Progressive Algorithms for Domination and Independence

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Abstract

We consider a generic algorithmic paradigm that we call *progressive exploration*, which can be used to develop simple and efficient parameterized graph algorithms. We identify two model-theoretic properties that lead to efficient progressive algorithms, namely variants of the *Helly property* and *stability*. We demonstrate our approach by giving linear-time fixed-parameter algorithms for the DISTANCE-r DOMINATING SET problem (parameterized by the solution size) in a wide variety of restricted graph classes, such as powers of nowhere dense classes, map graphs, and (for $r = 1$) biclique-free graphs. Similarly, for the DISTANCE-r INDEPENDENT SET problem the technique can be used to give a linear-time fixed-parameter algorithm on any nowhere dense class. Despite the simplicity of the method, in several cases our results extend known boundaries of tractability for the considered problems and improve the best known running times.

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1 Introduction

It is widely believed that many important algorithmic graph problems cannot be solved efficiently on general graphs. Consequently, a natural question is to identify the most general classes of graphs on which a given problem can be solved efficiently. Structural graph theory offers a wealth of concepts that can be used to design efficient algorithms for generally intractable problems on restricted graph classes. An important result in this area states that every property of graphs expressible in monadic second-order logic can be tested in linear time on every class of bounded treewidth [3]. Similarly, every property expressible in first-order logic can be tested in almost linear time on every nowhere dense graph class [11].

Nowhere denseness is an abstract notion of uniform sparseness in graphs, which is the foundational definition of the theory of sparse graphs; see the monograph of Nešetřil and Ossona de Mendez [14] for an overview. Formally, a graph class $\mathscr{C}$ is *nowhere dense* if for every $r \in \mathbb{N}$, one cannot obtain arbitrary large cliques by contracting disjoint connected subgraphs of radius at most r in graphs from $\mathscr{C}$. Many well-studied classes of sparse graphs are nowhere dense, for instance the class of planar graphs, any class of graphs with a fixed bound on the maximum degree, or any class of graphs excluding a fixed (topological) minor, are nowhere dense. Furthermore, under certain closure conditions, nowhere denseness constitutes the frontier of parameterized tractability for natural classes of problems. For instance, while the first-order model-checking problem is fixed-parameter tractable on every nowhere dense class $\mathscr{C}$ [11], on every subgraph-closed class $\mathscr{D}$ that is not nowhere dense, it is as hard as on the class of all graphs [5, 8]. Similar lower bounds are known for many individual problems, e.g. for the Distance-r Independent Set problem and the Distance-r Dominating Set problem, on subgraph-closed classes which are not nowhere dense [7, 16].

Towards the goal of extending the border of algorithmic tractability for the above mentioned problems beyond graph classes that are closed under taking subgraphs, we study a very simple and generic algorithmic paradigm that we call *progressive exploration*, described below.

This general idea can be applied to a *parameterized subset problem*, in which we are given a graph G and parameter k, and the goal is to determine if G satisfies a property of the form *"there exists a candidate of size k which agrees with every witness"*. Here, the notions of *candidates, witnesses,* and *agreeing* depend on the problem under consideration. For example, when considering the existence of a distance-r dominating set of size k, candidates are sets S of size k, witnesses are single vertices, and a candidate S *agrees* with every vertex in distance at most r from a vertex in S.

Another way of viewing a problem as above is to consider the bipartite "agreement" graph, whose left part consists of candidates, right part consists of witnesses, and adjacency denotes agreeing. Then the problem is to determine if the right part of this bipartite graph has some common neighbor. Note that the size of the bipartite graph is usually polynomial in terms of the size of the input graph G, where the exponent of the polynomial is the parameter k that we are interested in. In particular, if we are aiming at fixed-parameter tractability, we cannot even afford constructing the entire bipartite graph.

To solve a problem as above, a progressive exploration algorithm proceeds in rounds $i = 1, 2, \ldots$, where each round i finishes with constructing a candidate S_i and a set of witnesses W_i. In round i, we seek a candidate S_i that agrees with every vertex in the union of the previously constructed witness sets $W_1, W_2, \ldots, W_{i-1}$. If no such S_i exists, we can terminate and answer that there is no solution. On the other hand, if the found candidate S_i agrees with every witness in the graph, we also terminate and return it as a solution. Otherwise, we find another set W_i of witnesses, such that each of the candidates $S_1, \ldots, S_{i-1}, S_i$ found so far does not agree with one of the witnesses in W_i, and proceed to the next round. In this way, we progressively explore the whole solution space, while constructing more and more problematic witness sets that the future candidates must agree with, until we either find a solution or enough witnesses to certify that no solution exists.

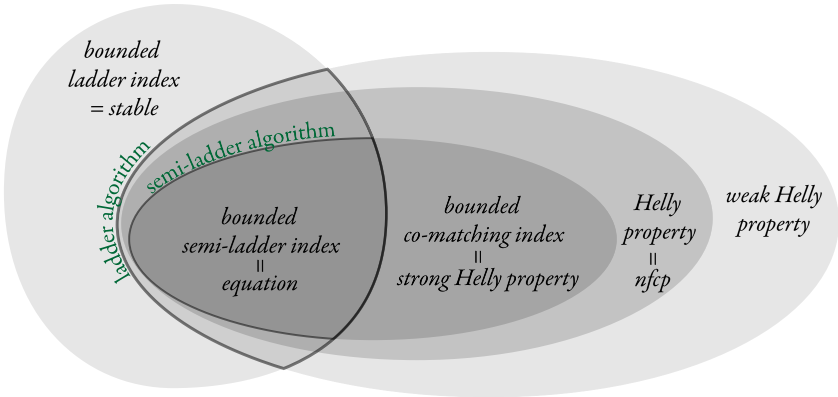

Figure 1 The figure depicts various properties of classes of bipartite graphs, which are introduced in Section 2. Domination- and independence-type problems studied in Section 3 reduce to the problem of determining whether the right part of a given bipartite graph has a common neighbor. In Section 4 two algorithms for the latter problem are devised, and their domains of applicability are marked above. The *ladder algorithm* has a larger domain, but requires a more powerful access oracle and has higher running time. Finally, we apply these algorithms to specific graph classes, yielding new fixed-parameter tractability results for domination- and independence-type problems.

Progressive graph exploration is a generic approach to solving graph problems, which so far rather resembles a wishful-thinking heuristic than a viable algorithmic methodology. Such algorithms can be applied to any input graph, however, a priori there are multiple problem-dependent details to be filled.

First, in order to implement the iteration, we should efficiently compute candidates S_i and small witness sets W_i in every round. Second, to analyze the running time we need to give an upper bound on the number of rounds in which the algorithm terminates. If we can guarantee that each round can be implemented efficiently and that the number of rounds is bounded in the parameter k, we immediately obtain a fixed-parameter algorithm for the problem under consideration.

In this work we study properties of algorithmic problems and graphs that ensure these desired features. Inspired by notions from model theory, we identify combinatorial properties of the arising bipartite graphs which lead to efficient progressive exploration algorithms. The properties that guarantee the existence of small witness sets are variants of the *Helly property*, called *nfcp* in model theory. The property that guarantees that the progressive exploration algorithms stop after a bounded number of rounds is the model-theoretic notion of *stability*. Under these conditions, for problems formulated using short distances in the graph, we are able to implement progressive exploration efficiently, yielding fast and simple fixed-parameter algorithms. See the caption in Figure 1 for an overview of the paper.

We demonstrate our approach by applying it to the Distance-r Dominating Set problem and the Distance-r Independent Set problem on a variety of restricted graph classes. More precisely, we prove that:

- For every $r \in \mathbb{N}$ and graph class $\mathscr{C}$ that is either nowhere dense, or is a power of a nowhere dense class, or is the class of map graphs, the Distance-r Dominating Set problem on any graph $G \in \mathscr{C}$ can be solved in time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$. Here and throughout the paper, $\|G\|$ denotes the total number of vertices and edges in a graph G.
- For every $t \in \mathbb{N}$, the Dominating Set problem on any $K_{t,t}$-free graph G can be solved in time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$; here, a graph is $K_{t,t}$-free if it does not contain the complete bipartite graph $K_{t,t}$ as a subgraph.

- For every $r \in \mathbb{N}$ and nowhere dense graph class $\mathscr{C}$, the Distance-r Independent Set problem on any graph $G \in \mathscr{C}$ can be solved in time $f(k) \cdot \|G\|$, for some function f.

Actually, for the last result, we also give a different algorithm with running time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$, which however does not rely on the concept of progressive exploration and uses some black-boxes from the theory of sparsity.

Our results extend the limits of tractability for Distance-r Dominating Set and Distance-r Independent Set and, in some cases, improve the best known running times. We include a comprehensive comparison with the existing literature at the end of Section 4. However, let us stress here a key point: all our algorithms are derived in a generic way using the idea of progressive exploration, hence they are very easy to implement and they do not use any algorithmic black-boxes that depend on the class from which the input is drawn. In fact, properties of the considered classes are used only when analyzing the running time.

2 Complexity-measures for bipartite graphs

In this section, we define the basic notions used in this paper, related to various complexity measures associated with bipartite graphs. We work with the following notion of bipartite graphs that is not standard in graph theory but suits the model theoretic context very well.

A *bipartite graph* is a triple $G = (L, R, E)$, where L and R are two sets of vertices, called the *left part* and *right part*, respectively, and $E \subseteq L \times R$ is a binary relation whose elements are called *edges*. Hence, bipartite graphs with parts L, R correspond bijectively to binary relations with domain L and codomain R. Note that each bipartite graph has a uniquely determined left and right part. Also, those parts are not necessarily disjoint.

Ladders, semi-ladders, and co-matchings. We now define various complexity measures for bipartite graphs, based on the size of a largest "obstruction" found in a given bipartite graph. There are several types of obstructions, leading to different complexity measures. We start with defining the various types of obstructions. Let $G = (L, R, E)$ be a bipartite graph. Two sequences, $a_1, \ldots, a_n \in L$ and $b_1, \ldots, b_n \in R$, form:

- a *co-matching* of order n in G if we have $(a_i, b_j) \in E \iff i \neq j$, for all $i, j \in \{1, \ldots, n\}$;
- a *ladder* of order n in G if we have $(a_i, b_j) \in E \iff i > j$, for all $i, j \in \{1, \ldots, n\}$; and
- a *semi-ladder* of order n in G if $(a_i, b_j) \in E$ for all $i, j \in \{1, \ldots, n\}$ with $i > j$, and $(a_i, b_i) \notin E$ for all $i \in \{1, \ldots, n\}$.

Note that in case of a semi-ladder we do not impose any condition for $i < j$. Observe that any ladder of order n and any co-matching of order n are also semi-ladders of order n.

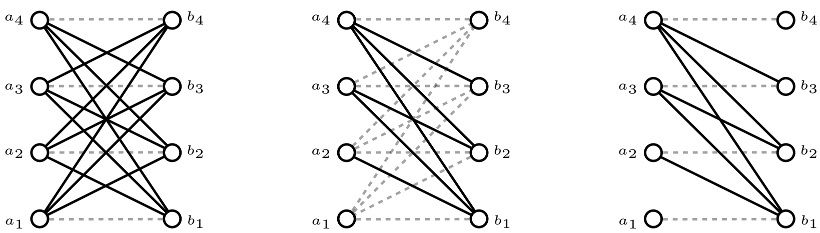

Figure 2 A co-matching, a ladder, and a semi-ladder of order 4, respectively. Dashed gray lines represent non-edges.

The *co-matching index* of a bipartite graph is the maximum order of a co-matching that it contains. A class of bipartite graphs has *bounded* co-matching index if the supremum of the co-matching indices of its members is finite. We define analogous notions for the *ladder index* and the *semi-ladder index*, in the expected way.

In this paper, we will often not care about the precise bounds on the indices of graphs, and it will only matter whether the respective index is bounded in a given class. Bounded ladder and semi-ladder indices correspond notions from model theory (see paragraph on formulas below). We will later relate the property of having a bounded co-matching index to a variant of the *Helly property*. Using a simple Ramsey argument, we now observe that boundedness of the semi-ladder index is equivalent to boundedness of both the co-matching and the ladder index. Let us first state Ramsey's theorem in the form used in this paper.

▶ **Theorem 1** (Ramsey's theorem). *For all $c, \ell \in \mathbb{N}$ there exists a number $R^c(\ell)$ with the following property. If the edges of a complete graph on $R^c(\ell)$ vertices are colored using c colors, then there is a set of ℓ vertices which is* monochromatic*, that is, all edges with both endpoints in this set are of the same color.*

The standard proof of Ramsey's theorem yields an upper bound of $R^c(\ell) \leqslant c^{c\ell-1}$ for $c \geqslant 2$. From now on, we adopt the notation $R^c(\ell)$ for the multicolored Ramsey numbers as described in Theorem 1. The proofs of all statements marked with ♠ can be found in the full version of the paper [10].

▶ **Lemma 2** (♠). *A class of bipartite graphs has finite semi-ladder index if and only if both its ladder index and its co-matching index are finite.*

Helly property and its variants. Let $p \in \mathbb{N}$ and let $G = (L, R, E)$ be a bipartite graph. We say that a subset $B \subseteq R$ is *covered* by a subset $A \subseteq L$ if there exists a vertex $a \in A$ which is adjacent to all the vertices of B. Then subsets A and B have the *p-Helly property* if either B is covered by A, or B contains a subset of size at most p that is not covered by A. We shall say that G has:

- the *weak p-Helly property* if L and R have the p-Helly property;
- the *p-Helly property* if L and B have the p-Helly property, for all $B \subseteq R$; and
- the *strong p-Helly property* if all $A \subseteq L$ and $B \subseteq R$ have the p-Helly property.

We say that a class $\mathscr{C}$ of bipartite graphs has the (weak/strong) Helly property if there is some $p \in \mathbb{N}$ such that all graphs in $\mathscr{C}$ have the (weak/strong) p-Helly property.

It turns out that the strong p-Helly property corresponds precisely to having co-matching index at most p.

▶ **Lemma 3** (♠). *Let $p \in \mathbb{N}$ and G be a bipartite graph. Then G has the strong p-Helly property if and only if it has co-matching index at most p.*

In the following paragraphs we will see specific examples of classes of bipartite graphs satisfying variants of the (weak/strong) Helly property.

Bipartite graphs defined by formulas. We construct bipartite graphs using logical formulas. In principle, we could consider formulas of any logic, but in this paper we only consider first-order logic in the vocabulary of graphs, i.e., using the binary relation symbol E representing edges, the binary relation symbol $=$ representing equality, and logical constructs $\vee, \wedge, \neg, \rightarrow, \forall, \exists$. E.g, the property $\mathrm{dist}(x, y) \leqslant 5$, expressing that x and y are at distance at most 5 in a graph G, can be expressed by a first-order formula using four existential quantifiers.

We write $\bar{x}$ to represent a tuple of variables (with every variable appearing only once). If V is a set, then $V^{\bar{x}}$ denotes the set of all assignments mapping variables in $\bar{x}$ to V. Let $\varphi(\bar{x}; \bar{y})$ be a formula with free variables partitioned into two disjoint tuples, $\bar{x}$ and $\bar{y}$. Given any graph G with vertex set V, the formula φ induces a bipartite graph $\varphi(G)$ with left part $V^{\bar{x}}$, right part $V^{\bar{y}}$, and where $\bar{a} \in V^{\bar{x}}$ and $\bar{b} \in V^{\bar{y}}$ are adjacent if and only if $\varphi(\bar{a}; \bar{b})$ holds in G. If $\mathscr{C}$ is a class of graphs and $\varphi(\bar{x}; \bar{y})$ is a formula, then by $\varphi(\mathscr{C})$ we denote the class of all bipartite graphs $\varphi(G)$, for $G \in \mathscr{C}$. We say that φ has *bounded ladder index* on a class $\mathscr{C}$ if the class $\varphi(\mathscr{C})$ has bounded ladder index; similarly for the co-matching and the semi-ladder index. The same definitions apply if instead of graphs we consider logical structures over some fixed signature, and $\varphi(\bar{x}; \bar{y})$ is a formula over that signature. For simplicity, we consider only graphs in this paper.

We note that the various indices are preserved by adding spurious free variables to formulas. Precisely, let $\varphi(\bar{x}; \bar{y})$ be a first-order formula and let $\varphi'(\bar{x}'; \bar{y}')$ be the same formula, but having extra free variables, i.e., $\bar{x}$ is a subtuple of $\bar{x}'$ and $\bar{y}$ is a subtuple of $\bar{y}'$. Then, for any graph G, the ladder index of $\varphi(G)$ is equal to the ladder index of $\varphi'(G)$, although the two bipartite graphs $\varphi(G)$ and $\varphi'(G)$ may differ. The same holds for all the other properties studied in this paper: co-matching index, semi-ladder index, (weak/strong) Helly property.

The next lemma shows that a positive boolean combination (a boolean combination in which no negations are used) of formulas with bounded semi-ladder index also has bounded semi-ladder index. The proof uses a Ramsey argument.

► **Lemma 4** (♠)**.** *Let $\varphi_1(\bar{x}; \bar{y}), \ldots, \varphi_k(\bar{x}; \bar{y})$ be formulas and let $\psi(\bar{x}; \bar{y})$ be a positive boolean combination of $\varphi_1, \ldots, \varphi_k$. Suppose G is a graph such that $\varphi_1(G), \ldots, \varphi_k(G)$ have semi-ladder index smaller than ℓ. Then $\psi(G)$ has semi-ladder index smaller than $R^k(\ell)$.*

We remark that the property of having bounded ladder index is preserved by taking arbitrary boolean combinations, not just positive ones. Finally, the analogue of Lemma 4 fails for the co-matching index if positive boolean combinations are considered, but still holds if we restrict attention to conjunctions of atomic formulas.

We will later need the following variant of Lemma 4, which provides a sharper bound for formulas of a special form. As usual in this work, the proof relies on a Ramsey-like argument.

► **Lemma 5** (♠)**.** *Let $\psi(\bar{x}; \bar{y}) = \bigvee_{j=1}^{k} \varphi(\bar{x}^j; \bar{y})$ for some $k \geqslant 2$, where $\varphi(\bar{x}; \bar{y})$ is a formula and $\bar{x}^1, \ldots, \bar{x}^k$ are permutations of $\bar{x}$. Suppose G is a graph such that $\varphi(G)$ has semi-ladder index smaller than ℓ. Then $\psi(G)$ has semi-ladder index smaller than $k^{\ell-1}$.*

Stability theory. Many of the combinatorial properties of bipartite graphs introduced above correspond to properties of formulas studied in model theory, specifically, in its modern branch called *stability theory*. Very roughly, stability theory studies how various combinatorial obstructions affect the logical complexity of the considered first-order theory. Stability theory will not be used in this paper, but for bibliographic completeness, we present a dictionary relating the notions introduced above to the notions studied there. Fix a class of structures $\mathscr{C}$; in model theory; usually $\mathscr{C}$ is the class of all models of a fixed first-order theory. A first-order formula $\varphi(\bar{x}; \bar{y})$ is called *stable* (with respect to $\mathscr{C}$) if $\varphi(\mathscr{C})$ has bounded ladder index [21, Chapter I.2]. It is called *nfcp* if $\varphi(\mathscr{C})$ has the Helly property [21, Chapter II.4]. It is called an *equation* if $\varphi(\mathscr{C})$ has bounded semi-ladder index [19, 15].

We also remark that the properties of bipartite graphs that we consider can be viewed as properties of set systems: a bipartite graph $G = (L, R, E)$ gives rise to a family $\mathcal{F}$ of subsets of R, consisting of neighborhoods of elements of L. The p-Helly property is usually formulated for set systems $\mathcal{F}$, requiring that every minimal subfamily of $\mathcal{F}$ with an empty intersection has at most p sets in it. The semi-ladder index corresponds to the maximal length of an inclusion-chain in the family of intersections of the sets in $\mathcal{F}$.

Stability in nowhere dense graph classes. The classes of bipartite graphs with bounded ladder index that are relevant to this paper are provided by the following result, due to Podewski and Ziegler [20].

▶ **Theorem 6** ([20], cf. [1, 18]). *Let $\mathscr{C}$ be a nowhere dense class of graphs and let $\varphi(\bar{x};\bar{y})$ be a first-order formula. Then φ has bounded ladder index on $\mathscr{C}$.*

The above result was originally stated for superflat graphs, and using the notion of stability. The proof relied on non-constructive model-theoretic methods, such as the compactness theorem. The connection with nowhere denseness was observed by Adler and Adler [1], and a proof providing explicit bounds was given by the last three authors in [18]. The observation of Adler and Adler is the starting point of this work, as it brings to light the connection between model theory and computer science which is further studied in this paper.

3 Domination and independence problems

We consider subset problems, where in a given graph we look for a solution S of size k that satisfies some property, whose dissatisfaction can be witnessed by a small subset of vertices W. Moreover, checking a candidate solution S against a witness W can be expressed in first-order logic. Thus, a problem of interest can be expressed by a sentence of the form $\exists_{\bar{x}}\forall_{\bar{y}}\varphi(\bar{x};\bar{y})$, for a suitable formula $\varphi(\bar{x};\bar{y})$, where $\bar{x}$ is a tuple of k variables that represent a candidate S, while $\bar{y}$ is a tuple of ℓ variables that represent a witness W.

▶ **Example 7.** The Distance-r Dominating Set problem for parameter k can be expressed as above using the formula $\delta_r^k(\bar{x};y) = \bigvee_{i=1}^{k} \delta_r(x_i,y)$, where $\delta_r(x,y)$ is a formula that checks whether $\mathrm{dist}(x,y) \leqslant r$, and $\bar{x} = (x_1,\ldots,x_k)$. Similarly, the Distance-r Independent Set problem for parameter k can be expressed using the formula $\eta_r^k(\bar{x};y) = \bigwedge_{1\leqslant i<j\leqslant k} \eta_r(x_i,x_j,y)$, where $\eta_r(x,x',y)$ is a formula that checks whether $\mathrm{dist}(x,y)+\mathrm{dist}(x',y) > r$.

Observe that a graph G satisfies the sentence $\exists_{\bar{x}}\forall_{\bar{y}}\varphi(\bar{x};\bar{y})$ if and only if the right part of the bipartite graph $H = \varphi(G)$ is covered by the left side, i.e., all vertices in the right part (witnesses) have a common neighbor (solution). We call this abstract problem – checking whether the right part of a given bipartite graph H is covered by the left side – the Coverage problem.

Note that the size of the bipartite graph $\varphi(G)$ is polynomial in the size of G, where the exponent depends on the number of free variables in φ, which is usually the parameter we are interested in. As we are aiming at fixed-parameter algorithms, we cannot afford to even construct the whole bipartite graph $\varphi(G)$. Therefore, we will design algorithms that solve the Coverage problem using an oracle access to the bipartite graph graph $H = \varphi(G)$, where the oracle calls will be implemented using subroutines on the original graph G. The running time of these algorithms, expressed in terms of the number of oracle calls, will be bounded only in terms of quantities (ladder indices, numbers governing Helly property, etc.) related to the class of bipartite graphs $\varphi(\mathscr{C})$, where $\mathscr{C}$ is the considered class of input graphs.

Therefore, to obtain an algorithm for solving the initial problem on a given graph class $\mathscr{C}$ we proceed in two steps:

- Prove that the class $\varphi(\mathscr{C})$ has a suitable Helly-type property and bounded ladder index.
- Design an algorithm for Coverage, for input bipartite graphs with suitable Helly-type properties and bounded ladder index, that uses only a bounded number of oracle calls.

In Section 4 we give two such algorithms solving Coverage: the *Semi-ladder Algorithm*, and the *Ladder Algorithm*. The Semi-ladder Algorithm requires that H has bounded semi-ladder index, whereas the Ladder Algorithm requires that H has bounded ladder index and the weak p-Helly property, for some fixed p. Note that by Lemmas 2 and 3, boundedness of the semi-ladder index is equivalent to boundedness of the ladder index and having the strong p-Helly property, for some fixed p, so the prerequisites for the Semi-ladder Algorithm are stronger than for the Ladder Algorithm. See Figure 1 for an overview.

We postpone the discussion of the algorithms to Section 4, and for now we focus on exhibiting the suitable properties for various classes of bipartite graphs. Slightly more precisely, we prove that on certain graph classes, formulas corresponding to domination-type problems have bounded semi-ladder index, while those corresponding to independence-type problems have the weak Helly property and bounded ladder index. Hence, in the first case we will apply the Semi-ladder Algorithm, and in the second – the Ladder Algorithm.

Distance formulas and domination-type problems. We shall prove fixed-parameter tractability results not only for distance-r domination, but for a more general class of domination-type problems. Those can be expressed by suitable formulas, as explained next.

▶ **Definition 8.** *For $r \in \mathbb{N}$, let $\delta_r(x, y)$ be the formula checking whether dist$(x, y) \leqslant r$. A* distance formula *is a formula $\varphi(\bar{x}; \bar{y})$ which is a boolean combination of* atoms *of the form $\delta_r(x, y)$, where the variable x occurs in $\bar{x}$, the variable y occurs in $\bar{y}$, and $r \in \mathbb{N}$ is any number. The* radius *of a distance formula is the maximal number r occurring in its atoms, whereas its* size *is the number of atoms occurring in it. A distance formula is* positive *if it is a positive boolean combination of atoms.*

A domination-type property *is a sentence ψ of the form $\exists_{\bar{x}}\forall_{\bar{y}}\, \varphi(\bar{x}; \bar{y})$, where φ is a positive distance formula. A* domination-type problem *is the computational problem of determining whether a given graph G satisfies a given domination-type property.*

▶ **Example 9.** Fix $r \in \mathbb{N}$ and let $\bar{x} = (x_1, \ldots, x_k)$ be a k-tuple of variable. Then the formula $\delta_r^k(\bar{x}; y)$ considered in Example 7 is a positive distance formula, hence the problem defined by the domination-type property $\exists_{\bar{x}}\forall_{\bar{y}}\, \delta_r^k(\bar{x}; y)$ (aka Distance-r Dominating Set) is a domination-type problem. Similarly, formulas $\varphi(\bar{x}; y)$ expressing the following properties give raise to natural domination-type problems:

- y is at distance at most r from at least two of the vertices $x_1, \ldots, x_k$; and
- the sum $\mathrm{dist}(x_1, y) + \mathrm{dist}(x_2, y) + \ldots + \mathrm{dist}(x_k, y)$ is at most r.

On the other hand, the formula $\eta_r^k(\bar{x}; y)$ considered in Example 7 is a distance formula, but it is not positive, and hence it does not yield a domination-type property.

From Lemmas 4 and 5 and the remark about spurious variables not affecting the semi-ladder index, we immediately obtain the following.

▶ **Corollary 10.** *Let $\varphi(\bar{x}; \bar{y})$ be a positive distance formula of radius r and size s. If G is a graph such that the semi-ladder index of $\delta_q(G)$ is smaller than ℓ for all $q \leqslant r$, then the semi-ladder index of $\varphi(G)$ is smaller than $R^s(\ell)$. Moreover, if $\varphi = \delta_r^k$ as defined in Example 7 and $k \geqslant 2$, then the semi-ladder index of $\varphi(G)$ is smaller than $k^{\ell-1}$.*

Domination problems. We first consider domination-type problems and prove that they have bounded semi-ladder indices on any nowhere dense class. This result can actually be extended beyond nowhere denseness: to powers of nowhere dense classes, to map graphs, and to $K_{t,t}$-free graphs for radius $r = 1$. We define the former two concepts next.

For a graph G and $s \in \mathbb{N}$, let G^s denote the graph with the same vertex set as G, where two vertices are adjacent if and only if their distance in G is at most s. If $\mathscr{D}$ is a graph class, then $\mathscr{D}^s$ denotes the class $\{G^s : G \in \mathscr{D}\}$. Note that a power of a nowhere dense class is not necessarily nowhere dense, e.g., the square of the class of stars is the class of complete graphs.

A graph G is a map graph if one can assign to each vertex of G a closed, arc-connected region in the plane so that the interiors of regions are pairwise disjoint and two vertices of G are adjacent if and only if their regions share at least one point on their boundaries. Note that map graphs are not necessarily planar and may contain arbitrarily large cliques, as four or more regions may share a single point on their boundaries.

The following result will be used in the next section to obtain fixed-parameter tractability of domination-type problems over graph classes described above.

▶ **Theorem 11** (♠). *For any $r \in \mathbb{N}$ and nowhere dense graph class $\mathscr{C}$, the formula $\delta_r(x, y)$ has bounded semi-ladder index on $\mathscr{C}$. The same holds also when $\mathscr{C} = \mathscr{D}^s$ for some nowhere dense class $\mathscr{D}$ and $s \in \mathbb{N}$, when $\mathscr{C}$ is the class of map graphs, and when $r = 1$ and $\mathscr{C}$ is the class of $K_{t,t}$-free graphs, for any fixed $t \in \mathbb{N}$.*

For the case when $\mathscr{C}$ is nowhere dense we utilize the well-known characterization of nowhere denseness via *uniform quasi-wideness* [13], which we recall below.

▶ **Definition 12.** *We say that a graph class $\mathscr{C}$ is* uniformly quasi-wide *if for all $r \in \mathbb{N}$, there are $s_r \in \mathbb{N}$ and $N_r : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, every graph $G \in \mathscr{C}$, and every vertex subset $W \subseteq V(G)$ of size larger than $N_r(k)$, there exist disjoint vertex subsets $S \subseteq V(G)$ and $A \subseteq W$ such that $|S| \leqslant s_r$, $|A| > k$, and A is distance-r independent in $G - S$.*

▶ **Theorem 13** ([12, 14]). *A graph class $\mathscr{C}$ is nowhere dense if and only if it is uniformly quasi-wide.*

The nowhere dense case of Theorem 11 is therefore encapsulated in the following lemma.

▶ **Lemma 14** (♠). *For every $r \in \mathbb{N}$ and uniformily quasi-wide class $\mathscr{C}$, the class $\delta_r(\mathscr{C})$ has bounded semi-ladder index.*

We now give a very rough sketch the proof of Lemma 14. If for some $G \in \mathscr{C}$ the graph $\delta_r(G)$ has semi-ladder index ℓ, then in G we have vertices $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ such that $\mathrm{dist}(a_i, b_j) \leqslant r$ for all $i > j$ and $\mathrm{dist}(a_i, b_i) > r$ for all i. Then provided ℓ is huge, by uniform quasi-wideness, we can find a large subset $A \subseteq \{a_1, \ldots, a_\ell\}$ of vertices that "communicate" with each other only through a set S of constant size – all paths of length at most $2r$ between vertices of A pass through S. Now the vertices from A have pairwise different distance-r neighborhoods within $\{b_1, \ldots, b_\ell\}$, but only a limited number of possible interactions with S (measured up to distance r). This quickly leads to a contradiction if A is large enough. The cases when $\mathscr{C}$ is a power of a nowhere dense class and when $\mathscr{C}$ is the class of map graphs follow as simple corollaries from the result for nowhere dense classes. The case when $\mathscr{C}$ is the class of $K_{t,t}$-free graphs is a simple observation: a large semi-ladder in $\delta_1(G)$ enforces a large biclique in G.

We remark that the above argument is similar to the reasoning that shows that graphs from a fixed nowhere dense class admit small *distance-r domination cores*: subsets of vertices whose distance-r domination forces distance-r domination of the whole graph. This property was first proved implicitly by Dawar and Kreutzer in their FPT algorithm for Distance-r Dominating Set on any nowhere dense class [4], also using uniform quasi-wideness. We refer the reader to [17, Chapter 3, Section 5] for an explicit exposition.

Having established boundedness of the semi-ladder index of $\delta_r(x,y)$ on a class $\mathscr{C}$, we can use Corollary 10 to extend this to any positive distance formula. Therefore, by Theorem 11, Corollary 10, and Lemma 3 we immediately obtain the following.

▶ **Corollary 15.** *Let $\mathscr{C}$ and r be as in Theorem 11 and let $\varphi(\bar{x};\bar{y})$ be a positive distance formula of radius at most r. Then the class $\varphi(\mathscr{C})$ has bounded semi-ladder index, so in particular it has the strong Helly property.*

In fact, Corollary 10 provides a better control of the semi-ladder index of $\varphi(\mathscr{C})$ in terms of the semi-ladder index of $\delta_r(\mathscr{C})$ and the size of φ. In the next section we will use these more refined bounds for a precise analysis of the running times.

Note that Corollary 15 does not generalize to arbitrary first-order formulas. Indeed, if $\mathscr{C}$ is the class of all edgeless graphs and $\varphi(x;y)$ is the formula $x \neq y$, then $\varphi(\mathscr{C})$ is the class of all complements of matchings, which does not even have the weak Helly property.

Independence problems. We now move to the Distance-r Independent Set problem: deciding whether a given graph contains a distance-r independent set of size k. This property is most naturally expressed using an existential sentence, and not as a sentence of the form $\exists_{\bar{x}}\forall_{\bar{y}}\varphi(\bar{x};\bar{y})$. However, in Example 7 we gave a suitable formula $\eta_r^k(\bar{x};y)$ that expresses the problem: the trick is to phrase the property that $x_1,\ldots,x_k$ are pairwise at distance more than r by saying that for every vertex y, for all $1 \leqslant i < j \leqslant k$ the sum of distances from y to x_i and x_j is larger than r. Thus, a vertex y that does not satisfy this condition may serve as a witness that a given tuple $\bar{x}$ does not form a distance-r independent set.

In the full version of the paper we prove the following.

▶ **Theorem 16 (♠).** *Let $\mathscr{C}$ be a nowhere dense class of graphs and let $k,r \in \mathbb{N}$. Then the class $\eta_r^k(\mathscr{C})$ has the weak p-Helly property, for some $p \in \mathbb{N}$ depending on k, r, and $\mathscr{C}$.*

It is easy to see that for any $k \geqslant 2$ and $r \geqslant 1$, the formula $\eta_r^k(x;y)$ does not have the strong Helly property on the class $\mathscr{C}$ of edgeless graphs. Thus, in general we cannot hope for boundedness of the semi-ladder index of $\eta_r^k(\mathscr{C})$ and use the Semi-Ladder Algorithm.

The proof of Theorem 16 is actually very different from the proof of Theorem 11, and presents a novel contribution of this work. Instead of uniform quasi-wideness, we use the characterization of nowhere denseness via the *Splitter game* [11]. The idea is that in case a graph $G \in \mathscr{C}$ does not have a distance-r independent set of size k, there is a small witness of this: a set W of size bounded in terms of k, r, and $\mathscr{C}$ such that for every vertex subset S of size k, some path of length at most r connecting two vertices of S crosses W. This exactly corresponds to the notion of witnessing expressed by η_r^k. Such a witness W is constructed recursively along Splitter's strategy tree in the Splitter game in G. We use the condition that G does not have a distance-r independent set of size k to prove that we can find a small (in terms of $k, r, \mathscr{C}$) set of "representative" moves of the Connector. Trimming the strategy tree to those moves bounds its size in terms of $k, r, \mathscr{C}$, yielding the desired upper bound on the witness size.

We remark that our proof of Theorem 16 can actually be turned into an algorithm for the Distance-r Independent Set problem on any nowhere dense class $\mathscr{C}$ with running time of $2^{\mathcal{O}(k\log k)} \cdot \|G\|$. However, this algorithm is much more complicated than the Ladder algorithm that we explain in the next section, and in particular it uses some black-box results from the theory of nowhere dense graph classes.

4 Algorithms

In this section, we present two algorithms solving the Coverage problem on a given bipartite graph. The bipartite graph can be only accessed via restricted access oracles; we start with presenting this model. We then describe the two algorithms. Then we show how the oracles can be efficiently implemented in the special cases relevant to this paper. This, in combination with the results from the previous section, will allow us to obtain the desired algorithmic statements about domination and independence problems on restricted graph classes.

The access oracles. We consider the following model of an algorithmic search for a solution in a bipartite graph representing the search space. Consider a bipartite graph $H = (L, R, E)$, where the left side L is the set of *candidates* and the right side R is the set of *witnesses*. An edge between a candidate $a \in L$ and a witness $b \in R$ is interpreted as that a and b *agree*: b agrees that a is a solution. Expressed in those terms, Coverage is the problem of finding a *solution*: a candidate which agrees with all witnesses. We will use the terminology of candidates, witnesses, solutions, and agreeing as explained above, as this facilitates the understanding of the algorithms for Coverage in terms of the original problems.

As we explained, the considered bipartite graph H will typically be of the form $\varphi(G)$ for some formula $\varphi(\bar{x}; \bar{y})$ expressing the considered problem. Thus, H shall represent the whole search space, so we allow our algorithms a restricted access to H via the following *oracles*.

Candidate Oracle: Given a set of witnesses $B \subseteq R$, the oracle either returns a candidate $a \in L$ that agrees with all witnesses of B, or concludes that no such candidate exists.

Weak Witness Oracle: Given a candidate $a \in L$, the oracle either concludes that a is a solution, or returns a witness $b \in R$ that does not agree with a.

Strong Witness Oracle: Given a set of candidates $A \subseteq L$ and a number $p \in \mathbb{N}$, the oracle either finds a set of witnesses $P \subseteq R$ such that $|P| \leqslant p$ and every candidate of A does not agree with some witness from P, or concludes that no such set P exists.

Note that the Weak Witness Oracle can be simulated by the Strong Witness Oracle applied to $A = \{a\}$. We now provide the two algorithms for Coverage announced in Section 3.

Semi-ladder Algorithm. The Semi-ladder Algorithm proceeds in a number of rounds, where each round consists of two steps: first the *Candidate Step*, and then the *Witness Step*. Also, the algorithm maintains a set B of witnesses gathered so far, initially set to be empty. The steps are defined as follows:

Candidate Step: Apply the Candidate Oracle to find a candidate $a \in L$ that agrees with all the witnesses in B. If no such candidate exists, terminate the algorithm returning that no solution exists. Otherwise, proceed to the Witness Step.

Witness Step: Apply the Weak Witness Oracle to find a witness $b \in R$ that does not agree with a. If there is no such witness, terminate the algorithm and return a as the solution. Otherwise, add b to B and proceed to the next round.

The correctness of the algorithm is obvious, while the running time can be bounded by the immediate observation that if the Semi-ladder Algorithm performs ℓ full rounds, then the candidates $a_1, \ldots, a_\ell \in L$ discovered in consecutive rounds, together with the witnesses $b_1, \ldots, b_\ell \in R$ added to B in consecutive rounds, form a semi-ladder in H.

▶ **Corollary 17.** *The Semi-ladder Algorithm applied to a graph H with semi-ladder index ℓ terminates after performing at most ℓ full rounds. Consequently, it uses at most $\ell + 1$ Candidate Oracle Calls, each involving a set of witnesses B with $|B| \leqslant \ell$, and at most ℓ Weak Witness Oracle Calls.*

Ladder algorithm. As before, the Ladder Algorithm maintains the set B of witnesses gathered so far, but also the set A of candidates found so far. The algorithm is also given a parameter $p \in \mathbb{N}$. Again, the algorithm proceeds in rounds, each consisting of the Candidate step and the Witness step, with the following description:

Candidate Step: Apply the Candidate Oracle to find a candidate $a \in L$ that agrees with all the witnesses in B. If no such candidate exists, terminate the algorithm returning that no solution exists. Otherwise, add a to A and proceed to the Witness step.

Witness Step: Apply the Strong Witness Oracle to set A and parameter p, yielding either a set of witnesses $P \subseteq R$ such that $|P| \leqslant p$ and every candidate from A does not agree with some witness from P, or a conclusion that no such set P exists. In the former case, add P to B and proceed to the next round. In the latter case, terminate the algorithm returning that a solution exists.

Note that the algorithm actually never finds a solution, but only may claim its existence in the Witness Step, and this claim is not substantiated by having a concrete solution in hand. However, the observation is that assuming the weak p-Helly property, the structure discovered by the algorithm is sufficient to deduce the existence of a solution.

▶ **Lemma 18 (♠).** *The Ladder Algorithm applied with parameter p in a bipartite graph with the weak p-Helly property is always correct.*

Finally, we show that if H has ladder index bounded by ℓ, then the algorithm terminates in a number of rounds bounded in terms of ℓ and p. For this we observe that during its execution, the algorithm in fact constructs a ladder in an auxiliary bipartite graph H' with candidates a on the left side and sets of witnesses P on the right side, and the ladder index of H' can be bounded in terms of p and the ladder index of H using a Ramsey argument.

▶ **Lemma 19 (♠).** *The Ladder Algorithm applied with parameter p to a bipartite graph H with ladder index smaller than ℓ terminates after performing less than $R^p(2\ell)$ full rounds.*

▶ **Corollary 20.** *The Ladder Algorithm applied with parameter p to a graph H with ladder index smaller than ℓ and the weak p-Helly property, always returns the correct answer and terminates after performing at most $q = R^p(\ell) - 1$ full rounds. Consequently, it uses at most $q + 1$ Candidate Oracle Calls, each involving a set of witnesses B with $|B| \leqslant pq$, and at most q Strong Witness Oracle Calls, each involving a set of candidates A with $|A| \leqslant q$.*

Implementing the oracles. The last missing ingredient for obtaining our algorithmic results is an efficient implementation of the oracles for bipartite graphs of the form $\varphi(G)$, where G is the input graph and $\varphi(\bar{x}, \bar{y})$ is a formula expressing the considered problem. We describe such an implementation whenever φ is a distance formula.

We use the concept of *distance profiles* and *distance profile complexity*. Let G be a graph and let S be a set of its vertices. For a vertex v of G, the *distance-r profile* of v on S, denoted $\mathrm{profile}_r^{G,S}(v)$, is the function mapping S to $\{0, 1, \ldots, r, \infty\}$ such that for $s \in S$,

$$\mathrm{profile}_r^{G,S}(v)(s) = \begin{cases} \mathrm{dist}_G(v,s) & \text{if } \mathrm{dist}_G(v,s) \leqslant r, \\ \infty & \text{otherwise.} \end{cases}$$

The *distance-r profile complexity* of G is the function from $\mathbb{N}$ to $\mathbb{N}$ defined as

$$\nu_r^G(m) = \max_{S \subseteq V,\, |S| \leqslant m} |\{\mathrm{profile}_r^{G,S}(v) \colon v \in V(G)\}|.$$

That is, this is the maximum possible number of different functions from S to $\{0, 1, \ldots, r, \infty\}$ realized as distance-r profiles on S of vertices of G, over all vertex subsets S of size at most m. For a graph class $\mathscr{C}$, we denote $\nu_r^{\mathscr{C}}(m) = \sup_{G \in \mathscr{C}} \nu_r^G(m)$.

Note that for any graph G and $r, m \in \mathbb{N}$ we have $\nu_r^G(m) \leqslant (r+2)^m$, as this is the total number of functions from a set of size m to $\{0, 1, \ldots, r, \infty\}$. This bound is exponential in m, however it is known that on nowhere dense classes an almost linear bound holds.

▶ **Lemma 21** ([9]). *Let $\mathscr{C}$ be a nowhere dense class of graphs. Then for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a constant $c_{r,\varepsilon}$ such that $\nu_r^{\mathscr{C}}(m) \leqslant c_{r,\varepsilon} \cdot m^{1+\varepsilon}$ for all $m \in \mathbb{N}$.*

We remark that the conclusion of Lemma 21 still holds when $\mathscr{C}$ is any fixed power of a nowhere dense class, and when $\mathscr{C}$ is the class of map graphs. Moreover, when $\mathscr{C}$ is the class of $K_{t,t}$-free graphs for some $t \in \mathbb{N}$, then $\nu_1^{\mathscr{C}}(m) \leqslant \mathcal{O}(m^t)$.

We are ready to give implementations for the oracles. The main idea is that because we are working with a distance formula, when looking for, say, a candidate that agrees with all witnesses in a set B, the only information that is relevant about any vertex is its distance-r profile on the set S consisting of all vertices appearing in the tuples of B. Hence, there are only $\nu_r^G(|S|)$ different "types" of vertices, and instead of checking all k-tuples of vertices in the graph, we can check all k-tuples of types.

▶ **Lemma 22** (♠). *Fix a distance formula $\varphi(\bar{x}; \bar{y})$ of radius r and with $|\bar{x}| = c$ and $|\bar{y}| = d$. Then for an input graph $G = (V, E)$, there are implementations of oracle calls in $\varphi(G)$ that achieve the following running times:*

- *Candidate Oracle: time $\mathcal{O}(|B| \cdot \|G\| + |B| \cdot \nu_r^G(d|B|)^c)$ for a call to $B \subseteq V^{\bar{y}}$;*
- *Weak Witness Oracle: time $\mathcal{O}(\|G\| + \nu_r^G(c)^d)$ for a call to $\bar{a} \in V^{\bar{x}}$;*
- *Strong Witness Oracle: time $\mathcal{O}(|A| \cdot \|G\| + |A| \cdot \nu_r^G(c|A|)^{pd})$ for a call to $A \subseteq V^{\bar{x}}$ and $p \in \mathbb{N}$.*

Algorithmic consequences. We are ready to present our algorithmic corollaries, promised in Section 1. Throughout this section, when stating parameterized running times we use k to denote the target size of a solution (i.e., distance-r dominating or independent set). We start with the domination problems.

▶ **Theorem 23.** *Fix $r \in \mathbb{N}$ and let $\mathscr{C}$ be a class of graphs such that for each $q \leqslant r$, the class $\delta_q(\mathscr{C})$ has finite semi-ladder index. Then, for any positive distance formula $\varphi(\bar{x}; \bar{y})$ of radius at most r and size k, the domination-type problem corresponding to φ can be solved on $\mathscr{C}$ in time $f(k) \cdot \|G\|$, for some function f.*

Proof. W.l.o.g. we can assume that $|\bar{x}|, |\bar{y}| \leqslant k$. Let $\ell \in \mathbb{N}$ be such that $\delta_q(\mathscr{C})$ has semi-ladder index smaller than ℓ, for all $q \leqslant r$. Given a graph G, we apply the Semi-Ladder Algorithm for the Coverage problem in the graph $\varphi(G)$ with implementations of oracles provided by Lemma 22. By Lemma 4 we conclude the semi-ladder index of $\varphi(\mathscr{C})$ is bounded by $R^k(\ell)$. Now the claimed running time follows immediately from Corollary 17 and Lemma 22. ◀

▶ Remark 24. By Corollary 17 and Lemma 22, the running time is actually $\mathcal{O}(p \cdot \nu_r^{\mathscr{C}}(p)^k \cdot \|G\|)$, where p is the semi-ladder index of $\varphi(G)$. By Lemma 4, we have that $p \leqslant R^k(\ell)$, which is upper-bounded by $k^{k\ell-1}$ for $k \geqslant 2$. Combining this with the trivial upper bound $\nu_r^{\mathscr{C}}(p) \leqslant (r+2)^p$ yields $f(k) \leqslant 2^{2^{\mathcal{O}(k \log k)}}$, where r and ℓ are considered fixed constants. However, if a priori we know for the graph class $\mathscr{C}$ that $\nu_r^{\mathscr{C}}(m)$ is polynomial in m, instead of exponential, then by the analysis above we obtain $f(k) \leqslant 2^{\mathcal{O}(k^2 \log k)}$. Finally, by Lemma 5, for $\varphi = \delta_r^k$ – the formula corresponding to the Distance-r Dominating Set problem – we can use a sharper bound of $p \leqslant k^{\ell-1}$. Thus, for this case we obtain an upper bound of $f(k) \leqslant 2^{\mathrm{poly}(k)}$ in the general setting, and $f(k) \leqslant 2^{\mathcal{O}(k \log k)}$ when $\nu_r^{\mathscr{C}}(m)$ is polynomial in m.

Now, using Theorem 23 together with combinatorial results stated in Section 2 we immediately obtain the algorithmic results promised in Section 1. Note that the results hold not only for Distance-r Dominating Set, but even for every domination-type problem of fixed radius r and size k that is considered the parameter.

▶ **Theorem 25.** *Fix $r \in \mathbb{N}$. Then any domination-type problem defined by a positive distance formula of size k and radius at most r can be solved in time $2^{\mathcal{O}(k^2 \log k)} \cdot \|G\|$ on any graph class $\mathscr{C}$ such that either $\mathscr{C}$ is nowhere dense, or $\mathscr{C} = \mathscr{D}^s$ for a nowhere dense class $\mathscr{D}$ and some $s \in \mathbb{N}$, or $\mathscr{C}$ is the class of map graphs, or $r = 1$ and $\mathscr{C}$ is the class of $K_{t,t}$-free graphs for some fixed $t \in \mathbb{N}$. Moreover, if this domination-type problem is* Distance-r Dominating Set *for parameter k, then the running time can be improved to $2^{\mathcal{O}(k \log k)} \cdot \|G\|$.*

Proof. By Theorem 11, the class $\delta_r(\mathscr{C})$ has finite semi-ladder index. By Lemma 21 and its strengthenings (see the comment below Lemma 21) $\nu_r^{\mathscr{C}}(m)$ is bounded by a polynomial in m. Hence, we may apply Theorem 23; the claimed running times follow from the remark following it. ◀

We now move to the independence problems, for which we apply the Ladder algorithm.

▶ **Theorem 26.** *Let $r \in \mathbb{N}$ and let $\mathscr{C}$ be a class of graphs such that for any $k \in \mathbb{N}$, the class $\eta_r^k(\mathscr{C})$ has ladder index smaller than $\ell(k)$ and has the weak $p(k)$-Helly property, for some functions $\ell, p \colon \mathbb{N} \to \mathbb{N}$. Then the* Distance-$r$ Independent Set *problem on $\mathscr{C}$ can be solved in time $f(k) \cdot \|G\|$, for some function f.*

Proof. Given a graph G, we apply the Ladder Algorithm in the graph $\eta_r^k(\mathscr{C})$ with implementations of oracles provided by Lemma 22. The correctness of the algorithm and the running time bound follow directly from Corollary 20 and Lemma 22, where we may set $f(k) = \mathcal{O}(R^{p(k)}(2\ell(k)) \cdot \nu_r^{\mathscr{C}}(p(k) \cdot R^{p(k)}(2\ell(k)))^{p(k)\cdot k})$. ◀

▶ **Theorem 27.** *For any $r \in \mathbb{N}$ and nowhere dense class $\mathscr{C}$, the* Distance-r Independent Set *problem on $\mathscr{C}$ can be solved in time $f(k) \cdot \|G\|$, for some function f.*

Proof. By Theorems 6 and 16, for every $k \in \mathbb{N}$ there are constants $\ell, p \in \mathbb{N}$, depending on k, such that the class $\eta_r^k(\mathscr{C})$ has ladder index bounded by ℓ and has the weak p-Helly property. This allows us to apply Theorem 26. ◀

5 Discussion of related results

Fixed-parameter tractability of both Distance-r Dominating Set and Distance-r Independent Set on any nowhere dense class follows from the general model-checking result for first-order logic of Kreutzer et al. [11]. The algorithms derived in this manner have running time $f(k) \cdot n^{1+\varepsilon}$ for any fixed $\varepsilon > 0$ and some function f, where n is the number of vertices of the input graph. In fact, an algorithm with running time $f(k) \cdot n^{1+\varepsilon}$ for the Distance-r Independent Set problem is one of the intermediate results used in [11]. A close inspection of this algorithm reveals that the polynomial factor is in fact $\|G\|$, improving the claimed $n^{1+\varepsilon}$, however this is not explicit in [11]. For the Distance-r Dominating Set problem, its fixed-parameter tractability on any nowhere dense class was established earlier by Dawar and Kreutzer [4], but their algorithm had at least a quadratic polynomial factor in the running time bound.

As far as Distance-r Dominating Set on powers of nowhere dense classes is concerned, we remark that the result provided in Theorem 25 would *not* follow immediately from applying the algorithm on the graph before taking the power, for radius rs instead of r. The reason is that the input consists only of the graph G^s, and it is completely unclear how to algorithmically find the preimage G if we are dealing with an arbitrary nowhere dense class $\mathscr{D}$. To the best of our knowledge, this result is a completely new contribution.

Regarding map graphs, the fixed-parameter tractability of the Distance-r Dominating Set problem on this class of graphs was established by Demaine et al. [6]. However, they use the recognition algorithm for map graphs of Thorup [23] to draw a map model of the graph; this algorithm has an estimated running time of at least $\mathcal{O}(n^{120})$ [2] and not all technical details have been published. Another way of obtaining a fixed-parameter algorithm would be to use the fact that map graphs have *locally bounded rankwidth*; however, again achieving linear running time would be difficult due to the need of computing branch decompositions with approximately optimum rankwidth, for which the best known algorithms have cubic running time. In contrast, as we have shown, the Semi-ladder Algorithm solves the problem in linear fixed-parameter time without the need of having a map model provided.

Finally, the fixed-parameter tractability of Dominating Set on $K_{t,t}$-free graphs, where both k and t are considered parameters, was established by Telle and Villanger [22]. Thus, Theorem 25 reproves this result and also improves upon the running time: from $2^{\mathcal{O}(k^{t+2})} \cdot \|G\|$ of [22] to $2^{\mathcal{O}(k \log k)} \cdot \|G\|$.

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