


# On the Stretch Factor of Polygonal Chains

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## Abstract

Let  $P = (p_1, p_2, \dots, p_n)$  be a polygonal chain. The *stretch factor* of  $P$  is the ratio between the total length of  $P$  and the distance of its endpoints,  $\sum_{i=1}^{n-1} |p_i p_{i+1}| / |p_1 p_n|$ . For a parameter  $c \geq 1$ , we call  $P$  a *c-chain* if  $|p_i p_j| + |p_j p_k| \leq c |p_i p_k|$ , for every triple  $(i, j, k)$ ,  $1 \leq i < j < k \leq n$ . The stretch factor is a global property: it measures how close  $P$  is to a straight line, and it involves all the vertices of  $P$ ; being a *c-chain*, on the other hand, is a *fingerprint*-property: it only depends on subsets of  $O(1)$  vertices of the chain.

We investigate how the *c-chain* property influences the stretch factor in the plane: (i) we show that for every  $\varepsilon > 0$ , there is a noncrossing *c-chain* that has stretch factor  $\Omega(n^{1/2-\varepsilon})$ , for sufficiently large constant  $c = c(\varepsilon)$ ; (ii) on the other hand, the stretch factor of a *c-chain*  $P$  is  $O(n^{1/2})$ , for every constant  $c \geq 1$ , regardless of whether  $P$  is crossing or noncrossing; and (iii) we give a randomized algorithm that can determine, for a polygonal chain  $P$  in  $\mathbb{R}^2$  with  $n$  vertices, the minimum  $c \geq 1$  for which  $P$  is a *c-chain* in  $O(n^{2.5} \text{ polylog } n)$  expected time and  $O(n \log n)$  space.

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## 1 Introduction

Given a set  $S$  of  $n$  point sites in the plane, what is the best way to connect  $S$  into a *geometric network (graph)*? This question has motivated researchers for a long time, going back as far as the 1940s, and beyond [19, 35]. Numerous possible criteria for a good geometric network



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have been proposed, perhaps the most basic being the *length*. In 1955, Few [20] showed that for any set of  $n$  points in a unit square, there is a traveling salesman tour of length at most  $\sqrt{2n} + 7/4$ . This was improved to at most  $0.984\sqrt{2n} + 11$  by Karloff [23]. Similar bounds also hold for the shortest spanning tree and the shortest rectilinear spanning tree [13, 16, 21]. Besides length, two further key factors in the quality of a geometric network are the *vertex dilation* and the *geometric dilation* [31], both of which measure how closely shortest paths in a network approximate the Euclidean distances between their endpoints.

The *dilation* (also called *stretch factor* [29] or *detour* [1]) between two points  $p$  and  $q$  in a geometric graph  $G$  is defined as the ratio between the length of a shortest path from  $p$  to  $q$  and the Euclidean distance  $|pq|$ . The *dilation* of the graph  $G$  is the maximum dilation over all pairs of vertices in  $G$ . A graph in which the dilation is bounded above by  $t \geq 1$  is also called a  $t$ -*spanner* (or simply a *spanner* if  $t$  is a constant). A complete graph in Euclidean space is clearly a 1-spanner. Therefore, researchers focused on the dilation of graphs with certain additional constraints, for example, noncrossing (i.e., plane) graphs. In 1989, Das and Joseph [15] identified a large class of plane spanners (characterized by two simple local properties). Bose et al. [6] gave an algorithm that constructs for any set of planar sites a plane 11-spanner with bounded degree. On the other hand, Eppstein [18] analyzed a fractal construction showing that  $\beta$ -*skeletons*, a natural class of geometric networks, can have arbitrarily large dilation.

The study of dilation also raises algorithmic questions. Agarwal et al. [1] described randomized algorithms for computing the dilation of a given path (on  $n$  vertices) in  $\mathbb{R}^2$  in  $O(n \log n)$  expected time. They also presented randomized algorithms for computing the dilation of a given tree, or cycle, in  $\mathbb{R}^2$  in  $O(n \log^2 n)$  expected time. Previously, Narasimhan and Smid [30] showed that an  $(1 + \varepsilon)$ -approximation of the stretch factor of any path, cycle, or tree can be computed in  $O(n \log n)$  time. Klein et al. [24] gave randomized algorithms for a path, tree, or cycle in  $\mathbb{R}^2$  to count the number of vertex pairs whose dilation is below a given threshold in  $O(n^{3/2+\varepsilon})$  expected time. Cheong et al. [12] showed that it is NP-hard to determine the existence of a spanning tree on a planar point set whose dilation is at most a given value. More results on plane spanners can be found in the monograph dedicated to this subject [31] or in several surveys [8, 17, 29].

We investigate a basic question about the dilation of polygonal chains. More precisely, we ask how the dilation between the endpoints of a polygonal chain (which we will call the *stretch factor*, to distinguish it from the more general notion of dilation) is influenced by *fingerprint* properties of the chain, i.e., by properties that are defined on  $O(1)$ -size subsets of the vertex set. Such fingerprint properties play an important role in geometry, where classic examples include the *Carathéodory property*<sup>1</sup> [26, Theorem 1.2.3] or the *Helly property*<sup>2</sup> [26, Theorem 1.3.2]. In general, determining the effect of a fingerprint property may prove elusive: given  $n$  points in the plane, consider the simple property that every 3 points determine 3 distinct distances. It is unknown [9, p. 203] whether this property implies that the total number of distinct distances grows superlinearly in  $n$ .

Furthermore, fingerprint properties appear in the general study of *local versus global properties of metric spaces* that is highly relevant to combinatorial approximation algorithms that are based on mathematical programming relaxations [5]. In the study of dilation,

<sup>1</sup> Given a finite set  $S$  of points in  $d$  dimensions, if every  $d + 2$  points in  $S$  are in convex position, then  $S$  is in convex position.

<sup>2</sup> Given a finite collection of convex sets in  $d$  dimensions, if every  $d + 1$  sets have nonempty intersection, then all sets have nonempty intersection.

interesting fingerprint properties have also been found. For example, a (continuous) curve  $C$  is said to have the *increasing chord property* [14, 25] if for any points  $a, b, c, d$  that appear on  $C$  in this order, we have  $|ad| \geq |bc|$ . The increasing chord property implies that  $C$  has (geometric) dilation at most  $2\pi/3$  [33]. A weaker property is the *self-approaching property*: a (continuous) curve  $C$  is self-approaching if for any points  $a, b, c$  that appear on  $C$  in this order, we have  $|ac| \geq |bc|$ . Self-approaching curves have dilation at most 5.332 [22] (see also [3]), and they have found interesting applications in the field of graph drawing [4, 7, 32].

We introduce a new natural fingerprint property and see that it can constrain the stretch factor of a polygonal chain, but only in a weaker sense than one may expect; we also provide algorithmic results on this property. Before providing details, we give a few basic definitions.

**Definitions.** A *polygonal chain*  $P$  in the Euclidean plane is specified by a sequence of  $n$  points  $(p_1, p_2, \dots, p_n)$ , called its *vertices*. The chain  $P$  consists of  $n - 1$  line segments between consecutive vertices. We say  $P$  is *simple* if only consecutive line segments intersect and they only intersect at their endpoints. Given a polygonal chain  $P$  in the plane with  $n$  vertices and a parameter  $c \geq 1$ , we call  $P$  a *c-chain* if for all  $1 \leq i < j < k \leq n$ , we have

$$|p_i p_j| + |p_j p_k| \leq c |p_i p_k|. \quad (1)$$

Observe that the  $c$ -chain condition is a fingerprint condition that is not really a local dilation condition – it is more a combination between the local chain substructure and the distribution of the points in the subchains.

The *stretch factor*  $\delta_P$  of  $P$  is defined as the dilation between the two end points  $p_1$  and  $p_n$  of the chain:

$$\delta_P = \frac{\sum_{i=1}^{n-1} |p_i p_{i+1}|}{|p_1 p_n|}.$$

Note that this definition is different from the more general notion of dilation (also called *stretch factor* [29]) of a graph which is the maximum dilation over all pairs of vertices. Since there is no ambiguity in this paper, we will just call  $\delta_P$  the stretch factor of  $P$ .

For example, the polygonal chain  $P = ((0, 0), (1, 0), \dots, (n, 0))$  is a 1-chain with stretch factor 1; and  $Q = ((0, 0), (0, 1), (1, 1), (1, 0))$  is a  $(\sqrt{2} + 1)$ -chain with stretch factor 3.

Without affecting the results, the floor and ceiling functions are omitted in our calculations. For a positive integer  $t$ , let  $[t] = \{1, 2, \dots, t\}$ . For a point set  $S$ , let  $\text{conv}(S)$  denote the convex hull of  $S$ . All logarithms are in base 2, unless stated otherwise.

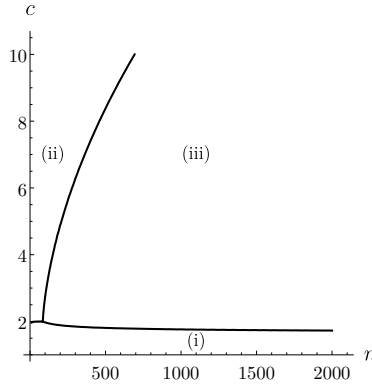
**Our results.** We deduce three upper bounds on the stretch factor of a  $c$ -chain  $P$  with  $n$  vertices (Section 2). In particular, we have (i)  $\delta_P \leq c(n - 1)^{\log c}$ , (ii)  $\delta_P \leq c(n - 2) + 1$ , and (iii)  $\delta_P = O(c^2 \sqrt{n - 1})$ .

From the other direction, we obtain the following lower bound (Section 3): For every  $c \geq 4$ , there is a family  $\mathcal{P}_c = \{P^k\}_{k \in \mathbb{N}}$  of simple  $c$ -chains, so that  $P^k$  has  $n = 4^k + 1$  vertices and stretch factor  $(n - 1)^{\frac{1 + \log(c - 2) - \log c}{2}}$ , where the exponent converges to  $1/2$  as  $c$  tends to infinity. The lower bound construction does not extend to the case of  $1 < c < 4$ , which remains open.

Finally, we present two algorithmic results (Section 4): (i) A randomized algorithm that decides, given a polygonal chain  $P$  in  $\mathbb{R}^2$  with  $n$  vertices and a threshold  $c > 1$ , whether  $P$  is a  $c$ -chain in  $O(n^{2.5} \text{ polylog } n)$  expected time and  $O(n \log n)$  space. (ii) As a corollary, there is a randomized algorithm that finds, for a polygonal chain  $P$  with  $n$  vertices, the minimum  $c \geq 1$  for which  $P$  is a  $c$ -chain in  $O(n^{2.5} \text{ polylog } n)$  expected time and  $O(n \log n)$  space.

**2 Upper Bounds**

At first glance, one might expect the stretch factor of a  $c$ -chain, for  $c \geq 1$ , to be bounded by some function of  $c$ . For example, the stretch factor of a 1-chain is necessarily 1. We derive three upper bounds on the stretch factor of a  $c$ -chain with  $n$  vertices in terms of  $c$  and  $n$  (cf. Theorems 1–3); see Fig. 1 for a visual comparison between the bounds. For large  $n$ , the bound in Theorem 1 is the best for  $1 \leq c \leq 2^{1/2}$ , while the bound in Theorem 3 is the best for  $c > 2^{1/2}$ . In particular, the bound in Theorem 1 is tight for  $c = 1$ . The bound in Theorem 2 is the best for  $c \geq 2$  and  $n \leq 111c^2$ .



**Figure 1** The values of  $n$  and  $c$  for which (i) Theorem 1, (ii) Theorem 2, and (iii) Theorem 3 give the current best upper bound.

Our first upper bound is obtained by a recursive application of the  $c$ -chain property. It holds for any positive distance function that may not even satisfy the triangle inequality.

► **Theorem 1.** For a  $c$ -chain  $P$  with  $n$  vertices, we have  $\delta_P \leq c(n - 1)^{\log c}$ .

**Proof.** We prove, by induction on  $n$ , that

$$\delta_P \leq c^{\lceil \log(n-1) \rceil}, \tag{2}$$

for every  $c$ -chain  $P$  with  $n \geq 2$  vertices. In the base case,  $n = 2$ , we have  $\delta_P = 1$  and  $c^{\lceil \log(2-1) \rceil} = 1$ . Now let  $n \geq 3$ , and assume that (2) holds for every  $c$ -chain with fewer than  $n$  vertices. Let  $P = (p_1, \dots, p_n)$  be a  $c$ -chain with  $n$  vertices. Then, applying (2) to the first and second half of  $P$ , followed by the  $c$ -chain property for the first, middle, and last vertex of  $P$ , we get

$$\begin{aligned} \sum_{i=1}^{n-1} |p_i p_{i+1}| &\leq \sum_{i=1}^{\lceil n/2 \rceil - 1} |p_i p_{i+1}| + \sum_{i=\lceil n/2 \rceil}^{n-1} |p_i p_{i+1}| \\ &\leq c^{\lceil \log(\lceil n/2 \rceil - 1) \rceil} (|p_1 p_{\lceil n/2 \rceil}| + |p_{\lceil n/2 \rceil} p_n|) \\ &\leq c^{\lceil \log(\lceil n/2 \rceil - 1) \rceil} \cdot c |p_1 p_n| \\ &\leq c^{\lceil \log(n-1) \rceil} |p_1 p_n|, \end{aligned}$$

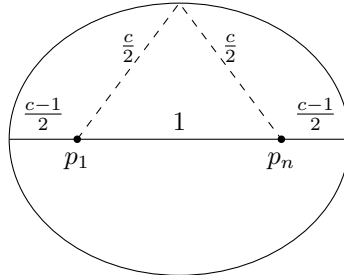
so (2) holds also for  $P$ . Consequently,

$$\delta_P \leq c^{\lceil \log(n-1) \rceil} \leq c^{\log(n-1)+1} = c \cdot c^{\log(n-1)} = c(n - 1)^{\log c},$$

as required. ◀

Our second bound interprets the  $c$ -chain property geometrically and makes use of the fact that  $P$  resides in the Euclidean plane.

► **Theorem 2.** *For a  $c$ -chain  $P$  with  $n$  vertices, we have  $\delta_P \leq c(n - 2) + 1$ .*



■ **Figure 2** The entire chain  $P$  lies in an ellipse with foci  $p_1$  and  $p_n$ .

**Proof.** Without loss of generality, assume that  $|p_1p_n| = 1$ . Since  $P$  is a  $c$ -chain, for every  $1 < j < n$ , we have  $|p_1p_j| + |p_jp_n| \leq c|p_1p_n| = c$ . If we fix the points  $p_1$  and  $p_n$ , then every  $p_j$  lies in an ellipse  $E$  with foci  $p_1$  and  $p_n$ , for  $1 < j < n$ , see Figure 2. The diameter of  $E$  is its major axis, whose length is  $c$ . Since  $E$  contains all vertices of the chain  $P$ , we have  $|p_1p_2|, |p_{n-1}p_n| \leq \frac{c+1}{2} \leq c$  and  $|p_jp_{j+1}| \leq c$  for all  $1 < j < n - 1$ . Therefore the stretch factor of  $P$  is bounded above by

$$\begin{aligned} \delta_P &= \frac{\sum_{j=1}^{n-1} |p_jp_{j+1}|}{|p_1p_n|} = |p_1p_2| + |p_{n-1}p_n| + \sum_{j=2}^{n-2} |p_jp_{j+1}| \\ &\leq \frac{c+1}{2} + \frac{c+1}{2} + c(n-3) = c(n-2) + 1, \end{aligned}$$

as required. ◀

Our third upper bound uses a volume argument to bound the number of long edges in  $P$ .

► **Theorem 3.** *Let  $P = (p_1, \dots, p_n)$  be a  $c$ -chain, for some constant  $c \geq 1$ , and let  $L = \sum_{i=1}^{n-1} |p_i p_{i+1}|$  be its length. Then  $L = O(c^2 \sqrt{n-1}) |p_1 p_n|$ , hence  $\delta_P = O(c^2 \sqrt{n-1})$ .*

**Proof.** We may assume that  $p_1p_n$  is a horizontal segment of unit length. By the argument in the proof of Theorem 2, all points  $p_i$  ( $i = 1, \dots, n$ ) are contained in an ellipse  $E$  with foci  $p_1$  and  $p_n$ , where the major axis of  $E$  has length  $c$ . Let  $U$  be the minimal axis-aligned square containing  $E$ ; its side is of length  $c$ .

We set  $x = 8c^2/\sqrt{n-1}$ ; and let  $L_0$  and  $L_1$  be the sum of lengths of all edges in  $P$  of length at most  $x$  and more than  $x$ , respectively. By definition, we have  $L = L_0 + L_1$  and

$$L_0 \leq (n-1)x = (n-1) \cdot 8c^2/\sqrt{n-1} = 8c^2\sqrt{n-1}. \tag{3}$$

We shall prove that  $L_1 \leq 8c^2\sqrt{n-1}$ , implying  $L \leq 2x(n-1) = O(c^2\sqrt{n-1})$ . For this, we further classify the edges in  $L_1$  according to their lengths: For  $\ell = 0, 1, \dots, \infty$ , let

$$P_\ell = \{p_i : 2^\ell x < |p_i p_{i+1}| \leq 2^{\ell+1} x\}. \tag{4}$$

Since all points lie in an ellipse of diameter  $c$ , we have  $|p_i p_{i+1}| \leq c$ , for all  $i = 0, \dots, n-1$ . Consequently,  $P_\ell = \emptyset$  when  $c \leq 2^\ell x$ , or equivalently  $\log(c/x) \leq \ell$ .

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We use a volume argument to derive an upper bound on the cardinality of  $P_\ell$ , for  $\ell = 0, 1, \dots, \lfloor \log(c/x) \rfloor$ . Assume that  $p_i, p_k \in P_\ell$ , and w.l.o.g.,  $i < k$ . If  $k = i + 1$ , then by (4),  $2^\ell x < |p_i p_k|$ . Otherwise,

$$2^\ell x < |p_i p_{i+1}| < |p_i p_{i+1}| + |p_{i+1} p_k| \leq c |p_i p_k|, \text{ or } \frac{2^\ell x}{c} < |p_i p_k|.$$

Consequently, the disks of radius

$$R = \frac{2^\ell x}{2c} = \frac{4 \cdot 2^\ell c}{\sqrt{n-1}} \quad (5)$$

centered at the points in  $P_\ell$  are interior-disjoint. The area of each disk is  $\pi R^2$ . Since  $P_\ell \subset U$ , these disks are contained in the  $R$ -neighborhood  $U_R$  of the square  $U$ , i.e., the Minkowski sum  $R + U$ . For  $\ell \leq \log(c/x)$ , we have  $2^\ell x \leq c$ , hence  $R = \frac{2^\ell x}{2c} \leq \frac{c}{2c} = \frac{1}{2} \leq \frac{c}{2}$ . Then we can bound the area of  $U_R$  from above as follows:

$$\text{area}(U_R) < (c + 2R)^2 \leq (2c)^2 = 4c^2. \quad (6)$$

Since  $U_R$  contains  $|P_\ell|$  interior-disjoint disks of radius  $R$ , we obtain

$$|P_\ell| \leq \frac{\text{area}(U_R)}{\pi R^2} < \frac{4c^2}{\pi R^2} = \frac{16c^4}{\pi 2^{2\ell} x^2}. \quad (7)$$

For every segment  $p_{i-1} p_i$  with length more than  $x$ , we have that  $p_i \in P_\ell$ , for some  $\ell \in \{0, 1, \dots, \lfloor \log(c/x) \rfloor\}$ . The total length of these segments is

$$\begin{aligned} L_1 &\leq \sum_{\ell=0}^{\lfloor \log(c/x) \rfloor} |P_\ell| \cdot 2^{\ell+1} x < \sum_{\ell=0}^{\lfloor \log(c/x) \rfloor} \frac{16c^4}{\pi 2^{2\ell} x^2} \cdot 2^{\ell+1} x = \sum_{\ell=0}^{\lfloor \log(c/x) \rfloor} \frac{32c^4}{\pi 2^\ell x} \\ &< \frac{32c^4}{\pi x} \sum_{\ell=0}^{\infty} \frac{1}{2^\ell} = \frac{64c^4}{\pi x} = \frac{8c^2}{\pi} \cdot \sqrt{n-1}, \end{aligned}$$

as required. Together with (3), this yields  $L \leq 8(1 + c^2/\pi) \cdot \sqrt{n-1}$ .  $\blacktriangleleft$

### 3 Lower Bounds

We now present our lower bound construction, showing that the dependence on  $n$  for the stretch factor of a  $c$ -chain cannot be avoided.

► **Theorem 4.** *For every constant  $c \geq 4$ , there is a set  $\mathcal{P}_c = \{P^k\}_{k \in \mathbb{N}}$  of simple  $c$ -chains, so that  $P^k$  has  $n = 4^k + 1$  vertices and stretch factor  $(n-1)^{\frac{1+\log(c-2)-\log c}{2}}$ .*

By Theorem 3, the stretch factor of a  $c$ -chain in the plane is  $O((n-1)^{1/2})$  for every constant  $c \geq 1$ . Since

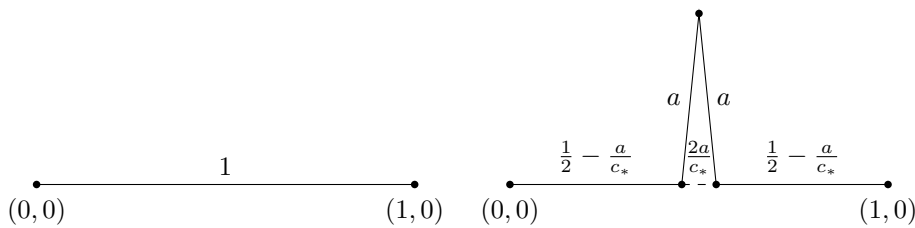
$$\lim_{c \rightarrow \infty} \frac{1 + \log(c-2) - \log c}{2} = \frac{1}{2},$$

our lower bound construction shows that the limit of the exponent cannot be improved. Indeed, for every  $\varepsilon > 0$ , we can set  $c = \frac{2^{2\varepsilon+1}}{2^{2\varepsilon-1}}$ , and then the chains above have stretch factor  $(n-1)^{\frac{1+\log(c-2)-\log c}{2}} = (n-1)^{1/2-\varepsilon} = \Omega(n^{1/2-\varepsilon})$ .

We first construct a family  $\mathcal{P}_c = \{P^k\}_{k \in \mathbb{N}}$  of polygonal chains. Then we show, in Lemmata 5 and 6, that every chain in  $\mathcal{P}_c$  is simple and indeed a  $c$ -chain. The theorem follows since the claimed stretch factor is a consequence of the construction.

**Construction of  $\mathcal{P}_c$ .** The construction here is a generalization of the iterative construction of the *Koch curve*; when  $c = 6$ , the result is the original Cesàro fractal (which is a variant of the Koch curve) [10]. We start with a unit line segment  $P^0$ , and for  $k = 0, 1, \dots$ , we construct  $P^{k+1}$  by replacing each segment in  $P^k$  by four segments such that the middle three points achieve a stretch factor of  $c_* = \frac{c-2}{2}$  (this choice will be justified in the proof of Lemma 6). Note that  $c_* \geq 1$ , since  $c \geq 4$ .

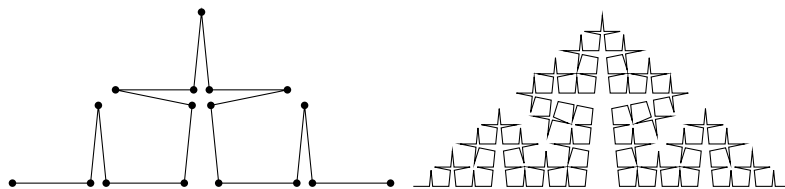
We continue with the details. Let  $P^0$  be the unit line segment from  $(0, 0)$  to  $(1, 0)$ ; see Figure 3 (left). Given the polygonal chain  $P^k$  ( $k = 0, 1, \dots$ ), we construct  $P^{k+1}$  by replacing each segment of  $P^k$  by four segments as follows. Consider a segment of  $P^k$ , and denote its length by  $\ell$ . Subdivide this segment into three segments of lengths  $(\frac{1}{2} - \frac{a}{c_*})\ell$ ,  $\frac{2a}{c_*}\ell$ , and  $(\frac{1}{2} - \frac{a}{c_*})\ell$ , respectively, where  $0 < a < \frac{c_*}{2}$  is a parameter to be determined later. Replace the middle segment with the top part of an isosceles triangle of side length  $a\ell$ . The chains  $P^0$ ,  $P^1$ ,  $P^2$ , and  $P^4$  are depicted in Figures 3 and 4.



■ **Figure 3** The chains  $P^0$  (left) and  $P^1$  (right).

Note that each segment of length  $\ell$  in  $P^k$  is replaced by four segments of total length  $(1 + \frac{2a(c_*-1)}{c_*})\ell$ . After  $k$  iterations, the chain  $P^k$  consists of  $4^k$  line segments of total length  $(1 + \frac{2a(c_*-1)}{c_*})^k$ .

By construction, the chain  $P^k$  (for  $k \geq 1$ ) consists of four scaled copies of  $P^{k-1}$ . For  $i = 1, 2, 3, 4$ , let the  $i$ th subchain of  $P^k$  be the subchain of  $P^k$  consisting of  $4^{k-1}$  segments starting from the  $((i-1)4^{k-1} + 1)$ th segment. By construction, the  $i$ th subchain of  $P^k$  is similar to the chain  $P^{k-1}$ , for  $i = 1, 2, 3, 4$ .<sup>3</sup> The following functions allow us to refer to these subchains formally. For  $i = 1, 2, 3, 4$ , define a function  $f_i^k : P^k \rightarrow P^k$  as the identity on the  $i$ th subchain of  $P^k$  that sends the remaining part(s) of  $P^k$  to the closest endpoint(s) along this subchain. So  $f_i^k(P^k)$  is similar to  $P^{k-1}$ . Let  $g_i : \mathcal{P}_c \setminus \{P^0\} \rightarrow \mathcal{P}_c$  be a piecewise defined function such that  $g_i(C) = \sigma^{-1} \circ f_i^k \circ \sigma(C)$  if  $C$  is similar to  $P^k$ , where  $\sigma : C \rightarrow P^k$  is a similarity transformation. Applying the function  $g_i$  on a chain  $P^k$  can be thought of as “cutting out” its  $i$ th subchain.



■ **Figure 4** The chains  $P^2$  (left) and  $P^4$  (right).

<sup>3</sup> Two geometric shapes are *similar* if one can be obtained from the other by translation, rotation, and scaling; and are *congruent* if one can be obtained from the other by translation and rotation.

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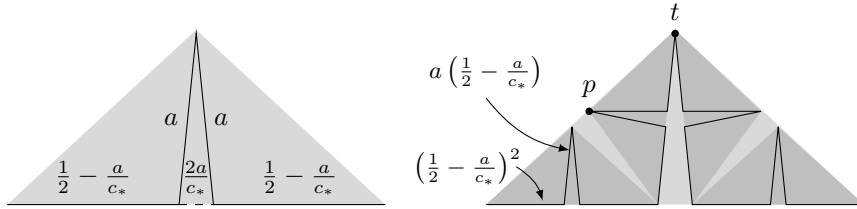
Clearly, the stretch factor of the chain monotonically increases with the parameter  $a$ . However, if  $a$  is too large, the chain is no longer simple. The following lemma gives a sufficient condition for the constructed chains to avoid self-crossings.

► **Lemma 5.** *For every constant  $c \geq 4$ , if  $a \leq \frac{c-2}{2c}$ , then every chain in  $\mathcal{P}_c$  is simple.*

**Proof.** Let  $T = \text{conv}(P^1)$ . Observe that  $T$  is an isosceles triangle; see Figure 5 (left). We first show the following:

▷ **Claim.** If  $a \leq \frac{c-2}{2c}$ , then  $\text{conv}(P^k) = T$  for all  $k \geq 1$ .

**Proof.** We prove the claim by induction on  $k$ . It holds for  $k = 1$  by definition. For the induction step, assume that  $k \geq 2$  and that the claim holds for  $k - 1$ . Consider the chain  $P^k$ . Since it contains all the vertices of  $P^1$ ,  $T \subset \text{conv}(P^k)$ . So we only need to show that  $\text{conv}(P^k) \subset T$ .



■ **Figure 5** Left: Convex hull  $T$  of  $P^1$  in light gray; Right: Convex hulls of  $g_i(P^2)$ ,  $i = 1, 2, 3, 4$ , in dark gray, are contained in  $T$ .

By construction,  $P^k \subset \bigcup_{i=1}^4 \text{conv}(g_i(P^k))$ ; see Figure 5 (right). By the inductive hypothesis,  $\text{conv}(g_i(P^k))$  is an isosceles triangle similar to  $T$ , for  $i = 1, 2, 3, 4$ . Since the bases of  $\text{conv}(g_1(P^k))$  and  $\text{conv}(g_4(P^k))$  are collinear with the base of  $T$  by construction, due to similarity, they are contained in  $T$ . The base of  $\text{conv}(g_2(P^k))$  is contained in  $T$ . In order to show  $\text{conv}(g_2(P^k)) \subset T$ , by convexity, it suffices to ensure that its apex  $p$  is also in  $T$ . Note that the coordinates of the top point is  $t = (1/2, a\sqrt{c_*^2 - 1}/c_*)$ , so the supporting line  $\ell$  of the left side of  $T$  is

$$y = \frac{2a\sqrt{c_*^2 - 1}}{c_*}x, \text{ and}$$

$$p = \left( \frac{1}{2} - \frac{a}{2c_*} - \frac{a^2(c_*^2 - 1)}{c_*^2}, \left( \frac{a}{2c_*} + \frac{a^2}{c_*^2} \right) \sqrt{c_*^2 - 1} \right).$$

By the condition of  $a \leq \frac{c-2}{2c} = \frac{c_*}{2(c_*+1)}$  in the lemma,  $p$  lies on or below  $\ell$ . Under the same condition, we have  $\text{conv}(g_3(P^k)) \subset T$  by symmetry. Then  $P^k \subset \bigcup_{i=1}^4 \text{conv}(g_i(P^k)) \subset T$ . Since  $T$  is convex,  $\text{conv}(P^k) \subset T$ . So  $\text{conv}(P^k) = T$ , as claimed. ◁

We can now finish the proof of Lemma 5 by induction. Clearly,  $P^0$  and  $P^1$  are simple. Assume that  $k \geq 2$ , and  $P^{k-1}$  is simple. Consider the chain  $P^k$ . For  $i = 1, 2, 3, 4$ ,  $g_i(P^k)$  is similar to  $P^{k-1}$ , hence simple by the inductive hypothesis. Since  $P^k = \bigcup_{i=1}^4 g_i(P^k)$ , it is sufficient to show that for all  $i, j \in \{1, 2, 3, 4\}$ , where  $i \neq j$ , a segment in  $g_i(P^k)$  does not intersect any segments in  $g_j(P^k)$ , unless they are consecutive in  $P^k$  and they intersect at a common endpoint. This follows from the above claim together with the observation that for  $i \neq j$ , the intersection  $g_i(P^k) \cap g_j(P^k)$  is either empty or contains a single vertex which is the common endpoint of two consecutive segments in  $P^k$ . ◀



In the remainder of this section, we assume that

$$a = \frac{c - 2}{2c} = \frac{c_*}{2(c_* + 1)}. \tag{8}$$

Under this assumption, all segments in  $P^1$  have the same length  $a$ . Therefore, by construction, all segments in  $P^k$  have the same length

$$a^k = \left( \frac{c_*}{2(c_* + 1)} \right)^k.$$

There are  $4^k$  segments in  $P^k$ , with  $4^k + 1$  vertices, and its stretch factor is

$$\delta_{P^k} = 4^k \left( \frac{c_*}{2(c_* + 1)} \right)^k = \left( \frac{2c_*}{c_* + 1} \right)^k.$$

Consequently,  $k = \log_4(n - 1) = \frac{\log(n-1)}{2}$ , and

$$\delta_{P^k} = \left( \frac{2c_*}{c_* + 1} \right)^{\frac{\log(n-1)}{2}} = \left( \frac{2c - 4}{c} \right)^{\frac{\log(n-1)}{2}} = (n - 1)^{\frac{1 + \log(c-2) - \log c}{2}},$$

as claimed. To finish the proof of Theorem 4, it remains to show the constructed polygonal chains are indeed  $c$ -chains.

► **Lemma 6.** *For every constant  $c \geq 4$ ,  $\mathcal{P}_c$  is a family of  $c$ -chains.*

We first prove a couple of facts that will be useful in the proof of Lemma 6. We defer an intuitive explanation until after the formal statement of the lemma.

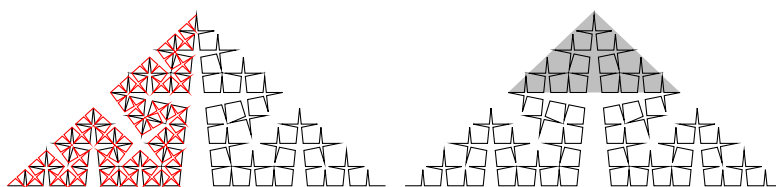
► **Lemma 7.** *Let  $k \geq 1$  and let  $P^k = (p_1, p_2, \dots, p_n)$ , where  $n = 4^k + 1$ . Then the following hold:*

- (i) *There exists a sequence  $(q_1, q_2, \dots, q_m)$  of  $m = 2 \cdot 4^{k-1}$  points in  $\mathbb{R}^2$  such that the chain  $R^k = (p_1, q_1, p_2, q_2, \dots, p_m, q_m, p_{m+1})$  is similar to  $P^k$ .*
- (ii) *For  $k \geq 2$ , define  $g_5 : \mathcal{P}_c \setminus \{P^0, P^1\} \rightarrow \mathcal{P}_c$  by*

$$g_5(P^k) = (g_3 \circ g_2(P^k)) \cup (g_4 \circ g_2(P^k)) \cup (g_1 \circ g_3(P^k)) \cup (g_2 \circ g_3(P^k)).$$

*Then  $g_5(P^k)$  is similar to  $P^{k-1}$ .*

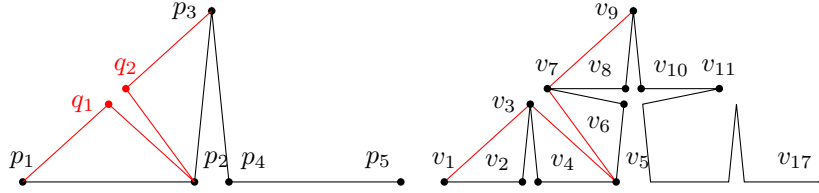
Part (i) of Lemma 7 says that given  $P^k$ , we can construct a chain  $R^k$  similar to  $P^k$  by inserting one point between every two consecutive points of the left half of  $P^k$ , see Figure 6 (left). Part (ii) says that the “top” subchain of  $P^k$  that consists of the right half of  $g_2(P^k)$  and the left half of  $g_3(P^k)$ , see Figure 6 (right), is similar to  $P^{k-1}$ .



■ **Figure 6** Left: Chain  $P^k$  with the scaled copy of itself  $R^k$  (in red); Right: Chain  $P^k$  with its subchain  $g_5(P^k)$  marked by its convex hull.

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**Proof of Lemma 7.** For (i), we review the construction of  $P^k$ , and show that  $R^k$  and  $P^k$  can be constructed in a coupled manner. In Figure 7 (left), consider  $P^1 = (p_1, p_2, p_3, p_4, p_5)$ . Recall that all segments in  $P^1$  are of the same length  $a = \frac{c_*}{2(c_*+1)}$ . The isosceles triangles  $\Delta p_1 p_2 p_3$  and  $\Delta p_1 p_3 p_5$  are similar. Let  $\sigma : \Delta p_1 p_3 p_5 \rightarrow \Delta p_1 p_2 p_3$  be the similarity transformation. Let  $q_1 = \sigma(p_2)$  and  $q_2 = \sigma(p_4)$ . By construction, the chain  $R^1 = (p_1, q_1, p_2, q_2, p_3)$  is similar to  $P^1$ . In particular, all of its segments have the same length. So the isosceles triangle  $\Delta p_1 q_1 p_2$  is similar to  $\Delta p_1 p_3 p_5$ . Moreover, its base is the segment  $p_1 p_2$ , so  $\Delta p_1 q_1 p_2$  is precisely  $\text{conv}(g_1(P^2))$ , see Figure 7 (right).



■ **Figure 7** Left: the chains  $P^1$  and  $R^1$  (red); Right: the chains  $P^2$  and  $R^1$  (red).

Write  $P^2 = (v_1, v_2, \dots, v_{17})$ , then  $v_3 = q_1$  by the above argument and  $v_7 = q_2$  by symmetry. Now  $\Delta v_1 v_2 v_3$ ,  $\Delta v_3 v_4 v_5$ ,  $\Delta v_5 v_6 v_7$ , and  $\Delta v_7 v_8 v_9$  are four congruent isosceles triangles, all of which are similar to  $\Delta v_1 v_9 v_{17}$ , since the angles are the same. Repeat the above procedure on each of them to obtain  $R^2 = (v_1, u_1, v_2, u_2, \dots, v_8, u_8, v_9)$ , which is similar to  $P^2$ . Continue this construction inductively to get the desired chain  $R^k$  for any  $k \geq 1$ .

For (ii), see Figure 7 (right). By definition,  $g_5(P^2)$  is the subchain  $(v_7, v_8, v_9, v_{10}, v_{11})$ . Observe that the segments  $v_7 v_8$  and  $v_{10} v_{11}$  are collinear by symmetry. Moreover, they are parallel to  $v_1 v_{17}$  since  $\angle v_7 v_8 v_9 = \angle v_1 v_5 v_9$ . So  $g_5(P^2)$  is similar to  $P^1$ ; see Figure 7 (left). Then for  $k \geq 2$ ,  $g_5(P^k)$  is the subchain of  $P^k$  starting at vertex  $v_7$ , ending at vertex  $v_{11}$ . By the construction of  $P^k$ ,  $g_5(P^k)$  is similar to  $P^{k-1}$ . ◀

Due to space constraints, the proof of Lemma 6 is deferred to the full version.

### 4 Algorithm for Recognizing $c$ -Chains

In this section, we design a randomized Las Vegas algorithm to recognize  $c$ -chains. More precisely, given a polygonal chain  $P = (p_1, \dots, p_n)$ , and a parameter  $c \geq 1$ , the algorithm decides whether  $P$  is a  $c$ -chain, in  $O(n^{2.5} \text{polylog } n)$  expected time. By definition,  $P = (p_1, \dots, p_n)$  is a  $c$ -chain if  $|p_i p_j| + |p_j p_k| \leq c |p_i p_k|$  for all  $1 \leq i < j < k \leq n$ ; equivalently,  $p_j$  lies in the ellipse of major axis  $c$  with foci  $p_i$  and  $p_k$ . Consequently, it suffices to test, for every pair  $1 \leq i < k \leq n$ , whether the ellipse of major axis  $c|p_i p_k|$  with foci  $p_i$  and  $p_k$  contains  $p_j$ , for all  $j, i < j < k$ . For this, we can apply recent results from geometric range searching.

► **Theorem 8.** *There is a randomized algorithm that can decide, for a polygonal chain  $P = (p_1, \dots, p_n)$  in  $\mathbb{R}^2$  and a threshold  $c > 1$ , whether  $P$  is a  $c$ -chain in  $O(n^{2.5} \text{polylog } n)$  expected time and  $O(n \log n)$  space.*

Agarwal, Matoušek and Sharir [2, Theorem 1.4] constructed, for a set  $S$  of  $n$  points in  $\mathbb{R}^2$ , a data structure that can answer ellipse range searching queries: it reports the number of points in  $S$  that are contained in a query ellipse. In particular, they showed that, for every  $\varepsilon > 0$ , there is a constant  $B$  and a data structure with  $O(n)$  space,  $O(n^{1+\varepsilon})$  expected

preprocessing time, and  $O\left(n^{1/2} \log^B n\right)$  query time. The construction was later simplified by Matoušek and Patáková [27]. Using this data structure, we can quickly decide whether a given polygonal chain is a  $c$ -chain.

**Proof of Theorem 8.** Subdivide the polygonal chain  $P = (p_1, \dots, p_n)$  into two subchains of equal or almost equal sizes,  $P_1 = (p_1, \dots, p_{\lceil n/2 \rceil})$  and  $P_2 = (p_{\lceil n/2 \rceil}, \dots, p_n)$ ; and recursively subdivide  $P_1$  and  $P_2$  until reaching 1-vertex chains. Denote by  $T$  the recursion tree. Then,  $T$  is a binary tree of depth  $\lceil \log n \rceil$ . There are at most  $2^i$  nodes at level  $i$ ; the nodes at level  $i$  correspond to edge-disjoint subchains of  $P$ , each of which has at most  $n/2^i$  edges. Let  $W_i$  be the set of subchains on level  $i$  of  $T$ ; and let  $W = \bigcup_{i \geq 0} W_i$ . We have  $|W| \leq 2n$ .

For each polygonal chain  $Q \in W$ , construct an ellipse range searching data structure  $\text{DS}(Q)$  described above [2] for the vertices of  $Q$ , with a suitable parameter  $\varepsilon > 0$ . Their overall expected preprocessing time is

$$\sum_{i=0}^{\lceil \log n \rceil} 2^i \cdot O\left(\left(\frac{n}{2^i}\right)^{1+\varepsilon}\right) = O\left(n^{1+\varepsilon} \sum_{i=0}^{\lceil \log n \rceil} \left(\frac{1}{2^i}\right)^\varepsilon\right) = O\left(n^{1+\varepsilon}\right),$$

their space requirement is  $\sum_{i=0}^{\lceil \log n \rceil} 2^i \cdot O(n/2^i) = O(n \log n)$ , and their query time at level  $i$  is  $O\left(\left(n/2^i\right)^{1/2} \text{polylog}(n/2^i)\right) = O\left(n^{1/2} \text{polylog } n\right)$ .

For each pair of indices  $1 \leq i < k \leq n$ , we do the following. Let  $E_{i,k}$  denote the ellipse of major axis  $c|p_i p_k|$  with foci  $p_i$  and  $p_k$ . The chain  $(p_{i+1}, \dots, p_{k-1})$  is subdivided into  $O(\log n)$  maximal subchains in  $W$ , using at most two subchains from each set  $W_i$ ,  $i = 0, \dots, \lceil \log n \rceil$ . For each of these subchains  $Q \in W$ , query the data structure  $\text{DS}(Q)$  with the ellipse  $E_{i,k}$ . If all queries are positive (i.e., the count returned is  $|Q|$  in *all* queries), then  $P$  is a  $c$ -chain; otherwise there exists  $j$ ,  $i < j < k$ , such that  $p_j \notin E_{i,k}$ , hence  $|p_i p_j| + |p_j p_k| > c|p_i p_k|$ , witnessing that  $P$  is not a  $c$ -chain.

The query time over all pairs  $1 \leq i < k \leq n$  is bounded above by

$$\begin{aligned} \binom{n}{2} \sum_{i=0}^{2^{\lceil \log n \rceil}} O\left(\left(n/2^i\right)^{1/2} \text{polylog}(n/2^i)\right) &= \binom{n}{2} \cdot O\left(n^{1/2} \text{polylog } n\right) \\ &= O\left(n^{2.5} \text{polylog } n\right). \end{aligned}$$

This subsumes the expected time needed for constructing the structures  $\text{DS}(Q)$ , for all  $Q \in W$ . So the overall running time of the algorithm is  $O\left(n^{2.5} \text{polylog } n\right)$ , as claimed. ◀

In the decision algorithm above, only the construction of the data structures  $\text{DS}(Q)$ ,  $Q \in W$ , uses randomization, which is independent of the value of  $c$ . The parameter  $c$  is used for defining the ellipses  $E_{i,k}$ , and the queries to the data structures; this part is deterministic. Hence, we can find the optimal value of  $c$  by Meggido's parametric search [28] in the second part of the algorithm.

Meggido's technique reduces an optimization problem to a corresponding decision problem at a polylogarithmic factor increase in the running time. An optimization problem is amenable to this technique if the following three conditions are met [34]: (1) the objective function is monotone in the given parameter; (2) the decision problem can be solved by evaluating bounded-degree polynomials, and (3) the decision problem admits an efficient parallel algorithm (with polylogarithmic running time using polynomial number of processors). All three conditions hold in our case: The area of each ellipse with foci in  $S$  monotonically increases with  $c$ ; the data structure of [27] answers ellipse range counting queries by evaluating

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polynomials of bounded degree; and the  $\binom{n}{2}$  queries can be performed in parallel. Alternatively, Chan's randomized optimization technique [11] is also applicable. Both techniques yield the following result.

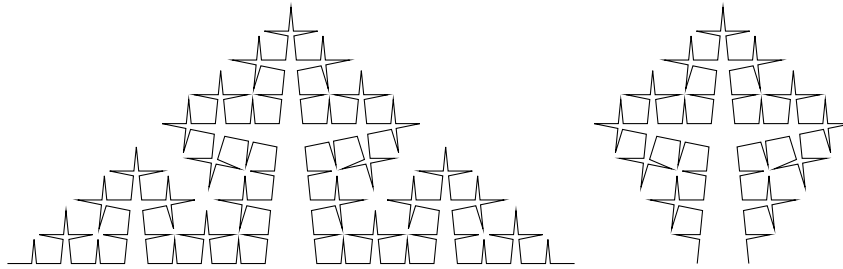
► **Corollary 9.** *There is a randomized algorithm that can find, for a polygonal chain  $P = (p_1, \dots, p_n)$  in  $\mathbb{R}^2$ , the minimum  $c \geq 1$  for which  $P$  is a  $c$ -chain in  $O(n^{2.5} \text{ polylog } n)$  expected time and  $O(n \log n)$  space.*

We remark that, for  $c = 1$ , the test takes  $O(n)$  time: it suffices to check whether points  $p_3, \dots, p_n$  lie on the line spanned by  $p_1 p_2$ , in that order.

### 5 Concluding Remarks

We end with some final observations and pointers for further research.

1. For  $k \geq 1$ , let  $P_*^k = g_2(P^k) \cup g_3(P^k)$ , see Figure 8 (right). It is easy to see that  $P_*^k$  is a  $c$ -chain with  $n = 4^k/2 + 1$  vertices and has stretch factor  $\sqrt{c(c-2)/8(n-1)}^{\frac{1+\log(c-2)-\log c}{2}}$ . Since  $\sqrt{c(c-2)/8} \geq 1$  for  $c \geq 4$ , this improves the result of Theorem 4 by a constant factor. Since this construction does not improve the exponent, and the analysis would be longer (requiring a case analysis without new insights), we omit the details.



■ **Figure 8** The chains  $P^4$  (left) and  $P_*^4$  (right).

2. If  $c$  is used instead of  $c_* = (c-2)/2$  in the lower bound construction, then the condition  $c \geq 4$  in Theorem 4 can be replaced by  $c \geq 1$ , and the bound can be improved from  $(n-1)^{\frac{1+\log(c-2)-\log c}{2}}$  to  $(n-1)^{\frac{1+\log c - \log(c+1)}{2}}$ . However, we were unable to prove that the resulting  $P^k$ 's,  $k \in \mathbb{N}$ , are  $c$ -chains, although a computer program has verified that the first few generations of them are indeed  $c$ -chains.
3. The volume argument in Theorem 3 easily generalizes to higher dimensions. If  $P$  be a  $c$ -chain in  $\mathbb{R}^d$  for fixed  $c \geq 1$  and  $d \geq 2$ , then  $\delta_P = O(c^2(n-1)^{1-1/d})$ . It is interesting to find out whether extra dimension(s) allows one to achieve a larger stretch factor.
4. The upper bounds in Theorem 1–3 are valid regardless of whether the chain is crossing or not. On the other hand, the lower bound in Theorem 4 is given by noncrossing chains. A natural question is whether a sharper upper bound holds if the chains are required to be noncrossing. More specifically, can the exponent of  $n$  in the upper bound be reduced to  $1/2 - \varepsilon$ , where  $\varepsilon > 0$  depends on  $c$ ?
5. Our algorithm in Section 4 can recognize  $c$ -chains with  $n$  vertices in  $O(n^{2.5} \text{ polylog } n)$  expected time and  $O(n \log n)$  space, using ellipse range searching data structures. It is likely that the running time can be improved in the future, perhaps at the expense of increased space, when suitable time-space trade-offs for semi-algebraic range searching become available. The existence of such data structures is conjectured [2], but currently remains open.

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