On the Expressivity of Linear Recursion Schemes

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Abstract

We investigate the expressive power of higher-order recursion schemes (HORS) restricted to linear types. Two formalisms are considered: multiplicative additive HORS (MAHORS), which feature both linear function types and products, and multiplicative HORS (MHORS), based on linear function types only.

For MAHORS, we establish an equi-expressivity result with a variant of tree-stack automata. Consequently, we can show that MAHORS are strictly more expressive than first-order HORS, that they are incomparable with second-order HORS, and that the associated branch languages lie at the third level of the collapsible pushdown hierarchy.

In the multiplicative case, we show that MHORS are equivalent to a special kind of pushdown automata. It follows that any MHORS can be translated to an equivalent first-order MHORS in polynomial time. Further, we show that MHORS generate regular trees and can be translated to equivalent order-0 HORS in exponential time. Consequently, MHORS turn out to have the same expressive power as 0-HORS but they can be exponentially more concise.

Our results are obtained through a combination of techniques from game semantics, the geometry of interaction and automata theory.

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1 Introduction

Higher-order recursion schemes (HORS) have recently emerged as a promising technique for model-checking higher-order programs [17]. Linear higher-order recursion schemes (LHORS) were introduced in [5] to facilitate a finer analysis of HORS by mixing intuitionistic and linear types. In this paper, we investigate the expressivity of their purely linear fragment.

First, we consider *multiplicative additive* HORS (MAHORS), which in addition to the linear function types ($\multimap$) feature product types (&), and thus allow for sharing but not re-use. We show that MAHORS are equivalent to a tree-generating variant of tree-stack automata (TSA), originally introduced to capture multiple context-free languages in the word language setting [7]. The translation from MAHORS to TSA amounts to representing the game semantics of MAHORS in the spirit of abstract machines derived from Girard's Geometry of Interaction (GoI) [11, 6]. The GoI view of computation makes it possible to interpret computation as a token machine that traverses a graph strongly related to the syntactic structure of the term. Somewhat suprisingly, so far this nearly automata-theoretic

$$\frac{}{\Gamma, x : \varphi \mid \Delta \vdash x : \varphi} \qquad \frac{}{\Gamma \mid \Delta, x : \varphi \vdash x : \varphi} \qquad \frac{\Gamma \mid \Delta \vdash t : \varphi_1 \,\&\, \varphi_2}{\Gamma \mid \Delta \vdash \pi_i\, t : \varphi_i} \qquad \frac{}{\Gamma \mid \Delta \vdash \bot : \varphi}$$

$$\frac{\Gamma \mid \Delta_1 \vdash t : \varphi \multimap \psi \qquad \Gamma \mid \Delta_2 \vdash u : \varphi}{\Gamma \mid \Delta_1, \Delta_2 \vdash t\, u : \psi} \qquad \frac{\Gamma \mid \Delta, x : \kappa \vdash t : \varphi}{\Gamma \mid \Delta \vdash \lambda x^{\kappa}.t : \kappa \multimap \varphi} \qquad \frac{\Gamma \mid \Delta \vdash t_i : \varphi_i \qquad (i \in \{1,2\})}{\Gamma \mid \Delta \vdash \langle t_1, t_2 \rangle : \varphi_1 \,\&\, \varphi_2}$$

Figure 1 Typing rules for the additive linear λ-calculus.

flavour of GoI has not been exploited to establish connections with automata models, and we believe we are the first to do so explicitly. As a consequence, we can conclude that the branch languages of trees generated by MAHORS are multiple context-free and, thus, that they belong to the third level of the collapsible pushdown hierarchy [12]. In addition, we show that MAHORS are strictly more expressive than first-order HORS[1], and that they are not comparable with second-order HORS.

Secondly, we consider *multiplicative* HORS (MHORS), featuring linear function types only. In this case, our earlier MAHORS-to-TSA translation specialises to a translation into a special kind of tree-generating pushdown automata (LPDA) in which reachable configurations must be reached in a unique run. We show that MHORS and LPDA are equi-expressive and, moreover, that any MHORS can be translated to an equivalent MHORS of order 1. Further, using reachability techniques for pushdown automata, we show that LPDA are equivalent to bounded pushdown automata that forget elements stored at the bottom of the stack after the stack height exceeds a certain depth. It follows that MHORS generate regular trees, though the MHORS representation may be exponentially more succinct than order-0 HORS.

2 Linear Recursion Schemes

In this section we introduce the object of study of this paper, MAHORS and MHORS.

The main ingredient of MAHORS is the *linear λ-calculus with products* – also called the *additive linear λ-calculus*, as the product is an additive connective in the sense of Linear Logic [10]. The following definitions follow [5], restricting type formers to linear connectives (note that [5] imposes some syntactic restrictions on the shape of types and terms that we can drop here to simplify presentation, as they play no role in the technical development).

Types are formed with the ground type o and the connectives $\multimap$ and $\&$. We define the **typed terms** directly by the typing rules of Figure 1. Typing judgments have the form $\Gamma \mid \Delta \vdash t : \varphi$, where Γ and Δ are two lists of variable declarations. Intuitively, Δ is the main context containing variables that can be used at most once (such terms are often called *affine* but we opt for the name *linear* nonetheless). In contrast, Γ comprises *duplicable* variables that may be reused at will, as witnessed by the application rule. In M(A)HORS, Γ will be used only for terminal and non-terminal symbols. Linear λ-terms are equipped with standard reduction rules; we write $\triangleright_\beta$ for β-reduction for functions and products, whose definition can be found *e.g.* in [5]. Any term t has a normal form, written $\mathrm{BT}(t)$.

Trees arise as ground-type terms typable in replicable contexts representing a ranked alphabet. Recall that in HORS, a symbol b of arity n is represented as a constant $\mathsf{b} : o \to \cdots \to o \to o$ with n arguments. Here, a ranked alphabet Σ may be represented in two distinct

[1] Type order is defined by $ord(o) = 0$ and $ord(\theta \to \theta') = \max(ord(\theta) + 1, ord(\theta'))$. The order of a HORS is the highest order of (the types of) its non-terminals.

ways: *multiplicatively*, with $\mathsf{b} : o \multimap \ldots \multimap o \multimap o$, or *additively*, with $\mathsf{b} : \&_n o \multimap o$, where $\&_n o$ stands for $o \& \cdots \& o$ (n copies)[2]. The choice does not impact how finite trees are represented: in both cases a $\triangleright_\beta$-normal $\Sigma \mid \_ \vdash t : o$ (if not $\bot$) must start with a variable from Σ with some arity n, followed by n $\triangleright_\beta$-normal sub-trees; *i.e.* it represents a tree (with certain branches possibly leading to $\bot$). The multiplicative vs additive distinction matters in the definition of schemes, though: with additive typing, resources (variables) may be shared when calculating two sub-branches of an infinite tree, which is disallowed with multiplicative typing.

Linear recursion schemes consist of a system of recursive equations, where each clause is given by a λ-term with a restricted shape. A term $\Gamma \mid \_ \vdash t : \varphi$ is called **applicative** if it is $\triangleright_\beta$-normal, and has the form $\lambda x_1^{\varphi_1} \ldots \lambda x_n^{\varphi_n}.t'$ where t' has no abstraction.

▶ **Definition 1.** *A **Multiplicative Additive Recursion Scheme (MAHORS)** is a 4-tuple $\mathcal{G} = \langle \Sigma, \mathcal{N}, \mathcal{R}, S \rangle$ where: (1) Σ is a ranked alphabet; (2) $\mathcal{N}$ is a finite set of typed **non-terminals**; we use upper-case letters $F, G, H, \ldots$ to range over them. We denote the type of F by $\mathcal{N}(F)$ and write $F : \mathcal{N}(F)$; (3) $S \in \mathcal{N}$ is a distinguished **start symbol** of type o; and (4) $\mathcal{R}$ is a function associating to each F in $\mathcal{N}$ an applicative term $\Sigma, \mathcal{N} \mid \_ \vdash \mathcal{R}(F) : \mathcal{N}(F)$, with Σ represented additively. A **MHORS** is defined as a MAHORS where Σ is represented multiplicatively and the typing of $\mathcal{N}$ does not involve products.*

If $\mathcal{G} = \langle \Sigma, \mathcal{N}, \mathcal{R}, S \rangle$ is a MAHORS, then for each $F \in \mathcal{N}$ and $n \in \mathbb{N}$ there is $\Sigma \mid \_ \vdash \mathsf{unf}_n(F) : \mathcal{N}(F)$ defined by $\mathsf{unf}_0(F) = \bot$ and $\mathsf{unf}_{n+1}(F) = \mathcal{R}(F)[\mathsf{unf}_n(G)/G \mid G \in \mathcal{N}]$. The family $(\mathsf{unf}_i(F))_{i\in\mathbb{N}}$ forms a chain for $\leq$ defined as usual by $\bot \leq t$, closed by congruence. As evaluation is monotone, $(\mathrm{BT}(\mathsf{unf}_i(F))_{i\in\mathbb{N}}$ also forms a chain, hence it has a lub which may be defined as the ideal completion of finite normal terms $\Sigma \mid \_ \vdash t : o$ ordered by $\leq$. We may then define $\mathrm{BT}(\mathcal{G}) = \bigsqcup_{i\in\mathbb{N}} \mathrm{BT}(\mathsf{unf}_i(S))$, the **infinite tree generated by** $\mathcal{G}$.

Our schemes comprise an explicit divergence symbol $\bot$. This is unusual, but does not affect expressivity as it could always be defined with a new non-terminal with rule $\mathcal{R}(\Omega) = \Omega$. Finally, we identify silently trees and terms $\Sigma \mid \_ \vdash t : o$.

3 Finite Memory Game Semantics and Geometry of Interaction

Game semantics is a semantic technique to give a compositional interpretation of higher-order programs [14]. By presenting higher-order computation as a *game* between two players embodying the program and its execution environment (Player for the program, Opponent for the environment), it effectively reduces higher-order computation to an exchange of tokens between terms. At first forgetting recursion, we briefly review the interpretation of the linear λ-calculus with products in *simple games*, then introduce its refined interpretation as finite-memory strategies, which will inform the translation of M(A)HORS to TSA.

3.1 Games and strategies

A **game** is a tuple $A = \langle M_A, \lambda_A, P_A \rangle$ where M_A is a set of *moves*, $\lambda_A : M_A \to \{O, P\}$ is a *polarity function* (we write $M_A^O = \lambda_A^{-1}(\{O\})$ and $M_A^P = \lambda_A^{-1}(\{P\})$), and $P_A \subseteq M_A^*$ is a non-empty prefix-closed set of *valid plays*, whose elements are O-starting and alternating: if $s = s_1 \ldots s_n \in P_A$, then $\lambda_A(s_1) = O$ and $\lambda_A(s_i) \neq \lambda_A(s_{i+1})$. We write $\epsilon \in P_A$ for the empty play and $s \sqsubseteq s'$ for the prefix ordering.

[2] [5] considers also intermediate typings, but this does not contribute extra expressivity.

$$\begin{array}{ccccccc}
(o_1 & \multimap & o_2) & \multimap & o_3 & \multimap & o_4 \\
 & & & & & & \circ_4 \\
 & & \bullet_2 & & & & \\
\circ_1 & & & & & & \\
 & & & & \bullet_3 & &
\end{array}$$

Figure 2 A play on $[\![(o \multimap o) \multimap o \multimap o]\!]$.

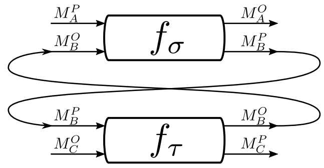


Figure 3 Composition of history-free skeletons.

Games represent *types*. Plays in a game for a type φ represent executions on φ following (for this paper) a call-by-name evaluation strategy. For instance, Figure 2 shows a play in the game for $(o \multimap o) \multimap o \multimap o$, read from top to bottom. We use indices on atom occurrences and moves for disambiguation, but the usual convention in game semantics is to signify the identity of moves simply by their position under the corresponding type component. After Opponent ($\circ$, the environment) starts computation by the initial move on the right, Player ($\bullet$, the program) responds by interrogating its function argument. Opponent, playing for this argument, calls its argument. Player terminates by calling its second argument. This play is, in fact, the maximal play of the interpretation of $\lambda f^{o \multimap o}.\, \lambda x^o.\, f\, x : (o \multimap o) \multimap o \multimap o$.

Each type φ may be interpreted as a game $[\![\varphi]\!]$. The game $[\![o]\!]$ has $M_{[\![o]\!]} = \{\circ\}$ with $\lambda(o) = O$, and $P_{[\![o]\!]} = \{\epsilon, \circ\}$. To match the type constructor $\multimap$, the **linear arrow game** $A \multimap B$ has as moves the tagged disjoint union $M_{A \multimap B} = M_A + M_B = \{1\} \times M_A \cup \{2\} \times M_B$ with polarity $\lambda_{A \multimap B}(1, a) = \overline{\lambda_A(a)}$ and $\lambda_{A \multimap B}(2, b) = \lambda_B(b)$, where $\overline{O} = P$ and $\overline{P} = O$. The plays $P_{A \multimap B}$ include all O-starting, alternating sequences $s \in M^*_{A \multimap B}$ such that the *restrictions* $s \upharpoonright A \in M_A^*$ and $s \upharpoonright B \in M_B^*$, defined in the obvious way, are in P_A and P_B respectively. Hence, $A \multimap B$ can be viewed as playing the two games A and B in parallel, with the polarity reversed in A, in such a way that any play must start in B and Player is able to switch between the components. With these definitions the reader can check that $[\![(o \multimap o) \multimap o \multimap o]\!] = ([\![o]\!] \multimap [\![o]\!]) \multimap ([\![o]\!] \multimap [\![o]\!])$ includes four moves corresponding to the four atom occurrences, and has only two maximal plays: the one in Figure 2, and $\circ_4 \bullet_3$.

The **tensor game** $A \otimes B$ has moves $M_{A \otimes B} = M_A + M_B$, polarity $\lambda_{A \otimes B}(1, a) = \lambda_A(a)$ and $\lambda_{A \otimes B}(2, b) = \lambda_B(b)$, and plays are those $s \in M^*_{A \otimes B}$ that are alternating, O-starting and such that $s \upharpoonright A \in P_A$ and $s \upharpoonright B \in P_B$. Dually to $\multimap$, it follows from the definition that here only O can change between components. The **product game** $A \,\&\, B$ has the same moves and polarity as $A \otimes B$, but only the plays where either $s \upharpoonright A$ or $s \upharpoonright B$ is empty. Hence, with their first move, Opponent fixes the component in which the rest of the game will be played.

A **strategy** σ on A, written $\sigma : A$, is $\sigma \subseteq P_A^{ev}$ (writing P_A^{ev} for the set of even-length plays) which is non-empty, closed under even-length prefix, and *deterministic*, in the sense that if $sab, sab' \in \sigma$, then $b = b'$. The interpretation of terms yields strategies; for instance

$$[\![\lambda f^{o \multimap o}.\, \lambda x^o.\, f\, x : (o \multimap o) \multimap o \multimap o]\!] = \{\epsilon,\ \circ_4 \bullet_2,\ \circ_4 \bullet_2 \circ_1 \bullet_3\}$$

is a strategy on $[\![(o \multimap o) \multimap o \multimap o]\!]$ with moves following the naming convention of Figure 2.

The interpretation of terms exploits a number of constructions on strategies. In particular, to compute the **composition** of $\sigma : A \multimap B$ and $\tau : B \multimap C$ we first let σ, τ interact by considering all sequences in $(M_A + M_B + M_C)^*$ whose restrictions to A, B and B, C are respectively in σ and τ; and then project those to $P_{A \multimap C}$ to obtain $\tau \circ \sigma : A \multimap C$. We omit the details [14]. Overall, the structure needed to interpret the linear λ-calculus with products is succinctly summarized by stating that games and strategies form a *symmetric monoidal closed category with products* [14] – to any $\_ \mid x_1 : \varphi_1, \ldots, x_n : \varphi_n \vdash t : \varphi$ this lets us associate $[\![t]\!] : \bigotimes_{1 \leq i \leq n} [\![\varphi_i]\!] \multimap [\![\varphi]\!]$ in such a way that this is invariant under reduction – note however that in this paper, we avoid the categorical language as much as possible.

$$
\begin{array}{cccccc}
(o_1 \multimap & o_2) \multimap & ((o_3 \multimap & o_4) \;\&\; & (o_5 \multimap & o_6)) \\
 & & & \circ_4 & & \\
 & \bullet_2 & & & & \\
\circ_1 & & & & & \\
 & & \bullet_3 & & &
\end{array}
\qquad\qquad
\begin{array}{cccccc}
(o_1 \multimap & o_2) \multimap & ((o_3 \multimap & o_4) \;\&\; & (o_5 \multimap & o_6)) \\
 & & & & & \circ_6 \\
 & \bullet_2 & & & & \\
\circ_1 & & & & & \\
 & & & & \bullet_5 &
\end{array}
$$

Figure 4 The two maximal plays of contraction on $[\![o \multimap o]\!]$.

3.2 History-free and finite memory strategies

A strategy $\sigma : A$ is **history-free** if its behaviour only depends on the last move, *i.e.* there is a partial function $f : M_A^O \rightharpoonup M_A^P$ such that for all $s \in \sigma$, for all $sa \in P_A$, we have $sab \in \sigma$ iff $f(a)$ is defined and $b = f(a)$. It is key in *AJM games* [1] that, without products, terms yield history-free strategies. If $\sigma : A$ is history-free, it is characterized by the corresponding partial function $f : M_A^O \rightharpoonup M_A^P$, known as its *history-free skeleton*. For instance, the strategy $[\![\lambda f^{o\multimap o}.\, \lambda x^o.\, f\, x]\!]$ with a unique maximal play in Figure 2, has history-free skeleton $\{\circ_4 \mapsto \bullet_2,\ \circ_1 \mapsto \bullet_3\}$.

One can also directly interpret terms as history-free skeletons: this is usually referred to as *Geometry of Interaction* [11], which has close ties with game semantics [3]. In particular, composition of history-free strategies can be performed directly on skeletons. If $\sigma : A \multimap B$ and $\tau : B \multimap C$ are history-free, their history-free skeletons, which have the types

$$f_\sigma : M_A^P + M_B^O \rightharpoonup M_A^O + M_B^P \qquad\qquad f_\tau : M_B^P + M_C^O \rightharpoonup M_B^O + M_B^P,$$

may be composed via *feedback* on B, pictured in Figure 3. For any Opponent move in $A \multimap C$, we apply the corresponding function f_σ or f_τ. As long as the response is in B, we keep applying f_σ and f_τ alternately. This process may stay in B forever (a *livelock*, in which case the composition $f_{\tau\circ\sigma}$ is undefined), but otherwise we eventually get a Player move in $A \multimap C$ as required; defining a partial function $f_{\tau\circ\sigma} : M_{A\multimap C}^O \rightharpoonup M_{A\multimap C}^P$. One may visualize a token entering on the left carrying an Opponent move, then bouncing in B until it eventually exits on the right. Other constructions used in the interpretation may be presented similarly, altogether giving (for the linear λ-calculus) a presentation of evaluation through a finite automaton called a *token machine*, where a token enters through an Opponent move, and bounces through the term until it eventually exits, giving the result of computation [18].

This is our starting point to represent evaluation of M(A)HORS via an automaton. However, there is an issue: strategies for linear λ-terms *with products* are not in general history-free. For instance, Figure 4 displays the two maximal plays of a *contraction/duplication* strategy $[\![\lambda f^{o\multimap o}.\, \langle f, f\rangle : (o \multimap o) \multimap ((o \multimap o) \,\&\, (o \multimap o))]\!]$. It reacts to $\circ_1$ differently depending on the history. To account for this, one may replace partial functions $f : M_A^O \rightharpoonup M_A^P$ with $f : M_A^O \times \mathcal{M} \rightharpoonup M_A^P \times \mathcal{M}$, *i.e. transducers*, where $\mathcal{M}$, the *memory*, is a finite set (see the *memoryful geometry of interaction* of [13] – however, we are not aware of this being used to define finite memory strategies). We give below a definition in this spirit, adapted to ease the translation to TSA and to deal with the branching in M(A)HORS due to *terminal symbols*.

We fix a ranked alphabet Σ (the multiplicative/additive distinction plays no role here).

▶ **Definition 2.** *A* ***transducer*** $\mathcal{T}$ *on a game* A*, written* $\mathcal{T} : A$*, is* $\mathcal{T} = \langle \mathcal{M}_- \uplus \mathcal{M}_+, m_0, \delta_-, \delta_+\rangle$ *where* $\mathcal{M}_-$ *is a finite set of* ***passive memory states*** *with a distinguished* ***initial memory state*** $m_0 \in \mathcal{M}_-$*,* $\mathcal{M}_+$ *is a finite set of* ***active memory states****, and* ***transition functions****:*

$$
\begin{array}{rcrcl}
\delta_- & : & \mathcal{M}_- \times M_A^O & \to & \mathcal{M}_+ \\
\delta_+ & : & \mathcal{M}_+ & \rightharpoonup & \mathcal{M}_+ \;+\; \mathcal{M}_- \times M_A^P \;+\; \{\mathsf{b}(m_1, \ldots, m_{|\mathsf{b}|}) \mid m_i \in \mathcal{M}_+,\ \mathsf{b} \in \Sigma\}\,.
\end{array}
$$

$$\delta_+^{\mathcal{T}\odot\mathcal{S}}((m_{\mathcal{S}}^-, m_{\mathcal{T}}^+)) = \begin{cases} (m_{\mathcal{S}}^-, m') & \text{if } \delta_+^{\mathcal{T}}(m_{\mathcal{T}}^+) = m' \\ \mathsf{b}((m_{\mathcal{S}}^-, m_1), \dots, (m_{\mathcal{S}}^-, m_{|\mathsf{b}|})) & \text{if } \delta_+^{\mathcal{T}}(m_{\mathcal{T}}^+) = \mathsf{b}(m_1, \dots, m_{|\mathsf{b}|}) \\ ((m_{\mathcal{S}}^-, m_{\mathcal{T}}^-), (2, c)) & \text{if } \delta_+^{\mathcal{T}}(m_{\mathcal{T}}^+) = (m_{\mathcal{T}}^-, (2, c)) \\ (\delta_-^{\mathcal{S}}(m_{\mathcal{S}}^-, (2, b)), m_{\mathcal{T}}^-) & \text{if } \delta_+^{\mathcal{T}}(m_{\mathcal{T}}^+) = (m_{\mathcal{T}}^-, (1, b)) \end{cases}$$

$$\delta_+^{\mathcal{T}\odot\mathcal{S}}((m_{\mathcal{S}}^+, m_{\mathcal{T}}^-)) = \begin{cases} (m', m_{\mathcal{T}}^-) & \text{if } \delta_+^{\mathcal{S}}(m_{\mathcal{S}}^+) = m' \\ \mathsf{b}((m_1, m_{\mathcal{T}}^-), \dots, (m_{|\mathsf{b}|}, m_{\mathcal{T}}^-)) & \text{if } \delta_+^{\mathcal{S}}(m_{\mathcal{S}}^+) = \mathsf{b}(m_1, \dots, m_{|\mathsf{b}|}) \\ ((m_{\mathcal{S}}^-, m_{\mathcal{T}}^-), (1, a)) & \text{if } \delta_+^{\mathcal{S}}(m_{\mathcal{S}}^+) = (m_{\mathcal{S}}^-, (1, a)) \\ (m_{\mathcal{S}}^-, \delta_-^{\mathcal{T}}(m_-^{\mathcal{T}}, (1, b))) & \text{if } \delta_+^{\mathcal{S}}(m_{\mathcal{S}}^+) = (m_{\mathcal{S}}^-, (2, b)) \end{cases}$$

Figure 5 Positive transitions of the composition of strategic transducers.

Any transducer $\mathcal{T}$ on $[\![o]\!]$ will be called **closed**. Apart from the forced initial $\delta_-(m_0, \circ)$, it is a finite tree-generating automaton, producing a tree $\mathsf{Tree}(\mathcal{T})$. But in general transducers may play on arbitrary games. In passive states, a transducer is waiting for an Opponent move, while in active states, it is performing internal computation that may result in a terminal symbol or in a Player move and the transition to a passive state. If $\delta_+(m) = \mathsf{b}(m_1, \dots, m_{|\mathsf{b}|})$, it produces the terminal symbol b; exploring the ith child results in continuing with m_i.

Like strategies, transducers can be *composed*.

▶ **Definition 3.** *Let $\mathcal{S} = (\mathcal{M}_-^{\mathcal{S}} \uplus \mathcal{M}_+^{\mathcal{S}}, m_0^{\mathcal{S}}, \delta_-^{\mathcal{S}}, \delta_+^{\mathcal{S}}) : A \multimap B$ and $\mathcal{T} = (\mathcal{M}_-^{\mathcal{T}} \uplus \mathcal{M}_+^{\mathcal{T}}, m_0^{\mathcal{T}}, \delta_-^{\mathcal{T}}, \delta_+^{\mathcal{T}}) : B \multimap C$ be transducers. The transducer $\mathcal{T} \odot \mathcal{S}$ on game $A \multimap C$ has $\mathcal{M}_-^{\mathcal{T}\odot\mathcal{S}} = \mathcal{M}_-^{\mathcal{S}} \times \mathcal{M}_-^{\mathcal{T}}$ and $\mathcal{M}_+^{\mathcal{T}\odot\mathcal{S}} = \mathcal{M}_+^{\mathcal{S}} \times \mathcal{M}_-^{\mathcal{T}} \uplus \mathcal{M}_-^{\mathcal{S}} \times \mathcal{M}_+^{\mathcal{T}}$, with initial state $(m_0^{\mathcal{S}}, m_0^{\mathcal{T}})$. The* transition function *is defined via $\delta_-^{\mathcal{T}\odot\mathcal{S}}((m_{\mathcal{S}}^-, m_{\mathcal{T}}^-), (2, c)) = (m_{\mathcal{S}}^-, \delta_-^{\mathcal{T}}(m_{\mathcal{T}}^-, (2, c)))$, $\delta_-^{\mathcal{T}\odot\mathcal{S}}((m_{\mathcal{S}}^-, m_{\mathcal{T}}^-), (1, a)) = (\delta_-^{\mathcal{S}}(m_{\mathcal{S}}^-, (1, a)), m_{\mathcal{T}}^-)$, and positive transitions given in Figure 5.*

Besides composition, all operations on strategies used in the interpretation of the linear λ-calculus with products have a counterpart on transducers. Altogether, for any $\Sigma \mid x_1 : \varphi_1, \dots, x_n : \varphi_n \vdash t : \varphi$, this yields a transducer $\langle\!|t|\!\rangle : \otimes_{1 \le i \le n} [\![\varphi_i]\!] \multimap [\![\varphi]\!]$. In particular, if $\Sigma \mid \_ \vdash t : o$, this yields a *closed* transducer $\langle\!|t|\!\rangle : [\![o]\!]$. It is obtained directly by induction on syntax following denotational semantics, and in particular in polynomial time. We can prove:

▶ **Proposition 4.** *For any $\Sigma \mid \_ \vdash t : o$, $\mathsf{Tree}(\langle\!|t|\!\rangle) = \mathrm{BT}(t)$.*

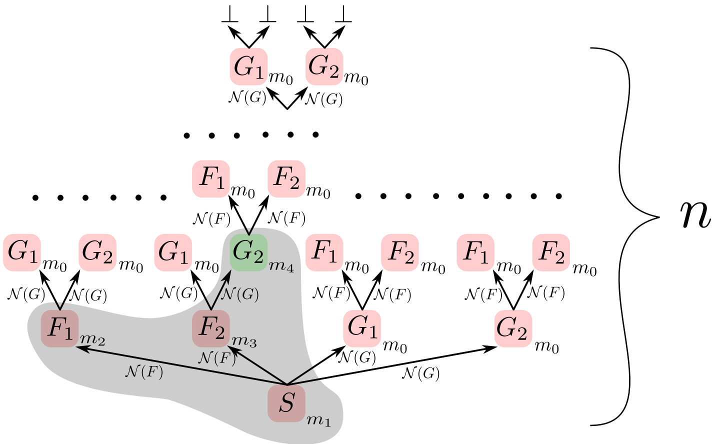
The proof works by linking transducers with game semantics. The simple game semantics presented above cannot directly deal with the presence of non-terminals replicable at will and the associated branching, so we must first extend it to "tree-generating game semantics". The details, though rather direct, are too lengthy for the paper, so we instead present the connection ignoring the terminal symbols.

Ignoring branching transitions, transducers generate strategies. Writing $m^- \overset{a}{\to} m^+$ when $\delta_-(m^-, a) = m^+$, $m_1^+ \to m_2^+$ when $\delta_+(m_1^+) = m_2^+$ and $m^+ \overset{b}{\to} m^-$ when $\delta_+(m^+) = (m^-, b)$; the set $\mathrm{Traces}(\mathcal{T})$ comprises all sequences $s_1 \dots s_{2n} \in M_A^*$ such that (with $m_0, \dots, m_n \in \mathcal{M}_-$)

$$m_0 \overset{s_1}{\to} \to^* \overset{s_2}{\to} m_1 \overset{s_3}{\to} \to^* \overset{s_4}{\to} m_2 \dots m_{n-1} \overset{s_{2n-1}}{\to} \to^* \overset{s_{2n}}{\to} m_n \,.$$

We say that $\mathcal{T}$ is a **strategic transducer** if for all $s \in \mathrm{Traces}(\mathcal{T}) \cap P_A$, if $sa \in P_A$ and $sab \in \mathrm{Traces}(\mathcal{T})$, then $sab \in P_A$. Then, $\mathrm{Traces}(\mathcal{T}) \cap P_A$ is a strategy written $\mathrm{Strat}(\mathcal{T})$. We say that $\sigma : A$ has **finite memory** if $\sigma = \mathrm{Strat}(\mathcal{T})$ for a strategic transducer $\mathcal{T}$. We also recover *history-free strategies* as those for which $\mathcal{M}_-$ is a singleton. For instance, the strategy in Figure 4 is generated using $\mathcal{M}_- = \{m_0, m_1\}$ and $\mathcal{M}_+ = \mathcal{M}_- \times M^O_{[\![o \multimap o]\!]}$, $\delta_-(m, a) = (m, a)$, $\delta_+(\_, \circ_4) = (m_0, \bullet_2)$, $\delta_+(\_, \circ_6) = (m_1, \bullet_2)$, $\delta_+(m_0, \circ_1) = (m_0, \bullet_3)$ and $\delta_+(m_1, \circ_1) = (m_1, \bullet_5)$.

Proposition 4 boils down to the fact that all constructions on transducers in the interpretation preserve strategic transducers, and match operations on strategies – for instance, $\mathrm{Strat}(\mathcal{T} \odot \mathcal{S}) = \mathrm{Strat}(\mathcal{T}) \circ \mathrm{Strat}(\mathcal{S})$. This entails that for all t, $\mathrm{Strat}(\langle\!|t|\!\rangle) = [\![t]\!]$. But for closed transducers $\langle\!|t|\!\rangle$ and tree-generating game semantics, $\mathsf{Tree}(\langle\!|t|\!\rangle) = \mathrm{Strat}(\langle\!|t|\!\rangle)$. Since game semantics is invariant under reduction, $[\![t]\!] = [\![\mathrm{BT}(t)]\!] = \mathrm{BT}(t)$, and Proposition 4 follows.



■ **Figure 6** Illustration of a state of the n-th unfolding.

4 Game Semantics to TSA

The previous section lets us associate, to any $\Sigma \mid \_ \vdash t : o$, a finite tree-generating automaton. We extend this with recursion in two steps: first we evaluate finite unfoldings using finite automata, and then we build a single automaton with additional memory (a *Tree Stack Automaton*) whose runs amount to dynamically exploring these finite unfoldings.

4.1 Unfolding recursive calls

Let us fix a M(A)HORS $\mathcal{G} = \langle \Sigma, \mathcal{N}, \mathcal{R}, S \rangle$. By definition, for each $F \in \mathcal{N}$ we have $\Sigma, \mathcal{N} \mid \_ \vdash \mathcal{R}(F) : \mathcal{N}(F)$. Let $N \in \mathbb{N}$ be such that for all $F, G \in \mathcal{N}$, G appears at most N times in $\mathcal{R}(F)$. For all $F \in \mathcal{N}$, we choose a term $\Sigma \mid \mathcal{N}_1, \dots, \mathcal{N}_N \vdash \mathcal{R}'(F) : \mathcal{N}(F)$ obtained by giving different names $G_1, \dots, G_p$ ($p \leq N$) to all occurrences of $G \in \mathcal{N}$ in $\mathcal{R}(F)$. How these names are assigned does not matter. Although $\mathcal{R}'$ differs from $\mathcal{R}$, it can be equivalently used to define the finite approximations of $\mathrm{BT}(\mathcal{G})$. For each $F \in \mathcal{N}$ and $n \in \mathbb{N}$, we redefine $\Sigma \mid \_ \vdash \mathsf{unf}_n(F) : \mathcal{N}(F)$ by setting $\mathsf{unf}_0(F) = \bot$, and $\mathsf{unf}_{n+1}(F) = \mathcal{R}'(F)[\mathsf{unf}_n(G)/G_i \mid G \in \mathcal{N}_i, 1 \leq i \leq N]$. Although defined differently, this gives the same result as in Section 2.

But, unlike the original unfolding, this one can be replicated with strategic transducers. For each $F \in \mathcal{N}$, the interpretation of the previous section yields a strategic transducer:

$$\langle \mathcal{R}'(F) \rangle : \bigotimes_{1 \leq i \leq N} \bigotimes_{G \in \mathcal{N}} [\![\mathcal{N}(G)]\!] \multimap [\![\mathcal{N}(F)]\!] .$$

The unfolding above can then be replicated as follows.

▶ **Proposition 5.** *Setting $\mathcal{T}_F^0 = \bot$ with all positive transitions undefined, and $\mathcal{T}_F^{n+1} = \langle \mathcal{R}'(F) \rangle \odot (\bigotimes_{1 \leq i \leq N} \bigotimes_{G \in \mathcal{N}} \mathcal{T}_G^n) : [\![\mathcal{N}(F)]\!]$, for all $n \in \mathbb{N}$, we have $\mathsf{Tree}(\mathcal{T}_S^n) = \mathrm{BT}(\mathsf{unf}_n(S))$.*

Proof. By the substitution lemma for symmetric monoidal closed categories with products, syntactic substitution matches composition in the denotational model. It follows by induction that for all $F \in \mathcal{N}$, for all $n \in \mathbb{N}$, $\langle \mathsf{unf}_n(F) \rangle$ and $\mathcal{T}_F^n$ are transducers generating the same finite memory strategy. By Proposition 4, $\mathsf{Tree}(\mathcal{T}_S^n) = \mathsf{Tree}(\langle \mathsf{unf}_n(S) \rangle) = \mathrm{BT}(\mathsf{unf}_n(S))$. ◀

Figure 6 displays the structure of transducer compositions arriving at the finite tree automaton $\mathcal{T}_S^n$, for a M(A)HORS $\mathcal{G}$ where $\mathcal{R}(S)$ has two occurrences of F and two occurrences of G, $\mathcal{R}(F)$ has two occurrences of G, and $\mathcal{R}(G)$ has two occurrences of F. Each node stands for the matching strategic transducer (corresponding to a non-terminal), edges represent compositions. Running $\mathcal{T}_S^n$ passes control between the composed transducers, with always exactly one active after the initial transition. Figure 6 shows a possible state during a run: the grey area marks nodes that have already been explored. Outside of the grey area, the (local) transducer memory must be m_0. The green node is active, and all others passive. Following the transition function of $\langle\mathcal{R}(G)\rangle$, we may next update the local memory m_4, produce a terminal and branch, or update to a passive state and send control up or down.

4.2 Tree Stack Automata

Now we give a single automaton with infinite memory whose bounded restrictions match the approximations above. It has a *stack* to deal with recursion, such that each state of the stack corresponds to a node in Figure 6. As these nodes stand for strategic transducers, they all have a finite memory. Accordingly, the automaton maintains a *store* associating, to each previously visited stack state/node, its *local memory*, accessed or updated only when visiting that node. We think of the store as a tree: the stack alphabet denotes *directions*, and stack values denote *positions* in the tree, *i.e. nodes* in (the infinite version of) Figure 6. *Pushes* and *pops* correspond to moving *up* and *down* the tree. Such an automata model is known as a *Tree Stack Automaton (TSA)* [7] – here, we introduce *tree-generating TSA*.

▶ **Definition 6.** *A tree-generating TSA $\mathcal{A}$ is a tuple $\langle\Sigma, Q, \Gamma, \mathcal{M}, \delta, q_0, \gamma_0, m_0\rangle$ where Σ is a ranked alphabet of* terminals*, Q is a set of* states*, Γ is a finite* stack alphabet*, $\mathcal{M}$ is a finite* memory alphabet*, $q_0 \in Q$ is the* starting state*, $\gamma_0 \notin \Gamma$ is the* bottom-of-stack marker *and $m_0 \in \mathcal{M}$ is the* initial local memory*. Letting $\Gamma^\bullet = \Gamma \uplus \{\gamma_0\}$, the transition function δ has type:*

$$\delta : Q \times \mathcal{M} \times \Gamma^\bullet \rightharpoonup Q + \{\mathsf{b}(q_1, \dots, q_{|\mathsf{b}|}) \mid q_i \in Q,\, \mathsf{b} \in \Sigma\} + Q \times \mathcal{M} \times (\{up_\gamma \mid \gamma \in \Gamma\} + \{down\}) .$$

Informally, the transitions operate as follows. Initially, only γ_0 is on the stack. Subsequently, given state q, local memory m, and top of the stack $\gamma \in \Gamma^\bullet$:

1. if $\delta(q, m, \gamma) = q'$, the automaton changes state to q', leaving the stack and local memory unchanged;
2. if $\delta(q, m, \gamma) = \mathsf{b}(q_1, \dots, q_{|\mathsf{b}|})$, it outputs $\mathsf{b} \in \Sigma$ and branches – to explore the ith child ($1 \le i \le |b|$) it proceeds to state q_i leaving other components unchanged;
3. if $\delta(q, m, \gamma) = (q', m', up_{\gamma'})$, it updates the local memory to m', changes state to q' and pushes γ' onto the stack / moves up in direction γ' (if this is the first visit to that node, its local memory is set to m_0);
4. if $\delta(q, m, \gamma) = (q', m', down)$, it updates the local memory to m' and the state to q', and then pops / moves down (we adopt the convention that γ_0 cannot be popped so, if $\gamma = \gamma_0$ in this case, the automaton blocks).

Running a TSA $\mathcal{A}$ produces a possibly infinite tree $\mathsf{Tree}(\mathcal{A})$.

In the degenerate case where $\mathcal{M} = \{m_0\}$, tree-generating TSAs turn out to be precisely tree-generating deterministic pushdown automata (PDA): the local memory cannot store information, so only the stack remains. In general, however, it is not hard to see that TSAs are Turing-complete; fortunately we will only need TSAs satisfying a further condition called *restriction* [7]. A tree-generating TSA is k**-restricted** if every node can be accessed from below at most k times. It is **restricted** if it is k-restricted for some $k \in \mathbb{N}$.

We implement the evaluation of a MAHORS $\mathcal{G}$ with a restricted TSA $\mathcal{A}(\mathcal{G})$ with states

$$\mathcal{Q} = (\sum_{F \in \mathcal{N}} M^O_{\otimes_{1 \le i \le N} \otimes_{G \in \mathcal{N}} [\![\mathcal{N}(G)]\!] \multimap [\![\mathcal{N}(F)]\!]}) + (\sum_{F \in \mathcal{N}} \mathcal{M}_+^{\langle\mathcal{R}'(F)\rangle}).$$

$$\begin{array}{rcll}
(\mathsf{Move}(F,a),(F,m),_) & \mapsto & \mathsf{State}(F,\delta^F_-(m,a)) & \\
(\mathsf{State}(F,m),_,_) & \mapsto & \mathsf{State}(F,m') & \text{if } \delta^F_+(m) = m' \\
(\mathsf{State}(F,m),_,_) & \mapsto & \mathsf{b}(\mathsf{State}(F,m_1),\dots,\mathsf{State}(F,m_{|\mathsf{b}|})) & \text{if } \delta^F_+(m) = \mathsf{b}(m_1,\dots,m_{|\mathsf{b}|}) \\
(\mathsf{State}(F,m),_,_) & \mapsto & (\mathsf{Move}(G,(2,a)),(F,m'),up_{(F,i)}) & \text{if } \delta^F_+(m) = (m',(1,i,G,a)) \text{ with } a \in M^O_{\mathcal{N}(G)} \\
(\mathsf{State}(G,m),_,(F,i)) & \mapsto & (\mathsf{Move}(F,(1,i,G,a)),(G,m'),down) & \text{if } \delta^G_+(m) = (m',(2,a)) \text{ with } a \in M^P_{\mathcal{N}(G)}
\end{array}$$

Figure 7 Transition function for the GoI TSA.

We use constructors Move and State to refer to elements from the left and right components of $\mathcal{Q}$ respectively. The *memory alphabet* is $\mathcal{M} = \sum_{F\in\mathcal{N}} \mathcal{M}_-^{\langle\mathcal{R}'(F)\rangle} / \equiv$, where $\equiv$ is the smallest equivalence relation with $(F,m_0) \equiv (G,m_0)$ for all $F,G \in \mathcal{N}$. We write m_0 for this equivalence class, providing the *initial memory state*. The *stack alphabet* is $\Gamma = N \times \mathcal{N}$ where N is the smallest integer such that all non-terminals have fewer than N occurrences in $\mathcal{R}(F)$, for all $F \in \mathcal{N}$. The start state is $q_0 = \mathsf{Move}(S,\circ)$ and the transition function is given in Figure 7.

The TSA $\mathcal{A}(\mathcal{G})$ is designed so that a run of stack size bounded by n simulates a run of $\mathcal{T}^n_S$. When in state $\mathsf{State}(F,m)$, the automaton is currently operating in a F node of $\mathcal{T}^n_S$ (as in Figure 6), performing internal computation following δ^F_+. If this internal computation produces a move, this move will be addressed either up or down the stack, depending of whether it is a Player move in $\mathcal{N}(F)$ (in which case we must move down), or an Opponent move in $\otimes_{1\le i\le N}\otimes_{G\in\mathcal{N}}[\![\mathcal{N}(G)]\!]$ (in which case we must move up, passing the control to a recursive call). If the state is $\mathsf{State}(G,m)$ and the top of the stack is (F,i), that means that we are currently running non-terminal G, which was called as the i-th occurrence of G in F. So the stack, together with the non-terminal symbol in the state, indicate the address of a node in Figure 6. When moving up or down the stack, we first change to a transient state $\mathsf{Move}(F,a)$ in which the automaton reads the input move using δ^F_- and resumes as above.

▶ **Theorem 7.** *For any MAHORS $\mathcal{G}$, there exists a restricted TSA $\mathcal{A}(\mathcal{G})$ (constructed in polynomial time) such that* $\mathsf{Tree}(\mathcal{A}(\mathcal{G})) = \mathrm{BT}(\mathcal{G})$.

Proof. For $n \ge 1$, write $\mathsf{Tree}_n(\mathcal{A}(\mathcal{G}))$ for the tree obtained from the truncated run-tree where the stack size is bounded by $n-1$ (where γ_0 has size 0). By construction, this truncated run-tree is weakly bisimilar to that of $\mathcal{T}^n_S$. In particular, $\mathsf{Tree}_n(\mathcal{A}(\mathcal{G})) = \mathsf{Tree}(\mathcal{T}^n_S) = \mathrm{BT}(\mathsf{unf}_n(S))$ by Proposition 5, so $\mathsf{Tree}(\mathcal{A}(\mathcal{G})) = \mathrm{BT}(\mathcal{G})$ by continuity.

This TSA is restricted: for any type φ, there is a bound on the length of plays in $P_{[\![\varphi]\!]}$ – in fact $M_{[\![\varphi]\!]}$ is finite, and plays in $P_{[\![\varphi]\!]}$ cannot use the same move twice. Let k be an upper bound to the maximal length of a play in $P_{\otimes_{G\in\mathcal{N}}[\![\mathcal{N}(G)]\!]}$. Then, $\mathcal{A}(\mathcal{G})$ is k-restricted. Indeed, fix a stack value $\gamma_{n+1}\gamma_n\dots\gamma_0$ with $\gamma_{n+1} = (F,i)$. Then, all transitions moving between $\gamma_{n+1}\dots\gamma_0$ and $\gamma_n\dots\gamma_0$ carry a move from $M_{\otimes_{G\in\mathcal{N}}[\![\mathcal{N}(G)]\!]}$. By construction, the sequence of such moves forms a play in $P_{\otimes_{G\in\mathcal{N}}[\![\mathcal{N}(G)]\!]}$. Hence, it is bounded by k. ◀

If the input scheme is an MHORS then each $\mathcal{R}'(F)$ is interpreted by a history-free strategy: $\mathcal{M}_-^{\langle\mathcal{R}'(F)\rangle}$ is a singleton. Consequently, $\mathcal{A}(\mathcal{G})$ has trivial memory and is in fact simply a PDA. This PDA is still k-restricted but also satisfies a stronger *linearity* property:

▶ **Lemma 8.** *Let $\mathcal{G}$ be an MHORS. Then the tree-generating PDA $\mathcal{A}(\mathcal{G})$ is **linear**, in the sense that the associated graph of reachable configurations is a tree.*

Proof. A strategic transducer on A is **reversible** if for each $a \in M^P_A$ there is at most one $m \in \mathcal{M}_+$ such that $\delta_+(m) = (\_,a)$ and for each $m \in \mathcal{M}_+$ there is at most one $(m',a) \in \mathcal{M}_- \times M^O_A$ such that $\delta_-(m',a) = m$ or at most one $m' \in \mathcal{M}_+$ such that $\delta(m') = m$, and the two possibilities are mutually exclusive. Reversible strategic transducers are closed under all operations used in the interpretation, hence if $\Sigma \mid \Delta \vdash t : A$ involves no product, $\langle t\rangle$ is *reversible* (this phenomenon is well-known in GoI [6]). This entails that $\mathcal{A}(\mathcal{G})$ is linear. ◀

5 TSA to MAHORS

In this section we show how to simulate a k-restricted TSA $\mathcal{A} = \langle \Sigma, Q, \Gamma, \mathcal{M}, \delta, q_0, \gamma_0, m_0 \rangle$ in MAHORS, i.e. we establish the converse of Theorem 7.

Let $B = |\Gamma|$ and $\Gamma = \{\gamma_1, \cdots, \gamma_B\}$. Nodes of the tree store will be represented using non-terminals $F^{u_1,\cdots,u_B;d}_{q,m,\gamma}$, where $(q, m, \gamma) \in Q \times \mathcal{M} \times \Gamma^\bullet$ represent the current state, node label and top of the stack respectively, d $(1 \le d \le k+1)$ is the number of times the node has already been visited from below and each u_j $(0 \le u_j \le k,\ 1 \le j \le B)$ is the number of times that the jth child has been visited from below. For brevity, we will write $\vec{u}$ instead of $u_1, \cdots, u_B$.

For $1 \le d \le k$, $F^{\vec{u};d}_{q,m,\gamma}$ has $B+1$ arguments: the first B arguments are used to simulate up_{γ_j} $(1 \le j \le B)$ and the last one corresponds to *down*. Each of the arguments is a Q-indexed tuple of continuations, so that projection can be used to select the right component to model the associated state change. When moving up the tree (up_{γ_j}), we call the jth argument passing as an argument another continuation that makes it possible to return (move down) later. Dually, when moving down the tree, we call the last argument passing as an argument a continuation that represents a further visit up. Using these ideas, one could code unrestricted TSA in an untyped setting, but we shall rely on carefully crafted types that allow, for each node, for up to k visits from below. In particular, if the automaton is moving down having visited a node k times from below, the corresponding upwards continuation for the $k+1$ visit is of type o, i.e. it is not usable for any future calls. The rules for $1 \le d \le k$ are summarised in the table below, using λ notation for brevity (for $F^{\vec{u};k+1}_{q,m,\gamma}$ we set $F^{\vec{u};k+1}_{q,m,\gamma} x_1 \cdots x_B = \bot$).

$\delta(q,m,\gamma)$	rule
q'	$F^{\vec{u};d}_{q,m,\gamma}\, x_1 \cdots x_B y = F^{\vec{u};d}_{q',m,\gamma}\, x_1 \cdots x_B y$
$\mathsf{b}(q_1, \cdots, q_{\lvert\mathsf{b}\rvert})$	$F^{\vec{u};d}_{q,m,\gamma}\, x_1 \cdots x_B y = \mathsf{b}\, \langle F^{\vec{u};d}_{q_1,m,\gamma}\, x_1 \cdots x_B y, \cdots, F^{\vec{u};d}_{q_{\lvert\mathsf{b}\rvert},m,\gamma}\, x_1 \cdots x_B y \rangle$
$(q', m', down)$	$F^{\vec{u};d}_{q,m,\gamma}\, x_1 \cdots x_B y = (\pi_{q'} y)\, \langle F^{\vec{u};d+1}_{q'',m',\gamma}\, x_1 \cdots x_B \mid q'' \in Q \rangle$
(q', m', up_{γ_j})	$F^{\vec{u};d}_{q,m,\gamma}\, x_1 \cdots x_B y = (\pi_{q'} x_j)\, \langle \lambda z^{\overline{T_j}}.F^{\vec{u}+e_j;d}_{q'',m',\gamma}\, x_1 \cdots x_{j-1} z x_{j+1} \cdots x_B y \mid q'' \in Q \rangle$

In the *down* case, note that the q'th component of y is used to model state change and that the continuation features m' instead of m to reflect the local memory update. Note also the change from d to $d+1$, which updates the count of visits from below.

In the *up* case, the q'th component of x_j is used to model state change and the direction of the upward move (γ_j). The use of the same γ on both sides captures the same position on the stack and m' is used on the rhs to simulate the local memory update. d does *not* change, because the continuation represents revisiting the node from above (rather than from below). However, once the node is revisited from above in the future, its jth child will have been visited $u_j + 1$ times from below: hence the change to u_j (we write $\vec{u} + e_j$ for $\vec{u}$ with the jth component incremented by 1). In the *up* case, we use a λ-term inside a rule to highlight the intention more clearly, this can be avoided by using an auxiliary non-terminal.

The start symbol $S : o$ has rule $S = (F^{0,\cdots,0;1}_{q_0,m_0,\gamma_0})\, N_1 \cdots N_B \langle \bot \mid q \in Q \rangle$. The divergent terms correspond to our convention that the automaton blocks when *down* is called at the root node. N_j $(1 \le j \le B)$ stands for $\langle N_{q,j} \mid q \in Q \rangle$, where $N_{q,j}$ are auxiliary non-terminals that represent nodes visited for the first time. They are subject to the rule $N_{q,j} y = F^{0,\cdots,0;1}_{q,m_0,\gamma_j} N_1 \cdots N_B y$.

The scheme depends on types of the form T_i $(0 \le i \le k)$ defined by $T_k = o$ and $T_i = \overline{(\overline{T_{i+1}} \multimap o)} \multimap o$, where $\overline{T}$ stands for $\&_{q \in Q} T$, i.e. $|Q|$ copies of T. In particular, we have $F^{\vec{u};d}_{q,m,\gamma} : \overline{T_{u_1}} \multimap \cdots \multimap \overline{T_{u_B}} \multimap T_{d-1}$ and $N_{q,j} : T_0$.

▶ **Theorem 9.** *For any restricted TSA, there exists an equivalent MAHORS (constructible in exponential time).*

In conjunction with Theorem 7, this shows that MAHORS and restricted TSA are equivalent.

6 Expressivity of MAHORS

It is easy to see that any (classic) first-order recursion scheme (1-HORS) can be viewed as a MAHORS, simply by giving the terminals types of the form $o \,\&\, \cdots \,\&\, o \multimap o$. Hence, MAHORS are at least as expressive as first-order HORS. Next, informed by results from the preceding sections, we can discuss their relationship with schemes of higher orders. Because our TSA model is a tree-generating variant of the automata from [7], which capture multiple context-free languages [21], we can immediately conclude the following.

▶ **Lemma 10.** *The branch language of a tree generated by a MAHORS is multiple context-free.*

Thanks to the Lemma, we can show that MAHORS and second-order HORS are incomparable.

▶ **Example 11.** There exists a second-order HORS, which is not equivalent to any MAHORS. For example, consider the 2-HORS given by: $S = Fb$, $Ff = a(f\$)(F(Gf))$, $Gfx = f(fx)$, where $a : o \to o \to o$, $b : o \to o$ and $\$: o$ are terminals and $F : (o \to o) \to o$ and $G : (o \to o) \to o \to o$ are non-terminals. The scheme generates an infinite tree whose finite branches correspond to the language $L = \{a^n b^{2^{n-1}} \$ \mid n \geq 1\}$. Because it is known that L is not multiple context-free [21, Lemma 3.5], it cannot be the branch language of a MAHORS by Lemma 10.

▶ **Example 12.** We give a MAHORS that is not equivalent to any second-order HORS, exploiting the fact that the language $L = \{w\#w\#w \mid w \in D\}$, where D is the Dyck language ($D = \epsilon \mid [D]D$), is not indexed [8] (see also page 2 of [16]). The MAHORS given below (using λ-syntax for brevity) has been obtained by lifting the grammar rules for D to triples of words, encoded with the type $\mathsf{T}_3 = ((o \multimap o) \multimap (o \multimap o) \multimap (o \multimap o) \multimap o) \multimap o$. Consequently, it generates a tree whose finite branches are the words of L prefixed by a segment of b's and followed by \$. The terminal $\mathsf{b} : (o \,\&\, o) \multimap o$ represents rule choice and the other terminals ($[,], \# : o \multimap o$, $\$: o$) are used to build the word. The scheme relies on the following non-terminals: $S : o$, $D : \mathsf{T}_3$, $K : (o \multimap o) \multimap (o \multimap o) \multimap (o \multimap o)$ and $I : o \multimap o$, which are subject to the following rules:

$$S = D(\lambda xyz.x(\#(y(\#(z\$))))), \qquad Kxyv = [(x(](yv))), \qquad Iv = v,$$
$$Df = \mathsf{b}\langle fIII, D(\lambda x_1y_1z_1.D(\lambda x_2y_2z_2.f(Kx_1x_2)(Ky_1y_2)(Kz_1z_2)))\rangle$$

If the scheme were equivalent to a 2-HORS, the language of its branches would be accepted by a 2-CPDA [12], i.e. it would be indexed [2]. However, indexed languages are closed under homomorphism, so L would be indexed too, because erasing b's and \$ is a homomorphism.

Lemma 10 identifies a strong restriction on branch languages of trees generated by MAHORS. Since multiple context-free languages form a subset of third-order collapsible pushdown languages [20], it is natural to ask whether every MAHORS might be equivalent to a third-order HORS. One could try to establish this, for example, by showing that, for every restricted TSA, there is an equivalent MAHORS that uses third-order types. Unfortunately, our proof of Theorem 7 uses types whose order grows linearly in the restriction parameter k. At the time of writing, we believe this necessary to capture the complexity of run-trees generated by our (infinite-)tree-generating TSA, though we are aware that similar hierarchies for (finite-)word languages and (finite-)tree languages do collapse, e.g. second-order abstract categorial grammars [19, 15]. The main difficulty that prevents us from translating TSA into MAHORS of order 3 is that there may be infinitely many (sub)runs that start from a given node, visit only nodes above it and return to the same node, and all such runs have to be captured in a single MAHORS. In contrast, for word languages, when TSA are seen as acceptors of finite words, it suffices to focus on the representation of a single run [7].

7 Multiplicative HORS (MHORS)

In this section we consider MHORS, i.e. &-free MAHORS. Recall from Lemma 8 that, for any MHORS, there exists an equivalent *linear* PDA (LPDA) $\langle \Sigma, Q, \Gamma, \delta, q_0, \gamma_0 \rangle$ with transition function $\delta : Q \times \Gamma^\bullet \rightharpoonup Q + \{\mathsf{b}(q_1, \ldots, q_{|\mathsf{b}|}) \mid q_i \in Q, \mathsf{b} \in \Sigma\} + Q \times (\{up_\gamma \mid \gamma \in \Gamma\} + \{down\})$ such that any reachable configuration must be reachable through a unique path. Next we prove the converse using first-order MHORS only. In combination with Lemma 8, this amounts to a polynomial-time translation from arbitrary MHORS to first-order MHORS.

In what follows, we view an LPDA as a pushdown system with a successor relation $\Rightarrow$, in order to exploit standard reachability techniques [4, 9]. We work with configurations of the form $(q, t) \in Q \times (\Gamma^\bullet)^*$. As we do not have the space to review all the necessary definitions, let us just recall that the techniques employ *multi-automata* over $\Gamma^\bullet$ to recognise sets of configurations. Multi-automata are finite-state machines with multiple initial states, one for each state of the analysed pushdown system. Let i_q be the initial state of a multi-automaton corresponding to $q \in Q$. Then a multi-automaton is said to recognise (q, t) if it accepts t once started from i_q (this corresponds to processing stack content top-down). In particular, we take advantage of the following facts.

- For any LPDA $\mathcal{A}$, there exists a multi-automaton $\mathcal{A}_{era}$, constructible in polynomial time, which captures erasable stack content, i.e. $\{(q, t) \in Q \times \Gamma^* \mid \exists q' \in Q. (q, t) \Rightarrow^* (q', \epsilon)\}$. Using terminology from [4], this corresponds to $pre^*(Q \times \{\epsilon\})$. Hence, given $\mathcal{A}$, one can calculate the relation $R_\mathcal{A} = \{(q, \gamma, q') \in Q \times \Gamma \times Q \mid (q, \gamma) \Rightarrow^* (q', \epsilon)\}$ in polynomial time.
- For any LPDA $\mathcal{A}$, there exists a multi-automaton $\mathcal{A}_{rea}$, constructible in polynomial time, which represents all configurations reachable from (q_0, γ_0), i.e. all $(q, t\gamma_0)$ such that $(q_0, \gamma_0) \Rightarrow^* (q, t\gamma_0)$. This corresponds to representing $post^*(\{(q_0, \gamma_0)\})$ [9].

▶ **Lemma 13.** *For any LPDA $\mathcal{A}$, there exists an equivalent MHORS (of order 1) and its construction can be carried out in polynomial time.*

Proof. The translation is similar to the PDA-to-1-HORS translation in [12] except that reachability analysis ($R_\mathcal{A}$) is used to identify places where variables actually get used. This is needed to produce a term that is linearly typable. ◀

Consequently, LPDA and MHORS are equivalent. We end this section by showing they generate regular trees. Our first lemma states that, if the stack of an LPDA grows sufficiently, there is a point after which elements lying below a certain level will no longer be accessible.

▶ **Lemma 14.** *Let $s \in \Gamma^*$. There exists a bound $H_s \geq 0$ such that, for any $t \in \Gamma^*$, if $(q, ts\gamma_0)$ is reachable and $|t| > H_s$ then there is no q' such that $(q, t) \Rightarrow^* (q', \epsilon)$.*

Proof. Consider $X = \{(q, s\gamma_0) \mid (q_0, \gamma_0) \Rightarrow^* (q, s\gamma_0)\}$. Observe that $0 \leq |X| \leq |Q|$. Because we work with an LPDA, there can be at most $|Q|$ runs from (q_0, γ_0) to X. Let H_s be the maximum stack height occurring in these runs (take 0 if $X = \varnothing$). Suppose $(q, ts\gamma_0)$ is reachable and $|t| > H_s$. If we had $(q, t) \Rightarrow^* (q', \epsilon)$ for some q' then there would be a run $(q_0, \gamma_0) \Rightarrow^* (q, ts\gamma_0) \Rightarrow^* (q', s\gamma_0)$ in which the stack height exceeds H_s (because it visits $(q, ts\gamma_0)$). This contradicts the choice of H_s. ◀

The above bound depends on s. We show that there is a uniform bound, polynomial with respect to the size of $\mathcal{A}$. First, given $s \in \Gamma^*$, the multi-automaton $\mathcal{A}_{rea}$ discussed earlier can be modified to represent $\{(q, t) \mid (q_0, \gamma_0) \Rightarrow^* (q, ts\gamma_0)\}$ simply by changing the accepting states of $\mathcal{A}_{rea}$ (to those from which an original accepting state is reachable via an $s\gamma_0$-labelled path). Let $\mathcal{A}^s_{rea}$ be the resultant automaton. Note that the size of $\mathcal{A}^s_{rea}$ is bounded by a polynomial in $|\mathcal{A}|$ that is independent of s, because the only difference between $\mathcal{A}^s_{rea}$ and $\mathcal{A}_{rea}$ is the set of accepting states, and its size bounded by $|Q|$.

Observe that $\{(q,t) \mid (q_0,\gamma_0) \Rightarrow^* (q, ts\gamma_0),\, (q,t) \Rightarrow^* (q',\epsilon) \text{ for some } q'\}$ is exactly the set of configurations that are represented by both $\mathcal{A}^s_{rea}$ and $\mathcal{A}_{era}$. Consider the product $\mathcal{A}'$ of the two multi-automata. By Lemma 14, $\mathcal{A}'$ cannot have reachable loops. Consequently, the longest word that it accepts from any initial state is bounded by the number of states of the automaton, which is polynomial in $|\mathcal{A}|$. As this reasoning is independent of s, we obtain:

▶ **Lemma 15.** *For any LPDA $\mathcal{A}$, there exists a bound H, polynomial in $|\mathcal{A}|$, such that, for any $s,t \in \Gamma^*$, if $(q, ts\gamma_0)$ is reachable and $|t| > H$ then there is no q' such that $(q,t) \Rightarrow^* (q',\epsilon)$.*

This implies that an LPDA can only use H top elements from its stack, i.e. its stack can be simulated by a finite state automaton, which is exponentially bigger. Because any 0-HORS is also an MHORS, MHORS and 0-HORS are equivalent, i.e. they generate exactly the regular trees. However, it is worth noting that MHORS may be more succinct.

▶ **Example 16.** The MHORS built from terminals $a, b : o \multimap o$, non-terminals $S : o, F_i : o \multimap o$ $(1 \le i \le n)$ with $S = F_n(bS)$, $F_0(x) = ax$ and $F_i(x) = F_{i-1}(F_{i-1}x)$ for $1 \le i \le n$ generates an infinite branch $(a^{2^n}b)^\omega$, which could only be generated by a 0-HORS of exponential size in n.

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