

# Improved Bounds for the Excluded-Minor Approximation of Treedepth

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## Abstract

Treedepth, a more restrictive graph width parameter than treewidth and pathwidth, plays a major role in the theory of sparse graph classes. We show that there exists a constant  $C$  such that for every integers  $a, b \geq 2$  and a graph  $G$ , if the treedepth of  $G$  is at least  $Cab \log a$ , then the treewidth of  $G$  is at least  $a$  or  $G$  contains a subcubic (i.e., of maximum degree at most 3) tree of treedepth at least  $b$  as a subgraph.

As a direct corollary, we obtain that every graph of treedepth  $\Omega(k^3 \log k)$  is either of treewidth at least  $k$ , contains a subdivision of full binary tree of depth  $k$ , or contains a path of length  $2^k$ . This improves the bound of  $\Omega(k^5 \log^2 k)$  of Kawarabayashi and Rossman [SODA 2018].

We also show an application for approximation algorithms of treedepth: given a graph  $G$  of treedepth  $k$  and treewidth  $t$ , one can in polynomial time compute a treedepth decomposition of  $G$  of width  $\mathcal{O}(kt \log^{3/2} t)$ . This improves upon a bound of  $\mathcal{O}(kt^2 \log t)$  stemming from a tradeoff between known results.

The main technical ingredient in our result is a proof that every tree of treedepth  $d$  contains a subcubic subtree of treedepth at least  $d \cdot \log_3((1 + \sqrt{5})/2)$ .

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## 1 Introduction

For an undirected graph  $G$ , the *treedepth* of  $G$  is the minimum height of a rooted forest whose ancestor-descendant closure contains  $G$  as a subgraph. Together with more widely known related width notions such as treewidth and pathwidth, it plays a major role in structural graph theory, in particular in the study of general sparse graph classes [8, 7, 6].

An important property of treedepth is that it admits a number of equivalent definitions. Following the definition of treedepth above, a *treedepth decomposition* of a graph  $G$  consists of a rooted forest  $F$  and an injective mapping  $f : V(G) \rightarrow V(F)$  such that for every  $uv \in E(G)$  the vertices  $f(u)$  and  $f(v)$  are in ancestor-descendant relation in  $F$ . The *width* of a treedepth decomposition  $(F, f)$  is the height of  $F$  (the number of vertices on the longest leaf-to-root path in  $F$ ) and the treedepth of  $G$  is the minimum possible height of a treedepth decomposition



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of  $G$ . A *centered coloring* of a graph  $G$  is an assignment  $\alpha : V(G) \rightarrow \mathbb{Z}$  such that for every connected subgraph  $H$  of  $G$ ,  $\alpha$  has a *center* in  $H$ : a vertex  $v \in V(H)$  of unique color, i.e.,  $\alpha(v) \neq \alpha(w)$  for every  $w \in V(H) \setminus \{v\}$ . A *vertex ranking* of a graph  $G$  is an assignment  $\alpha : V(G) \rightarrow \mathbb{Z}$  such that in every connected subgraph  $H$  of  $G$  there is a unique vertex of *maximum* rank (value  $\alpha(v)$ ). Clearly, each vertex ranking is a centered coloring. It turns out that the minimum number of colors (minimum size of the image of  $\alpha$ ) needed for a centered coloring and for a vertex ranking are equal and equal to the treedepth of a graph [6].

While there are multiple examples of algorithmic usage of treedepth in the theory of sparse graphs [7, 8], our understanding of the complexity of computing minimum width treedepth decompositions is limited. For a graph  $G$ , let  $\text{td}(G)$  and  $\text{tw}(G)$  denote the treedepth and the treewidth of  $G$ , respectively. An algorithm of Reidl, Rossmanith, Villaamil, and Sikdar [9] computes exactly the treedepth of an input graph  $G$  in time  $2^{\mathcal{O}(\text{td}(G) \cdot t)} n^{\mathcal{O}(1)}$ , given a tree decomposition of  $G$  of width  $t$ . Combined with the classic constant-factor approximation algorithm for treewidth that runs in  $2^{\mathcal{O}(\text{tw}(G))} n^{\mathcal{O}(1)}$  time [10], one obtains an exact algorithm for treedepth running in time  $2^{\mathcal{O}(\text{td}(G)\text{tw}(G))} n^{\mathcal{O}(1)}$ . No faster exact algorithm is known.

For approximation algorithms, the following folklore lemma (presented with full proof in [4]) is very useful.

► **Lemma 1.** *Given a graph  $G$  and a tree decomposition  $(T, \beta)$  of  $G$  of maximum bag size  $w$ , one can in polynomial time compute a treedepth decomposition of  $G$  of width at most  $w \cdot \text{td}(T)$ .*

Using Lemma 1, one can obtain an approximation algorithm for treedepth with a cheap tradeoff trick.<sup>1</sup>

► **Lemma 2.** *Given a graph  $G$ , one can in polynomial time compute a treedepth decomposition of  $G$  of width  $\mathcal{O}(\text{td}(G) \cdot \text{tw}(G)^2 \log \text{tw}(G))$ .*

**Proof.** Let  $n = |V(G)|$ . Using the polynomial-time approximation algorithm for treewidth [3], compute a tree decomposition  $(T, \beta)$  of  $G$  of width  $t = \mathcal{O}(\text{tw}(G) \sqrt{\log \text{tw}(G)})$  and  $\mathcal{O}(n)$  bags. For every integer  $1 \leq k \leq (\log n)/t$ , use the algorithm of [9] to check in polynomial time if the treedepth of  $G$  is at most  $k$ . Note that if this is the case, the algorithm finds an optimal treedepth decomposition and we can conclude. Otherwise, we have  $\log n \leq \text{td}(G) \cdot t$  and we apply Lemma 1 to  $G$  and  $(T, \beta)$  obtaining a treedepth decomposition of  $G$  of width

$$\mathcal{O}(t \log n) \leq \mathcal{O}(\text{td}(G) \cdot t^2) \leq \mathcal{O}(\text{td}(G) \cdot \text{tw}(G)^2 \log \text{tw}(G)). \quad \blacktriangleleft$$

Lemma 2 is the only polynomial approximation algorithm for treedepth running in polynomial time we are aware of.

A related topic to exact and approximation algorithms computing minimum-width treedepth decomposition is the study of obstructions to small treedepth. Dvořák, Giannopoulou, and Thilikos [2] proved that every minimal graph of treedepth  $k$  has the number of vertices at most double-exponential in  $k$ . More recently, Kawarabayashi and Rossman showed an excluded-minor theorem for treedepth.

► **Theorem 3** ([4]). *There exists a universal constant  $C$  such that for every integer  $k$  every graph of treedepth at least  $Ck^5 \log^2 k$  is either of treewidth at least  $k$ , contains a subdivision of a full binary tree of depth  $k$  as a subgraph, or contains a path of length  $2^k$ .*

<sup>1</sup> This trick has been observed and communicated to us by Michał Pilipczuk. We thank Michał for allowing us to include it in this paper.

While neither the results of [2] nor [4] have a direct application in the approximability of treedepth, these topics are tightly linked with each other and we expect that a finer understanding of treedepth obstructions is necessary to provide more efficient algorithms computing or approximating the treedepth of a graph.

### Our results

Our main result is the following statement, improving upon the work of Kawarabayashi and Rossman [4].

► **Theorem 4.** *Let  $G$  be a graph of treewidth  $\text{tw}(G)$  and treedepth  $\text{td}(G)$ . Then there exists a subcubic tree  $H$  that is a subgraph of  $G$  and is of treedepth  $\Omega(\text{td}(G)/(\text{tw}(G) \log \text{tw}(G)))$ .*

In other words, Theorem 4 states that there exists a constant  $C$  such that for every graph  $G$  and integers  $a, b \geq 2$ , if the treedepth of  $G$  is at least  $Cab \log a$ , then the treewidth of  $G$  is at least  $a$  or  $G$  contains a subcubic tree of treedepth  $b$ . Since every subcubic tree of treedepth  $d$  contains either a simple path of length  $2^{\Omega(\sqrt{d})}$  or a subdivision of a full binary tree of depth  $\Omega(\sqrt{d})$  [4], we have the following corollary.

► **Corollary 5.** *Let  $G$  be a graph of treewidth  $\text{tw}(G)$  and treedepth  $\text{td}(G)$ . Then for some*

$$h = \Omega\left(\sqrt{\text{td}(G)/(\text{tw}(G) \log \text{tw}(G))}\right)$$

*$G$  contains either a simple path of length  $2^h$  or a subdivision of a full binary tree of depth  $h$ .*

*Consequently, there exists an absolute constant  $C$  such that for every integer  $k \geq 1$  and a graph  $G$  of treedepth at least  $Ck^3 \log k$ , either*

- *$G$  has treewidth at least  $k$ ,*
- *$G$  contains a subdivision of a full binary tree of depth  $k$  as a subgraph, or*
- *$G$  contains a path of length  $2^k$ .*

In other words, Corollary 5 improves the bound  $k^5 \log^2 k$  of Kawarabayashi and Rossman [4] to  $k^3 \log k$ . We remark here that there are subcubic trees of treedepth  $\Omega(h^2)$  that contains neither a path of length  $2^h$  nor a subdivision of a full binary tree of depth  $h$ ,<sup>2</sup> and thus the quadratic loss between the statements of Theorem 4 and Corollary 5 is necessary.

Inside the proof of Theorem 4 we make use of the following lemma that may be of independent interest. This lemma is the main technical improvement upon the work of Kawarabayashi and Rossman [4].

► **Lemma 6.** *Every tree of treedepth  $d$  contains a subcubic subtree of treedepth at least  $\frac{\log((1+\sqrt{5})/2)}{\log(3)}d$ .*

*Furthermore, such a subtree can be found in polynomial time.*

The techniques developed to prove Theorem 4 have some implications on the approximability of treedepth. We improve upon Lemma 2 as follows.

► **Theorem 7.** *Given a graph  $G$ , one can in polynomial time compute a treedepth decomposition of  $G$  of width  $\mathcal{O}(\text{td}(G) \cdot \text{tw}(G) \log^{3/2} \text{tw}(G))$ .*

<sup>2</sup> It is straightforward to deduce such an example from the proof of [4]. We provide such an example in Section 6.

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The result of Kawarabayashi and Rossman [4] has been also an important ingredient in the study of *linear colorings* [5]. A coloring  $\alpha : V(G) \rightarrow \mathbb{Z}$  of a graph  $G$  is a *linear coloring* if for every (not necessarily induced) path  $P$  in  $G$  there exists a vertex  $v \in V(P)$  of unique color  $\alpha(v)$  on  $P$ . Clearly, each centered coloring is a linear coloring, but the minimum number of colors needed for a linear coloring can be much smaller than the treedepth of a graph. Kun et al. [5] provided a polynomial relation between the treedepth and the minimum number of colors in a linear coloring; by replacing their usage of [4] by our result (and using an improved bound for the excluded grid theorem [1]) we obtain an improved bound.

► **Theorem 8.** *There exists a polynomial  $p$  such that for every integer  $k$  and graph  $G$ , if the treedepth of  $G$  is at least  $k^{19}p(\log k)$ , then every linear coloring of  $G$  requires at least  $k$  colors.*

The previous bound of [5] is  $k^{190}p(\log k)$ .

After proving Lemma 6 in Section 2, we prove Theorem 4 in Section 3. Theorem 7 is proven in Section 3 while Theorem 8 is proven in Section 5. The symbol  $\log_p$  stands for base- $p$  logarithm and  $\log$  stands for  $\log_2$ . We denote  $\varphi = \frac{1+\sqrt{5}}{2}$ ; note that  $\varphi$  is chosen in a way so that  $\varphi^2 = \varphi + 1$  and  $\varphi > 1$ .

### 2 Subcubic subtrees of trees of large treedepth

This section focuses on proving Lemma 6.

Schäffer [11] proved that there is a linear time algorithm for finding a vertex ranking with minimum number of colors of a tree  $T$ . We follow [5] for a good description of its properties.

In original Schäffer's algorithm ranks are starting from 1, however for the ease of exposition let us assume that ranks are starting from 0. That is, the algorithm constructs a vertex ranking  $\alpha : V(T) \rightarrow \{0, 1, 2, \dots\}$  trying to minimize the maximum value attained by  $\alpha$ . Assume that  $T$  is rooted in an arbitrary vertex and for every  $v \in V(T)$  let  $T_v$  be the subtree rooted at  $v$ .

Of central importance to Schäffer's algorithm are what we will refer to as *rank lists*. For a rooted tree  $T$ , the rank list  $L(T)$  for vertex ranking  $\alpha$  consists of these ranks  $i$  for which there exists a path  $P$  starting from the root and ending in a vertex  $v$  with  $\alpha(v) = i$  such that for every  $u \in V(P) \setminus \{v\}$  we have  $\alpha(u) < \alpha(v)$ , that is,  $v$  is the unique vertex of maximum rank on  $P$ . More formally:

► **Definition 9.** *For a vertex ranking  $\alpha$  of tree  $T$ , the rank list of  $T$ , denoted  $L(T)$ , can be defined recursively as  $L(T) = L(T \setminus T_v) \cup \{\alpha(v)\}$  where  $v$  is the vertex of maximum rank in  $T$ .*

Schäffer's algorithm arbitrarily roots  $T$  and builds the ranking from the leaves to the root of  $T$ , computing the rank of each vertex from the rank lists of each of its children. For brevity, we denote  $L(v) = L(T_v)$  for every  $v$  in  $T$ .

► **Proposition 10.** *Let  $\alpha$  be a vertex ranking of  $T$  produced by Schäffer's algorithm and let  $v \in T$  be a vertex with children  $u_1, \dots, u_l$ . If  $x$  is the largest integer appearing on rank lists of at least two children of  $v$  (or  $-1$  if all such rank lists are pairwise disjoint) then  $\alpha(v)$  is the smallest integer satisfying  $\alpha(v) > x$  and  $\alpha(v) \notin \bigcup_{i=1}^l L(u_i)$ .*

For a node  $v \in V(T)$ , and vertex ranking  $\alpha$ , the following potential is pivotal to the analysis of Schäffer's algorithm. Let  $l_0 > l_1 > \dots > l_{|L(v)|-1}$  be the elements of  $L(v)$  sorted in decreasing order.

$$\zeta(v) = \sum_{r \in L(v)} 3^r = \sum_{i=0}^{|L(v)|-1} 3^{l_i}.$$

When we write  $\zeta(T)$  for some tree  $T$  we refer to  $\zeta(s)$  where  $s$  is a root of  $T$ . For our purposes, we will also use a skewed version of potential function with a different base

$$\sigma(v) = \sum_{i=0}^{|L(v)|-1} \varphi^{l_i-i},$$

where again  $l_0 > l_1 > \dots > l_{|L(v)|-1}$  are elements of  $L(v)$  sorted in decreasing order. Throughout this section, when focusing on one node  $v \in V(T)$ , we use notation that  $l_i$  is  $i$ -th element of set  $L(v)$  when sorted in decreasing order and when 0-based indexed.

Let us start with proving following two bounds that estimate  $\text{td}(T)$  in terms of  $\zeta(T)$  and  $\sigma(T)$ .

▷ **Claim 11.**  $\log_\varphi(\sigma(T)) \geq \text{td}(T) - 1$ .

*Proof.* We know that  $L(T)$  is nonempty and its biggest element is equal to  $\text{td}(T) - 1$  (we need to subtract one because we use nonnegative numbers as ranks, not positive). Therefore we have

$$\sigma(T) = \sum_{i=0}^{|L(T)|-1} \varphi^{l_i-i} \geq \varphi^{l_0} = \varphi^{\text{td}(T)-1}.$$

Hence,  $\log_\varphi(\sigma(T)) \geq \text{td}(T) - 1$ , as desired. ◁

▷ **Claim 12.**  $\log_3(\zeta(T)) + \log_3(2) < \text{td}(T)$ .

*Proof.* We have that

$$\zeta(T) = \sum_{r \in L(v)} 3^r \leq \sum_{r=0}^{\text{td}(T)-1} 3^r = \frac{3^{\text{td}(T)} - 1}{2},$$

$$2\zeta(T) \leq 3^{\text{td}(T)} - 1 < 3^{\text{td}(T)}$$

$\log_3(2) + \log_3(\zeta(T)) < \text{td}(T)$ . ◁

We are ready to prove Lemma 6. Given tree  $T$  we want to produce a subcubic (i.e., maximum degree at most 3) tree  $S$  which is a subtree of  $T$  and that fulfills  $\text{td}(S) > \text{td}(T) \log_3(\varphi)$ .

Let us start our algorithm by arbitrary rooting  $T$  and computing rank lists using Schäffer's algorithm. Then for every vertex  $v \in T$  we define  $C(v)$  as a set of two children of  $v$  that have the biggest value of  $\zeta$  in case  $v$  has at least two children, or all children otherwise. Let us now define forest  $F$  whose vertex set is the same as vertex set of  $T$  where for every  $v$  we put edges between  $v$  and all elements of  $C(v)$ . Clearly this is a forest consisting of subcubic trees which are subtrees of  $T$  (where *subtree* is understood as subgraph, not necessarily as some vertex  $t$  along with all its descendants in a rooted tree). Let  $S$  be a tree of this forest containing root of  $T$ . We claim that  $S$  is that subcubic subtree of  $T$  we are looking for. Note that computing  $F$  and thus  $S$  can be trivially done in polynomial time. Hence, we are left with proving that  $\text{td}(S) > \text{td}(T) \log_3(\varphi)$ .

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Let us root every tree of  $F$  in a vertex that was closest to root of  $T$  in  $T$ . Then compute rank lists for these trees using Schäffer's algorithm. So now, for every vertex we have two rank lists, one for  $T$  and one for  $F$ . Let us now denote these second ranklists as  $\tilde{L}(v)$  for  $v \in V(T)$  and let us define function  $\tilde{\zeta}$  which will be similar potential function as  $\zeta$ , but operating on rank lists  $\tilde{L}(v)$  instead of  $L(v)$ . Following claim will be crucial.

▷ **Claim 13.** For every  $v \in V(T)$  it holds that  $\tilde{\zeta}(v) \geq \sigma(v)$ .

We first verify that Claim 13 implies Lemma 6.

**Proof of Lemma 6.** Using also Claims 12 and 11 we infer that

$$\begin{aligned}
 \text{td}(S) &> \log_3(\tilde{\zeta}(S)) + \log_3(2) && \text{by Claim 12 for } S \\
 &\geq \log_3(\sigma(T)) + \log_3(2) && \text{by Claim 13} \\
 &= \log_\varphi(\sigma(T)) \cdot \log_3(\varphi) + \log_3(2) && \text{logarithm base change} \\
 &\geq (\text{td}(T) - 1) \cdot \log_3(\varphi) + \log_3(2) && \text{by Claim 11 for } T \\
 &= \text{td}(T) \cdot \log_3(\varphi) - \log_3(\varphi) + \log_3(2) > \text{td}(T) \cdot \log_3(\varphi). && \blacktriangleleft
 \end{aligned}$$

Thus it remains to prove Claim 13. To this end, we prove two auxiliary inequalities.

▷ **Claim 14.** For every  $v \in V(T)$  it holds that  $\tilde{\zeta}(v) \geq 1 + \sum_{s \in C(v)} \tilde{\zeta}(s)$

*Proof.* We express every  $\tilde{\zeta}(x)$  for  $x \in \{v\} \cup C(v)$  as a sum of powers of 3 and count how many times each power occurs on both sides of this claimed inequality. Consider a summand  $3^c$ . If  $c > \alpha(v)$  then, by the choice of  $\alpha(v)$ ,  $3^c$  appears at most once on the right side and if it appears there, then it appears on the left side as well, so contributions of summands of form  $3^c$  for  $c > \alpha(v)$  to both sides are equal. The summand  $3^{\alpha(v)}$  appears once on the left side and does not appear on the right side. For  $c < \alpha(v)$ , the summands of form  $3^c$  appear at most twice in  $\sum_{s \in C(v)} \tilde{\zeta}(s)$ , so their contribution to right side can be bounded from above by  $\sum_{c=0}^{\alpha(v)-1} 2 \cdot 3^c = 3^{\alpha(v)} - 1$ , so in fact  $3^{\alpha(v)}$  from the left side contributes at least as much as remaining summands from the right side. This finishes the proof of the claim. ◀

▷ **Claim 15.** For every  $v \in V(T)$  it holds that  $\sigma(v) \leq 1 + \sum_{s \in C(v)} \sigma(s)$

*Proof.* Recall that by the definition  $C(v)$  is a set of two children of  $v$  in  $T$  with the biggest values of  $\zeta$  or a set of all children of  $v$  in case it has less than two of them. Observe that having bigger value of  $\zeta(v)$  is another way of expressing having the set  $L(v)$  bigger lexicographically when sorted in decreasing order.

If  $v$  is a leaf then  $C(v)$  is empty and  $\sigma(v) = 1$ , so the inequality is obvious. Henceforth we focus on a vertex  $v$  that is not a leaf. In our proof following equation will come handy:

$$\varphi = \sum_{i=0}^{\infty} \varphi^{-2i}$$

It holds since  $\sum_{i=0}^{\infty} \varphi^{-2i} = \frac{1}{1-\varphi^{-2}} = \frac{\varphi^2}{\varphi^2-1} = \frac{\varphi^2}{\varphi} = \varphi$ .

Let us now analyze  $L(v)$ . It consists of some prefix  $P$  of values that appeared exactly once in children of  $v$ , then  $\alpha(v)$  and then nothing (when enumerated from the biggest to the smallest). Let us now denote by  $A_i$  intersection of  $L(u_i)$  and  $P$ , where  $u_i$  is  $i$ -th child of  $v$  when sorted in nonincreasing order by their values  $\zeta(u_i)$  (1-based). We distinguish two cases:

**Case 1:  $A_2$  is nonempty.** If  $A_2$  is nonempty then in particular it means that  $v$  has at least two children. Let us denote the biggest element of  $L(u_2)$  by  $d$ . We have that  $d \in P$ , but  $d$  is not the biggest element of  $P$ . Its contribution to  $\sigma(u_2)$  is  $\varphi^d$ , however its contribution to  $\sigma(v)$  is at most  $\varphi^{d-1}$  (because of the skew and since  $d$  is not the biggest element of  $P$ ). Contribution to  $\sigma(v)$  of elements smaller than  $d$  can be bounded from above by  $\varphi^{d-3} + \varphi^{d-5} + \dots$ . We know that  $d = l_j$  for some  $j$ , where  $j \geq 1$  and  $L(v)$  consists of elements  $l_0 > l_1 > \dots > l_{|L(v)-1|}$ . We have that  $l_k \in L(u_1)$  for  $k < j$  and that  $l_j \geq l_i + (i - j)$  for  $i \geq j$ , so  $l_i - i \leq l_j - j - 2(i - j) = d - j - 2(i - j)$ . We can deduce that

$$\begin{aligned} \sigma(v) &= \sum_{i=0}^{|L(v)|-1} \varphi^{l_i-i} = \sum_{i=0}^{j-1} \varphi^{l_i-i} + \sum_{i=j}^{|L(v)|-1} \varphi^{l_i-i} \leq \sigma(u_1) + \sum_{i=j}^{|L(v)|-1} \varphi^{d-j-2(i-j)} \\ &\leq \sigma(u_1) + \varphi^{d-j} \sum_{i=j}^{\infty} \varphi^{-2(i-j)} = \sigma(u_1) + \varphi^{d-j} \sum_{i=0}^{\infty} \varphi^{-2i} = \sigma(u_1) + \varphi^{d-j+1} \\ &\leq \sigma(u_1) + \varphi^d \leq \sigma(u_1) + \sigma(u_2) < 1 + \sigma(u_1) + \sigma(u_2), \end{aligned}$$

which is what we wanted to prove.

**Case 2:  $A_2$  is empty.** Let us now introduce a few variables:

- $d$  - the biggest integer number smaller than  $\alpha(v)$  that is not an element of  $L(u_1)$ . We know that elements from  $d + 1$  to  $\alpha(v) - 1$  belong to  $L(u_1)$ .
- $k$  - shorthand for number of these elements (which is equal to  $\alpha(v) - 1 - d$ ).  $k$  can be zero, but cannot be negative.
- $g$  - the number of elements of  $L(v)$  that are bigger than  $\alpha(v)$ .

Then from the definition of  $\alpha(v)$  either

- $d = -1$ ; or
- $v$  has at least two children and  $L(u_2)$  contains a number that is at least  $d$ .

Because of that we have  $1 + \sum_{s \in C(v)} \sigma(s) \geq \sigma(u_1) + \varphi^d$ . We know that  $\sum_{s \in C(v)}$  is either  $\sigma(u_1)$  or  $\sigma(u_1) + \sigma(u_2)$ , depending on whether  $v$  has only one child or more. If  $d = -1$  then  $1 \geq \varphi^d$  and stated inequality holds. If  $d \neq -1$  then  $u_2$  exists and  $\sigma(u_2) \geq \varphi^d$ .

Note that either  $k > 0$  or  $g > 0$ , because if  $k = g = 0$  then  $d = \alpha(v) - 1$  and  $L(u_1)$  cannot contain elements bigger than  $\alpha(v)$  (because  $g = 0$ ), cannot contain  $\alpha(v)$  (from the definition of  $\alpha(v)$ ) and cannot contain  $\alpha(v) - 1$  (since  $d = \alpha(v) - 1$ ), so its biggest element is at most  $d - 1$ . If  $d = -1$  then it means that  $v$  is a leaf, but we already assumed it is not one. However, if  $v$  has at least two children and  $L(u_2)$  contains a number that is at least  $d$ , then it contradicts the assumption that  $\zeta(u_1) \geq \zeta(u_2)$ . So indeed it holds that  $k > 0$  or  $g > 0$  and therefore  $k + g \geq 1$ .

We have that

$$\sigma(v) - \sigma(u_1) \leq \varphi^{\alpha(v)-g} - (\varphi^{\alpha(v)-g-1} + \varphi^{\alpha(v)-g-3} + \dots + \varphi^{\alpha(v)-g-2k+1}),$$

which is because summands coming from numbers bigger than  $\alpha(v)$  in  $L(v)$  and  $L(u_1)$  cancel out ( $A_2$  is empty, so all elements of  $L(v)$  different than  $\alpha(v)$  come from  $L(u_1)$ ) and new rank  $\alpha(v)$  contributes  $\varphi^{\alpha(v)-g}$  to  $\sigma(v)$  whereas  $L(u_1)$  contains numbers from  $d + 1$  up to  $\alpha(v) - 1$  and their contribution to  $\sigma(u_1)$  is  $\varphi^{\alpha(v)-g-1} + \varphi^{\alpha(v)-g-3} + \dots + \varphi^{\alpha(v)-g-2k+1}$ . We conclude that  $\sigma(v) - \sigma(u_1) \leq \varphi^{-g}(\varphi^{\alpha(v)} - (\varphi^{\alpha(v)-1} + \varphi^{\alpha(v)-3} + \dots + \varphi^{\alpha(v)-2k+1}))$ .

On the other hand since  $\varphi^2 = \varphi + 1$  we have that

$$\begin{aligned} \varphi^{\alpha(v)} &= \varphi^{\alpha(v)-1} + \varphi^{\alpha(v)-2} = \varphi^{\alpha(v)-1} + \varphi^{\alpha(v)-3} + \varphi^{\alpha(v)-4} = \dots = \\ &= (\varphi^{\alpha(v)-1} + \varphi^{\alpha(v)-3} + \dots + \varphi^{\alpha(v)-2k+1}) + \varphi^{\alpha(v)-2k}. \end{aligned}$$



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Because of that we have

$$\sigma(v) - \sigma(u_1) \leq \varphi^{-g} \cdot \varphi^{\alpha(v) - 2k} = \varphi^{\alpha(v) - 2k - g} = \varphi^{\alpha(v) - (\alpha(v) - 1 - d) - k - g} = \varphi^{d+1 - (k+g)} \leq \varphi^d.$$

From that we conclude that  $\sigma(v) \leq \sigma(u_1) + \varphi^d \leq 1 + \sum_{s \in C(v)} \sigma(s)$ , what concludes proof of this claim.  $\triangleleft$

Now, having claims 15 and 14 proven, we can wrap our reasoning up. If  $v$  is a leaf then  $\sigma(v) = \tilde{\zeta}(v) = 1$ . If  $v$  is not a leaf then we know that  $\sigma(v) \leq 1 + \sum_{s \in C(v)} \sigma(s)$  and  $\tilde{\zeta}(v) \geq 1 + \sum_{s \in C(v)} \tilde{\zeta}(s)$ , so by straightforward induction we get that  $\sigma(v) \leq \tilde{\zeta}(v)$  for every  $v \in V(T)$ , as desired by Claim 13.

### 3 Proof of Theorem 4

Let  $G$  be a nonempty graph and let  $r = \text{td}(G)/(\text{tw}(G) + 1)$ . Recall that our goal is to show existence of a subcubic tree  $H$  being a subgraph of  $G$  such that  $\text{td}(H)$  is  $\Omega(r/\log(\text{tw}(G) + 1))$ . Without loss of generality we may assume that  $G$  is connected, as otherwise we focus on the connected component of  $G$  of maximum treedepth. Also, the statement is trivial for  $\text{tw}(G) \leq 1$  (when  $G$  is a tree) and when  $r \leq 2$ , so assume otherwise.

We consider a greedy tree decomposition of  $G$ , as defined in [4]. A greedy tree decomposition is a tree decomposition that can be also interpreted as a treedepth decomposition.

Recall that a tree decomposition of a graph  $G$  is a pair  $(T, \beta)$  where  $T$  is a rooted tree and  $\beta : V(T) \rightarrow 2^{V(G)}$  is such that for every  $v \in V(G)$  the set  $\{t \in V(T) \mid v \in \beta(t)\}$  induces a connected nonempty subtree of  $T$  and for every  $uv \in E(G)$  there exists  $t \in V(T)$  with  $u, v \in \beta(t)$ . A tree decomposition  $(T, \beta)$  of a graph  $G$  is *greedy* if

1.  $V(T) = V(G)$ ,
2. for every  $uv \in E(G)$ , the nodes  $u$  and  $v$  in  $T$  are in ancestor-descendant relation in  $T$ , and
3. for every vertex  $u \in V(T)$  and its child  $v$  there is some descendant  $w$  of  $v$  in  $T$  such that  $uw \in E(G)$ .

Lemma 3.6. in [4] states that for every connected graph  $G$  there exists a greedy tree decomposition  $(T, \beta)$  of width  $\text{tw}(G)$ , that is, all bags  $\beta(t)$  for  $t \in V(T)$  have size bounded by  $\text{tw}(G) + 1$ . Let  $(T, \beta)$  be such a decomposition of our graph  $G$ . By Lemma 1 we get that  $\text{td}(T) \geq r$ , as otherwise we would be able to construct treedepth decomposition of too low treedepth.

In the remainder of the proof we show the following lemma.

► **Lemma 16.** *Let  $G$  be a connected graph,  $(T, \beta)$  be a greedy tree decomposition of  $G$ , and let  $\tau \geq 2$  be such that  $|\beta(t)| \leq \tau$  for every  $t \in V(T)$ . Then  $G$  contains a subcubic tree of treedepth  $\Omega(\text{td}(T)/\log \tau)$ .*

Theorem 4 follows immediately from Lemma 16 applied to the tree decomposition  $(T, \beta)$  of  $G$ . Thus, it remains to prove Lemma 16.

To this end, we first apply Lemma 6 to tree  $T$  and obtain a subcubic tree  $S$  such that

$$\text{td}(S) \geq r \cdot \log_3(\varphi). \tag{1}$$

Second, we apply the core part of the reasoning of Kawarabayashi and Rossman [4]. The construction of Section 5 of [4] can be encapsulated in the following lemma.



► **Lemma 17** (Section 5 of [4]). *Let  $(T, \beta)$  be a greedy tree decomposition of graph  $G$  and let  $\tau = \max_{t \in V(T)} |\beta(t)|$ . Then for every subcubic subtree  $S$  of  $T$  there exists a subtree  $F$  of  $G$  such that  $V(S) \subseteq V(F)$  and the maximum degree of  $F$  is bounded by  $\tau + 2$ .*

By application of Lemma 17 to our decomposition  $(T, \beta)$  and subtree  $S$  we get a tree  $F$  in  $G$ , which has large treedepth, as we show in a moment. To this end, we need the following simple bound on treedepth of trees.

► **Lemma 18.** *For every tree  $H$  with maximum degree bounded by  $d \geq 2$  it holds that*

$$\log_d |V(H)| \leq \text{td}(T) \leq 1 + \log_2 |V(H)|.$$

**Proof.** We use the following equivalent recursive definition of treedepth: Treedepth of an empty graph is 0, treedepth of a disconnected graph equals the maximum of treedepth over its connected components, while for nonempty connected graphs  $G$  we have  $\text{td}(G) = 1 + \min_{v \in V(G)} \text{td}(G - v)$ .

For the lower bound, for  $k \geq 1$  let  $f_d(k)$  be the maximum possible number of vertices of a tree of maximum degree at most  $d$  and treedepth at most  $k$ . Clearly,  $f_d(1) = 1$ . Since removing a single vertex from a tree of maximum degree at most  $d$  results in at most  $d$  connected components, we have that

$$f_d(k+1) \leq 1 + d \cdot f_d(k).$$

Consequently, we obtain by induction that

$$f_d(k) \leq d^k - 1.$$

This proves the lower bound. For the upper bound, note that in every tree  $T$  there exists a vertex  $v \in V(T)$  such that every connected component of  $T - \{v\}$  has at most  $|V(T)|/2$  vertices. Consequently, if we define  $g(n)$  to be the maximum possible treedepth of an  $n$ -vertex tree, then  $g(1) = 1$  and we have that

$$g(n) \leq 1 + g(\lfloor n/2 \rfloor).$$

This proves the upper bound. ◀

By (1) and Lemma 18 we get that  $|V(S)| \geq 2^{r \cdot \log_3(\varphi) - 1}$ . This implies that also

$$|V(F)| \geq 2^{r \cdot \log_3(\varphi) - 1}. \quad (2)$$

As  $S$  is subcubic, by Lemma 17 we know that the maximum degree of  $F$  is bounded by  $\text{tw}(G) + 3$ . Therefore Lemma 18 and (2) jointly imply that

$$\text{td}(F) \geq \log_{(\text{tw}(G)+3)} 2^{r \cdot \log_3(\varphi) - 1} \geq \frac{r \cdot \log_3(\varphi) - 1}{\log(\text{tw}(G) + 3)} \geq \frac{\log_3(\varphi)}{20} \cdot r / \log(\text{tw}(G) + 1). \quad (3)$$

Here, the last inequality follows from the assumptions  $r > 2$  and  $\text{tw}(G) \geq 2$ ; note that the constant 20 is sufficiently large constant for the estimations to work.

As tree  $F$  is not necessarily subcubic, we apply one more time Lemma 6 and get a subcubic subtree  $H$  of  $F$  such that

$$\text{td}(H) \geq \text{td}(F) \cdot \log_3(\varphi) \geq \frac{\log_3(\varphi)^2}{20} \cdot r / \log(\text{tw}(G) + 1), \quad (4)$$

which finishes the proof of Lemma 16 and of Theorem 4.

#### 4 Proof of Theorem 7

**Proof of Theorem 7.** Without loss of generality we can assume that the input graph  $G$  is connected. As in the proof of Lemma 2, we apply the polynomial-time approximation algorithm for treewidth [3], to compute a tree decomposition  $(T_0, \beta_0)$  of  $G$  with  $\mathcal{O}(n)$  nodes of  $T_0$  and  $|\beta(t)| \leq \tau$  for every  $t \in V(T_0)$  and some  $\tau = \mathcal{O}(\text{tw}(G)\sqrt{\log \text{tw}(G)})$ . As discussed in [4], one can in polynomial time turn  $(T_0, \beta_0)$  into a greedy tree decomposition  $(T, \beta)$  of  $G$  without increasing the maximum size of a bag, that is, still  $|\beta(t)| \leq \tau$  for every  $t \in V(T)$ . We apply Lemma 1 to  $(T, \beta)$ , returning a treedepth decomposition of  $G$  of width at most  $\tau \cdot \text{td}(T) = \mathcal{O}(\text{td}(T)\text{tw}(G)\sqrt{\log \text{tw}(G)})$ .

It remains to bound  $\text{td}(T)$ . Lemma 16 asserts that  $G$  contains a subcubic tree  $H$  of treedepth  $\Omega(\text{td}(T)/\log \tau)$ . Therefore  $\text{td}(T) = \mathcal{O}(\text{td}(H) \log \tau) = \mathcal{O}(\text{td}(G) \log \text{tw}(G))$  and thus the width of the computed treedepth decomposition is  $\mathcal{O}(\text{td}(G)\text{tw}(G) \log^{3/2} \text{tw}(G))$ . This finishes the proof of Theorem 7.  $\blacktriangleleft$

#### 5 Proof of Theorem 8

Here we show how to assemble the proof of Theorem 8 from Theorem 4, a number of intermediate results of [5], and an improved excluded grid theorem due to Chuzhoy and Tan [1]:

► **Theorem 19** ([1]). *There exists a polynomial  $p_{\text{GMT}}$  such that for every integer  $k$  if a graph  $G$  has treewidth at least  $k^9 p_{\text{GMT}}(\log k)$  then  $G$  contains a  $k \times k$  grid as a minor.*

The following two results were proven in [5].

► **Lemma 20** ([5]). *If a graph  $G$  contains a  $k \times k$  grid as a minor, then every linear coloring of  $G$  requires  $\Omega(\sqrt{k})$  colors.*

► **Lemma 21** ([5]). *If  $G$  is a tree of treedepth  $d$  and maximum degree  $\Delta$ , then every linear coloring of  $G$  requires at least  $d/\log_2(\Delta)$  colors.*

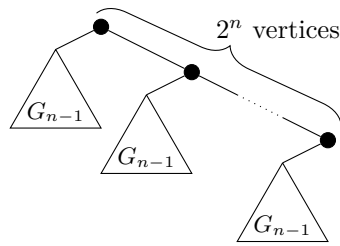
Recall that Theorem 4 asserts that there exists a constant  $C$  such that for every graph  $G$  and integers  $a, b \geq 2$ , if the treedepth of  $G$  is at least  $Cab \log a$ , then either the treewidth of  $G$  is at least  $a$  or  $G$  contains a subcubic tree of treedepth at least  $b$ . Applying this theorem to  $a = \theta(k^2)$  and  $b = k \log_2(3)$ , one obtains that if the treedepth of  $G$  is  $\Omega(k^{19} p_{\text{GMT}}(\log k) \log k)$ , then  $G$  contains either a  $\theta(k^2) \times \theta(k^2)$  grid minor or a subcubic tree of treedepth at least  $k \log_2(3)$ . In the first outcome, Lemma 20 gives the desired number of colors of a linear coloring, while in the second outcome the same result is obtained from Lemma 21. This concludes the proof of Theorem 8.

#### 6 An example of a tree with treedepth quadratic in the height of the binary tree or logarithm of a length of a path

In this section we provide a construction of a family of trees  $(G_n)_{n \geq 1}$  such that

1. The tree  $G_n$  does not contain a path with  $2^{n+2}$  vertices.
2. The tree  $G_n$  does not contain a subdivision of a full binary tree of depth  $n + 2$ .
3. The treedepth of  $G_n$  is at least  $\binom{n+1}{2}$ .

We will consider each tree  $G_n$  as a rooted tree. The tree  $G_1$  consists of a single vertex. For  $n \geq 2$ , the tree  $G_n$  is defined recursively as follows. We take a path  $P_n$  with  $2^n$  vertices and for each  $v \in V(P_n)$  we create a copy  $G_n^v$  of  $G_{n-1}$  and attach its root to  $v$ . We root  $G_n$  in one of the endpoints of  $P_n$ ; see Figure 1. We now proceed with the proof of the properties of  $G_n$ .



■ **Figure 1** Construction of  $G_n$ .

Since every path in  $G_n$  is contained in at most two root-to-leaf paths (not necessarily edge-disjoint), to show Property (1) it suffices to show the following.

► **Lemma 22.** *Every root-to-leaf path in  $G_n$  contains less than  $2^{n+1}$  vertices.*

**Proof.** We prove the statement by induction on  $n$ . For  $n = 1$  the statement is straightforward. For the inductive step, observe that every root-to-leaf path in  $G_n$  consists of a subpath of  $P_n$  (which has  $2^n$  vertices) and a root-to-leaf path in one of the copies  $C_n^v$  of  $G_{n-1}$  (which has less than  $2^n$  vertices by the inductive assumption). ◀

We say that a subtree  $H$  of  $G_n$  that is a subdivision of a full binary tree of height  $h \geq 1$  is *aligned* if  $h = 1$  or  $h \geq 2$  and the closest to the root vertex of  $H$  is of degree 2 in  $H$  and its deletion breaks  $H$  into two subtrees containing a subdivision of a full binary tree of height  $h - 1$ . In other words, an aligned subtree has the same ancestor-descendant relation as the tree  $G_n$ . Observe that any subtree  $H_0$  of  $G_n$  that is a subdivision of a full binary tree of height  $h \geq 2$  contains a subtree that is an aligned subdivision of a full binary tree of height  $h - 1$ . Therefore, to prove Property (2), it suffices to show the following.

► **Lemma 23.**  *$G_n$  does not contain an aligned subdivision of a full binary tree of height  $n + 1$ .*

**Proof.** We prove the claim by induction on  $n$ . It is straightforward for  $n = 1$ . For  $n \geq 2$ , let  $H$  be such an aligned subtree of  $G_n$  and let  $w$  be the closest to the root of  $G_n$  vertex of  $H$ . If  $w \in V(C_n^v)$  for some  $v \in V(P_n)$ , then  $H$  is completely contained in  $C_n^v$ , which is a copy of  $G_{n-1}$ . Otherwise,  $w \in V(P_n)$  and thus one of the components of  $H - \{w\}$  lies in  $C_n^w$ . However, this component contains an aligned subdivision of a full binary tree of height  $n$ . In both cases, we obtain a contradiction with the inductive assumption. ◀

We are left with the treedepth lower bound of Property (3). To this end, we consider the following families of trees. For integers  $a, b \geq 1$ , the family  $\mathcal{G}_{a,b}$  contains all trees  $H$  that are constructed from a path  $P_H$  with at least  $2^a$  vertices by attaching, for every  $v \in V(P_H)$ , a tree  $T_v$  of treedepth at least  $b$  by an edge to  $v$ . We show the following.

► **Lemma 24.** *For every  $H \in \mathcal{G}_{a,b}$  we have  $\text{td}(H) \geq a + b$ .*

**Proof.** We prove the lemma by induction on  $a$ . For  $a = 1$  we have  $\text{td}(H) \geq a + 1$  and  $H$  contains two vertex-disjoint subtrees of treedepth at least  $b$  each. Assume then  $a > 1$  and  $H \in \mathcal{G}_{a,b}$ . Then for every  $v \in V(H)$ ,  $H - v$  contains a connected component that contains a subtree belonging to  $\mathcal{G}_{a-1,b}$ . This finishes the proof. ◀

We show Property (3) by induction on  $n$ . Clearly,  $\text{td}(G_1) = 1 = \binom{1+1}{2}$ . Consider  $n \geq 2$ . Since the treedepth of  $G_{n-1}$  is at least  $\binom{n}{2}$ , we have that  $G_n \in \mathcal{G}_{n, \binom{n}{2}}$ . By Lemma 24, we have that

$$\text{td}(G_n) \geq n + \binom{n}{2} = \binom{n+1}{2}.$$

This finishes the proof of Property (3).

## 7 Conclusions

We have provided improved bounds for the excluded minor approximation of treedepth of Kawarabayashi and Rossman [4]. Our main result, Theorem 4, is close to being optimal in the following sense: as witnessed by the family of trees, if one considers the measure  $r := \text{td}(G)/\text{tw}(G)$ , one cannot hope to find a tree in  $G$  of treedepth larger than  $r$ . We pose getting rid of the  $\log(\text{tw}(G) + 1)$  factor in Theorem 4 as an open problem. Improving the  $Ck^3 \log k$  bound of Corollary 5 to  $Ck^{3-\varepsilon}$  for some  $\varepsilon > 0$  seems much more challenging.

Our main result can be applied to a polynomial-time treedepth approximation algorithm, improving upon state-of-the-art tradeoff trick. As a second open problem, we ask for a polynomial-time or single-exponential in treedepth parameterized algorithm for constant or polylogarithmic approximation of treedepth.

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