

Some Variations on Lyndon Words

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Abstract

In this paper we compare two finite words u and v by the lexicographical order of the infinite words u^ω and v^ω . Informally, we say that we compare u and v by the infinite order. We show several properties of Lyndon words expressed using this infinite order. The innovative aspect of this approach is that it allows to take into account also non trivial conditions on the prefixes of a word, instead that only on the suffixes. In particular, we derive a result of Ufnarovskij [V. Ufnarovskij, *Combinatorial and asymptotic methods in algebra*, 1995] that characterizes a Lyndon word as a word which is greater, with respect to the infinite order, than all its prefixes. Motivated by this result, we introduce the prefix standard permutation of a Lyndon word and the corresponding (left) Cartesian tree. We prove that the left Cartesian tree is equal to the left Lyndon tree, defined by the left standard factorization of Viennot [G. Viennot, *Algèbres de Lie libres et monoïdes libres*, 1978]. This result is dual with respect to a theorem of Hohlweg and Reutenauer [C. Hohlweg and C. Reutenauer, *Lyndon words, permutations and trees*, 2003].

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1 Introduction

Let A be a totally ordered alphabet. A word w is called a *Lyndon word* if for each nontrivial factorization $w = uv$, one has $w < v$ (here $<$ is the lexicographical order). Lyndon words were introduced in [19].

A well-known theorem of Lyndon states that every finite word w can be decomposed in a unique way as a nonincreasing product $w = \ell_1 \ell_2 \cdots \ell_n$ of Lyndon words. This theorem, which is a combinatorial counterpart of the famous theorem of Poincaré-Birkhoff-Witt, provides an example of a factorization of the free monoid (see [18]). It has also many algorithmic applications and it may be computed in an efficient way. Indeed, Duval proposed in [11] a linear-time algorithms to compute it, while Apostolico and Crochemore proposed in [1] a $O(\lg n)$ -time parallel algorithm.

The (right) *Lyndon tree* of a Lyndon word w corresponds recursively to the following *right standard factorization* of w , when no reduced to a single letter: w can be written as $w = uv$ where v is the longest proper nonempty suffix of w which is a Lyndon word. The word u is then also a Lyndon word. Remark that one can also define a *left standard factorization* of a Lyndon word, and then a left Lyndon tree (cf. [24] and [5]).



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On the other hand, one can associate to a Lyndon word w the *Cartesian tree* corresponding to its suffix standard permutation (also commonly known as the inverse suffix array of w). Hohlweg and Reutenauer have proved that the (right) Lyndon tree of a Lyndon word is equal to its Cartesian tree (see [15]). This connection is useful for the computation of runs in a word (see, e.g. [9, 8, 2, 3]).

In this paper we consider a new approach that uses infinite words: the relation between two finite words u and v is determined by the lexicographical order of the infinite words u^ω and v^ω (where $u^\omega = uuu\cdots$). Informally, we say that we *compare u and v using the infinite order*.

Note that one can have $u < v$ but $u^\omega > v^\omega$. For instance, if $a < b$, one has $ab < aba$ but $(aba)^\omega < (ab)^\omega$.

This new relation between words has been used in some important results in combinatorics on words, as, for instance in the bijection of Gessel and Reutenauer (cf. [13]), which is at the basis of some extensions of the Burrows-Wheeler transform (see [20] and [16]).

We show that several properties of Lyndon words can be expressed by using the infinite order. We prove (Corollary16) that a word w is a Lyndon word if and only if $w^\omega < v^\omega$ for each proper suffix v of w , i.e., w is smaller, with respect to the infinite order, than all its proper suffixes. Moreover, we show that in the classical factorization theorem by Lyndon (every word can be factorized in a unique way as a non-increasing product of Lyndon words) the product is non-increasing also with respect to the infinite order. We also deduce (Theorems 21 and 23) new characterizations of the first and of the last factor of the factorization in Lyndon words.

The innovative aspect of this new approach in the study of Lyndon words is that it takes into account also conditions on the prefixes of a word, instead that only on the suffixes. In particular, we derive (Corollary17) a result of Ufnarovskij (cf. [23]) that characterizes a Lyndon word as a word which is greater, with respect to the infinite order, than all its proper prefixes.

In the last section, motivated by the Ufnarovskij's Theorem, we show that the ordering of the prefixes of a word according to the infinite order is non-trivial, and this leads to the notion *prefix array* of a word. We then introduce the *prefix standard permutation* of a word (the inverse of the prefix array) and the corresponding left Cartesian tree. We prove, as a result which is dual with respect to that of Hohlweg and Reutenauer, that the left Cartesian tree of a Lyndon words is equal to its left Lyndon tree (Theorem 36).

Some of the results of Sections 3 and 4 can be extended by considering a *generalized order* relation, i.e., an order in which the comparison between two words depends on the length of their common prefix. A word w is called a *generalized Lyndon word* if $w^\omega < v^\omega$ (with respect the generalized order) for each proper suffix v of w . Generalized Lyndon words have been introduced in [22] and their theory has been further developed in [10]. A very special case of generalized order is given by the *alternating order*, in which the comparison between two words depends on the *parity* of the length of their common prefix. The generalized Lyndon words with respect to the alternating order are called *Galois words*. A bijection, similar to that of Gessel and Reutenauer, for the alternating order, has been proved in [12]: it leads to the definition of the *Alternate Burrows-Wheeler Transform* (ABWT), that has been also studied in [14].

2 Definitions and notations

For undefined notation we refer to [17] and [18]. We denote by A a finite alphabet, by A^* the free monoid and by A^+ the free semigroup. Elements of A^* are called *words* and the identity element, denoted by 1 is called the *empty word*. We say that a word u is a *factor* of

the word w if $w = xuy$ for some words x, y ; u is a *prefix* (resp. *suffix*) if $x = 1$ (resp. $y = 1$); it is *nontrivial* if $u \neq 1$ and *proper* if $u \neq w$. We say that $w = ps$ is a *nontrivial factorization* of w whenever p, s are both nonempty. The length of a word $w = a_1a_2 \cdots a_n$, where $a_i \in A$ for all i , is equal to n and it is denoted by $|w|$.

A *period* of a word $a_1a_2 \cdots a_n$ is a natural integer p such that $a_i = a_{i+p}$ for any i such that $i, i+p \in \{1, \dots, n\}$; it is called a *nontrivial period* if $0 < p < n$. A word having a nontrivial period is called *periodic*.

We say that v is a *fractional power* of u if $u = u_1u_2$ and $v = u^k u_1$ for some nonnegative integer k . In this case, one writes $v = u^r$, where $r = k + |u_1|/|u|$ is a positive rational. Note that for $k = 0$ (or $r < 1$) this means that v is a prefix of u . Fractional powers are also known as *sesquipowers* (see, e.g., [21]).

We say that the v is a *strict fractional power* of u if v is a fractional power of u and, with the notations above, $k \geq 1$ or, equivalently, that $r \geq 1$. In this case u is a prefix of v .

► **Example 1.** Let $u = abcdef$. Then $u^{2/3} = abcd$ and $u^{5/3} = abcdefabcd$. The last one is, in particular, a strict fractional power of u .

We denote by A^ω the set of sequences over A , also called *infinite words*; such a sequence $(a_n)_{n \geq 1}$ is also written $a_1a_2a_3 \cdots$. If w is a (finite) word of length $n \geq 1$, w^ω denotes the infinite word having w as a prefix and of period n .

We denote by $A^\infty = A^* \cup A^\omega$ the set of finite and infinite words.

A *border* of a word w of length n is a word which is simultaneously a nontrivial proper prefix and suffix of w . A word is called *unbordered* if it has no border. It is well-known that a word has a border if and only if it is periodic.

3 Infinite order on finite words

Given an order $<$ on the alphabet A , we can define the *lexicographical order* $<_{lex}$ (or simply $<$ when it is clear from the context) on A^∞ in the following way: $u <_{lex} v$ if either u is a proper prefix of v (in which case u must be in A^*) or we may write $u = pau'$, $v = pbv'$ for some words $p \in A^*$, $u', v' \in A^\infty$ and some letters $a, b \in A$ such that $a < b$.

► **Definition 2.** Let s, t be two distinct elements of A^ω such that we have a factorization $s = u_1 \cdots u_k s_0$ with u_1, \dots, u_k finite nonempty words and s_0 is an infinite word. We say that the comparison between s and t takes place within u_k if $u_1 \cdots u_{k-1}$ is a prefix of t , but $u_1 \cdots u_k$ is not. If moreover u_1, u_2, \dots, u_k are letters we say that the comparison takes place at position k .

Note that, when the comparison takes place within u_k , one may write $t = u_1 \cdots u_{k-1} u'_k t'$, for some $t' \in A^\omega$ and $u'_k \neq u_k$ such that $|u'_k| = |u_k|$.

Suppose that u, v are finite nonempty words. The following fact is well-known: one has $u^\omega = v^\omega$ if and only if u, v are power of a common word, and this is true if and only if u and v commute (see, for instance, [18, Corollary 6.2.5]).

In the sequel we compare two finite words u and v by comparing the infinite words u^ω and v^ω , with respect to the lexicographical order. Informally, we say that we *compare u and v with respect to the infinite order*.

Note that, given two nonempty finite words u, v , it is not true that $u^\omega < v^\omega \Leftrightarrow u < v$, as shown in the following example.

► **Example 3.** Let us consider the two words $u = b$, $v = ba$. Since u is a prefix of v we have $u < v$. Nevertheless $u^\omega = bbb \cdots > bababa \cdots = v^\omega$.

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In the next section we will show that this equivalence holds for Lyndon words (Theorem 20).

Even though we use the term "infinite order", when $u^\omega \neq v^\omega$ the comparison between u^ω and v^ω takes place at a position bounded by a function of the lengths of u and v , as stated by the following lemma, which is a reformulation of the well known Fine and Wilf theorem (see, for instance, [17]).

► **Lemma 4.** *Let u, v be nonempty words such that $u^\omega \neq v^\omega$. Then the comparison between u^ω and v^ω takes place at a position $k \leq |u| + |v| - \gcd(|u|, |v|)$.*

The following example shows that the bound given in the previous lemma is tight.

► **Example 5.** Let $u = abaab$ and $v = abaababa$. Then,

$$u^\omega = abaababaabab\dots \quad \text{and} \quad v^\omega = abaababaabaa\dots$$

One has that $v^\omega < u^\omega$ and that the comparison takes place at position $12 = |u| + |v| - 1$.

► **Lemma 6.** *Let u, v be nonempty words such that $u^\omega \neq v^\omega$. Then the comparison between u^ω and v^ω takes place within the first factor v of v^ω if and only if v is not a fractional power of u .*

Proof. The comparison between the two infinite words takes place within the first v if and only if the two prefixes of length $|v|$ of u^ω and v^ω are different. The conclusion follows from the fact that v is a fractional power of u if and only if v is a prefix of u^ω . ◀

► **Lemma 7.** *Let $s, t \in A^\omega$ be as in Definition 2 (and the sentence following it). Then $s < t$ (resp. $s > t$) implies $u_1 \cdots u'_k s' < u_1 \cdots u_k t'$ (resp. $u_1 \cdots u'_k s' > u_1 \cdots u_k t'$) for any infinite words s', t' .*

► **Lemma 8.** *Let u, v be nonempty finite words such that $u^\omega < v^\omega$ and let x, y be two finite words. Then*

- (i) *if neither u or v is a prefix of the other, then $(ux)^\omega < (vy)^\omega$;*
- (ii) *if v is not a fractional power of u , then $(u^{k+1}x)^\omega < (vy)^\omega$, where k is the largest integer such that u^k is a prefix of v . In particular $u^\omega < (vy)^\omega$.*

Proof. In case (i), the comparison between the two infinite words takes place within the prefix of length $\min(|u|, |v|)$. Hence we conclude using Lemma 7.

Suppose now that the hypothesis of (ii) holds. Then we can write $u = u'au_1$ and $v = u^k u' b v_1$, with $u' \in A^*$, $a, b \in A$ such that $a \neq b$, and $u_1, v_1 \in A^*$. Let $m = |u^k u'|$. Since $u^\omega = u^k u' a u_1 u^\omega$ and since $u^\omega < v^\omega$, we have that $a <_{m+1} b$. The two infinite words u^ω and $(u^{k+1}x)^\omega$ share the same prefix of length $m + 1$, and the same do v^ω and $(vy)^\omega$. Thus the comparison between $(u^{k+1}x)^\omega$ and $(vy)^\omega$ takes place at position $m + 1$. Since $a <_{m+1} b$, we can conclude. ◀

We use several times the following observation: the opposite order $\tilde{<}$ of an order $<$ is also a lexicographical order.

► **Example 9.** Let $<$ be the usual lexicographical order on $\{a, b\}$, that is such that $a < b$. Then $\tilde{<}$ is defined by $b \tilde{<} a$.

► **Theorem 10.** *Let u, v be finite nonempty words such that $u^\omega \neq v^\omega$, and s, t be infinite words in $\{u, v\}^\omega$. Then*

$$u^\omega < v^\omega \iff us < vt.$$

Proof. Without loss of generality, we may assume that u, v are primitive.

Suppose that $u^\omega < v^\omega$. Let us first consider the case when v is not a fractional power of u . Then we can write $u = u_1 a u_2$, $v = u^k u_1 b v_2$ with letters $a \neq b$ and $k \geq 0$. Then $u^\omega = u^k u_1 a \dots$ and $v^\omega = u^k u_1 b \dots$. Since $u^\omega < v^\omega$, we must have $a < b$. Note that by hypothesis, one has either $us = u^\omega$ or $us = u^l v \dots$, with $l \geq 1$. In both cases us begins by u^{k+1} . Thus $us = u^k u_1 a \dots$. Moreover vt begins by v and hence $vt = u^k u_1 b \dots$. Therefore $us < vt$.

Suppose now that v is a strict fractional power of u . We can write $v = u^k u_1$, $u = u_1 u_2$, for some $k \geq 1$, and u_1, u_2 nonempty finite words (since $u^\omega \neq v^\omega$). In particular, we have $u_1 u_2 \neq u_2 u_1$ since u is primitive. Then $u^\omega = u^k u_1 u_2 u_1 \dots$ and $v^\omega = u^k u_1 u_1 u_2 \dots$ (since $k \geq 1$). From the inequality $u^\omega < v^\omega$, we deduce that $u_2 u_1 < u_1 u_2$. We claim that $us = u^k u_1 u_2 u_1 \dots$: indeed, either $s = u^l v \dots$, $1 \leq l \leq \infty$, so that u^{k+2} is a prefix of us , implying the claim; or $s = v \dots$, so that us begins by $uv = uu^k u_1 = u^k u_1 u_2 u_1$ and the claim is true, too. Moreover $vt = u^k u_1 u_1 u_2 \dots$. Therefore $us < vt$.

It remains the case where v is a proper prefix of u . Then u is either not a fractional power of v , or u is a strict fractional power of v . In this case, we have $v^\omega \tilde{<} u^\omega$. Hence the previous arguments imply that $vt \tilde{<} us$ and therefore $us < vt$.

Suppose now that $u^\omega < v^\omega$ does not hold. Since $u^\omega \neq v^\omega$, we have $u^\omega \tilde{<} v^\omega$. From the previous arguments it follows that $us \tilde{<} vt$, hence $vt < us$ and therefore $us < vt$ does not hold. \blacktriangleleft

Using the previous theorem we can prove the following result (part of the it is stated, in a more general context, in [22], see also [10]).

► **Corollary 11.** *The following conditions are equivalent for nonempty words $u, v \in A^*$:*

- (1) $u^\omega < v^\omega$;
- (2) $(uv)^\omega < v^\omega$;
- (3) $u^\omega < (vu)^\omega$;
- (4) $(uv)^\omega < (vu)^\omega$;
- (5) $u^\omega < (uv)^\omega$;
- (6) $(vu)^\omega < v^\omega$.

Proof. It follows from Theorem 10 that Condition (1) is equivalent to each of the Conditions (2), (3) and (4).

Condition (5) is equivalent to condition (3): indeed $u^\omega < (vu)^\omega \Leftrightarrow uu^\omega < u(vu)^\omega \Leftrightarrow u^\omega < (uv)^\omega$. Similarly, condition (6) is equivalent to condition (2): indeed, $(uv)^\omega < v^\omega \Leftrightarrow v(uv)^\omega < vv^\omega \Leftrightarrow (vu)^\omega < v^\omega$. \blacktriangleleft

Note that previous corollary implies a result proved by Bergman in [4, Lemma 5.1] (see also [23, p.34 and pp.101–102]).

► **Corollary 12.** *Let u, v be two finite words such that $u^\omega < v^\omega$. Then*

$$u^\omega < (uv)^\omega < (vu)^\omega < v^\omega. \quad (1)$$

Another immediate consequence of Corollary 11 is that one can define the “infinite order” without any explicit use of infinite words, as stated by the following corollary.

► **Corollary 13.** *Let u, v be two finite words. Then $u^\omega < v^\omega$ if and only if $uv < vu$.*

4 Lyndon words

In this section we show some fundamental properties of Lyndon words by using the infinite order, instead of the classical lexicographical order. Moreover, the infinite order allows to introduce an innovative point of view about Lyndon words, by taking into account conditions on the prefixes of a word (instead that only on the suffixes). In particular, we derive a result by Ufnarovskij (Corollary 17) that characterizes a Lyndon word as a word which is greater (with respect to the infinite order) than its proper prefixes.

Let us start with the classical definition of Lyndon words in terms of the usual lexicographical order (see, for instance, [11]).

► **Definition 14.** *A word w is a Lyndon word if any one of the following three equivalent conditions holds:*

- (i) *for any nontrivial factorization $w = uv$, $u < v$,*
- (ii) *for any nontrivial factorization $w = uv$, $uv < v$,*
- (iii) *for any nontrivial factorization $w = uv$, $uv < vu$.*

The following theorem provides a characterization of Lyndon words by using the infinite order.

► **Theorem 15.** *A word w is a Lyndon word if and only if, for any nontrivial factorization $w = uv$, any one of the six equivalent conditions of Corollary 11 holds.*

Proof. From Corollary 13 it follows that condition (iii) of Definition 14 is equivalent to condition (1) of Corollary 11. ◀

In next corollary (part of whose has already been proved, in a more general context, in [22, Proposition 2.1], see also [10]) we highlight conditions (1) and (2) of Corollary 11. Even though such conditions show some formal similarity with the conditions (i) and (ii) of Definition 14, they are essentially different.

► **Corollary 16.** *A word w is a Lyndon word if and only if one of the following condition is satisfied for any nontrivial factorization $w = uv$:*

1. $u^\omega < v^\omega$.
2. $w^\omega < v^\omega$.

The following result, due to Ufnarovskij (see also [23, Theorem 2, p.35]) follows from Theorem 15 and condition (5) of Corollary 11. It provides a characterization of a Lyndon word that takes into account its prefixes, instead than its suffixes.

► **Corollary 17 (Ufnarovskij).** *A word w is a Lyndon word if and only if for any nontrivial factorization $w = ps$, one has $p^\omega < w^\omega$.*

► **Example 18.** The word $w = aabab$ is a Lyndon word. We have $a^\omega = (aa)^\omega < (aba)^\omega < (aab)^\omega < w^\omega$.

The following result is classical (see, for instance [17]).

► **Theorem 19.** *Each word in A^* can be factorized in a unique way as a nonincreasing product of Lyndon words.*

The term "nonincreasing" in the previous theorem is referred to the classical lexicographical order. The following theorem (cf. [6, Theorem 8]) shows that the factorization in Lyndon words is non-increasing also with respect to the infinite order.

► **Theorem 20.** *Let u, v be two Lyndon words. Then $u < v$ if and only if $u^\omega < v^\omega$,*

Proof. Suppose that $u < v$. If u is not a prefix of v , then obviously $u^\omega < v^\omega$.

If u is a prefix of v , then $v = uy$, for some nonempty word y . From $u < v$ and $v < y$ (since v is a Lyndon word) we get $u^2 < uy = v$, and thus $u^\omega < v^\omega$.

To prove the converse implication, let us suppose that $u^\omega < v^\omega$. By contradiction, suppose $v < u$. By the first part of the proof we would have $v^\omega < u^\omega$, which gives us a contradiction. ◀

In the following theorem we characterize the last element of the factorization in Lyndon words by using the infinite order.

► **Theorem 21.** *Let $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i Lyndon words such that $\ell_1^\omega \geq \ell_2^\omega \geq \cdots \geq \ell_n^\omega$. Then ℓ_n is the shortest among all nontrivial suffixes s of w such that s^ω is minimum.*

Proof. Let z be the shortest among all nontrivial suffixes of w such that s^ω is minimum. If $w = z$ then w is a Lyndon word and $\ell_1 = \ell_n = z$.

Otherwise, we can write $w = uz$. Consider the factorization of u in Lyndon words: $u = \ell'_1 \ell'_2 \cdots \ell'_k$, with $\ell'_1^\omega \geq \ell'_2^\omega \geq \cdots \geq \ell'_k^\omega$. By hypothesis, $(\ell'_k z)^\omega > z^\omega$. Then, using Corollary 11, one has $\ell'_k z > z$. It follows that $\ell'_1 \ell'_2 \cdots \ell'_k z$ is a non-increasing factorization of w in Lyndon words. Since the factorization is unique, we have that $z = \ell_n$. ◀

► **Example 22.** Let $w = ababaab$. Its non-increasing factorization into Lyndon words is $w = (ab)(ab)(aab)$. One can check that $(aab)^\omega < (abaab)^\omega < w^\omega < (ab)^\omega < (baab)^\omega < (babaab)^\omega < b^\omega$.

In the previous results we gave a characterization of the last element of the factorization of a word in Lyndon words. Now, we focus on the first factor. This result is motivated by point 1 of Corollary 16: the fact that a word w is not a Lyndon word implies the existence of a prefix u such that $u^\omega \geq v^\omega$, where v is the corresponding suffix. If one chooses the shortest prefix satisfying this property, this turns out to be the first factor in the Lyndon factorization. In the same vein, it is motivated by Ufnarovskij's Theorem (Corollary 17 above).

► **Theorem 23.** *Let $w = \ell_1 \ell_2 \cdots \ell_n$ be the nonincreasing factorization into Lyndon words of a finite nonempty word w .*

1. *The word ℓ_1 is the shortest nontrivial prefix p of w such that, when writing $w = ps$, one has either $s = 1$ or $p^\omega \geq s^\omega$.*
2. *The word ℓ_1 is the shortest nontrivial prefix p of w such that $p^\omega \geq w^\omega$.*

In order to prove Theorem 23 we need a preliminary result which refines Corollary 11 in the case of the usual lexicographical order.

Note that, for any infinite words s, t such that $s < t$, with $<$ the classical order, and for any finite word w , one has $ws < wt$.

► **Corollary 24.** *If $\ell_1, \ell_2, \dots, \ell_n$, with $n \geq 2$, are Lyndon words such that $\ell_1^\omega \geq \ell_2^\omega \geq \cdots \geq \ell_n^\omega$, then $\ell_1^\omega \geq (\ell_2 \cdots \ell_n)^\omega$.*

Proof. The case $n = 2$, it is trivial. Let consider the case $n \geq 3$. By induction hypothesis we have $\ell_2^\omega \geq (\ell_3 \cdots \ell_n)^\omega$. From Corollary 11 it follows that $\ell_2^\omega \geq (\ell_2 \cdots \ell_n)^\omega$. Hence, $\ell_1^\omega \geq (\ell_2 \cdots \ell_n)^\omega$. ◀

In order to prove Theorem 23 let us recall the well-known fact that all Lyndon words are unbordered (see, for instance, [7]).

Proof of Theorem 23. Let us prove the first assertion. When $n = 1$, then $w = \ell_1$ is a Lyndon word and the result is true by point 1 of Corollary 16.

Suppose now that $n \geq 2$. Then, by Corollary 24, we have $\ell_1^\omega \geq (\ell_2 \cdots \ell_n)^\omega$. Let p be a nontrivial prefix of w shorter than ℓ_1 . Thus, we have a nontrivial factorization $\ell_1 = pq$ for some $q \neq 1$. By Corollary 16, we know that $p^\omega < q^\omega$. Since ℓ_1 is unbordered, q cannot be a fractional power of p . Thus, by point (ii) of Lemma 8, one has $p^\omega < (q\ell_2 \cdots \ell_n)^\omega$, which prove the first part of the theorem.

The second assertion just follows from the first one. Indeed, using Corollary 11, we have that if $s \neq 1$, then $p^\omega \geq s^\omega$ is equivalent to $p^\omega \geq (ps)^\omega$. ◀

► **Example 25.** Let $w = ababaab$. As seen in Example 22, its nonincreasing factorization into Lyndon words is $w = (ab)(ab)(aab)$. One can check that $(ab)^\omega > w^\omega > (abaab)^\omega$ while $a^\omega < w^\omega < (babaab)^\omega$.

5 Left Lyndon tree and prefix standardization

In [15], the authors associate with each Lyndon word w an increasing tree, based on the suffixes of w ; they show that the completion of this tree, with leaves appropriately labeled by the letters of w , is equal to the tree obtained by iterating the right standard factorization of w (equivalently the Lie bracketing associated with w).

In this section we give a similar construction and result, based on the prefixes of w instead that on the suffixes. This construction is motivated by Ufnarovskij's Theorem (Corollary 17).

5.1 Left Lyndon tree

In [24] (cf. also [5]) Viennot introduced the notion of left standard factorization of a Lyndon word. Let us consider a Lyndon word w having length at least 2. The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

The following is a well-known result (see [24, p. 14]).

► **Proposition 26.** *Let $w = uv$ be a Lyndon word with its left standard factorization. Then both u and v are Lyndon words and $u < v$. Moreover, either v is a letter, or given its left standard factorization $v = v_1v_2$ one has $v_1 \leq u$.*

► **Corollary 27.** *Let u, v, v_1 and v_2 as in Proposition 26. Then v_1 is a prefix of u .*

Proof. By Proposition 26 we have $v_1 \leq u < v_1v_2$. By a classical property of lexicographical order, this implies that $u = v_1m$, for a certain word m such that $m < v_2$. Thus v_1 is a prefix of u . ◀

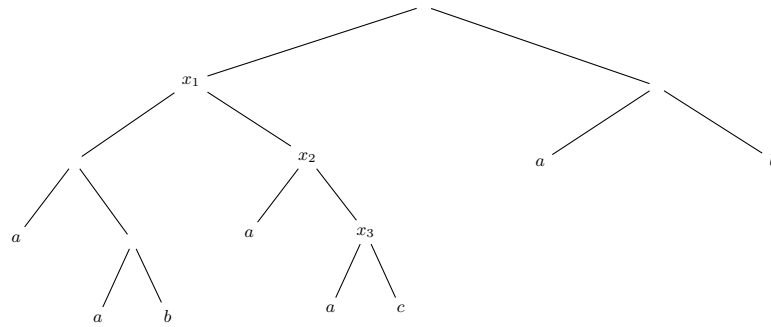
► **Example 28.** Let us consider the Lyndon word $w = abaacab$ on the alphabet $\{a, b, c\}$ with $a < b < c$. Its left standard factorization is $w = (abaac)(ab)$. The suffix $v = ab$ is a Lyndon word with left standard factorization $(a)(b)$ and one has $a \leq abaac$.

The *free magma* $M(A)$ over A is the set of *complete trees* over A defined recursively as follows:

- each letter is a tree;
- if t_1, t_2 are trees, then (t_1, t_2) is a tree.

We will use the classical notions of *root*, *internal node* and *leaf* for a tree.

There is a canonical surjective mapping φ from $M(A)$ onto A^+ defined as follows: given a tree t , its image $\varphi(t)$, called its *foliage*, is defined recursively by:



■ **Figure 1** The left Lyndon tree of $w = aabaacab$.

- $\varphi(a) = a$ for any a in A ;
- $\varphi((t_1, t_2)) = \varphi(t_1)\varphi(t_2)$ for any two trees t_1, t_2 .

In other words, $\varphi(t)$ is obtained by vertically projecting the leaves of t onto a horizontal line.

► **Example 29.** The foliage of the tree in Figure 1 is the word $aabaacab$.

To each Lyndon word in A^+ we can associate a complete tree $\mathcal{L}(w)$ in $M(A)$, called the *left Lyndon tree* of w , defined recursively as follows:

- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.

It is clear by the definition that $\varphi(\mathcal{L}(w)) = w$.

► **Example 30.** The left Lyndon tree associated to the Lyndon word $w = aabaacab$ is shown in Figure 1 (disregarding the labels of the internal nodes).

5.2 Prefix standardization

Consider an alphabet A and a total order $<$ on A . We define an order \prec on the free monoid as follows: $u \prec v$ if either

- $u^\omega < v^\omega$, or
- $u^\omega = v^\omega$ (which means that u, v are power of the same word) and if u is longer than v .

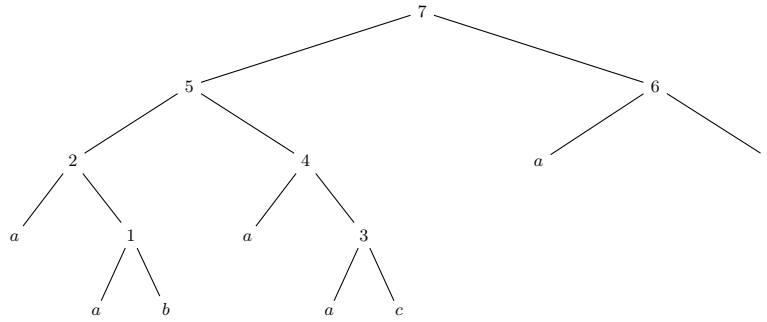
► **Example 31.** Let us consider the above order on $\{a, b\}^*$ induced by $a < b$. One has $aa \prec a \prec ab \prec ba \prec b$.

Let w be a word of length $n + 1$ on a totally ordered alphabet. Let us consider the sequence $p_1, p_2, \dots, p_n, p_{n+1} = w$ of its nonempty prefixes, in increasing length.

Motivated by Ufnarovskij's Theorem (Corollary 17), we call *prefix standard permutation* of w the unique permutation $\sigma \in \mathfrak{S}_{n+1}$ such that

$$p_{\sigma^{-1}(1)} \prec p_{\sigma^{-1}(2)} \prec \dots \prec p_{\sigma^{-1}(n)} \prec p_{\sigma^{-1}(n+1)}.$$

In other words, we number each letter in w by $1, 2, \dots, n, n + 1$ as follows: 1 for the letter where the \prec -smallest prefix ends, 2 for the second smallest one in the order \prec , and so on. Then σ is the represented by the word w translated in this new alphabet. We denote such a permutation by $\text{pstd}(w)$.



■ **Figure 2** The tree $\mathcal{C}(aabaacab)$.

► **Example 32.** The prefix standard permutation of the word $w = aabaacab$ is the permutation $\text{pstd}(w) = 21543768$. Indeed one can check that $aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca \prec aabaac \prec w$.

► **Remark 33.** In [15] the authors define the “suffix standard permutation” of a word. In literature, this is also commonly known as “inverse suffix array”. Therefore, our prefix standard permutation could also be called “inverse prefix array”.

An *injective word* is a word without repetition of letters. Each permutation in \mathfrak{S}_n can be seen as a complete injective word, that is an injective word in the ordered alphabet $\{1 < 2 < \dots < n\}$, having all n letters as factors.

With each injective word α on a totally ordered alphabet, and in particular with each permutation $\alpha \in \mathfrak{S}_n$, we can associate bijectively a binary noncomplete labeled *decreasing tree* (i.e., such that each node is smaller than its father) defined recursively as follows:

- the root is n , the maximum letter in α ;
- its left and right subtrees (which may be empty) are the trees associated to u and v respectively, where u, v are injective words in $\{1, 2, \dots, n\}$ such that $\alpha = unv$.

The inverse bijection is obtained by vertically projecting the labels on the horizontal line.

► **Example 34.** The decreasing tree associated to $\alpha = 2154376$ is the tree shown in Figure 2 restricted to its internal nodes.

Let us now associate a tree with a Lyndon word, called the *left Cartesian tree* of w , and denoted $\mathcal{C}(w)$, as follows (see also [9] and the references there). To each letter $a \in A$ we define $\mathcal{C}(a) = a$. Otherwise, let $\sigma = \text{pstd}(w) \in \mathfrak{S}_{n+1}$, with $|w| = n + 1$, and let $\alpha \in \mathfrak{S}_n$ be the permutation obtained by removing the last digit in σ (which is $n + 1$, because of Corollary 17). Using the construction before we obtain a noncomplete binary decreasing tree t^* having n nodes. We define $\mathcal{C}(w)$ as the complete binary tree having t^* as tree of internal nodes and the letters of w as leaves in order from left to right, i.e., such that $\varphi(t) = w$.

► **Example 35.** The tree $\mathcal{C}(w)$ for $w = aabaacab$ is shown in Figure 2.

For our purpose, we give an alternative construction of $\mathcal{C}(w)$. Consider the sequence of proper prefixes (p_1, p_2, \dots, p_n) of w , viewed as a word of length n on the alphabet A^* , totally ordered by \prec . Since it is an injective word, we can associate with it a decreasing tree as before; call it $t^*(w)$. Also as before, we can complete $t^*(w)$ to a tree $t(w)$, having the letters of w as leaves in order from left to right, i.e., such that $\varphi(t(w)) = w$. The completed tree $t(w)$, disregarding the labels of the internal nodes, coincides with $\mathcal{C}(w)$. Indeed, the word α coincides with the previous injective word, up to the unique increasing order isomorphism from $\{1, \dots, n\}$ into the set of proper prefixes of w , sending i to the i -th prefix for the order \prec .

5.3 Equivalence of the trees

From the constructions seen before it is clear that $\varphi(\mathcal{L}(w)) = \varphi(\mathcal{C}(w))$ for every Lyndon word w . We actually have a stronger result.

► **Theorem 36.** *Let w be a Lyndon word. The trees $\mathcal{L}(w)$ and $\mathcal{C}(w)$ are equal.*

To prove the theorem we need some intermediate result. Let x be an internal node of some planar binary complete tree $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2)$. We call *left subtrees sequence* with respect to the tree \mathfrak{t} and the node x , denoted $\text{lss}(\mathfrak{t}, x)$, the sequence of subtrees of \mathfrak{t} hanging at the left of the path from the root to x , that is the sequence of subtrees of \mathfrak{t} recursively defined as follows:

- if x is the root, then $\text{lss}(\mathfrak{t}, x) = (\mathfrak{t}_1)$;
- if x is in \mathfrak{t}_1 , then $\text{lss}(\mathfrak{t}, x) = \text{lss}(\mathfrak{t}_1, x)$;
- if x is in \mathfrak{t}_2 , then $\text{lss}(\mathfrak{t}, x) = (\mathfrak{t}_1, \text{lss}(\mathfrak{t}_2, x))$.

► **Lemma 37.** *Let w be a Lyndon word and let x be an internal node of $\mathcal{L}(w)$. Let $\text{lss}(\mathfrak{t}, x) = (\mathfrak{t}_1, \dots, \mathfrak{t}_n)$ and let ℓ_i be the foliage of the tree \mathfrak{t}_i for each i . Then all ℓ_i are Lyndon words. Moreover, ℓ_{i+1} is a prefix of ℓ_i for each $1 \leq i \leq n-1$.*

Proof. The fact that each ℓ_i is a Lyndon word follows from the more general fact that the foliage of each subtree of $\mathcal{L}(w)$ is a Lyndon word, as follows recursively from the construction of $\mathcal{L}(w)$.

We prove the other assertion by induction on the size of the tree. If x is the root, then $n = 1$ and there is nothing to prove.

Let us suppose that x is not the root of the tree and let us write $\mathcal{L}(w) = (\mathfrak{t}', \mathfrak{t}'')$.

If x is an internal node of \mathfrak{t}' , then $\text{lss}(\mathfrak{t}, x) = \text{lss}(\mathfrak{t}', x)$, and we can conclude by induction.

Suppose now that x is an internal node of \mathfrak{t}'' , and let $\mathfrak{t}'' = (\mathfrak{s}_1, \mathfrak{s}_2)$. Thus $\text{lss}(\mathfrak{t}, x) = (\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_n)$, with $\mathfrak{t}_1 = \mathfrak{t}'$ and $(\mathfrak{t}_2, \dots, \mathfrak{t}_n) = \text{lss}(\mathfrak{t}'', x)$ (note that $n \geq 2$). By induction it is enough to show that ℓ_2 is a prefix of ℓ_1 . Let v_1, v_2 be respectively the foliages of $\mathfrak{s}_1, \mathfrak{s}_2$. Then by the property of the left standard factorization and by the construction of the tree $\mathcal{L}(w)$, we have that v_1 is a prefix of ℓ_1 . Now either x is in \mathfrak{s}_2 and $\mathfrak{t}_2 = \mathfrak{s}_1$ and $\ell_2 = v_1$ or x is in \mathfrak{s}_1 and ℓ_2 is a proper prefix of v_1 . In both cases ℓ_2 is a prefix of ℓ_1 . ◀

► **Example 38.** Let us consider the tree \mathfrak{t} in Figure 1 and its internal node x_3 . The left subtrees sequence $\text{lss}(\mathfrak{t}, x_3)$ is equal to $(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3)$, where \mathfrak{t}_i is the subtree of \mathfrak{t} having root x_i . The foliages of the three subtrees are respectively the words $\ell_1 = aab$, $\ell_2 = a$ and $\ell_3 = a$. Each of them is a prefix of the previous one.

► **Lemma 39.** *Let ℓ_1, \dots, ℓ_n , be Lyndon words such that ℓ_{i+1} is a prefix of ℓ_i for each $i = 1, \dots, n-1$. Let $\ell_n = \ell'_n \ell''_n$ be the left standard factorization of ℓ_n . Then*

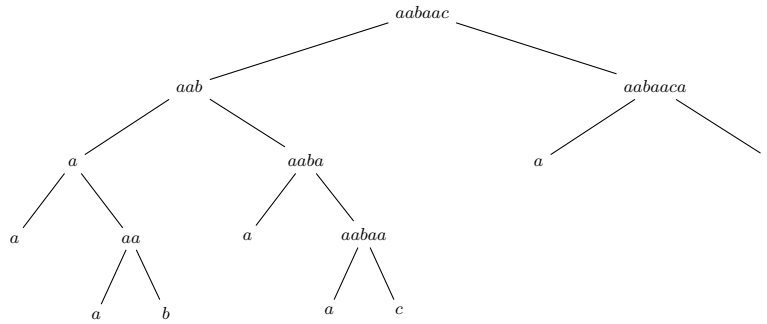
- (i) $(\ell_1 \cdots \ell_{n-1} \ell'_n)^\omega < (\ell_1 \cdots \ell_n)^\omega$;
- (ii) moreover, if $n \geq 2$, one has $(\ell_1 \cdots \ell_n)^\omega \leq (\ell_1 \cdots \ell_{n-1})^\omega$.

Proof. If $n = 1$, we have $\ell_1^\omega < \ell''_1^\omega$ by Corollary 16 and $\ell_1^\omega < (\ell'_1 \ell''_1)^\omega$ by Corollary 11. Hence $\ell_1^\omega < \ell_1^\omega$.

Suppose now that $n \geq 2$. Since both $(\ell_1 \cdots \ell_{n-1} \ell'_n)^\omega$ and $(\ell_1 \cdots \ell_n)^\omega$ start with the word $\ell_1 \cdots \ell_{n-1} \ell'_n$, we have

$$(\ell_1 \cdots \ell_{n-1} \ell'_n)^\omega < (\ell_1 \cdots \ell_n)^\omega \iff (\ell_1 \cdots \ell_{n-1} \ell'_n)^\omega < \ell''_n (\ell_1 \cdots \ell_n)^\omega.$$

Since ℓ_n is a prefix of ℓ_1 , the last inequality is of the form $\ell_n s_0 < \ell''_n t_0$, with $s_0, t_0 \in A^\omega$, and this is true since $\ell_n < \ell''_n$ by Corollary 16 and because ℓ_n is not a prefix of ℓ''_n .



■ **Figure 3** Variant of the left Lyndon tree of $w = abaacab$ labeling the internal nodes with their left foliage.

Let us now prove point (ii). If $\ell_1 = \ell_2 = \dots = \ell_n$, we trivially have $(\ell_1 \dots \ell_n)^\omega = (\ell_1 \dots \ell_{n-1})^\omega$. Thus, let us suppose that $\ell_1 \neq \ell_n$. Since both terms of the inequality start with $\ell_1 \dots \ell_{n-1}$, we have

$$(\ell_1 \dots \ell_n)^\omega \leq (\ell_1 \dots \ell_{n-1})^\omega \iff \ell_n(\ell_1 \dots \ell_n)^\omega \leq (\ell_1 \dots \ell_{n-1})^\omega.$$

Since ℓ_n is a prefix of ℓ_1 , and $\ell_n \neq \ell_1$ we can write $\ell_1 = \ell_n v$ for a nonempty finite word v . Thus, the last inequality is of the form $\ell_n \ell_1 s_0 \leq \ell_n v t_0$ with $s_0, t_0 \in A^\omega$, and this is equivalent to $\ell_1 s_0 \leq v t_0$, which is true because of Corollary 16. ◀

Before proving the main result let us introduce the following notation. Let $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2)$ be a complete binary labeled tree. To each internal node x of \mathfrak{t} we associate the word $g_{\mathfrak{t}}(x)$, called the *left foliage* of x in \mathfrak{t} , as follows:

- if x is the root of \mathfrak{t} , then $g_{\mathfrak{t}}(x) = \varphi(\mathfrak{t}_1)$;
- if x is in \mathfrak{t}_1 , then $g_{\mathfrak{t}}(x) = g_{\mathfrak{t}_1}(x)$;
- if x is in \mathfrak{t}_2 , then $g_{\mathfrak{t}}(x) = \varphi(\mathfrak{t}_1)g_{\mathfrak{t}_2}(x)$.

For further use, we note that the length of $g_{\mathfrak{t}}(x)$ is equal to the number of leaves located at the left of x in \mathfrak{t} .

Also, we use the following result.

► **Lemma 40.** *Let \mathfrak{t} be a complete binary labeled tree and x an internal node of \mathfrak{t} . Then $g_{\mathfrak{t}}(x) = \ell_1 \dots \ell_n$, where ℓ_1, \dots, ℓ_n are the foliages of the trees hanging at the left on the path from the root to x .*

Proof. Let $\text{lss}(\mathfrak{t}, x) = (\mathfrak{t}_1, \dots, \mathfrak{t}_n)$. We have to show that $g_{\mathfrak{t}}(x) = \varphi(\mathfrak{t}_1) \dots \varphi(\mathfrak{t}_n)$. This follows easily from the recursive definition of both *lls* and *g*. ◀

► **Example 41.** The label shown in Figure 3 is obtained by relabeling the internal nodes of the tree $\mathcal{L}(w)$ in Figure 1, using the left foliage function.

Proof of Theorem 36. Let us consider the two trees $\mathcal{L}(w)$ and $\mathcal{C}(w)$. Note first that they have the same foliage. So, in order to prove the equality, it is enough to show that the two trees obtained from $\mathcal{L}(w)$ and $\mathcal{C}(w)$ by removing the leaves (that is, considering only the internal nodes) are equal.

We actually show that by labeling the internal nodes of $\mathcal{L}(w)$ using the function $g_{\mathfrak{t}}$, we obtain the same tree as $\mathcal{C}(w)$. For this it is enough to show that the labeling $g_{\mathfrak{t}}$ is decreasing, and that its projection is exactly the sequence (p_1, \dots, p_n) of nonempty prefixes of w . Indeed, the decreasing tree associated with an injective word on a totally alphabet is unique.

Let us consider two internal nodes x and y of \mathfrak{t} , such that y is a child of x . We show that $g_{\mathfrak{t}}(y) \prec g_{\mathfrak{t}}(x)$, i.e., that either

- $g_{\mathfrak{t}}(y)^{\omega} < g_{\mathfrak{t}}(x)^{\omega}$, or
- $g_{\mathfrak{t}}(y)^{\omega} = g_{\mathfrak{t}}(x)^{\omega}$ and $|g_{\mathfrak{t}}(y)| > |g_{\mathfrak{t}}(x)|$.

Suppose first that y is a right child of x and denote by h_1, \dots, h_n the foliagees of the subtrees in the sequence $\text{lss}(\mathfrak{t}, y)$. We have $n \geq 2$, and by Lemma 40 $g_{\mathfrak{t}}(x) = h_1 \cdots h_{n-1}$, and $g_{\mathfrak{t}}(y) = h_1 \cdots h_n$. Thus $g_{\mathfrak{t}}(y) \prec g_{\mathfrak{t}}(x)$ follows from Lemmata 37 and 39 and from the fact that $|g_{\mathfrak{t}}(y)| > |g_{\mathfrak{t}}(x)|$.

Suppose now that y is a left child of x . Let $\text{lss}(\mathfrak{t}, x) = (\mathfrak{t}_1, \dots, \mathfrak{t}_n)$, with $\mathfrak{t}_n = (\mathfrak{t}'_n, \mathfrak{t}''_n)$. Then $\text{lss}(\mathfrak{t}, y) = (\mathfrak{t}_1, \dots, \mathfrak{t}_{n-1}, \mathfrak{t}'_n)$. Let h_i be the foliage of \mathfrak{t}_i for each $1 \leq i \leq n$, and h'_n be the foliage of \mathfrak{t}'_n . Thus $g_{\mathfrak{t}}(x) = h_1 \cdots h_n$ and $g_{\mathfrak{t}}(y) = h_1 \cdots h_{n-1} h'_n$. Thus $g_{\mathfrak{t}}(x) \prec g_{\mathfrak{t}}(y)$ by the Lemmata 37 and 39.

It remains to show that the projection is exactly (p_1, \dots, p_n) . This follows from the fact the length of $g_{\mathfrak{t}}(x)$ is equal to the number of leaves located at the left of a given node x ; hence $g_{\mathfrak{t}}(x)$ is the prefix of w of length this number. We conclude because the lengths of the successive projections of the internal nodes increase by 1 from left to right. ◀

► **Example 42.** Let $w = aabaacab$. The tree $\mathcal{L}(w)$ with each internal node x labeled by $g_{\mathcal{L}(w)}(x)$ (as described in the proof of Theorem 36) is shown in Figure 3.

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