

# Planted Models for $k$ -Way Edge and Vertex Expansion

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## Abstract

Graph partitioning problems are a central topic of study in algorithms and complexity theory. Edge expansion and vertex expansion, two popular graph partitioning objectives, seek a 2-partition of the vertex set of the graph that minimizes the considered objective. However, for many natural applications, one might require a graph to be partitioned into  $k$  parts, for some  $k \geq 2$ . For a  $k$ -partition  $S_1, \dots, S_k$  of the vertex set of a graph  $G = (V, E)$ , the  $k$ -way edge expansion (resp. vertex expansion) of  $\{S_1, \dots, S_k\}$  is defined as  $\max_{i \in [k]} \Phi(S_i)$ , and the balanced  $k$ -way edge expansion (resp. vertex expansion) of  $G$  is defined as

$$\min_{\{S_1, \dots, S_k\} \in \mathcal{P}_k} \max_{i \in [k]} \Phi(S_i),$$

where  $\mathcal{P}_k$  is the set of all balanced  $k$ -partitions of  $V$  (i.e each part of a  $k$ -partition in  $\mathcal{P}_k$  should have cardinality  $|V|/k$ ), and  $\Phi(S)$  denotes the edge expansion (resp. vertex expansion) of  $S \subset V$ . We study a natural planted model for graphs where the vertex set of a graph has a  $k$ -partition  $S_1, \dots, S_k$  such that the graph induced on each  $S_i$  has large expansion, but each  $S_i$  has small edge expansion (resp. vertex expansion) in the graph. We give bi-criteria approximation algorithms for computing the balanced  $k$ -way edge expansion (resp. vertex expansion) of instances in this planted model.

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## 1 Introduction

The complexity of computing various graph expansion parameters are central open problems in theoretical computer science, and in spite of many decades of intensive research, they are yet to be fully understood [6, 5, 22, 7, 13, 40]. A central problem in the study of graph partitioning is that of computing the sparsest edge cut in a graph. For a graph  $G = (V, E)$ , we define the *edge expansion* of a set  $S$  of vertices, denoted by  $\phi(S)$  as

$$\phi(S) \stackrel{\text{def}}{=} \frac{|E(S, V \setminus S)|}{|S||V \setminus S|} |V|, \quad (1.1)$$



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where  $E(S, V \setminus S) \stackrel{\text{def}}{=} \{\{u, v\} \in E \mid u \in S, v \in V \setminus S\}$ . The edge expansion of the graph  $G$  is defined as  $\phi_G \stackrel{\text{def}}{=} \min_{S \subset V} \phi(S)$ . Related to this is the notion of the vertex expansion of a graph. For a graph  $G = (V, E)$ , we define the *vertex expansion* of a set  $S$  of vertices, denoted by  $\phi^V(S)$  as

$$\phi^V(S) \stackrel{\text{def}}{=} \frac{|N(S) \cup N(V \setminus S)|}{|S||V \setminus S|} |V|, \quad (1.2)$$

where  $N(S) \stackrel{\text{def}}{=} \{v \in V \setminus S \mid \exists u \in S \text{ such that } \{u, v\} \in E\}$ . The vertex expansion of the graph  $G$  is defined as  $\phi_G^V \stackrel{\text{def}}{=} \min_{S \subset V} \phi^V(S)$ . A few other related notions of vertex expansion have been studied in the literature, we discuss them in Section 1.4. We also give a brief description of related works in Section 1.4.

### Graph $k$ -partitioning

The vertex expansion and edge expansion objectives seek a 2-partition of the vertex set of the graph. However, for many natural applications, one might require a graph to be partitioned into  $k$  parts, for some  $k \geq 2$ . Let us use  $\Phi$  to denote either  $\phi$  (edge expansion) or  $\phi^V$  (vertex expansion). For a  $k$ -partition  $S_1, \dots, S_k$  of the vertex set, the  *$k$ -way edge/vertex expansion* of  $\{S_1, \dots, S_k\}$  is defined as

$$\Phi^k(S_1, \dots, S_k) \stackrel{\text{def}}{=} \max_{i \in [k]} \Phi(S_i),$$

and the  *$k$ -way edge/vertex expansion* of  $G$  is defined as

$$\Phi_G^k \stackrel{\text{def}}{=} \min_{\{S_1, \dots, S_k\} \in \mathcal{P}_k} \Phi^k(S_1, \dots, S_k),$$

where  $\mathcal{P}_k$  is the set of all  $k$ -partitions of the vertex set. Optimizing these objective function is useful when one seeks a  $k$ -partition where each part has small expansion. The edge expansion version of this objective has been studied in [26, 23, 21], etc., and the vertex expansion version of this objective has been studied in [12]; see Section 1.4 for a brief summary of the related work.

For many NP-hard optimization problems, simple heuristics work very well in practice, for e.g. SAT [9], sparsest cut [18, 19], etc. One possible explanation for this phenomenon could be that instances arising in practice have some inherent structure that makes them “easy”. Studying natural random/semi-random families of instances, and instances with planted solutions has been a fruitful approach towards understanding the structure of easy instances, and in modelling instances arising in practice, especially for graph partitioning problems [33, 29, 30, 28] (see Section 1.4 for a brief survey). Moreover, studying semi-random and planted instances of a problem can be used to better understand what aspects of a problem make it “hard”. Therefore, in an effort to better understand the complexity of graph  $k$ -partitioning problems, we study the  $k$ -way edge and vertex expansion of a natural planted model of instances. We give bi-criteria approximation algorithms for instances from these models.

## 1.1 $k$ -way planted models for expansion problems

We study the following model of instances.

► **Definition 1.1** ( $k$ -Part-edge). *An instance of  $k$ -Part-edge( $n, k, \varepsilon, \lambda, d, r$ ) is generated as follows.*

1. Let  $V$  be a set of  $n$  vertices. Partition  $V$  into  $k$  sets  $\{S_1, S_2, \dots, S_k\}$ , with  $|S_t| = n/k$  for every  $t \in [k]$ . For each  $t \in [k]$ , add edges between arbitrarily chosen pairs of vertices in  $S_t$  to form an arbitrary roughly  $d$ -regular (formally, the degree of each vertex should lie in  $[d, rd]$ ) graph of spectral gap (defined as the second smallest eigenvalue of the normalized Laplacian matrix of the graph, see Section 2.1 for definition) at least  $\lambda$ .
2. For all  $i, j \in [k]$ , add edges between arbitrarily chosen pairs of vertices in  $S_i \times S_j$  such that  $\phi_G(S_i) \leq \varepsilon rd \forall i \in [k]$ .
3. (Monotone Adversary) For each  $t \in [k]$ , add edges between any number of arbitrarily chosen pairs of vertices within  $S_t$ .

Output the resulting graph  $G$ .

Analogously, we define the vertex expansion model.

► **Definition 1.2** ( $k$ -Part-vertex). An instance of  $k$ -Part-vertex( $n, k, \varepsilon, \lambda, d, r$ ) is generated as follows.

1. Let  $V$  be a set of  $n$  vertices. Partition  $V$  into  $k$  sets  $\{S_1, S_2, \dots, S_k\}$ , with  $|S_t| = n/k$  for every  $t \in [k]$ . For each  $t \in [k]$ , add edges between arbitrarily chosen pairs of vertices in  $S_t$  to form an arbitrary roughly  $d$ -regular (formally, the degree of each vertex should lie in  $[d, rd]$ ) graph of spectral gap (defined as the second smallest eigenvalue of the normalized Laplacian matrix of the graph, see Section 2.1 for definition) at least  $\lambda$ .
2. For each  $t \in [k]$ , partition  $S_t$  into  $T_t$  and  $S_t \setminus T_t$  such that  $|T_t| \leq \varepsilon n/k$ . Add edges between any number of arbitrarily chosen pairs of vertices in  $\cup_{i \in [k]} T_i$ .
3. (Monotone Adversary) For each  $t \in [k]$ , add edges between any number of arbitrarily chosen pairs of vertices within  $S_t$ .

Output the resulting graph  $G$ .

The only difference between  $k$ -Part-edge and  $k$ -Part-vertex is in the expansion of the sets. In step 2 of Definition 1.1, we ensured that  $\phi(S_i) \leq \varepsilon rd \forall i \in [k]$ <sup>1</sup>. In step 2 of Definition 1.2, the definition ensures that  $\phi^V(S_i) \leq \varepsilon k \forall i \in [k]$ .

Both these models can be viewed as the generalization to  $k$ -partitioning of models studied in the literature for 2-partitioning problems for edge expansion [29], etc. and vertex expansion [28], etc. These kinds of models can be used to model communities in networks, where  $k$  is the number of communities. The intra-community connections are typically stronger than the inter-community connections. This can be modelled by requiring  $S_i$  to have large expansion (see Theorem 1.3 and Theorem 1.4 for how large a  $\lambda$  is needed compared to  $\varepsilon$ ). Our work for  $k > 2$  can be used to study more general models of communities than the case of  $k = 2$ .

## 1.2 Our Results

We give bi-criteria approximation algorithms for the instances generated from the  $k$ -Part-edge and  $k$ -Part-vertex models. We define OPT as follows

$$\text{OPT} \stackrel{\text{def}}{=} \min_{\{P_1, \dots, P_k\} \in \tilde{\mathcal{P}}_k} \Phi^k(P_1, \dots, P_k),$$

where  $\Phi$  is  $\phi$  for  $k$ -Part-edge, and  $\phi^V$  for  $k$ -Part-vertex, and  $\tilde{\mathcal{P}}_k$  is the set of all *balanced*  $k$ -partitions of the vertex-set, i.e. for each  $\{P_1, \dots, P_k\} \in \tilde{\mathcal{P}}_k$ , we have  $|P_i| = n/k \forall i \in [k]$ . We note that in  $k$ -Part-edge,  $\text{OPT} \leq \varepsilon rd$ , and in  $k$ -Part-vertex,  $\text{OPT} \leq \varepsilon k$ .

<sup>1</sup> Since  $\phi(S)$  measures the weight of edges leaving  $S$  (see (1.1)), it is often more useful to compare edge expansion to some quantity related to the degrees of the vertices inside  $S$ . Therefore, in step 2 of Definition 1.1, we require  $\phi(S_i) \leq \varepsilon rd \forall i \in [k]$ , instead of  $\phi(S_i) \leq \varepsilon \forall i \in [k]$ .

► **Theorem 1.3.** *There exist universal constants  $c_1, c_2 \in \mathbb{R}^+$  satisfying the following: there exists a polynomial-time algorithm that takes as input a graph from the class  $k$ -Part-edge( $n, k, \varepsilon, \lambda, d, r$ ) with  $\varepsilon \leq \lambda/(800kr^3)$ , and outputs  $k$  disjoint sets of vertices  $W_1, \dots, W_k \subseteq V$ , that for each  $i \in [k]$  satisfy:*

1.  $|W_i| \geq c_1 n/k$ ,
2.  $\phi(W_i) \leq c_2 k \text{OPT}$ .

► **Theorem 1.4.** *There exist universal constants  $c_1, c_2 \in \mathbb{R}^+$  satisfying the following: there exists a polynomial-time algorithm that takes as input a graph from the class  $k$ -Part-vertex( $n, k, \varepsilon, \lambda, d, r$ ) with  $\varepsilon \leq \lambda/(800kr^3)$ , and outputs  $k$  disjoint sets of vertices  $W_1, \dots, W_k \subseteq V$ , that for each  $i \in [k]$  satisfy:*

1.  $|W_i| \geq c_1 n/k$ ,
2.  $\phi^V(W_i) \leq c_2 k \text{OPT}$ .

Note when  $k = \mathcal{O}(1)$ , Theorem 1.3 and Theorem 1.4 guarantee constant factor bi-criteria approximation algorithms. The currently best known approximation guarantees for general instances (i.e. worst case approximation guarantees) of  $k$ -way edge expansion problems are of the form  $\mathcal{O}(\text{OPT} \sqrt{\log n} f_1(k))$  or  $\mathcal{O}(\sqrt{\text{OPT}} f_2(k))$  where  $f_1(k), f_2(k)$  are some functions of  $k$ , and the currently best known approximation guarantees for general instances (i.e. worst case approximation guarantees) of  $k$ -way vertex expansion problems are of the form  $\mathcal{O}(\text{OPT} \sqrt{\log n} f_3(k))$  or  $\mathcal{O}(\sqrt{\text{OPT}} f_4(k, d))$  where  $f_3(k)$  is some functions of  $k$  and  $f_4$  is some function of  $k$  and the maximum vertex degree  $d$ . We survey these results in Section 1.4. Note that our bi-criteria approximation guarantees in Theorem 1.3 and Theorem 1.4 are multiplicative approximation guarantees and are independent of  $n$ .

The above theorem shows that it is possible to produce  $k$  disjoint subsets, each of size  $\Omega(n/k)$ , each with expansion a factor  $k$  away from that of the planted partition. While this may not form a partition of the vertex set, it is not difficult to show that with a loss of a factor of  $k$ , we can indeed get a true partition. This idea of moving from disjoint sets to a partition is well-known, and has been used before in other works (for e.g., [21]).

► **Corollary 1.5.** *There exist universal constants  $c_1, c_2 \in \mathbb{R}^+$  satisfying the following: there exists a polynomial-time algorithm that takes as input a graph from  $k$ -Part-edge( $n, k, \varepsilon, \lambda, d, r$ ) (resp.  $k$ -Part-vertex( $n, k, \varepsilon, \lambda, d, r$ )) with  $\varepsilon \leq \lambda/800kr^3$ , and outputs a  $k$ -partition  $\mathcal{P} = \{P_1, \dots, P_k\}$  of  $V$  such that:*

1. For each  $i \in [k]$ ,  $|P_i| \geq c_1 n/k$ ,
2. For each  $i \in [k]$ ,  $\phi(P_i) \leq c_2 k^2 \text{OPT}$  (resp.  $\phi^V(P_i) \leq c_2 k^2 \text{OPT}$ ).

We note that the above result approximates the  $k$ -way expansion of the best *balanced* partition in  $G$ . The proofs of the above results are given in Section 3.

### 1.3 Proof Overview

For proving Theorem 1.3 and Theorem 1.4 we use an SDP relaxation (see Section 2.2) similar to the one used by [23, 31], etc. For the case when  $k = 2$ , [29, 28] used slightly different SDP constraints, and showed that when  $S_1$  and  $S_2$  contain large edge expanders, the set of SDP solution vectors  $\{u_i : i \in V\}$  contain two sets  $L_1, L_2$  such that  $|L_1|, |L_2| = \Omega(n)$ ,  $L_1$  and  $L_2$  have small diameter, and the distance between  $L_1$  and  $L_2$  is  $\Omega(1)$ . The core of our analysis can be viewed as proving an analogue of this for  $k > 2$  (Proposition 3.3), however, this requires some new ideas. For  $i \in [k]$ , let  $\mu_i$  denote the mean of the vectors corresponding to the vertices in  $S_i$ . We use the expansion within  $S_i$ 's together with the SDP constraints

to show that for  $i, j \in [k]$ ,  $i \neq j$ , each  $\mu_i$  must have  $\Omega(n/k)$  vertices sufficiently close to it, and that  $\mu_i$  and  $\mu_j$  must be sufficiently far apart. This can be used to show the existence of  $k$  such sets  $L_1, \dots, L_k$ , such that for each  $i \in [k]$ ,  $L_i$  has sufficiently small diameter and  $L_i$  is sufficiently far from  $L_j \forall j \neq i$ . The proof of our structure theorem is similar in spirit to the proof of structure theorem of [39], but our final guarantees are very different, we discuss their work in more detail in Section 1.4.

If we can compute  $k$  such sets  $L_1, \dots, L_k$ , then using standard techniques, we can recover  $k$  sets having small expansion. In the case of  $k = 2$ , one could just guess a vertex from each these sets, and compute the two sets satisfying our requirements using standard techniques. For  $k > 2$ , guessing a vertex from each of the balls around  $\mu_i$  would also suffice to compute sets  $L_1, \dots, L_k$  satisfying our requirements. However, doing this naively would take time  $O(n^k)$ . To obtain an algorithm for this task whose running time is  $\mathcal{O}(\text{poly}(n, k))$ , we use a simple greedy algorithm (Algorithm 1) to iteratively compute the sets  $L_i$  such that  $L_i$  has sufficiently small diameter and is sufficiently far from  $L_j$  for all  $j < i$ . To ensure that this approach works, one has to ensure that at the start of iteration  $i + 1$ , the set of SDP vectors for the vertices in  $V \setminus \cup_{j=i}^k L_j$  has at least  $k - i$  clusters each of size  $\Omega(n/k)$  and having small diameter. We use our structural result to prove that this invariant holds in all iterations of the algorithm.

## 1.4 Related Work

[28] studied the 2-way vertex-expansion in  $k$ -Part-vertex for  $k = 2$ , and gave a constant factor bi-criteria approximation algorithm. Our proofs and results can be viewed as generalizing their result to  $k > 2$ . They also studied a stronger semi-random model, and gave an algorithm for exact recovery (i.e. a 1-approximation algorithm) w.h.p. [29] studied the 2-way edge-expansion in a model similar to  $k$ -Part-edge for  $k = 2$ , and gave a constant factor bi-criteria approximation algorithm. Our proofs and results can be viewed as generalizing their result to  $k > 2$ .

**$k$ -partitioning problems.** The minimum  $k$ -cut problem asks to find a  $k$ -partition of the vertex set which cuts the least number of edges; [43, 38, 42] all gave 2-approximation algorithms for this problem. A number of works have investigated  $k$ -way partitioning in the context of edge expansion. Bansal et al. [8] studied the problem of computing a  $k$ -partitioning  $S_1, \dots, S_k$  of the vertex set such that  $|S_i| = n/k$  for each  $i \in [k]$ , which minimizes  $\max_{i \in [k]} |E(S_i, V \setminus S_i)|$ . They give an algorithm which outputs a  $k$ -partition of the vertex set  $T_1, \dots, T_k$  such that  $|T_i| \leq (2 + \varepsilon)n/k$ , and  $\max_{i \in [k]} |E(T_i, V \setminus T_i)| \leq \mathcal{O}(\sqrt{\log n \log k}) \text{OPT}$ , where OPT denotes the cost of the optimal solution. There are also many connections between graph partitioning problems and graph eigenvalues. Let  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  denote the eigenvalues of the normalized Laplacian matrix of the graph. Typically, a different but related notion of edge expansion is used, which is defined as follows.

$$\phi'(S) \stackrel{\text{def}}{=} \frac{|E(S, V \setminus S)|}{\min \{\text{vol}(S), \text{vol}(V \setminus S)\}},$$

where  $\text{vol}(S)$  is defined as the sum of the degrees of the vertices in  $S$ . [25] gave an algorithm to find a  $k$ -partition which cuts at most  $\mathcal{O}(\sqrt{\lambda_k \log k})$  fraction of the edges. [21, 26] showed that for any  $k$  non-empty disjoint subsets  $S_1, \dots, S_k \subset V$ ,  $\max_{i \in [k]} \phi'(S_i) = \Omega(\lambda_k)$ . [21] (see also [26, 23]) gave an algorithm to find a  $(1 - \varepsilon)k$  partition  $S_1, \dots, S_{(1-\varepsilon)k}$  of the vertex set satisfying  $\max_i \phi'(S_i) = \mathcal{O}((1/\varepsilon^3) \sqrt{\lambda_k \log k})$  for any  $\varepsilon > 0$ , and a collection of  $k$  non-empty, disjoint subsets  $S_1, \dots, S_k \subset V$  satisfying  $\max_i \phi'(S_i) = \mathcal{O}(k^2 \sqrt{\lambda_k})$ . [23] gave an algorithm to find a partition of  $V$  into  $(1 - \varepsilon)k$  disjoint subsets  $S_1, S_2, \dots, S_{(1-\varepsilon)k}$ , such that  $\phi'(S_i) \leq \mathcal{O}(\sqrt{\log n \log k} \text{OPT})$ .

Given a parameter  $\delta$ , the small-set edge expansion problem asks to compute the set  $S \subset V$  have the least edge expansion among all sets of cardinality at most  $\delta|V|$  (or volume at most  $\delta \text{vol}(V)$ ). Bansal et al. [8] and Raghavendra et al. [41] gave a bi-criteria approximation algorithm for the small-set edge expansion problem. [23] gave an algorithm that outputs  $(1 - \varepsilon)k$  partition  $S_1, \dots, S_{(1-\varepsilon)k}$  such that  $\max_i \phi'(S_i) = \mathcal{O}(\text{poly}(1/\varepsilon)\sqrt{\log n \log k} \text{OPT})$ , where OPT is least value of  $\max_{i \in [k]} \phi'(S_i)$  over all  $k$ -partitions  $S_1, \dots, S_k$  of the vertex set. [23] also studied a balanced version of this problem, and gave bi-criteria approximation algorithms.

Let  $\rho_k(G)$  denote  $\min_{S_1, \dots, S_k} \max_{i \in [k]} \phi'(S_i)$  where the minimum is over sets of  $k$  non-empty disjoint subsets  $S_1, \dots, S_k \subset V$ . Kwok et al. [20] showed that for any  $l > k$ ,  $\rho_k(G) = \mathcal{O}(lk^6 \lambda_k / \sqrt{\lambda_l})$ . They also gave a polynomial time algorithm to compute non-empty disjoint sets  $S_1, \dots, S_k \subset V$  satisfying this bound. Combining this with the results of [21, 26], we get a  $\mathcal{O}(lk^6 / \sqrt{\lambda_l})$  approximation to the problem of computing  $k$  non-empty disjoint subsets  $S_1, \dots, S_k \subset V$  which have the least value of  $\max_{i \in [k]} \phi'(S_i)$ . Here the approximation factor depends on  $\lambda_l$ , but even in the best case when  $\lambda_l = \Omega(1)$  for some  $l = O(k)$ , the expression for the approximation guarantee reduces to  $\mathcal{O}(k^7)$ . They also show that for any  $l > k$  and any  $\varepsilon > 0$ , there is a polynomial time algorithm to compute non-empty disjoint subsets  $S_1, \dots, S_{(1-\varepsilon)k} \subset V$  such that  $\max_{i \in [(1-\varepsilon)k]} \phi'(S_i) = \mathcal{O}((l \log^2 k) / (\text{poly}(\varepsilon)k)) \lambda_k / \sqrt{\lambda_l}$ .

Peng et al. [39] define the family of well clustered graphs to be those graphs for which  $\lambda_{k+1} / \rho_k(G) = \Omega(k^2)$  (their structure theorem requires this ratio to be  $\Omega(k^2)$ , their algorithms require the separation to be larger, i.e.  $\Omega(k^3)$ ). They show that for such graphs, using the bottom  $k$  eigenvectors of the normalized Laplacian matrix, one can compute a  $k$ -partition which is close to the optimal  $k$ -partition for  $k$ -way edge expansion. They measure the closeness of their solution to the optimal solution in terms of the volume of the symmetric difference between the solution returned by their algorithm and the optimal solution. They start by showing that the vertex embedding of the graph into the  $k$ -dimensional space consisting of the bottom- $k$  eigenvectors is clustered. Our technique to prove our main structural result Proposition 3.3, which shows that the SDP solution is clustered, is similar in spirit. Firstly, we note that the results of [39] apply to edge expansion problems and not vertex expansion problems. Moreover, due to the action of the monotone adversary, the  $\lambda_{k+1}$  of instances from  $k$ -Part-edge could be very small in which case the results of [39] wouldn't be applicable.

[12] showed that for a hypergraph  $H = (V, E)$ , there exist  $(1 - \varepsilon)k$  disjoint subsets  $S_1, \dots, S_{(1-\varepsilon)k}$  of the vertex set such that  $\max_i \phi(S_i) = \mathcal{O}(k^2 \text{poly} \log(k) / e^{1.5}) \sqrt{\gamma_k \log r}$ , where  $r$  is the size of the largest hyperedge,  $\phi(S)$  denotes the hypergraph expansion of a set of vertices  $S$ ,  $\gamma_k$  is the  $k$ th smallest eigenvalue of the hypergraph Laplacian operator (we refer the reader to [12] for the definition of  $\phi(\cdot)$ ,  $\gamma_k$ , etc.) Combining these ideas from [12] with the ideas from [24], we believe it should be possible to obtain an algorithm that outputs  $(1 - \varepsilon)k$  disjoint subsets  $S_1, \dots, S_{(1-\varepsilon)k}$  such that  $\max_i \phi(S_i) = \mathcal{O}(k^2 \text{poly} \log(k) \text{poly}(1/\varepsilon)) \sqrt{\log n} \text{OPT}$ , where is OPT is least value of  $\max_{i \in [k]} \phi(S_i)$  over all  $k$ -partitions  $S_1, \dots, S_k$  of the vertex set. Using a standard reduction from vertex expansion in graphs to hypergraph expansion, we get analogs of the above mentioned results for vertex expansion in graphs.

**Vertex Expansion.** An alternative, common definition of vertex expansion that has been studied in the literature is  $\phi^{V,a}(S) \stackrel{\text{def}}{=} (|V||N(S)| / (|S||V \setminus S|))$ , and as before,  $\phi_G^{V,a} \stackrel{\text{def}}{=} \min_{S \subset V} \phi^{V,a}(S)$ . As Louis et al. [27] show, the computation  $\phi_G^V$  and  $\phi_G^{V,a}$  is equivalent upto constant factors.

Feige et al. [13] gave a  $\mathcal{O}(\sqrt{\log n})$ -approximation algorithm for computing the vertex expansion of a graph. Bobkov et al. [10] gave a Cheeger-type inequality for vertex expansion in terms of a parameter  $\lambda_\infty$ , which plays a role similar to  $\lambda_2$  in edge-expansion. Building on

this, Louis et al. [27] gave an SDP based algorithm to compute a set having vertex expansion at most  $\mathcal{O}\left(\sqrt{\phi_G^V \log d}\right)$  in graphs having vertex degrees at most  $d$ . This bound is tight upto constant factors [27] assuming the SSE hypothesis. Louis and Makarychev [24] gave a bi-criteria approximation for small-set vertex expansion.

**Edge Expansion.** Arora et al. [7] gave a  $\mathcal{O}(\sqrt{\log n})$ -approximation algorithm for computing the edge expansion of a graph. Cheeger's inequality [6, 5] says that  $\lambda_2/2 \leq \min_{S \subseteq V} \phi'(S) \leq \sqrt{2\lambda_2}$ .

**Stochastic Block Models and Semi-Random Models.** Stochastic Block Models (SBMs) are randomized instance-generation models based on the edge expansion objective and have been intensively studied in various works, starting with [16, 11, 17]. The goal is to identify and recover communities in a given random graph, where edges within communities appear with a probability  $p$  that is higher than the probability  $q$  of edges across communities. Both exact and approximate recovery guarantees for SBMs have been investigated using various algorithms [33, 35, 32, 1, 36, 37], leading to the resolution of a certain conjecture regarding for what range of model parameters are recovery guarantees are possible. While the above results deal mostly with the case of SBMs with two communities,  $k$ -way SBMs (for  $k > 2$  communities) have been studied in recent works [2, 3, 4].

Semi-Random Models allow instance generation using a combination of both random edges and some amount of monotone adversarial action (i.e. not change the underlying planted solution). SDP-based methods seem to work well in this regard, since they are robust to such adversarial action. Many variants of semi-random models for edge expansion have been studied in literature. Examples include works due to Feige and Kilian [14], Guedon and Vershynin [15], Moitra et al. [34], and Makarychev et al. [29, 30, 31]. [31] also allows for a small amount of non-monotone errors in their model. These works give approximate and exact recovery guarantees for a range of parameters in their respective models.

## 2 Preliminaries and Notation

### 2.1 Notation

We denote graphs by  $G = (V, E)$ , where the vertex set  $V$  is identified with  $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ . The vertices are indexed by  $i, j$ . For any  $S \subseteq V$ , we denote the induced subgraph on  $S$  by  $G[S]$ . Given  $i \in V$  and  $T \subseteq V$ , define  $N_T(i) \stackrel{\text{def}}{=} \{j \in T : \{i, j\} \in E\}$ , and  $N(i) = N_V(i)$ .

Given the normalized Laplacian  $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ , the *spectral gap* of  $G$  denoted by  $\lambda$ , is the second-smallest eigenvalue of  $\mathcal{L}$ . *Spectral expanders* are a family of graphs with  $\lambda$  at least some constant (independent of the number of vertices in  $G$ ).

Specific to graphs  $G$  generated in the  $k$ -Part-vertex and  $k$ -Part-edge models, let  $\mathcal{S} = \{S_1, \dots, S_t\}$  be the collection of sets for any  $i \in V$ , let  $S(i)$  denote the set  $S \in \mathcal{S}$  such that  $i \in S$ . For a single subset  $W \subseteq V$ , we define  $\partial W = \{i \in W : \exists j \notin W \text{ with } j \in N(i)\} \cup \{i \notin W : \exists j \in W \text{ with } j \in N(i)\}$ , i.e., the *symmetric vertex boundary* of the cut  $(W, V \setminus W)$ . We let  $E(\partial S)$  be the edges going across the cut  $(S, V \setminus S)$ , for any  $S \subseteq V$ . Given any  $k$ -partition of the vertex set  $\mathcal{W} = \{W_1, \dots, W_k\}$ , we define  $\partial \mathcal{W} = \cup_{i \in [k]} \partial W_i$  to be the set of boundary vertices on this partition, and  $E(\partial \mathcal{W}) = \cup_{i \in [k]} E(\partial W_i)$  to be the edges across this partition.

## 2.2 SDP for $k$ -way edge and vertex expansion

Our algorithms for both  $k$ -Part-edge and  $k$ -Part-vertex models use a natural semi-definite programming (SDP) relaxation for  $k$ -way expansion. The objective function we use is the “min-sum” objective in each case. For  $k$ -Part-vertex, it looks to minimize the number of boundary vertices in a balanced  $k$ -way partition of the vertex set, and correspondingly in  $k$ -Part-edge, the total number of edges across a balanced  $k$ -way partition of the vertex set.

For the  $k$ -Part-edge model, we use the following SDP relaxation.

► **SDP 2.1 (Primal).**  $k$ -Part-edge

$$\min_U \quad \frac{1}{2} \sum_{i,j \in E} U_{ii} + U_{jj} - 2U_{ij}$$

subject to

$$\begin{aligned} U_{ii} &= 1 & \forall i \in V \\ U_{ij} &\geq 0 & \forall i, j \in V \\ \sum_j U_{ij} &= n/k & \forall i \in V \\ U_{jj} &\geq U_{ij} + U_{jk} - U_{ik} & \forall i, j, k \in V \\ U &\succeq 0 \end{aligned}$$

► **SDP 2.2 (Primal).**  $k$ -Part-vertex

$$\min_U \quad \sum_{i \in V} \eta_i$$

subject to

$$\begin{aligned} \eta_i &\geq U_{ii} + U_{jj} - 2U_{ij} & \forall i, \forall j \in N(i) \\ U_{ii} &= 1 & \forall i \in V \\ U_{ij} &\geq 0 & \forall i, j \in V \\ \sum_j U_{ij} &= n/k & \forall i \in V \\ U_{jj} &\geq U_{ij} + U_{jk} - U_{ik} & \forall i, j, k \in V \\ U &\succeq 0 \end{aligned}$$

The intended integral solution for  $U$  in the SDP relaxation (SDP 2.2, SDP 2.1) for either model is  $U_{ij} = 1$ , if  $i, j$  lie in the same subset in the planted  $k$ -partition of  $V$ , and 0 otherwise. We can alternatively view the SDP variables as a set of vectors  $\{u_i \in \mathbb{R}^n\}_{i \in V}$ , satisfying  $u_i^T u_j = U_{ij}$ . These can be obtained by the Cholesky decomposition of the matrix  $U$ . Notice that the constraint  $\sum_j U_{ij} = n/k$  in the relaxations above is specific to  $k$ -way partitions with exactly  $n/k$  vertices in each partition, and hence is satisfied by both models for the integral solution. The second-to-last set of constraints in either SDP are called  $\ell_2^2$  triangle inequalities, and can be rephrased in the language of vectors as:

$$\|u_i - u_j\|^2 + \|u_k - u_j\|^2 \geq \|u_i - u_k\|^2 \quad \forall i, j, k \in V \quad (2.1)$$

It is easy to verify that these are satisfied by the ideal integral solution, corresponding to  $u_i = e_t$ , where  $i \in S_t$ .



For  $k$ -Part-edge, for every edge across the partition we accumulate a value of 1 in the SDP objective in the integral solution. Since every  $S_t$  has  $\phi(S_t) \leq \epsilon rd$ , we have:

$$\begin{aligned} |E(\partial S_t)| &\leq \epsilon rd \frac{n}{k} \cdot \left(1 - \frac{1}{k}\right) \leq \epsilon rd \frac{n}{k} \\ \implies 2 \left| \bigcup_{t=1}^k E(\partial S_t) \right| &\leq \epsilon rdn \end{aligned}$$

Since the number of edges going across the partition is at most<sup>2</sup>  $\epsilon rdn$ , this is an upper bound on the optimum of SDP 2.1.

For  $k$ -Part-vertex, the integral solution will further set,  $\eta_i = 2$  for any boundary vertex  $i$  of the partition  $\mathcal{S}$ , and  $\eta_i = 0$  if  $i$  is not a boundary vertex, yielding a primal objective value of  $2\epsilon n$ . Thus, the optimal value of SDP 2.2 is at most  $2\epsilon n$ .

Furthermore, if OPT is as defined in Section 1.2, then in either case we have that  $\text{SDP} \leq \text{OPT} \cdot n$ .

We introduce some notation regarding the SDP solution vectors  $\{u_i\}_{i \in V}$  that will be useful for proofs. Let  $d(i, j) \stackrel{\text{def}}{=} \|u_i - u_j\|^2$ . Due to inequalities (2.1),  $d(\cdot, \cdot)$  is a metric. Given a set  $L \subseteq V$ , define  $d(i, L) \stackrel{\text{def}}{=} \min_{j \in L} d(i, j)$ . The  $\ell_2^2$  diameter of  $L$  is  $\text{diam}(L) = \max_{i, j \in L} d(i, j)$ . A ball of  $\ell_2^2$  radius  $a$  around a point  $x \in \mathbb{R}^n$  is defined as  $B(x, a) \stackrel{\text{def}}{=} \{j \in V : d(j, x) \leq a\}$ .

Further proof-specific notations are defined as and when they are needed in the respective sections.

### 3 Bi-criteria Guarantees in the Planted Model

We now give a proof of Theorem 1.3, Theorem 1.4 and Corollary 1.5. The main idea is to show that the SDP solution is clustered around  $k$  disjoint balls, each of which have a significant overlap with a distinct  $S_i$ , for  $i \in [k]$ . We can then extract out  $k$  sets greedily using an  $\ell_1$  line embedding.

In what follows, it is convenient to view the variables in the primal SDP as being vectors  $u_i \in \mathbb{R}^n$  for each  $i \in V$  that satisfy  $u_i^T u_j = U_{ij}$ .

The missing proofs for the results in this section are given in the full version of the paper.

#### 3.1 Preliminary Lemmas

► **Lemma 3.1.** *Let  $\delta \leq 1/100$  and  $\alpha \leq 1$  be real numbers. Let  $\{u_i\}_{i \in V}$  be a feasible SDP solution vector set for SDP 2.1 or SDP 2.2. Suppose there exists a set  $L \subseteq V$  that satisfies:*

- (a)  $|L| \geq \alpha n$
- (b)  $\text{diam}(L) \leq \delta$ .

We have:

- (a) (Edge) *If  $\{u_i\}_{i \in V}$  is an optimal solution to SDP 2.1 with objective value  $\beta n$ , then there exists an  $i \in L$ , and  $a \in [\delta, 1/50]$  such that  $W \stackrel{\text{def}}{=} B(i, a)$  satisfies  $\phi(W) \leq \mathcal{O}(\beta/\alpha)$ .*
- (b) (Vertex) *If  $\{u_i\}_{i \in V}$  is an optimal solution to SDP 2.2 with objective value  $\beta n$ , then there exists an  $i \in L$ , and  $a \in [\delta, 1/50]$  such that  $W \stackrel{\text{def}}{=} B(i, a)$  satisfies  $\phi^V(W) \leq \mathcal{O}(\beta/\alpha)$ .*

<sup>2</sup> We use a slightly loose upper bound for convenience, to match up parameters in our proofs with the  $k$ -Part-vertex model.

Part (a) of the above lemma follows from standard arguments in edge-expansion literature. Part (b) is a slight modification of [28, Lemma 3.1]<sup>3</sup>. We defer both proofs to the full version of the paper.

We next show that if the SDP solution is clustered into  $k$  disjoint, well-separated balls of small diameter, then we can iteratively use Lemma 3.1 to find  $k$  disjoint sets, each with small vertex or edge expansion.

► **Lemma 3.2.** *Let  $\delta \leq \frac{1}{100}$  and  $k \in \mathbb{Z}$  be large enough. Suppose the optimal SDP solution vectors  $\{u_i\}_{i \in V}$  to SDP 2.1 (resp. SDP 2.2) yield an objective value of  $\beta n$  and satisfy the following properties:*

(a) *There exist disjoint sets  $L_1, L_2, \dots, L_k \subseteq V$ , with  $\text{diam}(L_t) \leq \delta$ ,*

(b) *For each  $t \in [k]$ , and for some constant  $\gamma$ , we have  $|L_t| \geq \gamma n/k$ ,*

(c) *For every  $t \neq t'$ ,  $d(L_t, L_{t'}) \geq 1/10$ .*

*Then, we can in polynomial time, find  $k$  disjoint sets  $W_1, \dots, W_k \subseteq V$  such that for every  $t \in [k]$ ,  $|W_t| \geq \gamma n/k$ , and  $\phi(W_t) \leq \mathcal{O}(\beta k/\gamma)$  (resp.  $\phi^V(W_t) \leq \mathcal{O}(\beta k/\gamma)$ ).*

### 3.2 Showing that the SDP solution is clustered

We next show that for any input instance from the class  $k$ -Part-edge or  $k$ -Part-vertex with appropriate parameters, every feasible set of SDP solution vectors are clustered. Using Lemma 3.2, we can then immediately conclude the proof of Theorem 1.3 and Theorem 1.4.

Our main technical result is the following proposition.

► **Proposition 3.3.** *Let  $\{u_i\}_{i \in V}$  be the optimal solution SDP 2.1 (resp. SDP 2.2) for an instance  $G$  from  $k$ -Part-edge( $n, k, \varepsilon, \lambda, d, r$ ) (resp.  $k$ -Part-vertex( $n, k, \varepsilon, \lambda, d, r$ )) with  $\varepsilon k r^3/\lambda \leq 1/800$ . Then, there exist sets  $L_1, \dots, L_k \subseteq V$  such that:*

(a)  $\text{diam}(L_t) \leq 1/100$ ,

(b)  $\forall t \in [k] : |L_t \cap S_t| \geq n/2k$ ,

(c)  $\forall t \neq t' : d(L_t, L_{t'}) \geq 1/10$ .

**Proof of Proposition 3.3.** We begin with the following lemma; the proof is given in the full version of the paper.

► **Lemma 3.4.** *Let  $\{u_i\}_{i \in V}$  be the optimal solution to the SDP for an instance  $G$  from  $k$ -Part-vertex or  $k$ -Part-edge. For each  $t \in [k]$ , let  $\mu_t = \mathbb{E}_{i \in S_t}[u_i]$ . The following holds:*

(a)  $\forall t \in [k] : \mathbb{E}_{j \in S_t}[\|\mu_t - u_j\|^2] \leq \frac{k\varepsilon r^3}{\lambda}$

(b)  $1 \geq \|\mu_t\|^2 \geq 1 - k\varepsilon r^3/\lambda$

(c)  $\forall t \neq t' : \mu_t^T \mu_{t'} \leq k\varepsilon r^3/\lambda$

We use this to prove Proposition 3.3. For each  $t \in [k]$ , define  $L_t \stackrel{\text{def}}{=} B(\mu_t, 1/400)$ . Clearly,  $\text{diam}(L_t) \leq 1/100$ .

Since the parameters for either  $k$ -Part model are assumed to satisfy  $\varepsilon k r^3/\lambda \leq 1/800$ , we have that for every  $t \in [k]$ , item (a) from Lemma 3.4 implies that  $\mathbb{E}_{j \in S_t}[\|\mu_t - u_j\|^2] \leq k\varepsilon r^3/\lambda \leq 1/800$ . We can now use Markov's inequality:

<sup>3</sup> References to the results and proofs in [28] are with respect to the full version of that paper, available currently as an arXiv preprint.

$$\begin{aligned}
\Pr_{j \in S_t} \left[ \|\mu_t - u_j\|^2 > \frac{1}{400} \right] &= \frac{|S_t \setminus (L_t \cap S_t)|}{|S_t|} && \dots \text{ since } L_t \stackrel{\text{def}}{=} B(\mu_t, 1/400) \\
\implies \frac{|L_t \cap S_t|}{|S_t|} &= 1 - \Pr_{j \in S_t} \left[ \|\mu_t - u_j\|^2 > \frac{1}{400} \right] \\
&\geq 1 - \frac{\mathbb{E}_{j \in S_t} [\|\mu_t - u_j\|^2]}{1/400} = \frac{1}{2} \\
\implies |L_t \cap S_t| &\geq \frac{n}{2k}
\end{aligned}$$

To prove item (c) of the lemma, we first prove the following claim:

▷ **Claim 3.5.**

$$\forall t \neq t' \quad \|\mu_t - \mu_{t'}\|^2 \geq \frac{9}{10}$$

Proof.

$$\begin{aligned}
\|\mu_t - \mu_{t'}\|^2 &= \|\mu_t\|^2 + \|\mu_{t'}\|^2 - 2\mu_t^T \mu_{t'} \\
&\geq 1 - \frac{k\varepsilon r^3}{\lambda} + 1 - \frac{k\varepsilon r^3}{\lambda} - 2 \times \frac{k\varepsilon r^3}{\lambda} && \dots \text{ using Lemma 3.4} \\
&\geq 1 - \frac{4k\varepsilon r^3}{\lambda} \geq \frac{19}{20} > \frac{9}{10} && \dots \text{ since } \frac{k\varepsilon r^3}{\lambda} \leq \frac{1}{800} \quad \triangleleft
\end{aligned}$$

From the definition of the sets  $\{L_t\}_{t \in [k]}$ , we will use the (plain Euclidean) triangle inequality and the above claim. Let  $t \neq t'$ . We know that  $d(L_t, L_{t'}) = d(i, i')$  for some  $i \in L_t$  and  $i' \in L_{t'}$ . Using this:

$$\begin{aligned}
d(L_t, L_{t'}) &= d(i, i') \\
&= \|u_i - u_{i'}\|^2 \\
&\geq (\|\mu_t - \mu_{t'}\| - \|\mu_t - u_i\| - \|\mu_{t'} - u_{i'}\|)^2 \\
&\quad \dots \text{ by triangle inequality on the point sequence } \mu_t \rightarrow i \rightarrow i' \rightarrow \mu_{t'} \\
&\geq \left( \|\mu_t - \mu_{t'}\| - \frac{1}{20} - \frac{1}{20} \right)^2 && \dots \text{ since } d(\mu_{t'}, i'), d(\mu_t, i) \leq \sqrt{\frac{1}{400}} = \frac{1}{20} \\
&\geq \left( \frac{9}{10} - \frac{1}{10} \right)^2 > \frac{1}{10}.
\end{aligned}$$

Using the above, we now infer the proof of Theorem 1.3 and Theorem 1.4.

**Proof of Theorem 1.3 and Theorem 1.4.** Consider the optimal SDP solution vectors  $\{u_i\}_{i \in V}$  for an instance  $G$  from  $k$ -Part-edge( $n, k, \varepsilon, \lambda, d, r$ ) (resp.  $k$ -Part-vertex( $n, k, \varepsilon, \lambda, d, r$ )), with the parameters satisfying the given conditions, and having an objective value of  $\beta n$ . Note that  $\beta \leq \text{OPT}$ , as the SDP is a relaxation. Using Proposition 3.3, we infer the existence of sets  $L_1, \dots, L_k$  satisfying the conditions given. The SDP solution thus satisfies all the conditions of Lemma 3.2, with  $\delta = \frac{1}{100}$  and  $\gamma = 1/2$ , and therefore, we can find in polynomial time,  $k$  disjoint subsets  $W_1, \dots, W_k$ :  $|W_t| \geq n/2k$ , and  $\phi(W_t) \leq \mathcal{O}(\beta k)$ , for every  $t \in [k]$  for  $k$ -Part-edge, or correspondingly  $\phi^V(W_t) \leq \mathcal{O}(\beta k)$  for  $k$ -Part-vertex. Algorithm 1 describes the steps in the algorithm explicitly. ◀

## 23:12 Planted Models for $k$ -Way Edge and Vertex Expansion

■ **Algorithm 1** Algorithm for rounding SDP solutions for  $k$ -Part-vertex ( $k$ -Part-edge) instances.

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**Input:**  $G = (V, E)$  from  $k$ -Part( $n, k, \varepsilon, \lambda, r$ ) and an optimal SDP solution  $\{u_i\}_{i \in V}$  on  $G$   
**Output:** Disjoint sets  $W_1, \dots, W_k \subseteq V$  with  $|W_t| \geq n/2k$

- 1:  $C \leftarrow \emptyset$
- 2: **for**  $t \in 1, \dots, k$  **do**
- 3:      $W_t \leftarrow \emptyset$
- 4:     **for**  $i \in V$  **do**
- 5:         **for**  $r \in [1/100, 1/50)$  **do** ▷ Can be done in a discrete fashion
- 6:              $\hat{W} \leftarrow B(i, r)$
- 7:             **if**  $|\hat{W}| < n/2k$  **or**  $\hat{W} \cap C \neq \emptyset$  **continue**
- 8:             (For  $k$ -Part-edge): **if**  $W_t = \emptyset$  **or**  $\phi(W_t) > \phi(\hat{W})$  **then**  $W_t \leftarrow \hat{W}$
- 9:             (For  $k$ -Part-vertex): **if**  $W_t = \emptyset$  **or**  $\phi^V(W_t) > \phi^V(\hat{W})$  **then**  $W_t \leftarrow \hat{W}$
- 10:         **end for**
- 11:     **end for**
- 12:      $C \leftarrow C \cup W_t$
- 13: **end for**
- 14: **return**  $W_1, \dots, W_t$

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**Proof of Corollary 1.5.** The proof for both parts uses a technique to move from disjoint sets to partitions used before, for instance in [21]. Since previous works use it for edge expansion already, we state the proof for  $k$ -Part-vertex first.

**For  $k$ -Part-vertex:** We start with the sets  $W_1, \dots, W_k$  from Theorem 1.4. From the definition of  $\phi^V$ , we have:

$$\begin{aligned} |\partial W_t| &= |N(W_t)| + |N(V \setminus W_t)| \\ &\leq \mathcal{O}(1) \cdot \text{OPT} \cdot k \cdot \frac{|W_t| |V \setminus W_t|}{n} = \mathcal{O}(k \cdot \text{OPT} |W_t|) \quad \forall t \in [k] \end{aligned}$$

Define the partition  $\mathcal{P} = \{P_1, \dots, P_k\}$  as follows:  $P_i = W_i$  if  $i \neq k$ , and  $P_k = V \setminus \bigcup_{i \in [k-1]} W_i$ . Clearly, we have:

$$|\partial P_k| \leq \left| \bigcup_{t=1}^{k-1} \partial W_t \right| \leq \mathcal{O}(k \cdot \text{OPT} \sum_{t=1}^{k-1} |W_t|) \leq \mathcal{O}(kn \cdot \text{OPT})$$

Above, the last inequality follows since the  $W_t$ 's are all disjoint. Since  $|P_k| \geq \Omega(n/k)$ , and  $|V \setminus P_k| \geq \Omega(n)$ , we infer that  $\phi^{V,k}(\mathcal{P}) \leq \phi^V(P_k) \leq \mathcal{O}(k^2 \cdot \text{OPT})$ .

**For  $k$ -Part-edge:** The proof is very similar to the preceding one for  $k$ -Part-vertex, except we work with edges. Again, from the definition of  $\phi$ , we have, for the sets given by Theorem 1.3:

$$|E(\partial W_t)| \leq \mathcal{O}(1) \cdot \text{OPT} \cdot k \cdot \frac{|W_t| |V \setminus W_t|}{n} = \mathcal{O}(k \cdot \text{OPT} |W_t|)$$

As before, we define  $\mathcal{P} = \{P_1, \dots, P_k\}$  as follows:  $P_i = W_i$  if  $i \neq k$ , and  $P_k = V \setminus \bigcup_{i \in [k-1]} W_i$ . From the above bound on  $|E(\partial W_t)|$ , we get that:

$$|E(\partial P_k)| = \mathcal{O}(k \cdot \text{OPT} \sum_{t=1}^{k-1} |W_t|) = \mathcal{O}(kn \cdot \text{OPT}),$$

giving that  $\phi^k(\mathcal{P}) = \mathcal{O}(k^2 \cdot \text{OPT})$ . ◀

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