complex with non-vanishing reduced Euler characteristic is known to be evasive. Applying tools from the "topological approach to evasiveness" which was introduced in the seminal paper of Khan, Saks and Sturtevant [FOCS 83], we prove that #IndSub(Φ) is #W[1]-hard for every monotone property Φ that does not hold on the Hamilton cycle as well as for some monotone properties that hold on the Hamilton cycle such as being triangle-free or not k-edge-connected for $k > 2$. Moreover, we show that for those properties #IndSub(Φ) can not be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f unless the Exponential Time Hypothesis (ETH) fails. In the final part of the paper, we investigate non-monotone properties and prove that #IndSub(Φ) is #W[1]-hard if Φ is any non-trivial modularity constraint on the number of edges with respect to some prime q or if Φ enforces the presence of a fixed isolated subgraph.

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms, Theory of computation → Problems, reductions and completeness, Mathematics of computing → Combinatorics, Mathematics of computing → Graph theory, General and reference → General literature

Keywords and phrases counting complexity, Euler characteristic, homomorphisms, parameterized complexity, simplicial complexes

Digital Object Identifier 10.4230/LIPIcs.IPEC.2018.24

Related Version `https://arxiv.org/abs/1807.01920`

Acknowledgements We are very grateful to Cornelius Brand, Radu Curticapean, Holger Dell and Johannes Lengler for helpful advice and fruitful discussions. Furthermore we wish to point out that Sloane's OEIS [1] was very useful in the course of this work.

[1] The second author was supported by the grants SNF-200020162928 and ERC-2017-AdG-786580-MACI.

13th International Symposium on Parameterized and Exact Computation (IPEC 2018).
Editors: Christophe Paul and Michał Pilipczuk; Article No. 24; pp. 24:1–24:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

In their work about the parameterized complexity of counting problems [11] Flum and Grohe introduced the parameterized analogue of the theory of computational counting as layed out by Valiant in his seminal paper about the complexity of computing the permanent [28]. Since then parameterized counting has evolved into a well-studied subfield of parameterized complexity theory. In particular, there has been remarkable progress in the classification of problems that require to count small structures in large graphs. It turned out that many families of such counting problems allow so-called dichotomy results, that is, every problem in the family is either fixed-parameter tractable or hard for the class #W[1] — the counting equivalent of W[1]. One result of that kind is the dichotomy for counting homomorphisms [10, 13]. Here one is given a graph H from a class of graphs $\mathcal{H}$ and an arbitrary graph G and the task is to compute the number of homomorphisms from H to G. When parameterized by $|H|$ this problem is fixed-parameter tractable if there exists a constant upper bound on the treewidth of graphs in $\mathcal{H}$ and #W[1]-hard otherwise. Similar results have been shown for the problems of counting subgraph embeddings [9], induced subgraphs [7] and locally injective homomorphisms [26]. As results like Ladner's theorem (see e.g. [19, 2]) rule out such dichotomies in the general case one might ask why all of the above problems indeed do allow such complexity classifications. The answer to that question was given very recently by Curticapean, Dell and Marx [8] who proved that, in some sense, all of those problems are the same. To this end, they defined the problem of computing linear combinations of homomorphisms which they called *graph motif parameters*. Here one is given a graph G and a function a of finite support that maps graphs to rational numbers and the task is to compute

$$\sum_{H} a(H) \cdot \#\mathsf{Hom}(H, G)\,, \tag{1}$$

where the sum is over all (unlabeled) simple graphs and $\#\mathsf{Hom}(H,G)$ denotes the number of homomorphisms from H to G. A result of Lovász (see e.g. Chapt. 5 in [20]) implies that the number of subgraph embeddings $\#\mathsf{Emb}(H,G)$ as well as the number of induced subgraphs $\#\mathsf{IndSub}(H,G)$ can be expressed as a linear combination of homomorphisms. In case of embeddings the result states that

$$\#\mathsf{Emb}(H,G) = \sum_{\rho \geq \emptyset} \mu(\emptyset, \rho) \cdot \#\mathsf{Hom}(H/\rho, G)\,, \tag{2}$$

where the sum is over the partition lattice of the vertices of H, μ is the Möbius function over that lattice and H/ρ is obtained from H by identifying vertices along ρ. Now, intuitively, the main result of Curticapean, Dell and Marx states that computing a linear combination of homomorphisms is precisely as hard as computing the hardest term in the linear combination. Together with the dichotomy for counting homomorphisms this implies that every problem expressible as a linear combination of homomorphisms is either fixed-parameter tractable or #W[1]-hard. The purpose of this work is a thorough investigation of the problem of counting induced subgraphs through the lense of the framework of graph motif parameters. Chen, Thurley and Weyer [7] proved that the problem $\#\mathsf{IndSub}(\mathcal{H})$ of, given a graph $H \in \mathcal{H}$ and an arbitrary graph G, computing $\#\mathsf{IndSub}(H,G)$, is fixed-parameter tractable when parameterized by $|H|$ if and only if $\mathcal{H}$ is finite and #W[1]-hard otherwise. While this result resolves the parameterized complexity of problems such as computing the number of induced cycles of length k,[2] it is not applicable to problems such as computing the number of connected induced subgraphs of size k. For this reason, Jerrum and Meeks [15, 14, 22, 16] introduced and studied the following problem: Let Φ be a (computable) graph property, then the problem $\#\mathsf{IndSub}(\Phi)$ asks, given a graph G and a natural number k, to count all induced subgraphs of size k in G that satisfy Φ.[3] In other words, the goal is to compute $\sum_{H\in\Phi_k}\#\mathsf{IndSub}(H,G)$, where Φ_k is the set of all (unlabeled) graphs with k vertices that satisfy Φ. The generality of $\#\mathsf{IndSub}(\Phi)$ allows to count almost arbitrary substructures in graphs, subsuming lots of parameterized counting problems that have been studied before, and hence the problem deserves a thorough complexity analysis with respect to the property Φ. Jerrum and Meeks proved it to be #W[1]-hard for the property of connectivity [15], for the property of having an even (or odd) number of edges [16] as well as for some other properties (see Section 1.2). As noted in [8], the theory of graph motif parameters immediately implies that for every property Φ, the problem $\#\mathsf{IndSub}(\Phi)$ is either fixed-parameter tractable or #W[1]-hard. However, for a concrete Φ it might be highly non-trivial to prove for which graphs H the term $\#\mathsf{Hom}(H,G)$ is contained with a non-zero coefficient in the equivalent expression as linear combination of homomorphisms. Unfortunately, this is precisely what needs to be done to find out whether $\#\mathsf{IndSub}(\Phi)$ is fixed-parameter tractable or #W[1]-hard. In our investigation we will focus on the coefficient of $\#\mathsf{Hom}(K_k,G)$, where K_k is the complete graph on k vertices. We will see that for monotone properties, non-zeroness of this coefficient is sufficient for the property to be evasive.

1.1 Results and techniques

The framework of graph motif parameters [8] implies that for every property Φ and natural number k, there exists a function a from graphs to rationals such that for all graphs G it holds that $\sum_{H\in\Phi_k}\#\mathsf{IndSub}(H,G)=\sum_H a(H)\cdot\#\mathsf{Hom}(H,G)$. Our most important observation is concerned with the coefficient of the complete graph.

▶ **Theorem 1** (Intuitive version)**.** *Let Φ, k and a be as above. Then it holds that $a(K_k)=0$ if and only if $\sum_{A\in\mathsf{E}_k^\Phi}(-1)^{\#A}=0$, where E_k^Φ is the set of all edge-subsets A of the labeled complete graph with k vertices such that Φ holds for the graph induced by A.*

We will provide an introduction to the theory of graph motif parameters as well as the proof of Theorem 1 in Section 3. In Section 4 we turn towards monotone properties, i.e., properties that are closed under the removal of edges, and observe that in this case the term $\sum_{A\in\mathsf{E}_k^\Phi}(-1)^{\#A}$ is equal to the reduced Euler characteristic $\hat{\chi}$ of the simplicial graph complex $\Delta(\Phi_k)$ of Φ_k. Recall that a simplicial complex is a set of sets that is closed under taking non-empty subsets and a simplicial graph complex is a simplicial complex whose elements are subsets of the edges of the labeled complete graph. We will make this formal in Section 2. As computing the number of cliques of size k is #W[1]-complete [11] and computing a linear combination of homomorphisms is precisely as hard as computing its hardest term [8], the complexity of $\#\mathsf{IndSub}(\Phi)$ is resolved whenever Φ is monotone and the reduced Euler characteristic of $\Delta(\Phi_k)$ is known to be non-zero for infinitely many k.

---

[2] This problem can be equivalently expressed as $\#\mathsf{IndSub}(\mathcal{C})$, where C is the class of all cycles.

[3] Strictly speaking, $\#\mathsf{IndSub}(\Phi)$ is the unlabeled version of p-#INDUCED SUBGRAPH WITH PROPERTY(Φ), both of which have been introduced in [15]. However, as Jerrum and Meeks point out, those problems are equivalent for graph properties that are invariant under relabeling of vertices (see Section 1.3.1 in [15]), which is true for all properties we are concerned with in this work.

▶ **Corollary 2.** *Let Φ be a monotone graph property such that $\hat{\chi}(\Delta(\Phi_k)) \neq 0$ for infinitely many k. Then the problem $\#\mathsf{IndSub}(\Phi)$ is $\#\mathrm{W}[1]$-hard.*

We will also obtain a matching lower bound under the Exponential Time Hypothesis (ETH) if the set of all k for which $\hat{\chi}(\Delta(\Phi_k)) \neq 0$ satisfies a certain density condition (see Section 2).

The (reduced) Euler characteristic is well-understood for many graph complexes. For example, Chapt. 10.5 in the book of Jonsson [17] provides a large list of graph properties, each of whose reduced Euler characteristics are non-zero infinitely often. For those properties Corollary 2 is hence applicable. The study of the (reduced) Euler characteristic is, among others, motivated by Karp's famous evasiveness conjecture, stating that every non-trivial monotone graph property is evasive. A property Φ_k on graphs with k vertices is evasive if every decision-tree algorithm that branches on the presence or absence of edges of a given graph G needs to perform $\binom{k}{2}$ branches in the worst case to correctly decide whether Φ_k holds on G. We refer the reader to Miller's survey [23] for a detailed introduction. While the conjecture is still unresolved, there has been a major breakthrough due to Khan, Saks and Sturtevant [18] who proved the conjecture to be true whenever k is a prime power. Their paper "A Topological Approach to Evasiveness" was, as the name suggests, the first one to use topological tools such as fixed-point complexes under group operations to prove evasiveness of a given graph complex. One of their results reads as follows:

▶ **Theorem 3** ([18][4])**.** *Let Φ_k be a non-trivial monotone graph property. If $\hat{\chi}(\Delta(\Phi_k)) \neq 0$ then Φ_k is evasive.*

Unfortunately, the converse of this theorem does not hold. A counterexample is given in Chapt. 10.6 in Jonsson's book [17]. Nevertheless it turns out that some tools of the topological approach to evasiveness suit as well for a topological approach to #W[1]-hardness of $\#\mathsf{IndSub}(\Phi)$. The most important ingredient in our proofs is a theorem that goes back to Smith [27] (see also [24] and Chapt. 3 in [3]), intuitively stating that, given a simplicial complex Δ and a p-power group Γ for some prime p that operates on the ground set of Δ in a way that leaves the complex stable, it holds that $\hat{\chi}(\Delta) \equiv \hat{\chi}(\Delta^\Gamma) \mod p$, where Δ^Γ is the fixed-point complex of Δ with respect to Γ. Again, this will be made formal in Section 2. Applying this theorem to a rather simple group, we will be able to prove our main result which reads as follows:

▶ **Theorem 4.** *Let Φ be a non-trivial monotone graph property. Then $\#\mathsf{IndSub}(\Phi)$ is $\#\mathrm{W}[1]$-hard and, assuming ETH, can not be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f if at least one of the following conditions is true*

1. *Φ is false for odd cycles.*
2. *Φ is true for odd anti-holes.*
3. *There exists $c \in \mathbb{N}$ such that for all H it holds that $\Phi(H) = 1$ if and only if H is not c-edge-connected.*
4. *There exists a graph F such that for all H it holds that $\Phi(H) = 1$ if and only if there is no homomorphism from F to H.*

We remark that Rivest and Vuillemin [25] implicitly proved that the reduced Euler characteristic of a graph complex does not vanish if the first condition is true. Furthermore

[4] In fact, Khan, Saks and Sturtevant show that any non-evasive complex is collapsible. However, every collapsible complex has a reduced Euler characteristic of zero (see e.g. [21]). Hence the contraposition implies the theorem as stated.

we note that (non-)triviality of a monotone property needs to be defined with some care to exclude properties that depend only on the number of vertices of a graph. Details are given in Section 4. Examples of properties that satisfy the first condition are the ones of being bipartite, cycle-free, disconnected and non-hamiltonian. One example for the second condition is the property of having a chromatic number smaller or equal than half of the size of the graph (rounded up) and the third condition includes the properties of exclusion of a fixed complete graph as a subgraph.

In Section 5 we turn to non-monotone properties and illustrate that Theorem 1 itself is a useful criterion when it comes to establishing #W[1]-hardness of $\#\mathsf{IndSub}(\Phi)$. In particular, we will prove hardness whenever Φ is a non-trivial modularity constraint on the number of edges with respect to some prime or if Φ enforces the presence of a fixed isolated subgraph.

1.2 Related work

Jerrum and Meeks introduced and studied the problem $\#\mathsf{IndSub}(\Phi)$ for the following properties. In [15] they prove the problem to be #W[1]-hard if Φ is the property of being connected, which immediately follows from Theorem 4 as $\#\mathsf{IndSub}(\Phi)$ and $\#\mathsf{IndSub}(\neg\Phi)$ are equivalent[5] and the property of being disconnected is monotone and false for every cycle. In [16] hardness is established for the property of having an even (or odd) number of edges, which follows from Theorem 1 as every term in the sum $\sum_{A\in \mathsf{E}_k^\Phi}(-1)^{\#A}$ will have the same sign. In [14] Jerrum and Meeks prove the problem to be #W[1]-hard whenever the edge-density of graphs in Φ_k grows asymptotically slower than k^2 and in [22] Meeks shows that whenever Φ is co-monotone, i.e., $\neg\Phi$ is monotone, and the set of (edge-)minimal elements of Φ has unbounded treewidth, the problem is hard as well. Those latter results are independent from ours in the sense that ours do not imply theirs and vice versa. One example of a property whose hardness does not follow from the results of Jerrum and Meeks is bipartiteness: The edge-densities of both, the properties of being bipartite and not bipartite grow asymptotically as fast as k^2 and the edge-minimal non-bipartite graphs are odd cycles, hence having treewidth 2. However hardness for the property of being bipartite follows from the first condition of Theorem 4 as odd cycles are not bipartite. Moreover, we point out that Meeks' reduction in [22] uses the Excluded Grid Theorem and hence does not imply a tight lower bound under ETH. Finally we remark that due to space constraints some proofs are only sketched or omitted and we refer the interested reader to the related version of this paper.

2 Preliminaries

First we will introduce some basic notions. Given a finite set S, we write $\#S$ for the cardinality of S. We say that a set $\mathcal{K}$ of natural numbers is *dense* if there exists a constant $c > 0$ such that for all but finitely many $k \in \mathbb{N}$ there exists $k' \in \mathcal{K}$ such that $k \leq k' \leq c \cdot k$. Given a function a from a (not necessarily finite) set S to rational numbers, the *support* of a is the set of elements $s \in S$ such that $a(s) \neq 0$. We write $\mathsf{supp}(a)$ for the support of a. Given a natural number k, we write $[k]$ for the set $\{0, \ldots, k-1\}$. Given a finite group Γ of order p^s for some prime p and natural number s, we say that Γ is a *p-power group*.

[5] We just need to substract one from $\binom{n}{k}$ to get the other.

Graph theory

In this work all graphs are considered to be undirected, simple and to not contain self-loops. Given a graph G we write $V(G)$ for the vertices and $E(G)$ for the edges of G. We denote the complete graph on ℓ vertices as K_ℓ. A *labeled* graph is a graph G with a bijective labeling $\ell : V(G) \to [\#V(G)]$ of the vertices and we will sloppily identify vertices with their labels. A *subgraph* of G is a graph obtained from G by deleting vertices (including incident edges) and/or edges. Given a subset $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is the graph with vertices S and edges $E(G) \cap S^2$. A *homomorphism* from a graph H to a graph G is a function $\varphi : V(H) \to V(G)$ that is edge-preserving, i.e. for every edge $\{u, v\} \in E(H)$ it holds that $\{\varphi(u), \varphi(v)\} \in E(G)$. We write $\mathsf{Hom}(H, G)$ for the set of all homomorphisms from H to G. A homomorphism φ is called an *embedding* if φ is injective. We write $\mathsf{Emb}(H, G)$ for the set of all embeddings from H to G. An *isomorphism* from a graph H to a graph G is a bijective homomorphism. We say that H and G are *isomorphic*, denoted by $H \cong G$, if such an isomorphism exists and we denote $\mathcal{G}$ as the set of all (isomorphism types of) graphs. An *automorphism* of a graph H is an isomorphism from H to H. We write $\mathsf{Aut}(H)$ for the set of all automorphisms of H. An embedding φ from H to G is called a *strong embedding* if for all vertices $u, v \in V(H)$ it holds that $\{u, v\} \in E(H) \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E(G)$. We write $\mathsf{StrEmb}(H, G)$ for the set of all strong embeddings from H to G. Given graphs H and G, we write $\mathsf{Sub}(H, G)$ for the set of all subgraphs of G that are isomorphic to H and $\mathsf{IndSub}(H, G)$ for the set of all induced subgraphs of G that are isomorphic to H.

▶ **Fact 5.** *For all graphs H and G it holds that $\#\mathsf{Emb}(H, G) = \#\mathsf{Sub}(H, G) \cdot \#\mathsf{Aut}(H)$ and that $\#\mathsf{StrEmb}(H, G) = \#\mathsf{IndSub}(H, G) \cdot \#\mathsf{Aut}(H)$.*

A *graph property* Φ is a function from graphs to $\{0, 1\}$ such that $\Phi(G) = \Phi(G')$ whenever G and G' are isomorphic. We say that Φ holds on G if $\Phi(G) = 1$ and we are not going not distinguish between Φ and the set of graphs for which Φ holds as this will be clear from the context. We write Φ_k for the set of all (isomorphism types of) graphs with k vertices on which Φ holds. For technical reasons we define E_k^Φ to be the set of all edge-subsets A of the labeled complete graph with k vertices such that Φ holds on the graph with the same vertices and edges A. A graph property is called *monotone* if it is closed under the removal of edges, that is, if G' is obtained from G by removing edges and Φ holds for G, then Φ holds for G' as well. A property is called *co-monotone* if its complement is monotone[6].

Transformation groups and simplical (graph) complexes

Let Ω be a finite set. A *simplicial complex* over the ground set Ω is a set Δ of non-empty subsets of Ω such that whenever a set A is contained in Δ and A' is a non-empty subset of A, then A' is contained in Δ as well. An element A of Δ is called a *simplex* and the *dimension* of A, denoted as $\mathsf{dim}(A)$, is defined to be $\#A - 1$. The *Euler characteristic* χ of a simplical complex Δ is defined to be $\chi(\Delta) := \sum_{i \geq 0} (-1)^i \cdot \#\{A \in \Delta \mid \mathsf{dim}(A) = i\}$ and the *reduced Euler characteristic* of Δ is defined to be $\hat{\chi}(\Delta) := 1 - \chi(\Delta)$.

▶ **Fact 6.** $\hat{\chi}(\Delta) = \sum_{i \geq 0} (-1)^i \cdot \#\{A \in \Delta \cup \{\emptyset\} \mid \#A = i\}$.

Given a simplicial complex Δ and a finite group Γ that operates on the ground set Ω of Δ, we say that Δ is a Γ-*simplicial complex* if the induced action of Γ on subsets of Ω

[6] We remark that in some literature, e.g. [22, 25], the notions of monotonicity and co-monotonicity are reversed.

preserves Δ. More precisely, if $A \in \Delta$ and $g \in \Gamma$ then the set $g \triangleright A := \{g \triangleright a \mid a \in A\}$ is contained in Δ as well. If this is the case we can define the *fixed-point complex* Δ^Γ as follows. Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be the orbits of Ω with respect to Γ. Then

$$\Delta^\Gamma := \left\{ S \subseteq \{1, \ldots, k\} \;\middle|\; S \neq \emptyset \wedge \bigcup_{i \in S} \mathcal{O}_i \in \Delta \right\}$$

The following theorem, which is due to Smith [27] (see also [24] and Chapt. 3 in [3]), will be of crucial importance in Section 4.

▶ **Theorem 7.** *Let Γ a group of order p^s for some prime p and natural number s and let Δ be a Γ-simplicial complex. Then $\chi(\Delta) \equiv \chi(\Delta^\Gamma) \mod p$ and hence $\hat{\chi}(\Delta) \equiv \hat{\chi}(\Delta^\Gamma) \mod p$.*

Now let Φ be a monotone graph property. Then $\mathsf{E}_k^\Phi \setminus \{\emptyset\}$ is a simplicial complex, called the *graph complex* of Φ_k. The ground set is the set of all edges of the complete labeled graph K_k on k vertices and we emphasize $\mathsf{E}_k^\Phi \setminus \{\emptyset\}$ being a simplicial complex for monotone properties by denoting it as $\Delta(\Phi_k)$. If Γ is any permutation group on the set $[k]$ then Γ induces a group operation on the ground set of Φ_k, i.e. the edges of the labeled complete graph of size k, by relabeling the vertices according to the group element. In particular, $\Delta(\Phi_k)$ is a Γ-simplicial complex as Φ_k is invariant under relabeling of vertices. We write $\Delta^\Gamma(\Phi_k)$ for the fixed-point complex $\Delta(\Phi_k)^\Gamma$ under this operation.

Parameterized (counting) complexity

We will follow the definitions of Chapt. 14 of the textbook of Flum and Grohe [12]. A *parameterized counting problem* is a function $F : \{0,1\}^* \to \mathbb{N}$ together with a computable parameterization $\kappa : \{0,1\}^* \to \mathbb{N}$. (F, κ) is called *fixed-parameter tractable* (FPT) if there exists a deterministic algorithm $\mathbb{A}$ and a computable function f such that $\mathbb{A}$ computes F in time $f(\kappa(x)) \cdot |x|^{O(1)}$ for any input x. Given two parameterized counting problems (F, κ) and (F', κ'), a *parameterized Turing reduction* from (F, κ) to (F', κ') is an FPT algorithm w.r.t. κ that has oracle access to F' and that on input x computes $F(x)$ with the additional restriction that there exists a computable function g such that for any oracle query y it holds that $\kappa'(y) \leq g(\kappa(x))$. We write $(F, \kappa) \leq^T_P (F', \kappa')$.

The parameterized counting problem $\#\mathsf{Clique}$ aks, given a graph G and a natural number k, to compute the number of complete subgraphs of size k in G and the problem is parameterized by k. The class $\#\mathrm{W}[1]$ contains all problems (F, κ) such that $(F, \kappa) \leq^T_P \#\mathsf{Clique}$ holds. Given a recursively enumerable class of graphs $\mathcal{H}$ the problems $\#\mathsf{Hom}(\mathcal{H})$, $\#\mathsf{Emb}(\mathcal{H})$, $\#\mathsf{Sub}(\mathcal{H})$, $\#\mathsf{StrEmb}(\mathcal{H})$ and $\#\mathsf{IndSub}(\mathcal{H})$ ask, given a graph $H \in \mathcal{H}$ and an arbitrary (unlabeled) graph G, to compute $\#\mathsf{Hom}(H, G)$, $\#\mathsf{Emb}(H, G)$, $\#\mathsf{Sub}(H, G)$, $\#\mathsf{StrEmb}(H, G)$ and $\#\mathsf{IndSub}(H, G)$, respectively. All problems are parameterized by $|H|$. As stated in the introduction, there are dichotomy results for each of the aforementioned problems [10, 13, 9, 7]. We emphasize on the following, which is crucial for the framework of graph motif parameters.

▶ **Theorem 8** ([10, 13])**.** *The problem $\#\mathsf{Hom}(\mathcal{H})$ is fixed-parameter tractable if there exists $b \in \mathbb{N}$ such that the treewidth*[7] *of every graph in $\mathcal{H}$ is bounded by b. Otherwise, the problem is $\#\mathrm{W}[1]$-hard.*

[7] We remark that the graph parameter of treewidth is not used explicitely in this work. Hence we refer the reader e.g. to Chapt. 11 in [12].

In this work we deal with a generalization of $\#\mathsf{IndSub}(\mathcal{H})$. Let Φ be a computable graph property. The problem $\#\mathsf{IndSub}(\Phi)$ asks, given a graph G and a number $k \in \mathbb{N}$ to compute $\sum_{H \in \Phi_k} \#\mathsf{IndSub}(H, G)$. The parameter is k.

3 Graph motif parameters

In [8] Curticapean, Dell and Marx generalized the problem $\#\mathsf{Hom}(\mathcal{H})$ to linear combinations, called *graph motif parameters*. To this end, let $\mathcal{A}$ be a recursively enumerable set of functions $a : \mathcal{G} \to \mathbb{Q}$ such that $\mathsf{supp}(a)$ is finite. Then the problem $\#\mathsf{Hom}(\mathcal{A})$ asks, given $a \in \mathcal{A}$ and a graph G, to compute $\sum_{H \in \mathcal{G}} a(H) \cdot \#\mathsf{Hom}(H, G)$. The parameter is the description length of a, denoted by $|a|$. Their main result states that computing a linear combination of homomorphisms is as hard as computing all terms with non-zero coefficients:

▶ **Theorem 9** ([8]). *There exists a deterministic algorithm that, on input a function $a : \mathcal{G} \to \mathbb{Q}$ with finite support, a graph $F \in \mathsf{supp}(a)$ and a graph G and given oracle access to the function $G \mapsto \sum_{H \in \mathcal{G}} a(H) \cdot \#\mathsf{Hom}(H, G)$, computes $\#\mathsf{Hom}(F, G)$ in time $g(|a|) \cdot \#V(G)^{O(1)}$ and additionally satisfies that the number of vertices of every graph G' for which the oracle is queried is of size bounded by $\max_{H \in \mathsf{supp}(a)} \#V(H) \cdot \#V(G)$.*

Using this result, Curticapean, Dell and Marx proved that the problem $\#\mathsf{Hom}(\mathcal{A})$ is fixed-parameter tractable if there is a constant upper bound on the treewidth of all graphs that occur in the support of a function $a \in \mathcal{A}$, and #W[1]-hard otherwise. After that they showed that all of the problems $\#\mathsf{Emb}(\mathcal{H})$, $\#\mathsf{StrEmb}(\mathcal{H})$, ... are expressible as linear combinations of homomorphisms, immediately implying the existence of dichotomy results for those problems. However, establishing a concrete criterion for fixed-parameter tractability requires to find out which graphs are contained in the support of a function a when the problem is translated to a linear combination of homomorphisms, and this can be highly non-trivial.

In what follows, we will establish a concrete criterion for properties Φ such that the coefficient of K_k is non-zero when the function $G \mapsto \sum_{H \in \Phi_k} \#\mathsf{IndSub}(H, G)$ is translated to a linear combination of homomorphisms. This is motivated by the fact that, in this case, Theorem 9 allows us to compute the number $\#\mathsf{Hom}(K_k, G)$ which is equal to $k!$ times the number of cliques of size k in G. As $\#\mathsf{Clique}$ can not be solved in time $f(k) \cdot \#V(G)^{o(k)}$ for any computable function f under the Exponential Time Hypothesis [5, 6], we will not only obtain #W[1]-hardness but also a tight lower bound under the lense of fine-grained complexity theory.

▶ **Theorem 10** (Theorem 1 restated). *Let Φ be a graph property, let k be a non-zero natural number and let $a : \mathcal{G} \to \mathbb{Q}$ be the function such that for all graphs G the following is true*

$$\sum_{H \in \Phi_k} \#\mathsf{IndSub}(H, G) = \sum_{H} a(H) \cdot \#\mathsf{Hom}(H, G) \,.$$

Then $|k! \cdot a(K_k)| = |\sum_{A \in \mathsf{E}_k^\Phi} (-1)^{\#A}|$.

Proof. Using the principle of inclusion-exclusion we can express the number of strong embeddings in terms of the number of embeddings (see e.g. Chapt. 5.2.3 in [20]):

$$\#\mathsf{StrEmb}(H, G) = \sum_{\substack{H' \supseteq H \\ V(H) = V(H')}} (-1)^{\#E(H') - \#E(H)} \cdot \#\mathsf{Emb}(H', G) \,, \tag{3}$$

where H' ranges over all graphs obtained from H by adding edges. Next we collect terms in (3) that correspond to isomorphic graphs. To this end we let $\#\{H' \supseteq H\}$ denote the number of possibilities to add edges to H such that the resulting graph is isomorphic to H'. Note that $\#\{K_k \supseteq H\} = 1$ if H has k vertices. We obtain

$$\#\mathsf{StrEmb}(H,G) = \sum_{H' \in \mathcal{G}} (-1)^{\#E(H')-\#E(H)} \cdot \#\{H' \supseteq H\} \cdot \#\mathsf{Emb}(H',G)\,. \tag{4}$$

Next we translate the number of embeddings to a linear combination of homomorphisms. This can be done using Möbius inversion[8] (see [8] or Chapt. 5.2.3 in [20]):

$$\#\mathsf{Emb}(H',G) = \sum_{\rho \geq \emptyset} \mu(\emptyset,\rho) \cdot \#\mathsf{Hom}(H'/\rho,G)\,, \tag{5}$$

where the sum and the Möbius function μ are over the partition lattice of $V(H')$ and H'/ρ is obtained from H' by contracting every pair of vertices that is contained in the same block in ρ. We observe that the coefficient of $\#\mathsf{Hom}(K_k,G)$ in the above sum is $\mu(\emptyset,\emptyset) = 1$ if H' is isomorphic to K_k and zero otherwise as every vertex contraction of a graph with k vertices that is not the complete graph can not result in the complete graph with k vertices. Hence the coefficient of $\#\mathsf{Hom}(K_k,G)$ in Equation (4) is precisely $(-1)^{\#E(K_k)-\#E(H)}$. Next we use Fact 5 and obtain that

$$\sum_{H \in \Phi_k} \#\mathsf{IndSub}(H,G) = \sum_{H \in \Phi_k} \#\mathsf{StrEmb}(H,G) \cdot \#\mathsf{Aut}(H)^{-1}\,. \tag{6}$$

It follows that the coefficient $a(K_k)$ of $\#\mathsf{Hom}(K_k,G)$ in Equation (6) satisfies

$$a(K_k) = \sum_{H \in \Phi_k} (-1)^{\#E(K_k)-\#E(H)} \cdot \#\mathsf{Aut}(H)^{-1}\,. \tag{7}$$

We now multiply this equation by $k!$, which we interpret as the number $\#\mathsf{Sym}_k$ of elements of the symmetric group of the k vertices. Taking also the absolute value on both sides allows us to drop the constant factor $(-1)^{\#E(K_k)}$ and we obtain

$$|k! \cdot a(K_k)| = \left| \sum_{H \in \Phi_k} (-1)^{\#E(H)} \cdot \frac{\#\mathsf{Sym}_k}{\#\mathsf{Aut}(H)} \right|\,. \tag{8}$$

For any graph H in the above sum choose a set A_0 of edges of the labeled complete graph K_k on k vertices such that the corresponding subgraph $G(A_0)$ is isomorphic to H. The group Sym_k acts on the vertices and thus on the edges of K_k and by the definition of a graph automorphism, the stabilizer of the set A_0 has exactly $\#\mathsf{Aut}(H)$ elements. On the other hand the orbit of A_0 under Sym_k is the collection of all sets A such that $G(A) \cong H$. By the Orbit Stabilizer theorem we have $\frac{\#\mathsf{Sym}_k}{\#\mathsf{Aut}(H)} = \#\{A \subseteq E(K_k) \mid G(A) \cong H\}$. Inserting in equation (8) we obtain $|k! \cdot a(K_k)| = |\sum_{H \in \Phi_k} \sum_{\substack{A \subseteq E(K_k) \\ G(A) \cong H}} (-1)^{\#E(H)}| = |\sum_{A \in \mathsf{E}_k^\Phi} (-1)^{\#A}|$. ◀

Theorem 10 implies the following sufficient criterion for hardness of $\#\mathsf{IndSub}(\Phi)$ which we will use in the remainder of the paper.

---

[8] We omit the formal introduction to Möbius inversion as we will only need that $\mu(\emptyset,\emptyset) = 1$. We refer the interested reader to [20], where the concept is introduced and Equation (5) is proved.

▶ **Corollary 11.** *Let Φ be a graph property and let $\mathcal{K} = \{k \in \mathbb{N} \mid \sum_{A \in \mathsf{E}_k^\Phi} (-1)^{\#A} \neq 0\}$. If $\mathcal{K}$ is infinite, then* $\#\mathsf{IndSub}(\Phi)$ *is* $\#\mathrm{W}[1]$*-hard. If additionally $\mathcal{K}$ is dense,* $\#\mathsf{IndSub}(\Phi)$ *can not be solved in time $f(k) \cdot \#V(G)^{o(k)}$ for any computable function f, unless ETH fails.*

Due to space constraints the proof is deferred to the related version of the paper. However, let us give the rough idea: By Theorem 9 and Theorem 10 there exists a (tight) reduction from the problem $\#\mathsf{Clique}(\mathcal{K})$ of counting cliques of size k where $k \in \mathcal{K}$ is promised.

As $\mathcal{K}$ is infinite we can apply a standard technique using colors and the inclusion-exclusion principle to reduce $\#\mathsf{Clique}$ to $\#\mathsf{Clique}(\mathcal{K})$, the former of which is $\#\mathrm{W}[1]$-complete and, assuming ETH, cannot be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f [5, 6]. This immediately yields $\#\mathrm{W}[1]$-hardness of $\#\mathsf{Clique}(\mathcal{K})$ and hence $\#\mathsf{IndSub}(\Phi)$. If additionally $\mathcal{K}$ is dense, the reduction is tight and the lower bound under ETH transfers as well.

4 Monotone properties

Recall that monotone graph properties are closed under the removal of edges. In what follows we assume every monotone graph property to hold on the independent set, i.e., the graph containing no edges, because otherwise the property would be trivially false. For technical reasons we say that a property is *non-trivial* if it is false on K_k for all but finitely many $k \in \mathbb{N}$.[9] Due to space constraints we might omit or only sketch proofs and defer the details to the related version of the paper. We start by refining Theorem 10 for monotone properties.

▶ **Lemma 12.** *Let Φ be a monotone graph property and let k be a non-zero natural number. Then it holds that $\sum_{A \in \mathsf{E}_k^\Phi} (-1)^{\#A} = \hat{\chi}(\Delta(\Phi_k))$.*

▶ **Corollary 13** (Corollary 2 restated)**.** *Let Φ be a monotone graph property and let*

$$\mathcal{K} = \{k \in \mathbb{N} \mid \hat{\chi}(\Delta(\Phi_k)) \neq 0\}\,.$$

If $\mathcal{K}$ is infinite, then $\#\mathsf{IndSub}(\Phi)$ *is* $\#\mathrm{W}[1]$*-hard. If additionally $\mathcal{K}$ is dense,* $\#\mathsf{IndSub}(\Phi)$ *can not be solved in time $f(k) \cdot \#V(G)^{o(k)}$ for any computable function f, unless ETH fails.*

Proof. Follows immediately from Lemma 12 and Corollary 11. ◀

The above criterion yields hardness of $\#\mathsf{IndSub}(\Phi)$ for every monotone graph property Φ whose graph complex is well-understood with respect to the (reduced) Euler characteristic. The thesis of Jonsson (see Chapt. 10.5 in [17]) provides a large list of graph complexes including e.g. disconnectivity, colorability and coverability, only to name a few, whose reduced Euler characteristics are non-zero infinitely often and to which Corollary 13 is hence applicable. We would also like to point out the work of Chakrabarti, Khot and Shi [4] who proved the reduced Euler characteristic of a large family of graph complexes to be odd. Their result will be used to prove the fourth condition of Theorem 4 and reads as follows — we state it in terms of homomorphisms.

▶ **Lemma 14** ([4])**.** *Let F be a graph and let $\Phi^{[F]}$ be the graph property that holds true on a graph G if and only if $\mathsf{Hom}(F, G) = \emptyset$, i.e., there is no homomorphism from F to G. Furthermore let $T_F := \min\{2^{2^t} - 1 \mid 2^{2^t} \geq \#V(F)\}$ and let $k \in \mathbb{N}$ such that $k \equiv 1 \mod T_F$. Then it holds that $\chi(\Phi_k^{[F]}) \equiv 0 \mod 2$ and hence $\hat{\chi}(\Phi_k^{[F]}) \equiv 1 \mod 2$.*

[9] This is required to exlude properties like $\Phi(G) = 0 \Leftrightarrow \#V(G) \equiv 1 \mod 2$ which indeed is monotone as it is closed under the removal of edges.

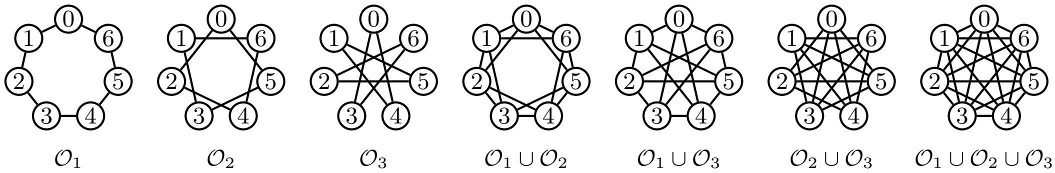

Figure 1 Non-empty unions of orbits on the operation of $\mathbb{Z}_7$ on the edge set of the labeled graph with 7 vertices. If Φ is trivially true then $\Delta^{\mathbb{Z}_7}(\Phi_7)$ contains all of the above subsets of orbits. If Φ holds only for bipartite graphs then none of the above subsets is contained in $\Delta^{\mathbb{Z}_7}(\Phi_7)$. If Φ is planarity then $\Delta^{\mathbb{Z}_7}(\Phi_7) = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$. More exotically, if Φ is the property of not being 5-edge-connected then $\Delta^{\mathbb{Z}_7}(\Phi_7)$ contains every subset of orbits except for $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$.

Unfortunately, as the proof of the above lemma shows, it is often quite tedious to argue about the (reduced) Euler characteristic of the graph complex induced by a more complicated property Φ and hence proving hardness of $\#\mathsf{IndSub}(\Phi)$. In the remainder of this section we will therefore demonstrate that Corollary 13 together with Theorem 7 yields a fruitful topological approach to prove #W[1]-hardness and conditional lower bounds for $\#\mathsf{IndSub}(\Phi)$, given that Φ is a monotone graph property. We outline the approach in the following lemma.

▶ **Lemma 15.** *Let Φ be a monotone graph property, let $\mathcal{K}$ be an infinite subset of $\mathbb{N}$ and let $\Gamma = \{\Gamma_k \mid k \in \mathcal{K}\}$ be a set of permutation groups such that for every $k \in \mathcal{K}$ the group Γ_k is a p_k-power group for some prime p_k. If $\hat{\chi}(\Delta^{\Gamma}(\Phi_k)) \not\equiv 0 \mod p_k$ holds for every $k \in \mathcal{K}$ then $\#\mathsf{IndSub}(\Phi)$ is #W[1]-hard. If additionally $\mathcal{K}$ is dense, $\#\mathsf{IndSub}(\Phi)$ can not be solved in time $f(k) \cdot \#V(G)^{o(k)}$ for any computable function f, unless ETH fails.*

Proof. Follows immediately from Corollary 13 and Theorem 7. ◀

Intuitively, Lemma 15 states that instead of analyzing $\hat{\chi}(\Delta(\Phi_k))$ which might be tedious, it suffices to prove that the reduced Euler characteristic of the fixed-point complex of $\Delta(\Phi_k)$ with respect to a p-power group is not 0 modulo p. For our purposes it will suffice to use the groups $\mathbb{Z}_p$ for prime numbers p, explained as follows. Recall that the ground set of $\Delta(\Phi_p)$ is the set of all edges of the labeled complete graph on p vertices. Now $b \in \mathbb{Z}_p$ is interpreted as a relabeling $x \mapsto x + b$ of the vertices[10], which induces an operation on the edges by mapping the edge $\{x, y\}$ to the edge $\{x + b, y + b\}$. We remark that this group was also used in an intermediate step in [18]. It can easily be verified that this mapping is a group operation. Furthermore $\Delta(\Phi_p)$ is a $\mathbb{Z}_p$-simplicial complex with respect to this operation as Φ is invariant under relabeling of vertices. Hence the fixed-point complex $\Delta^{\mathbb{Z}_p}(\Phi_p)$ is defined. Furthermore observe that every orbit of the group operation is an Hamilton cycle. We illustrate $\Delta^{\mathbb{Z}_7}(\Phi_7)$ for some properties Φ in Figure 1.

Note that, given a prime $p > 2$, the ground set of $\Delta^{\mathbb{Z}_p}(\Phi_p)$ consists of exactly $\frac{1}{2}(p-1)$ elements. In particular those are the Hamiltonian cycles $H_1 = (0, 1, 2, \dots)$, $H_2 = (0, 2, 4, \dots)$, $H_3 = (0, 3, 6, \dots)$, $\dots$, $H_{\frac{1}{2}(p-1)} = (0, \frac{1}{2}(p-1), p-1, \dots)$. Equivalently, H_i is the orbit of the (labeled) edge $\{0, i\}$ under the operation of $\mathbb{Z}_p$ for $i \in \{1, \dots, \frac{1}{2}(p-1)\}$ and it can easily be verified that those are all orbits of the group operation. In what follows, given a non-empty set $P \subseteq \{1, \dots, \frac{1}{2}(p-1)\}$, we write H_P for the graph with vertices (labeled with) $\{0, \dots, p-1\}$ and edges $\bigcup_{i \in P} H_i$.

▶ **Fact 16.** *Let P be non-empty subset of $\{1, \dots, \frac{1}{2}(p-1)\}$. Then $P \in \Delta^{\mathbb{Z}_p}(\Phi_p) \Leftrightarrow H_P \in \Phi_p$.*

[10] Here + is addition modulo p.

Now we have everything we need to prove our main result. We start with monotone properties that are false on odd cycles or true on odd antiholes.

▶ **Lemma 17.** *Let Φ be a non-trivial monotone graph property. If Φ does not hold on odd cycles or if Φ holds on odd anti-holes then there exists a constant $N \in \mathbb{N}$ such that $\hat{\chi}(\Delta^{\mathbb{Z}_p}(\Phi_p)) \not\equiv 0 \mod p$ for every prime $p > N$.*

Proof sketch. If Φ does not hold on odd cycles then $\Delta^{\mathbb{Z}_p}(\Phi_p) = \emptyset$ and hence we have that $\hat{\chi}(\Delta^{\mathbb{Z}_p}(\Phi_p)) = 1 - \chi(\Delta^{\mathbb{Z}_p}(\Phi_p)) = 1 - 0 = 1 \not\equiv 0 \mod p$. As Φ is non-trivial there exists $N \in \mathbb{N}$ such that $\Phi(K_k) = 0$ for all $k > N$. Now if Φ holds on odd anti-holes then $\Delta^{\mathbb{Z}_p}(\Phi_p) = \{P \mid \emptyset \subsetneq P \subsetneq \{1, \dots, \frac{1}{2}(p-1)\}\}$ for all $p > N$ since Φ is monotone and H_P is an anti-hole if and only if $\#P = \frac{p-1}{2} - 1$. Furthermore, Φ does not hold on $H_{\{1,\dots,\frac{1}{2}(p-1)\}} \cong K_p$ as $p > N$. Now it can be easily verified that $\hat{\chi}(\Delta^{\mathbb{Z}_p}(\Phi_p)) = (-1)^{\frac{1}{2}(p-1)+1} \not\equiv 0 \mod p$. ◀

We continue with one more exotic property which illustrates the utility of the topological approach by exploiting the simple structure of $\Delta^{\mathbb{Z}_p}(\Phi_p)$.

▶ **Lemma 18.** *Let $c \in \mathbb{N}$ be an arbitrary constant and let Φ be the graph property of being not $(c+1)$-edge-connected. Then $\hat{\chi}(\Delta^{\mathbb{Z}_p}(\Phi_p)) \not\equiv 0 \mod p$ for every prime $p > c+3$.*

Proof sketch. We rely on the observation that the graph H_P is $(c+1)$-edge-connected if and only if $\#P > \lfloor \frac{c}{2} \rfloor$. Hence $\Delta^{\mathbb{Z}_p}(\Phi_p) = \{P \subseteq \{1, \dots, \frac{1}{2}(p-1) \mid P \neq \emptyset \wedge \#P \leq \lfloor \frac{c}{2} \rfloor\}$. Now it can be easily verified that $\hat{\chi}(\Delta^{\mathbb{Z}_p}(\Phi_p)) = (-1)^{\lfloor \frac{c}{2} \rfloor} \cdot \binom{\frac{1}{2}(p-1)-1}{\lfloor \frac{c}{2} \rfloor} \not\equiv 0 \mod p$. ◀

Finally, Theorem 4 follows from Lemma 15, 14, 17 and 18. Details are given in the related version of the paper.

5 Non-monotone properties

In this section, we present #W[1]-hardness results for two non-monotone properties. For the first one, we generalize [16] where hardness was established for counting induced subgraphs with an even (or odd) number of edges. It turns out that any non-trivial modularity constraint with respect to a prime induces #W[1]-hardness. To this end let q be a prime and $\mathcal{Q}$ be a subset of $\{0, \dots, q-1\}$. Then the property $\mathsf{Mod}[q, \mathcal{Q}]$ holds on a graph H if and only if $(\#E(H) \bmod q) \in \mathcal{Q}$. For the second property, let F be a connected graph. Then the property $\mathsf{Iso}[F]$ holds on a graph H if and only if H contains an isolated subgraph that is isomorphic to F. Due to space constraints the proof of the following theorem is deferred to the related version of the paper.

▶ **Theorem 19.** *For all primes q, non-trivial subsets $\mathcal{Q}$ of $\{0, \dots, q-1\}$ and connected graphs F, the problems $\#\mathsf{IndSub}(\mathsf{Mod}[q, \mathcal{Q}])$ and $\#\mathsf{IndSub}(\mathsf{Iso}[F])$ are #W[1]-hard and can not be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f, unless ETH fails.*

6 Conclusion and future work

We used the framework of graph motif parameters to provide a sufficient criterion for #W[1]-hardness of $\#\mathsf{IndSub}(\Phi)$. For monotone properties Φ this amounts to the reduced Euler characteristic of the associated graph complex to be non-zero infinitely often. In particular, our results provide a fine-grained reduction from the problem of counting cliques of size k to counting induced subgraphs of size k with property Φ whenever Φ is monotone and $\hat{\chi}(\Delta(\Phi_k)) \neq 0$. Using a topological approach, we established hardness for a large class of non-trivial monotone graph properties. The obvious next question, whose answer would settle the parameterized complexity of $\#\mathsf{IndSub}(\Phi)$ for monotone properties completely, is whether for every non-trivial monotone property Φ the set of k such that $\hat{\chi}(\Delta(\Phi_k)) \neq 0$ is infinite.

References

1 OEIS Foundation Inc. (2018). The On-Line Encyclopedia of Integer Sequences. URL: `http://oeis.org`.

2 Karl R. Abrahamson, Rodney G. Downey, and Michael R. Fellows. Fixed-Parameter Intractability II (Extended Abstract). In *STACS 93, 10th Annual Symposium on Theoretical Aspects of Computer Science, Würzburg, Germany, February 25-27, 1993, Proceedings*, pages 374–385, 1993. `doi:10.1007/3-540-56503-5_38`.

3 Glen E Bredon. *Introduction to compact transformation groups*, volume 46. Academic press, 1972.

4 Amit Chakrabarti, Subhash Khot, and Yaoyun Shi. Evasiveness of Subgraph Containment and Related Properties. *SIAM J. Comput.*, 31(3):866–875, 2001. `doi:10.1137/S0097539700382005`.

5 Jianer Chen, Benny Chor, Mike Fellows, Xiuzhen Huang, David W. Juedes, Iyad A. Kanj, and Ge Xia. Tight lower bounds for certain parameterized NP-hard problems. *Inf. Comput.*, 201(2):216–231, 2005. `doi:10.1016/j.ic.2005.05.001`.

6 Jianer Chen, Xiuzhen Huang, Iyad A. Kanj, and Ge Xia. Strong computational lower bounds via parameterized complexity. *J. Comput. Syst. Sci.*, 72(8):1346–1367, 2006. `doi:10.1016/j.jcss.2006.04.007`.

7 Yijia Chen, Marc Thurley, and Mark Weyer. Understanding the Complexity of Induced Subgraph Isomorphisms. In *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part I: Tack A: Algorithms, Automata, Complexity, and Games*, pages 587–596, 2008. `doi:10.1007/978-3-540-70575-8_48`.

8 Radu Curticapean, Holger Dell, and Dániel Marx. Homomorphisms are a good basis for counting small subgraphs. In *Proceedings of the 49th ACM Symposium on Theory of Computing, STOC*, pages 210–223, 2017. `doi:10.1145/3055399.3055502`.

9 Radu Curticapean and Dániel Marx. Complexity of Counting Subgraphs: Only the Boundedness of the Vertex-Cover Number Counts. In *Proceedings of the 55th Annual Symposium on Foundations of Computer Science, FOCS*, pages 130–139, 2014. `doi:10.1109/FOCS.2014.22`.

10 Víctor Dalmau and Peter Jonsson. The complexity of counting homomorphisms seen from the other side. *Theoretical Computer Science*, 329(1):315–323, 2004.

11 Jörg Flum and Martin Grohe. The Parameterized Complexity of Counting Problems. *SIAM J. Comput.*, 33(4):892–922, 2004. `doi:10.1137/S0097539703427203`.

12 Jörg Flum and Martin Grohe. *Parameterized Complexity Theory (Texts in Theoretical Computer Science. An EATCS Series)*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.

13 Martin Grohe. The complexity of homomorphism and constraint satisfaction problems seen from the other side. *Journal of the ACM (JACM)*, 54(1):1, 2007.

14 Mark Jerrum and Kitty Meeks. Some Hard Families of Parameterized Counting Problems. *TOCT*, 7(3):11:1–11:18, 2015. `doi:10.1145/2786017`.

15 Mark Jerrum and Kitty Meeks. The parameterised complexity of counting connected subgraphs and graph motifs. *J. Comput. Syst. Sci.*, 81(4):702–716, 2015. `doi:10.1016/j.jcss.2014.11.015`.

16 Mark Jerrum and Kitty Meeks. The parameterised complexity of counting even and odd induced subgraphs. *Combinatorica*, 37(5):965–990, 2017. doi:10.1007/s00493-016-3338-5.

17 Jakob Jonsson. *Simplicial complexes of graphs*, volume 1928 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008. doi:10.1007/978-3-540-75859-4.

18 Jeff Kahn, Michael E. Saks, and Dean Sturtevant. A Topological Approach to Evasiveness. In *24th Annual Symposium on Foundations of Computer Science, Tucson, Arizona, USA, 7-9 November 1983*, pages 31–33, 1983. doi:10.1109/SFCS.1983.4.

19 Richard E. Ladner. On the Structure of Polynomial Time Reducibility. *J. ACM*, 22(1):155–171, 1975. doi:10.1145/321864.321877.

20 László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Society Providence, 2012.

21 Frank H. Lutz. Some Results Related to the Evasiveness Conjecture. *J. Comb. Theory, Ser. B*, 81(1):110–124, 2001. doi:10.1006/jctb.2000.2000.

22 Kitty Meeks. The challenges of unbounded treewidth in parameterised subgraph counting problems. *Discrete Applied Mathematics*, 198:170–194, 2016. doi:10.1016/j.dam.2015.06.019.

23 Carl A. Miller. Evasiveness of Graph Properties and Topological Fixed-Point Theorems. *Foundations and Trends in Theoretical Computer Science*, 7(4):337–415, 2013. doi:10.1561/0400000055.

24 Robert Oliver. Fixed-point sets of group actions on finite acyclic complexes. *Commentarii Mathematici Helvetici*, 50(1):155–177, 1975.

25 Ronald L. Rivest and Jean Vuillemin. On Recognizing Graph Properties from Adjacency Matrices. *Theor. Comput. Sci.*, 3(3):371–384, 1976. doi:10.1016/0304-3975(76)90053-0.

26 Marc Roth. Counting Restricted Homomorphisms via Möbius Inversion over Matroid Lattices. In *25th Annual European Symposium on Algorithms, ESA 2017, September 4-6, 2017, Vienna, Austria*, pages 63:1–63:14, 2017. doi:10.4230/LIPIcs.ESA.2017.63.

27 PA Smith. Fixed-point theorems for periodic transformations. *American Journal of Mathematics*, 63(1):1–8, 1941.

28 Leslie G. Valiant. The Complexity of Computing the Permanent. *Theor. Comput. Sci.*, 8:189–201, 1979. doi:10.1016/0304-3975(79)90044-6.