Guarded Teams: The Horizontally Guarded Case

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Abstract

Team semantics admits reasoning about large sets of data, modelled by sets of assignments (called teams), with first-order syntax. This leads to high expressive power and complexity, particularly in the presence of atomic dependency properties for such data sets. It is therefore interesting to explore fragments and variants of logic with team semantics that permit model-theoretic tools and algorithmic methods to control this explosion in expressive power and complexity.

We combine here the study of team semantics with the notion of *guarded logics*, which are well-understood in the case of classical Tarski semantics, and known to strike a good balance between expressive power and algorithmic manageability. In fact there are two strains of guardedness for teams. *Horizontal guardedness* requires the individual assignments of the team to be guarded in the usual sense of guarded logics. *Vertical guardedness*, on the other hand, posits an additional (or definable) hypergraph structure on relational structures in order to interpret a constraint on the component-wise variability of assignments within teams.

In this paper we investigate the horizontally guarded case. We study horizontally guarded logics for teams and appropriate notions of *guarded team bisimulation*. In particular, we establish *characterisation theorems* that relate invariance under guarded team bisimulation with guarded team logics, but also with logics under classical Tarski semantics.

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1 Introduction

Team semantics, which originates in the work of the model theorist Wilfrid Hodges [18], is based on the idea to evaluate logical formulae $\varphi(x_1, \dots, x_n)$ not for single assignments $s : \{x_1, \dots, x_n\} \to A$ from the free variables to elements of a structure $\mathfrak{A}$, but for *sets of such assignments*. These sets, which may have arbitrary size, are now called *teams*. The original motivation for team semantics has been to provide a compositional, model-theoretic semantics of the independence-friendly logic (IF-logic) [22], for which one previously only knew semantics based on either Skolem functions or on games of imperfect information. Team semantics has then become important as the mathematical basis of the modern logics of dependence and independence, which go back to the fundamental idea of Väänänen [25] to treat dependencies not as annotations of quantifiers (as in IF-logic), but as atomic properties of teams. Logics with team semantics for reasoning about dependence, independence, and imperfect information have meanwhile been established as a lively interdisciplinary research area, involving not just first-order logics, but also logics on the propositional and modal level, see e.g. [1].

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Team semantics admits reasoning about large sets of data, modelled by second-order objects such as sets of assignments, with a first-order syntax that does not explicitly refer to higher-order variables. In the presence of appropriate atomic team properties, such as dependence, inclusion and exclusion, or independence properties, team semantics can boost the expressiveness of first-order formalisms to the full power of existential second-order logic (denoted Σ^1_1) or, in the presence of further propositional operators such as different variants of implication or negation, even to full second-order logic (SO). Beyond logics for dependence and independence, team semantics may have broader applications. The idea of second-order reasoning with first-order syntax appears also in certain logics used for program verification (to reason about sets of variable assignments such as heaps, as in separation logic) or in quantum information theory (to reason about superpositions of basic states). Currently there are emerging new research directions that relate team semantics to such areas.

However, the ability of logics with team semantics to reason about second-order objects increases not only the expressive power, but also the complexity, and makes it much more difficult to understand the model theory of such formalisms and to handle them algorithmically. It is therefore relevant to explore fragments or variants of logics with team semantics that permit model-theoretic tools and algorithmic methods to control this explosion in expressive power and complexity. In this paper we explore a promising idea in this direction, namely the use of *guarded teams* and *guarded logics*. Guarded logics have been thoroughly investigated in the context of classical logical formalisms, and guarded fragments of first-order logic, fixed-point logics, and second-order logic have turned out to have very interesting and convenient model-theoretic and algorithmic properties, see e.g. [3, 4, 5, 12, 13, 14, 17, 19, 24] and our survey [15].

The basic guarded logic is the guarded fragment (GF) of first-order logic, introduced by Andréka, van Benthem and Németi [2]. It is defined by restricting existential and universal quantification in such a way that formulae only refer to *guarded tuples*, i.e., tuples of elements that occur together in some atomic fact. Syntactically, this means that quantifiers are used only in the form $\exists \bar{y}(\alpha \wedge \varphi)$ or $\forall \bar{y}(\alpha \rightarrow \varphi)$ where α is an atomic formula that must contain *all* free variables of φ (and possibly more). An important motivation for introducing the guarded fragment has been to explain and generalise the good algorithmic and model-theoretic properties of *modal logics* (see [6, 11] for background on modal logic). Recall that modal logic can be viewed as a fragment of first-order logic, via a standard translation that uses only two variables and a restricted kind of guarded quantification. The guarded fragment generalises the modal fragment enormously, dropping all restrictions (such as to use only two variables and only monadic and binary predicates), except the restriction that quantification must be guarded. It has turned out that almost all important algorithmic and model-theoretic properties of modal logic do indeed extend to the guarded fragment. In particular, the satisfiability problem for GF is decidable [2], GF has the finite model property, i.e., every satisfiable formula in the guarded fragment has a finite model [12], and moreover, GF has a generalised variant of the tree model property to the effect that every satisfiable formula has a model that admits a tree decomposition into guarded substructures, which in particular implies a bound on its tree width [12]. The tree model property paves the way to automata based algorithmic procedures for guarded logics. Further, GF admits efficient evaluation algorithms via model checking games of moderate size. There are similar results that hold for more powerful guarded logics, which are obtained either by a more liberal interpretation of guardedness (as in loosely guarded or clique guarded logics), by guarding negation instead of quantifiers, and/or by moving to guarded variants of stronger logics such as fixed-point or second-order logic.

The crucial model-theoretic tool to investigate guarded logics is the notion of a *guarded bisimulation* between two structures $\mathfrak{A}$ and $\mathfrak{B}$, which we view here as a set Z of pairs of guarded assignments (s, t) such that $s \mapsto t$ induces a local isomorphism, and Z satisfies appropriate back-and-forth properties (cf. Section 4). Results of fundamental importance for guarded logics are *characterisation theorems* such as the one due to Andréka, van Benthem and Németi [2], saying that *a first-order formula is invariant under guarded bisimulation if, and only if, it is equivalent to a formula of the guarded fragment*, in short $\mathsf{FO}/\sim_{\mathsf{g}} \equiv \mathsf{GF}$. There are several variants and extensions of this result, among them the finite model theory variant [24] (in which both the invariance statements, and the equivalence to a guarded formula are restricted to hold only over finite structures), as well as characterisation results for stronger guarded logics such as the one for fixed-point logic from [14]. See again [15] for a detailed discussion of guarded bisimulations in various contexts.

In this paper we aim at the development of the theory of *guarded teams* and *guarded logics with team semantics*. There are in fact two completely different variants of guarded teams. *Horizontal guardedness* requires all assignments in a team to be guarded in the usual sense. On this basis, we define horizontally guarded team semantics and horizontally guarded logics and relate them to the established classical framework of guarded logics. In particular, the good algorithmic properties of guarded fragments of first-order logic such as the decidability of the satisfiability problem, are easily seen to carry over to corresponding problems for horizontally guarded team semantics, such as the question whether a given guarded first-order formula is satisfiable by some nonempty team. However, corresponding questions for stronger guarded logics, involving atomic dependencies, are open.

To investigate the power of guarded team semantics we introduce and study two different notions of *guarded team bisimulation*, a weaker and a stronger one, and prove *characterisation theorems*, which relate formulae that are invariant under guarded team bisimulation to guarded team logics, but also to appropriate variants of logics with classical Tarski semantics. These are the core results of this paper. We remark that a loosely related characterisation theorem has been established in [20] for the much weaker context of *modal team semantics*.

Besides horizontal guardedness, there is also the rather different notion of *vertical guardedness* of a team, based on an additional (or definable) hypergraph structure on relational structures in order to interpret a constraint on the component-wise variability of the assignments in teams. When a team X is viewed as a table whose rows are the assignments of the team, then vertical guardedness imposes restraints to the effect that the columns, i.e. the value sets $X(x)$ are guarded. It turns out that this adds new and interesting second-order features to the team semantics of the resulting logics. However, due to space limitations we defer the development of this aspect to a future paper.

2 Team semantics

For a tuple $\bar{a} = (a_1, \ldots, a_n) \in A^n$ we denote by $[\bar{a}]$ the set $\{a_1, \ldots, a_n\}$ of its components. We write $\mathcal{P}(A)$ for the power set of A and set $\mathcal{P}^+(A) := \mathcal{P}(A) \setminus \{\varnothing\}$. An *assignment* to variables $x \in D$ into a set $A \neq \varnothing$ is a map $s \colon D \to A$. We write $s[x \mapsto a]$ for the assignment that extends, or updates, s by mapping x to a. We extend s in the obvious manner to tuples over D, and write $s(\bar{x})$ for $(s(x_1), \ldots, s(x_k)) \in A^k$ if $\bar{x} = (x_1, \ldots, x_k) \in D^k$, and similarly $s(d) := \{s(x) \colon x \in d\} \subseteq A$ for subsets $d \subseteq D$.

▶ **Definition 1.** A *team* is set X of assignments $s : D \to A$ with a common finite domain $D = \mathrm{dom}(X)$ of variables into a set A. Besides the empty team $\varnothing$ we also admit a unique team $\{\varnothing\}$ with empty domain and empty assignment. For every k-tuple $\bar{x}$ of variables from the domain of X, let $X(\bar{x}) := \{s(\bar{x}) \colon s \in X\} \subseteq A^k$ denote the set of values assumed by $\bar{x}$ in the team X. Thinking of an arbitrary but fixed enumeration of the finite domain of a team X as $\mathrm{dom}(X) = \{x_1, \ldots, x_k\}$ we often identify X with its *relational encoding* $[\![X]\!] := X(\bar{x}) = \{s(\bar{x}) \colon s \in X\} \subseteq A^k$.

Basic operations that possibly extend the domain of a given team to new variables are the unrestricted *generalisation* over A, $X[x \mapsto A] := \{s[x \mapsto a] \colon s \in X, a \in A\}$ and the *Skolem-extensions* $X[x \mapsto F] := \{s[x \mapsto a] \colon s \in X, a \in F(s)\}$ for any function $F \colon X \to \mathcal{P}^+(A)$. Note that $X[x \mapsto A] = X[x \mapsto F]$ for the constant function $F \colon s \mapsto A$, and that x may or may not be in the domain D of the original team X, but in any case the new team has domain $D \cup \{x\}$. Given teams X, Y with $D \subseteq \mathrm{dom}(X) \cap \mathrm{dom}(Y)$ we write $X \equiv_D Y$ if $(X \restriction D) = (Y \restriction D)$ where $(X \restriction D) := \{s \restriction D : s \in X\}$.

The traditional semantics (to which we refer as Tarski semantics) for first-order formulae $\varphi(\bar{x})$ is based on single assignments s whose domain must comprise the variables in $\mathrm{free}(\varphi)$; we write $\mathfrak{A} \models \varphi[s]$ for saying that $\mathfrak{A}$ satisfies φ with the assignment s.

The *team semantics* for $\mathsf{FO}(\tau)$ over τ-structures $\mathfrak{A}$ instead is defined by inductive clauses for the satisfaction relation $\mathfrak{A} \models_X \varphi$ saying that team X satisfies φ in $\mathfrak{A}$. Here X stands for a team in A with domain D for which we tacitly always assume that $\mathrm{dom}(X) \supseteq \mathrm{free}(\varphi)$.

- if φ is a literal then $\mathfrak{A} \models_X \varphi$ if $\mathfrak{A} \models \varphi[s]$ for all $s \in X$;
- $\mathfrak{A} \models_X \varphi_1 \wedge \varphi_2$ if $\mathfrak{A} \models_X \varphi_i$ for $i = 1, 2$;
- $\mathfrak{A} \models_X \varphi_1 \vee \varphi_2$ if $X = X_1 \cup X_2$ for two teams X_i such that $\mathfrak{A} \models_{X_i} \varphi_i$;
- $\mathfrak{A} \models_X \forall x \varphi$ if $\mathfrak{A} \models_Y \varphi$ for the team $Y = X[x \mapsto A]$;
- $\mathfrak{A} \models_X \exists x \varphi$ if $\mathfrak{A} \models_Y \varphi$ for some team Y of the form $Y = X[x \mapsto F]$, i.e., for some suitable Skolem extension $F \colon X \to \mathcal{P}(A) \setminus \{\varnothing\}$.

Note that there is no clause for negation (other than negation of atoms); we assume formulae to be written in negation normal form unless explicitly noted otherwise. It is not hard to see, through standard inductive arguments, that team semantics for FO satisfies the following principles:

Locality: whether $\mathfrak{A} \models_X \varphi$ is determined by $X \restriction \mathrm{free}(\varphi)$.
Downward closure: if $Y \subseteq X$, then $\mathfrak{A} \models_X \varphi$ implies $\mathfrak{A} \models_Y \varphi$.
Flatness: $\mathfrak{A} \models_X \varphi$ if, and only if, $\mathfrak{A} \models_{\{s\}} \varphi$ for all $s \in X$.
Empty team property: $\mathfrak{A} \models_\varnothing \varphi$ for all φ.
Union closure: if $\mathfrak{A} \models_{X_i} \varphi$ for all $i \in I$, then $\mathfrak{A} \models_X \varphi$ for $X := \bigcup_{i \in I} X_i$.

Other propositional connectives. For some purposes it is useful to consider a team semantic interpretation of other natural propositional connectives.

Implication: $\mathfrak{A} \models_X \psi \to \varphi$ if, for all teams $Y \subseteq X$ with $\mathfrak{A} \models_Y \psi$, also $\mathfrak{A} \models_Y \varphi$.
Intuitionistic disjunction: $\mathfrak{A} \models_X \psi \otimes \varphi$ if $\mathfrak{A} \models_X \psi$ or $\mathfrak{A} \models_X \varphi$.
Classical negation: $\mathfrak{A} \models_X \mathsf{non}\, \varphi$ if it is not the case that $\mathfrak{A} \models_X \varphi$.
Nonemptiness: this is is a nullary connective or logical constant without a classical counterpart, which we denote as NE, with $\mathfrak{A} \models_X \mathsf{NE}$ if $X \neq \varnothing$.
Falsum: this is a nullary connective as in the classical setting. Keeping in mind the empty team property, $\mathfrak{A} \models_X \bot$ for just $X = \varnothing$.

Regarding ordinary negation ($\neg$) in team semantics, which we here only allow at the atomic level, it is important to note that, in contrast to classical negation (non), it does *not* support the classical principle of the excluded middle (*tertium non datur*). Clearly there are teams that do not uniformly make up their mind between α and $\neg\alpha$ even for equalities or relational atoms α. The following logical equivalences between some of the above are easily proved, for any atomic α and arbitrary φ_i

$$\mathsf{NE} \equiv \mathsf{non}\,\bot, \quad \bot \equiv \alpha \wedge \neg\alpha, \quad \varphi_1 \otimes \varphi_2 \equiv \mathsf{non}\,(\mathsf{non}\,\varphi_1 \wedge \mathsf{non}\,\varphi_2).$$

These mark out classical negation as an augmentation of FO in the team semantic setting, which no longer satisfies any of the semantic criteria highlighted above, apart from locality. We denote as FO(non) the extension of FO with strong classical negation; corresponding notation for other choices of additional connectives, like FO(NE, $\otimes$), is self-explanatory.

Logics of dependence and independence. Team semantics is particularly important as the basis for logics that extend first-order logic by atomic team properties such as dependence, inclusion, exclusion, independence, and others. The best studied such logic is *dependence logic* [25], which extends first-order logic by dependency atoms of form $\text{dep}(\bar{x}, \bar{y})$, saying that the values for $\bar{y}$ are functionally dependent on (i.e. completely determined by) the values for $\bar{x}$. Other important such logics include different variants of *independence logics* [16], further *inclusion logic* FO($\subseteq$), which is based on inclusion dependencies ($\bar{x} \subseteq \bar{y}$) saying that every value for $\bar{x}$ in the team also occurs as a value for $\bar{y}$, and dually, *exclusion logic*, based on exclusion statements ($\bar{x} \mid \bar{y}$), saying that $\bar{x}$ and $\bar{y}$ have disjoint sets of values in the given team.

One way to describe the expressive power of a logic with team semantics is to relate it to some well-understood logic with classical Tarski semantics. One translates formulae $\varphi(\bar{x})$ from a logic $L(\tau)$ with team semantics into *sentences* φ^T, with Tarski semantics, of vocabulary $\tau \,\dot\cup\, \{T\}$ where T is an additional relation symbol for the team, such that for every structure $\mathfrak{A}$ and every team X we have that

$$\mathfrak{A} \models_X \varphi(\bar{x}) \iff (\mathfrak{A}, [\![X]\!]) \models \varphi^T,$$

where $[\![X]\!]$ is the relational encoding of the team X (see Definition 1). In all logics with team semantics that extend first-order formulae by atomic dependencies that are themselves first-order definable, and which do not make use of additional connectives beyond $\wedge, \vee$ and atomic negation, such a translation will always produce sentences in (a fragment of) existential second-order logic Σ^1_1. Understanding the expressive power of a logic L with team semantics thus means to identify the fragment of Σ^1_1 to which L is equivalent in the sense just described.

(1) Dependence logic and exclusion logic are equivalent to the fragment of Σ^1_1-sentences $\psi(T)$ in which the predicate T describing the team appears only negatively [21].

(2) Independence logic and inclusion-exclusion logic are equivalent with full Σ^1_1 (and thus can describe all NP-properties of teams) [9].

(3) Any fragment $L \subseteq$ FO, without any dependence properties, corresponds by flatness to the class $[L]^T$ of sentences of form $\forall\bar{x}(T\bar{x} \to \varphi(\bar{x}))$ where $\varphi(\bar{x}) \in L$ does not contain T.

(4) Inclusion logic is equivalent to the set of sentences of form $\forall\bar{x}(X\bar{x} \to \psi(X, \bar{x}))$, where $\psi(X, \bar{x})$ is a formula in the posGFP-fragment of least fixed-point logic, in which X occurs only positively [10].

In the presence of additional propositional connectives such as implication or strong negation, such translations may produce sentences in full second-order logic (SO), rather than its existential fragment Σ^1_1.

3 Horizontally guarded first-order logic

We deal with purely relational, finite vocabularies τ. Every interpretation of the relations in a τ-structure $\mathfrak{A} = (A, (R^{\mathfrak{A}})_{R\in\tau})$ induces a notion of *guarded subsets* and *guarded tuples*.

▶ **Definition 2.** The set $\mathbb{G}(\mathfrak{A})$ of *guarded subsets* of $\mathfrak{A}$ is the downward closure of the collection of all sets $[\bar{a}]$ for any atomic fact $R\bar{a}$ that holds in $\mathfrak{A}$, together with all singleton subsets $\{a\} \subseteq A$. Formally $\mathbb{G}(\mathfrak{A}) = \big\{B \subseteq [\bar{a}] \colon \bar{a} \in R^{\mathfrak{A}}, R \in \tau\big\} \cup \big\{\{a\} \colon a \in A\big\}$. A tuple $\bar{a} = (a_1, \dots, a_k) \in A^k$ is guarded in $\mathfrak{A}$ if the set of its components $[\bar{a}]$ is. An assignment $s : D \to A$ is guarded if $s(D) \in \mathbb{G}(\mathfrak{A})$ which is the case if, and only if, $s(\bar{x})$ is a guarded tuple for any tuple of variables from D. A relation $T \subseteq A^k$ is guarded in $\mathfrak{A}$ if it only consists of guarded tuples, and this is the case if, and only if, $\mathbb{G}((\mathfrak{A}, T)) = \mathbb{G}(\mathfrak{A})$, i.e. the expansion of $\mathfrak{A}$ by T does not introduce any new guarded subsets. A team X is *horizontally guarded* if it only consists of guarded assignments. We write $\mathbb{H}(\mathfrak{A})$ for the collection of all horizontally guarded teams over $\mathfrak{A}$.

Note that $\mathbb{H}(\mathfrak{A})$ is closed under subsets and restrictions of teams. The Skolem extensions $X[x \mapsto F]$ of a horizontally guarded team X will not in general be horizontally guarded, but we have the following.

▶ **Lemma 3.** *Every horizontally guarded team $X \in \mathbb{H}(\mathfrak{A})$ possesses, for every variable x, a unique maximal Skolem extension $Y = X[x \mapsto F] \in \mathbb{H}(\mathfrak{A})$.*

Proof. For $D = \mathrm{dom}(X) \setminus \{x\}$, put $Y := \{s[x \mapsto a] \colon s \in X, a \in A, s(D) \cup \{a\} \in \mathbb{G}(\mathfrak{A})\}$. This team is easily checked to be maximal among all teams $Y \in \mathbb{H}(\mathfrak{A})$ with $\mathrm{dom}(Y) = \mathrm{dom}(X) \cup \{x\}$ and $Y \equiv_D X$. ◀

We are now ready to introduce the *horizontally guarded team semantics* $\mathfrak{A} \models^{\mathsf{hg}}_X \varphi$ of first-order formulae for τ-structures $\mathfrak{A}$ and teams $X \in \mathbb{H}(\mathfrak{A})$. For its definition we modify the clauses in the standard definition of team semantics so as to restrict all relevant teams to $\mathbb{H}(\mathfrak{A})$. This modification is trivial for literals and conjunctions and obvious for disjunction since $\mathbb{H}(\mathfrak{A})$ is downward closed. For universal and existential quantification, the restriction is more interesting.

- $\mathfrak{A} \models^{\mathsf{hg}}_X \varphi_1 \vee \varphi_2$ if $X = X_1 \cup X_2$ for two teams $X_i \in \mathbb{H}(\mathfrak{A})$ such that $\mathfrak{A} \models^{\mathsf{hg}}_{X_i} \varphi_i$;
- $\mathfrak{A} \models^{\mathsf{hg}}_X \forall x\varphi$ if $\mathfrak{A} \models_Y \varphi$ for the maximal horizontally guarded Skolem extension of $X \upharpoonright (\mathrm{free}(\varphi) \setminus \{x\})$;
- $\mathfrak{A} \models^{\mathsf{hg}}_X \exists x\varphi$ if $\mathfrak{A} \models_Y \varphi$ for some horizontally guarded Skolem extension Y of $X \upharpoonright (\mathrm{free}(\varphi) \setminus \{x\})$.

Horizontally guarded first-order logic is the logic $\mathsf{FO}^{\mathsf{hg}}$ with usual syntax in negation normal form and semantics for horizontally guarded teams as defined above. $\mathsf{FO}^{\mathsf{hg}}$ satisfies the familiar properties of first-order team semantics, locality, downward closure, and flatness.

▶ **Lemma 4.** *For every $\varphi \in \mathsf{FO}^{\mathsf{hg}}$, every structure $\mathfrak{A}$, and every team $X \in \mathbb{H}(\mathfrak{A})$ with $\mathrm{free}(\varphi) \subseteq \mathrm{dom}(X)$, we have*
Locality: *$\mathfrak{A} \models^{\mathsf{hg}}_X \varphi \Leftrightarrow \mathfrak{A} \models^{\mathsf{hg}}_Y \varphi$ whenever $X \equiv_{\mathrm{free}(\varphi)} Y$.*
Downward closure: *If $Y \subseteq X$ and $\mathfrak{A} \models^{\mathsf{hg}}_X \varphi$, then also $\mathfrak{A} \models^{\mathsf{hg}}_Y \varphi$.*
Flatness: *$\mathfrak{A} \models^{\mathsf{hg}}_X \varphi$ if, and only if, $\mathfrak{A} \models^{\mathsf{hg}}_{\{s\}} \varphi$ for all $s \in X$.*

The difference between $\mathfrak{A} \models^{\mathsf{hg}}_{\{s\}} \varphi$ (in the sense of $\mathsf{FO}^{\mathsf{hg}}$) and $\mathfrak{A} \models \varphi[s]$ (in the sense of ordinary first-order logic) has nothing to do with team semantics but just with the implicit relativisation to guarded assignments in $\models^{\mathsf{hg}}$. In the classical first-order setting for single assignments, this corresponds to the (explicit) relativisation to guarded assignments in the guarded fragment $\mathsf{GF}(\tau) \subseteq \mathsf{FO}(\tau)$. And indeed, there is a straightforward translation also in the case of team semantics.

▶ **Definition 5.** The guarded fragment $\mathsf{GF} \subseteq \mathsf{FO}$ is the syntactic fragment of FO generated from atomic formulae by the boolean connectives and quantifications of the form $\exists\bar{y}\big(\alpha(\bar{x}\bar{y}) \wedge \varphi(\bar{x}\bar{y})\big)$, and, dually, $\forall\bar{y}\big(\alpha(\bar{x}\bar{y}) \rightarrow \varphi(\bar{x}\bar{y})\big)$, where $\varphi(\bar{x}\bar{y}) \in \mathsf{GF}$ has free variables among those listed in $\bar{x}\bar{y}$ and $\alpha(\bar{x}\bar{y})$ is an atomic formula in which all the listed variables occur. The formula α is called the *guard* of this quantification.[1] The semantics of GF is that of FO.

It is obvious that $\mathsf{GF}(\tau) \subseteq \mathsf{FO}(\tau)$ inherits from $\mathsf{FO}(\tau)$ the locality, downward closure and flatness properties for its team semantics. Note that $\mathsf{GF}(\tau)$ involves, in the first-order correspondent for universal guarded quantification, the use of implication as a propositional connective and recall its team semantics, which is downward closed by definition. As the classical equivalence $\alpha \rightarrow \varphi \equiv \neg\alpha \vee \varphi$ only involves the negation of an atomic guard α, it persists as an equivalence in team semantics. This remains true more generally in any context where the team semantics of $(\neg)\alpha$ is flat and that of φ is downward closed.

▶ **Lemma 6.** *Assume that $\varphi[\mathfrak{A}] = \{X : \mathfrak{A} \models_X \varphi\}$ is downward closed and that the team semantics of α and $\neg\alpha$ is flat. Then, for all teams X, $\mathfrak{A} \models_X \alpha \rightarrow \varphi \;\Leftrightarrow\; \mathfrak{A} \models_X \neg\alpha \vee \varphi$. The same logical equivalence holds in terms of horizontally guarded team semantics (keeping in mind that the class of horizontally guarded teams is downward closed).*

Recall that the set of guarded tuples $\bar{a} = (a_1, \dots, a_n) \in A^n$ is uniformly first-order definable in τ-structures $\mathfrak{A}$, for any fixed length $n \geqslant 1$ and fixed finite relational τ, by a formula

$$\mathsf{gd}(\bar{x}) := \bigwedge_{i \leqslant n} x_i = x_1 \;\vee\; \bigvee_{R \in \tau} \exists\bar{y}\Big(R\bar{y} \wedge \bigwedge_{i \leqslant n} \bigvee_{j \leqslant \mathrm{ar}(R)} x_i = y_j\Big).$$

In the following we regard, for every finite τ and every $\bar{x}$, the formula $\mathsf{gd}(\bar{x})$ as a new atomic formula and in particular also allow its negation $\neg\mathsf{gd}(\bar{x})$ which is correspondingly interpreted in the flat sense of $[\![X]\!] \cap \mathbb{G}(\mathfrak{A}) = \varnothing$:

$$\mathfrak{A} \models_X \neg\mathsf{gd}(\bar{x}) \;\text{ if }\; \mathfrak{A} \models \neg\mathsf{gd}[s(\bar{x})] \text{ for all } s \in X.$$

Note that, with this stipulation, $\mathsf{gd}(\bar{x})$ becomes a formula satisfying the requirements for α in Lemma 6, so that, for any $\varphi(\bar{x})$ whose team semantics is downward closed, we have the usual (classical) equivalence $\mathsf{gd}(\bar{x}) \rightarrow \varphi \equiv \neg\mathsf{gd}(\bar{x}) \vee \varphi$.

▶ **Proposition 7.** *For finite relational τ, there is a translation $\varphi(\bar{x}) \longmapsto \varphi^{\mathsf{hg}}(\bar{x})$ from $\mathsf{FO}(\tau)$ to $\mathsf{GF}(\tau)$ such that for all guarded assignments s, and all teams $X \in \mathbb{H}(\mathfrak{A})$,*

$$\mathfrak{A} \models^{\mathsf{hg}}_{\{s\}} \varphi \;\Leftrightarrow\; \mathfrak{A} \models \varphi^{\mathsf{hg}}[s] \quad \textit{and} \quad \mathfrak{A} \models^{\mathsf{hg}}_X \varphi \;\Leftrightarrow\; \mathfrak{A} \models_X \varphi^{\mathsf{hg}}.$$

An analogous translation works for extensions of FO and GF by arbitrary Σ^1_1-definable atomic dependence relations.

Proof. Define $\varphi \mapsto \varphi^{\mathsf{hg}}$ by induction on $\varphi \in \mathsf{FO}(\tau)$. The only non-trivial steps are those for the quantifiers. For $\varphi(\bar{x}) = \exists y\psi(\bar{x}y)$, put $\varphi^{\mathsf{hg}}(\bar{x}) := \exists y\big(\mathsf{gd}(\bar{x}y) \wedge \psi^{\mathsf{hg}}(\bar{x}y)\big)$, and for $\varphi(\bar{x}) = \forall y\psi(\bar{x}y)$, set

$$\varphi^{\mathsf{hg}}(\bar{x}) := \forall y\big(\mathsf{gd}(\bar{x}y) \rightarrow \psi^{\mathsf{hg}}(\bar{x}y)\big) \equiv \forall y\big(\neg\mathsf{gd}(\bar{x}y) \vee \psi^{\mathsf{hg}}(\bar{x}y)\big).$$

[1] If $\bar{x}\bar{y}$ consists of a single variable symbol z, α can be the equality $z = z$.

It is straightforward to verify that these stipulations support the equivalence claim for singleton teams. The equivalence claim for arbitrary teams follows by the flatness properties for $\mathsf{FO}^{\mathsf{hg}}$ and FO. ◀

One also checks that $\mathfrak{A} \models_X \varphi^{\mathsf{hg}} \Leftrightarrow \mathfrak{A} \models^{\mathsf{hg}}_X \varphi^{\mathsf{hg}}$, whence also $\mathfrak{A} \models^{\mathsf{hg}}_X \varphi \;\Leftrightarrow\; \mathfrak{A} \models^{\mathsf{hg}}_X \varphi^{\mathsf{hg}}$, so that the map $\cdot^{\mathsf{hg}}$ provides a normal form for FO w.r.t. its horizontally guarded team semantics.

Since most of the standard logics with team semantics have the empty team property, the natural variants of satisfiability problems in this context ask whether a given a formula φ admits a structure $\mathfrak{A}$ and a *nonempty* team X such that $\mathfrak{A} \models_X \varphi$. Notice that, by flatness, the question whether a guarded first-order formula $\varphi(\bar{x}) \in \mathsf{GF}$ is satisfiable in this sense is equivalent to asking whether $\varphi(\bar{x})$ is satisfiable in the usual sense of Tarski semantics, which is well-known to be decidable [2, 12].

4 Two notions of guarded team bisimulation

The natural notion of back and forth equivalence for guarded logics is *guarded bisimulation equivalence*. Just as the model theory of modal logics is governed by (modal) bisimulation equivalence, the nice model-theoretic properties of guarded logics are closely related to its invariance under guarded bisimulation equivalence $\sim_{\mathsf{g}}$ and its finite approximations $\sim^{\ell}_{\mathsf{g}}$. For a detailed discussion of guarded bisimulation and beyond, we refer to [15]. In the context investigated here it is convenient to view a guarded bisimulation between two structures $\mathfrak{A}$ and $\mathfrak{B}$ as a set Z of pairs (s,t) of guarded assignments that induce local isomorphisms between the two structures, and satisfy appropriate back and forth properties.

▶ **Definition 8.** A guarded bisimulation between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a set Z of pairs of guarded assignments $s : [\bar{x}] \to A$ and $t : [\bar{x}] \to B$, with $\mathrm{dom}(s) = \mathrm{dom}(t)$, such that, for all $(s,t) \in Z$:

- **(i)** $s \mapsto t$ induces a local isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. This means that for every atomic formulae α with $\mathrm{free}(\alpha) \subseteq \mathrm{dom}(s) = \mathrm{dom}(t)$ we have that $\mathfrak{A} \models \alpha[s] \iff \mathcal{B} \models \alpha[t]$.
- **(ii)** (*back*): for every guarded assignment t' into $\mathfrak{B}$ that coincides with t on $\mathrm{dom}(t') \cap \mathrm{dom}(t)$ there is a guarded assignment s' into $\mathfrak{A}$ that coincides with s on $\mathrm{dom}(s') \cap \mathrm{dom}(s)$ such that (s',t') is also in Z.
- **(iii)** (*forth*): for every guarded assignment s' into $\mathfrak{A}$ that coincides with s on $\mathrm{dom}(s') \cap \mathrm{dom}(s)$ there is a guarded assignment t' into $\mathfrak{B}$ that coincides with t on $\mathrm{dom}(t') \cap \mathrm{dom}(t)$ such that (s',t') is also in Z.

We write $\mathfrak{A}, s \sim_{\mathsf{g}} \mathfrak{B}, t$ if there is a guarded bisimulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(s,t) \in Z$. Further we write $\mathfrak{A} \sim_{\mathsf{g}} \mathfrak{B}$ if $\mathfrak{A}, \varnothing \sim_{\mathsf{g}} \mathfrak{B}, \varnothing$.

There is an obvious game-theoretic presentation of this in terms of *guarded bisimulation games* on $(\mathfrak{A}, \mathfrak{B})$ whose positions are the pairs (s,t) of guarded assignments that induce a local isomorphism as described above. Then the available moves for the first player, e.g. on the $\mathfrak{A}$-side, are to guarded assignments s' such that $s'(x) = s(x)$ for all variables $x \in \mathrm{dom}(s') \cap \mathrm{dom}(s)$. The second player then has to respond with a guarded assignment t' with $\mathrm{dom}(t') = \mathrm{dom}(s')$, such that $t'(x) = t(x)$ for $x \in \mathrm{dom}(t') \cap \mathrm{dom}(t)$, and (s',t') is again a valid position of the game.

Finite approximations $\sim^{\ell}_{\mathsf{g}}$ of $\sim_{\mathsf{g}}$ correspond to the existence of winning strategies for the second player for ℓ rounds in the guarded bisimulation game, and $\sim^{\omega}_{\mathsf{g}}$ is defined as the common refinement of the finite levels $\sim^{\ell}_{\mathsf{g}}$.

One obtains natural variants of the first-order Ehrenfeucht–Fraïssé Theorem for GF. The equivalence relations $\equiv^{\ell}_{\mathsf{GF}}$ and $\equiv_{\mathsf{GF}}$ are defined as levels of elementary equivalence in GF, where the ℓ in $\equiv^{\ell}_{\mathsf{GF}}$ refers to the nesting depth of guarded quantification (which is typically lower than the first-order quantifier rank, as guarded quantification may quantify over tuples in a single step). The relation $\equiv^{\infty}_{\mathsf{GF}}$ similarly denotes equivalence w.r.t. the infinitary variant of GF, with infinite disjunctions and conjunctions. For more details, we refer to [15].

▶ **Theorem 9** (Ehrenfeucht–Fraïssé Theorems for GF). *For finite relational vocabularies, and for every $\ell \in \mathbb{N}$:*

$$\mathfrak{A}, s \sim^{\ell}_{\mathrm{g}} \mathfrak{B}, t \iff \mathfrak{A}, s \equiv^{\ell}_{\mathsf{GF}} \mathfrak{B}, t \quad \text{and} \quad \mathfrak{A}, s \sim^{\omega}_{\mathrm{g}} \mathfrak{B}, t \iff \mathfrak{A}, s \equiv_{\mathsf{GF}} \mathfrak{B}, t.$$

Further, without restriction on the size of the vocabulary, $\mathfrak{A}, s \sim_{\mathrm{g}} \mathfrak{B}, t \iff \mathfrak{A}, s \equiv^{\infty}_{\mathsf{GF}} \mathfrak{B}, t$.

Just as in the classical first-order case (cf. [7, 8]) the implication from $\equiv^{\ell}_{\mathsf{GF}}$ to $\sim^{\ell}_{\mathrm{g}}$ relies on the fact that both equivalence relations have finite index, and that the $\sim^{\ell}_{\mathrm{g}}$-equivalence classes of $\mathfrak{A}, s$ are naturally definable by *characteristic formulae* $\chi^{\ell}_{\mathfrak{A},s} \in \mathsf{GF}$.

We now generalise the notion of guarded bisimulation equivalence from individual assignments to teams. It turns out that there are two different ways to do this, a basic one and a stronger one.

▶ **Definition 10.** *Guarded team bisimulation equivalence*, $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$ and its finite approximations $\mathfrak{A}, X \sim^{\ell}_{\mathrm{g}} \mathfrak{B}, Y$ are defined in a flat manner. Horizontally guarded teams $X \in \mathbb{H}(\mathfrak{A})$ and $Y \in \mathbb{H}(\mathfrak{B})$, with the same domain, are *guarded team bisimilar*, $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$ if for every $s \in X$ there is some $t \in Y$ such that $\mathfrak{A}, s \sim_{\mathrm{g}} \mathfrak{B}, t$, and vice versa. Guarded team ℓ-bisimilarity, $\mathfrak{A}, X \sim^{\ell}_{\mathrm{g}} \mathfrak{B}, Y$, is defined analogously.

Ordinary guarded bisimulation equivalences between individual assignments, like $\mathfrak{A}, s \sim_{\mathrm{g}} \mathfrak{B}, t$, are captured in this definition via the encodings of tuples as singleton teams: $\mathfrak{A}, \{s\} \sim_{\mathrm{g}} \mathfrak{B}, \{t\}$, and for naked structures, we have that $\mathfrak{A} \sim_{\mathrm{g}} \mathfrak{B}$ if, and only if, $\mathfrak{A}, \{\varnothing\} \sim_{\mathrm{g}} \mathfrak{B}, \{\varnothing\}$. For the empty team, however, the above definition says that $\mathfrak{A}, \varnothing \sim_{\mathrm{g}} \mathfrak{B}, \varnothing \not\sim_{\mathrm{g}} \mathfrak{B}, Y$ for any $Y \neq \varnothing$ and all τ-structures $\mathfrak{A}, \mathfrak{B}$. It readily follows from the definition that GF and $\mathsf{FO}^{\mathsf{hg}}$ are invariant under this notion of team bisimulation.

▶ **Proposition 11.** *If $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$ then $\mathfrak{A} \models_X \varphi \iff \mathfrak{B} \models_Y \varphi$ for any $\varphi \in \mathsf{GF}$, and $\mathfrak{A} \models^{\mathsf{hg}}_X \varphi \iff \mathfrak{B} \models^{\mathsf{hg}}_Y \varphi$ for every $\varphi \in \mathsf{FO}$.*

Beyond this essentially flat notion of guarded team bisimulation, there is a stronger one which focuses on the relational encoding of those teams as guarded relations.

▶ **Definition 12.** *Strong guarded team bisimulation equivalence*, $\mathfrak{A}, X \approx_{\mathrm{g}} \mathfrak{B}, Y$ is defined for teams $X \in \mathbb{H}(\mathfrak{A})$ and $Y \in \mathbb{H}(\mathfrak{B})$ with the same finite domain by the condition that $(\mathfrak{A}, [\![X]\!]) \sim_{\mathrm{g}} (\mathfrak{B}, [\![Y]\!])$, in terms of ordinary guarded bisimulation equivalence between the expansions of the two τ-structures by the relational encoding of the teams as $(\tau \dot\cup \{T\})$-structures for a new relation symbol T of the appropriate arity. Strong guarded team ℓ-bisimilarity, $\mathfrak{A}, X \approx^{\ell}_{\mathrm{g}} \mathfrak{B}, Y$, is analogously defined.

Obviously $\mathfrak{A}, X \approx_{\mathrm{g}} \mathfrak{B}, Y$ implies $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$, and $\mathfrak{A}, X \approx^{\ell+1}_{\mathrm{g}} \mathfrak{B}, Y$ implies $\mathfrak{A}, X \sim^{\ell}_{\mathrm{g}} \mathfrak{B}, Y$, for every $\ell \in \mathbb{N}$ (the formal offset of 1 in finite approximation levels is a consequence of the fact that $\approx^{0}_{\mathrm{g}}$ is trivial.) In particular, any team property that is $\sim_{\mathrm{g}}$ invariant, is also $\approx_{\mathrm{g}}$-invariant. We shall see in Sect. 7 that the converse fails in general.

5 The flatness of guarded team bisimulation

Horizontal guardedness is compatible with disjoint unions of relational structures (and teams). For any τ-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we have $\mathbb{G}(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \mathbb{G}(\mathfrak{A}_1) \cup \mathbb{G}(\mathfrak{A}_2)$ and hence also $\mathbb{H}(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \big\{X_1 \cup X_2 \colon X_i \in \mathbb{H}(\mathfrak{A}_i) \text{ for } i = 1, 2\big\}$.

Also, for any two pairs of τ-structures $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{B}_1, \mathfrak{B}_2$ with horizontally guarded teams $X_i \in \mathbb{H}(\mathfrak{A}_i)$ and $Y_i \in \mathbb{H}(\mathfrak{B}_i)$ such that $\mathfrak{A}_i, X_i \sim^\ell_{\mathrm{g}} \mathfrak{B}_i, Y_i$ for $i = 1, 2$, it follows that

$$(\mathfrak{A}_1 \oplus \mathfrak{A}_2), X_1 \cup X_2 \sim^\ell_{\mathrm{g}} (\mathfrak{B}_1 \oplus \mathfrak{B}_2), Y_1 \cup Y_2$$

and similarly for $\sim_{\mathrm{g}}$. The main point here is that every guarded assignment is fully contained in one component: correspondingly, whenever the first player makes a move in one of the disjoint unions that goes from one component to the other, so can the second player in the opposite structure, since the assumptions in particular guarantee $\mathfrak{A}_i \sim^\ell_{\mathrm{g}} \mathfrak{B}_i$ for the naked component structures.

Given a τ-structure $\mathfrak{A}$ and a team $X \in \mathbb{H}(\mathfrak{A})$, we write $\bigoplus_{s \in X}\big(\mathfrak{A}, \{s\}\big)$ for the disjoint union of copies of $\mathfrak{A}$ together with the singleton teams $\{s\}$ for the assignments $s \in X$. The following observation is immediate from the definition of guarded team bisimulation.

▶ **Proposition 13.** *For every structure $\mathfrak{A}$, we have $\mathfrak{A}, X \sim_{\mathrm{g}} \bigoplus_{s \in X}\big(\mathfrak{A}, \{s\}\big)$.*

As a further illustration of the interesting interplay between guarded team bisimulation, ordinary guarded bisimulation, flatness and downward closure, we may look at representatives for $\sim^\ell_{\mathrm{g}}$- or $\sim_{\mathrm{g}}$-classes that are built from singleton team configurations. We just formulate these considerations for $\sim^\ell_{\mathrm{g}}$, but the situation for $\sim_{\mathrm{g}}$ is analogous (as long as we are not concerned about definability in GF or $\mathsf{FO}^{\mathsf{hg}}$). Let $[\mathfrak{A}, X]^\ell$ denote the $\sim^\ell_{\mathrm{g}}$-class of the horizontally guarded team configuration $\mathfrak{A}, X$ within the class of τ-structures with horizontally guarded teams. Let $[\mathfrak{A}, s]^\ell$ similarly stand for the $\sim^\ell_{\mathrm{g}}$-class in the sense of ordinary guarded bisimulation within the class of τ-structures with guarded assignments. Then, for a singleton team $X = \{s\}$, we classically find

$$[\mathfrak{A}, s]^\ell = \big\{\mathfrak{B}, t \colon \mathfrak{B}, t \sim^\ell_{\mathrm{g}} \mathfrak{A}, s\big\} = \big\{\mathfrak{B}, t \colon \mathfrak{B}, Y \sim^\ell_{\mathrm{g}} \mathfrak{A}, \{s\},\ t \in Y\big\} = \big\{\mathfrak{B}, t \colon \mathfrak{B} \models \chi^\ell_{\mathfrak{A},s}[t]\big\}$$

where this definability relies on the characteristic formulae in the standard proof of the Ehrenfeucht–Fraïssé Theorem for GF, Theorem 9 above, and requires τ to be finite. On the other hand, in team terms,

$$\begin{aligned}[\mathfrak{A}, \{s\}]^\ell &= \big\{\mathfrak{B}, Y \colon \mathfrak{B}, Y \sim^\ell_{\mathrm{g}} \mathfrak{A}, \{s\}\big\} = \big\{\mathfrak{B}, Y \colon \mathfrak{B}, t \sim^\ell_{\mathrm{g}} \mathfrak{A}, s \text{ for all } t \in Y\big\} \\ &= \big\{\mathfrak{B}, Y \colon \mathfrak{B} \models_Y \chi^\ell_{\mathfrak{A},s}\big\},\end{aligned}$$

where the last equality relies on flatness (for the adequacy of the team semantic reading) and finiteness of τ (for the existence of the characteristic formulae χ, which we here use in negation normal form). For an arbitrary horizontally guarded team configuration $\mathfrak{A}, X$, correspondingly

$$\begin{aligned}[\mathfrak{A}, X]^\ell &= \big\{\mathfrak{B}, Y \colon \mathfrak{B}, Y \sim^\ell_{\mathrm{g}} \mathfrak{A}, X\big\} \\ &= \big\{\mathfrak{B}, \textstyle\bigcup_{s \in X} Y_s \colon Y_s \neq \varnothing \text{ and, for all } t \in Y_s, \mathfrak{B}, t \sim^\ell_{\mathrm{g}} \mathfrak{A}, s\big\} \\ &= \big\{\mathfrak{B}, Y \colon \mathfrak{B} \models_Y \textstyle\bigvee_{s \in X}\big(\mathsf{NE} \wedge \chi^\ell_{\mathfrak{A},s}\big)\big\}.\end{aligned}$$

Note that the nonemptiness condition indicates that (as expected) team guarded bisimulation equivalence does not respect downward closure. W.r.t. the last equality, finiteness of τ also ensures that the disjunction is finite up to logical equivalence.

The downward closure of $[\mathfrak{A}, X]^\ell$, on the other hand, has the simpler form

$$\begin{aligned} [\mathfrak{A}, X]^\ell \downarrow &= \{\mathfrak{B}, Y' \colon Y' \subseteq Y \text{ for some } \mathfrak{B}, Y \sim^\ell_{\mathsf{g}} \mathfrak{A}, X\} \\ &= \{\mathfrak{B}, \textstyle\bigcup_{s \in X} Y_s \colon \text{ for all } t \in Y_s, \mathfrak{B}, t \sim^\ell_{\mathsf{g}} \mathfrak{A}, s\} \\ &= \{\mathfrak{B}, Y \colon \mathfrak{B} \models_Y \textstyle\bigvee_{s \in X} \chi^\ell_{\mathfrak{A},s}\}, \end{aligned}$$

where definability according to the last equality again is only good for finite τ for the characteristic formulae $\chi^\ell_{\mathfrak{A},\bar{a}}(\bar{x}) \in \mathsf{GF}$. Their team semantic reading is again adequate by the flatness of $\mathsf{GF} \subseteq \mathsf{FO}$ and finiteness of τ also ensures that the disjunction is finite up to logical equivalence. As the team semantics for plain first-order logic $\mathsf{FO}(\tau)$ is flat, we also obtain the following observation.

▶ **Lemma 14.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures with teams X in A and Y in B such that for every $s \in X$ there is some $t \in Y$ such that $\mathfrak{A}, s \equiv^q_{\mathsf{FO}} \mathfrak{B}, t$, and vice versa. Then also $\mathfrak{A}, X \equiv^q_{\mathsf{FO}} \mathfrak{B}, Y$, i.e., the two team configurations are indistinguishable by $\mathsf{FO}(\tau)$-formulae $\varphi(\bar{x})$ of quantifier rank up to q.*

6 Expressive completeness for $\sim_{\mathsf{g}}$-invariance

The following upgrading of equivalences and semantic invariance conditions is a typical ingredient in model-theoretic proofs of expressive completeness of some (fragment of a) logic for some semantically characterised class of properties. Here we want to use it for team properties that do not distinguish between guarded team-bisimilar situations. In relating invariance of first-order definable team properties under full guarded team bisimilarity to invariance under a sufficiently fine finite approximation of ℓ-bisimilarity it also may be seen as a *compactness property*, albeit one which does not rely on the classical compactness theorem for first-order logic – as is amply demonstrated by the finite model theory reading.

Recall that, with a team X in A (with a particular fixed enumeration of its domain), we associate the relation $[\![X]\!] = \{s(\bar{x}) \colon s \in X\}$ over A. There are several natural notions of first-order equivalence (up to a given quantifier rank q or unrestricted) between τ-structures with teams. First,

$$\mathfrak{A}, X \equiv^q_{\mathsf{FO}} \mathfrak{B}, Y$$

holds if $\mathfrak{A}, X$ and $\mathfrak{B}, Y$ satisfy the same team properties that are definable in FO_q, FO with quantifier rank up to q, in terms of team semantics. Note that, due to flatness, $\mathfrak{A}, X \equiv^q_{\mathsf{FO}} \mathfrak{B}, Y$ just requires X and Y to agree on all those $\varphi(\bar{x}) \in \mathsf{FO}_q$ that are true or false across the whole team. Testing this for the characteristic formulae $\chi^q(\bar{x}) \in \mathsf{FO}_q$ (in negation normal form), which characterise the full FO_q-types of tuples over finite relational signatures, we see that $\mathfrak{A}, X \equiv^q_{\mathsf{FO}} \mathfrak{B}, Y$ implies that X and Y realise exactly the same FO_q-types of tuples. The seemingly stronger equivalence

$$\mathfrak{A}, X \equiv^q_{\mathsf{FO(non)}} \mathfrak{B}, Y$$

holds if $\mathfrak{A}, X$ and $\mathfrak{B}, Y$ are indistinguishable by $\mathsf{FO(non)}$-formulae of quantifier rank up to q. An analysis of the team semantics of $\mathsf{FO(non)}$ shows, however, that also $\mathfrak{A}, X \equiv^q_{\mathsf{FO(non)}} \mathfrak{B}, Y$ if, and only if, the two teams realise exactly the same FO_q-types of tuples. So these two notions of team equivalence coincide. (The flatness character of these team equivalences is analogous to that of guarded team bisimulation $\sim^\ell_g$; it similarly casts the classical notion of q-partial isomorphy $\simeq^q$ at the level of teams.)

$$(\mathfrak{A}, [\![X]\!]) \equiv^q_{\mathsf{FO}^T} (\mathfrak{B}, [\![Y]\!])$$

on the other hand refers to standard first-order semantics in the T-expansions and says that the $(\tau \dot\cup \{T\})$-structures $(\mathfrak{A}, [\![X]\!])$ and $(\mathfrak{B}, [\![Y]\!])$ satisfy the same first-order *sentences* up to quantifier rank q, which by the classical Ehrenfeucht–Fraïssé theorem for finite relational vocabularies is the same as q-partial isomorphy, $(\mathfrak{A}, [\![X]\!]) \simeq^q (\mathfrak{B}, [\![Y]\!])$. Simple Ehrenfeucht–Fraïssé arguments show that for singleton teams also this distinction does not matter. In specific situations below, this agreement can be extended further due to compatibility of $\simeq^q$ with disjoint sums of structures.

► **Lemma 15.** *For singleton teams $X = \{s\}$ in $\mathfrak{A}$ and $Y = \{t\}$ in $\mathfrak{B}$, with $s(\bar{x}) = \bar{a}$ and $t(\bar{x}) = \bar{b}$ such that $\mathfrak{A}, \bar{a} \equiv^q_{\mathsf{FO}} \mathfrak{B}, \bar{b}$, we also have $(\mathfrak{A}, X) = (\mathfrak{A}, \{s\}) \equiv^q_{\mathsf{FO(non)}} (\mathfrak{B}, \{t\}) = (\mathfrak{B}, Y)$ as well as $(\mathfrak{A}, [\![X]\!]) = (\mathfrak{A}, \{\bar{a}\}) \equiv^q_{\mathsf{FO}^T} (\mathfrak{B}, \{\bar{b}\}) = (\mathfrak{B}, [\![Y]\!])$.*

We next show that, for suitable ℓ, ordinary guarded team ℓ-bisimulation equivalence $\sim^\ell_{\mathsf{g}}$ can be upgraded to any one of these forms of first-order equivalence.

► **Lemma 16.** *For any ℓ that is sufficiently large in relation to q, any τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ with horizontally guarded teams $X \in \mathbb{H}(\mathfrak{A})$ and $Y \in \mathbb{H}(\mathfrak{B})$ such that $\mathfrak{A}, X \sim^\ell_{\mathsf{g}} \mathfrak{B}, Y$ admit $\sim_{\mathsf{g}}$-equivalent team configurations $\tilde{\mathfrak{A}}, \tilde{X} \sim_{\mathsf{g}} \mathfrak{A}, X$ and $\tilde{\mathfrak{B}}, \tilde{Y} \sim_{\mathsf{g}} \mathfrak{B}, Y$ such that simultaneously $\tilde{\mathfrak{A}}, \tilde{X} \equiv^q_{\mathsf{FO}} \tilde{\mathfrak{B}}, \tilde{Y}$, $\tilde{\mathfrak{A}}, \tilde{X} \equiv^q_{\mathsf{FO(non)}} \tilde{\mathfrak{B}}, \tilde{Y}$, and $(\tilde{\mathfrak{A}}, [\![\tilde{X}]\!]) \equiv^q_{\mathsf{FO}^T} (\tilde{\mathfrak{B}}, [\![\tilde{Y}]\!])$. Moreover, for finite $\mathfrak{A}$ and $\mathfrak{B}$, $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$ can be chosen finite.*

Proof. The proof essentially uses the flatness features of ordinary guarded team bisimulation $\sim_{\mathsf{g}}$ as expressed in Proposition 13 together with locality and flatness properties for first-order team semantics. This is combined with known model transformations that respect ordinary guarded bisimulation equivalence (guarded team bisimulation for singleton teams) and upgrade levels $\sim^\ell_{\mathsf{g}}$ to levels of ordinary first-order equivalence $\simeq^q$ or $\equiv^q_{\mathsf{FO}}$ between corresponding guarded tuples. The construction following [24] uses finite guarded bisimilar coverings, i.e., homomorphisms, with finite fibres, of the form $\pi\colon \mathfrak{A}^* \longrightarrow \mathfrak{A}$ such that for all guarded assignments s of $\mathfrak{A}^*$, $\mathfrak{A}^*, s \sim_{\mathsf{g}} \mathfrak{A}, \pi(s)$ due to natural guarded back-and-forth conditions for the map π. These finite coverings can be constructed such that, for any fixed level q there is a level ℓ such that for two such coverings $\pi\colon \mathfrak{A}^* \longrightarrow \mathfrak{A}$ and $\pi'\colon \mathfrak{B}^* \longrightarrow \mathfrak{B}$ and guarded assignments s and t,

$$(\dagger) \quad \mathfrak{A}, \pi(s) \sim^\ell_{\mathsf{g}} \mathfrak{B}, \pi(t) \quad \Rightarrow \quad \mathfrak{A}^*, s \simeq^q \mathfrak{B}^*, t.$$

The combined upgrading steps are illustrated in Figure 1. The first stage (1) corresponds to an application of Proposition 13, which scatters the members of the two teams so that no two members are in the same component. The second stage (2) is by means of guarded coverings according to the above. By (†) and Lemma 14 the resulting team configurations are in fact first-order team equivalent up to level q. For each individual matching pair of component structures $\mathfrak{A}^*, s(\bar{x}) \equiv^q_{\mathsf{FO}} \mathfrak{B}^*, t(\bar{x})$, by Lemma 15, the equivalence translates into the corresponding equivalence at the level of $\mathsf{FO(non)}$ as well as for the T-expansions: $\mathfrak{A}^*, s(\bar{x}) \simeq^q \mathfrak{B}^*, t(\bar{x})$ implies $(\mathfrak{A}^*, \{s(\bar{x})\}) \simeq^q (\mathfrak{B}^*, \{t(\bar{x})\})$ and the compositionality of the Ehrenfeucht–Fraïssé game under disjoint sums then shows the desired equivalences. ◄

► **Corollary 17.**

(a) *Let $\varphi(\bar{x}) \in \mathsf{FO}$ or $\varphi(\bar{x}) \in \mathsf{FO(non)}$ be invariant under guarded team bisimulation in the sense that $\mathfrak{A}, X \sim_{\mathsf{g}} \mathfrak{B}, Y$ implies $\mathfrak{A} \models_X \varphi \Leftrightarrow \mathfrak{B} \models_Y \varphi$ for any $X \in \mathbb{H}(\mathfrak{A})$ and $Y \in \mathbb{H}(\mathfrak{B})$. Then φ is in fact already invariant under guarded team ℓ-bisimulation $\sim^\ell_{\mathsf{g}}$ for some $\ell \in \mathbb{N}$, i.e. $\mathfrak{A}, X \sim^\ell_{\mathsf{g}} \mathfrak{B}, Y$ suffices to imply $\mathfrak{A} \models_X \varphi \Leftrightarrow \mathfrak{B} \models_Y \varphi$.*

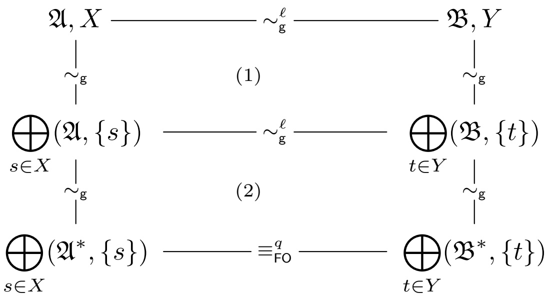

Figure 1 Upgrading of equivalences through structural transformations. The bottom rung simultaneously achieves q-equivalence at the level of $\mathsf{FO}(\mathsf{non})$ and FO^T (cf. Lemmas 15/16).

(b) *Any sentence* $\varphi \in \mathsf{FO}^T(\tau) = \mathsf{FO}(\tau \dot\cup \{T\})$ *that is invariant under guarded team bisimulation in the sense that* $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$ *implies* $(\mathfrak{A}, [\![X]\!]) \models \varphi \Leftrightarrow (\mathfrak{B}, [\![Y]\!]) \models \varphi$ *for any* $X \in \mathbb{H}(\mathfrak{A})$ *and* $Y \in \mathbb{H}(\mathfrak{B})$*, is in fact* $\sim_{\mathrm{g}}^{\ell}$*-invariant for suitable* ℓ *such that already* $\mathfrak{A}, X \sim_{\mathrm{g}}^{\ell} \mathfrak{B}, Y$ *implies* $(\mathfrak{A}, [\![X]\!]) \models \varphi \Leftrightarrow (\mathfrak{B}, [\![Y]\!]) \models \varphi$.

Both assertions also hold true in the sense of finite model theory, i.e., if both the assumption and the conclusion are limited to corresponding criteria for just finite structures $\mathfrak{A}$ *and* $\mathfrak{B}$.

Proof. In the diagram of Figure 1, φ is preserved along the vertical axes due to its preservation under guarded team bisimulation ($\sim_{\mathrm{g}}$-invariance). Overall, the detour through these guarded team bisimilar companions therefore shows that $\mathfrak{A} \models_X \varphi$ iff $\mathfrak{B} \models_Y \varphi$, or that $(\mathfrak{A}, [\![X]\!]) \models \varphi$ iff $(\mathfrak{B}, [\![Y]\!]) \models \varphi$, and thus establishes the desired $\sim_{\mathrm{g}}^{\ell}$-invariance. ◀

We are now ready to formulate two characterisation theorems for guarded team bisimulation and horizontally guarded team logics.

It follows from part (a) that any formula $\varphi(\bar{x}) \in \mathsf{FO}(\tau)$ that is invariant under ordinary guarded team bisimulation $\sim_{\mathrm{g}}$ is expressible in any logic with team semantic disjunction that is sufficiently expressive to define the classes

$$\begin{aligned} [\mathfrak{A}, \{s\}]^{\ell}\downarrow \;&=\; \big\{\mathfrak{B}, Y' \colon Y' \subseteq Y \text{ for some } \mathfrak{B}, Y \sim_{\mathrm{g}}^{\ell} \mathfrak{A}, \{s\}\big\} \\ &=\; \big\{\mathfrak{B}, Y \colon \mathfrak{B}, t \sim_{\mathrm{g}}^{\ell} \mathfrak{A}, s \text{ for all } t \in Y\big\} \end{aligned}$$

for every τ-structure $\mathfrak{A}$, every guarded assignment s, and every $\ell \in \mathbb{N}$. The reason is that in this case $\varphi(\bar{x})$, which we know to define both a flat team property and a $\sim_{\mathrm{g}}^{\ell}$-closed team property for suitable ℓ, is then logically equivalent to the formula $\bigvee\{\chi_{\mathfrak{A},s}^{\ell} \colon \mathfrak{A}, \{s\} \models \varphi\}$. Here the formulae $\chi_{\mathfrak{A},s}^{\ell} \in \mathsf{GF}(\tau)$ are from the proof of Theorem 9 and define, in team semantics, the downward closures of the $\sim_{\mathrm{g}}$-equivalence classes $[\mathfrak{A}, \{s\}]^{\ell}$. This conclusion holds true, in particular, for the logics $\mathsf{FO}(\tau)$ with horizontally guarded team semantics and for $\mathsf{GF}(\tau)$ with (horizontally guarded or general) team semantics. In other words, if $\mathcal{C}$ is a class of team configurations $\mathfrak{A}, X$ which is closed under $\sim_{\mathrm{g}}^{\ell}$ and downward closed and union closed for teams, then $\mathcal{C} = \big\{\mathfrak{B}, Y \colon \mathfrak{B} \models_Y \bigvee_{\mathfrak{A},X\in\mathcal{C}} \bigvee_{s\in X} \big(\chi_{\mathfrak{A},s}^{\ell}(\bar{x})\big)\big\}$, and, in terms of FO^T or GF^T,

$$\mathcal{C} = \Big\{\mathfrak{B}, Y \colon (\mathfrak{B}, [\![Y]\!]) \models \bigvee_{\mathfrak{A},X\in\mathcal{C}} \forall\bar{x}\Big(T\bar{x} \rightarrow \bigvee_{s\in X} \chi_{\mathfrak{A},s}^{\ell}(\bar{x})\Big)\Big\}.$$

▶ **Corollary 18** (First Characterisation Theorem for Guarded Team Semantics)**.**

$$\mathsf{FO}/{\sim_{\mathrm{g}}} \equiv \mathsf{FO}^{\mathsf{hg}} \equiv \mathsf{GF} \equiv [\mathsf{GF}]^T \equiv [\mathsf{FO}]^T/{\sim_{\mathrm{g}}}.$$

This, and the following characterisation theorem, are to be read in the sense that these logical formalisms define exactly the same properties of horizontally guarded teams. Recall that, for any $L \subseteq \mathsf{FO}$, $[L]^T$ denotes the set of all sentences $\forall \bar{x}(T\bar{x} \to \varphi(\bar{x}))$ (with classical Tarski semantics) such that $\varphi(\bar{x}) \in L$, so the last two equivalences follow from classical results. Similarly, part (b) of Corollary 17 shows the following for any team property expressed by a sentence $\varphi \in \mathsf{FO}(\tau \,\dot\cup\, \{T\})$ in the classical first-order semantics with appeal to the relational encoding $[\![X]\!]$ of teams X (which does not imply downward closure or flatness!). If φ is invariant under guarded team bisimulation $\sim_{\mathsf{g}}$, then it is equivalently expressible in any logic with ordinary disjunction that is sufficiently expressive to define all unions of classes

$$\begin{aligned}[][\mathfrak{A}, X]^{\ell} &= \big\{\mathfrak{B}, Y \colon \mathfrak{B}, Y \sim^{\ell}_{\mathsf{g}} \mathfrak{A}, X\big\} \\ &= \big\{\mathfrak{B}, \textstyle\bigcup_{s \in X} Y_s \colon Y_s \neq \varnothing \text{ and, for all } t \in Y_s, \mathfrak{B}, t \sim^{\ell}_{\mathsf{g}} \mathfrak{A}, s\big\},\end{aligned}$$

for every fixed $\ell \in \mathbb{N}$. As we saw above, the equivalence classes $[\mathfrak{A}, X]^{\ell}$ are definable, for instance in the extension of $\mathsf{FO}^{\mathsf{hg}}$ or GF by nonemptiness NE by formulae

$$\chi^{\ell}_{\mathfrak{A},X}(\bar{x}) = \bigvee_{s \in X} \big(\mathsf{NE} \wedge \chi^{\ell}_{\mathfrak{A},s}(\bar{x})\big) \equiv \bigvee_{s \in X} \big(\mathsf{non} \perp \wedge\, \chi^{\ell}_{\mathfrak{A},s}(\bar{x})\big)$$

derived from the classical characteristic formulae $\chi^{\ell}_{\mathfrak{A},s}$. In order to define the union of (finitely many) such classes, however, we need to invoke the strong intuitionistic disjunction $\otimes$ at the propositional level, as ordinary team disjunction would allow to mix team constituents (corresponding to unions of teams rather than an alternative between them). Now $\mathsf{FO}(\mathsf{non})^{\mathsf{hg}}$ and $\mathsf{GF}(\mathsf{non})$ are invariant under $\sim_{\mathsf{g}}$ by Proposition 11, and sufficiently expressive for the $\chi^{\ell}_{\mathfrak{A},s}(\bar{x})$ as well as to express nonemptiness ($\mathsf{NE} \equiv \mathsf{non} \perp$) and intuitionistic disjunction ($\varphi_1 \otimes \varphi_2 \equiv \mathsf{non}\,(\mathsf{non}\,\varphi_1 \wedge \mathsf{non}\,\varphi_2)$). If $\mathcal{C}$ is a class of team configurations $\mathfrak{A}, X$ composed of τ-structures $\mathfrak{A}$ for fixed finite τ, with teams $X \in \mathbb{H}(\mathfrak{A})$, which is $\sim^{\ell}_{\mathsf{g}}$-closed, then $\mathcal{C} = \big\{\mathfrak{B}, Y \colon \mathfrak{B} \models_Y \bigotimes_{\mathfrak{A},X \in \mathcal{C}} \bigvee_{s \in X} \big(\mathsf{NE} \wedge \chi^{\ell}_{\mathfrak{A},s}(\bar{x})\big)\big\}$, and in terms of FO^T or GF^T,

$$\mathcal{C} = \Big\{\mathfrak{B}, Y \colon (\mathfrak{B}, [\![Y]\!]) \models \bigvee_{\mathfrak{A},X \in \mathcal{C}} \left(\begin{array}{l} \bigwedge_{s \in X} \exists \bar{x}\big(T\bar{x} \,\wedge\, \chi^{\ell}_{\mathfrak{A},s}(\bar{x})\big) \\ \wedge\, \forall \bar{x}\big(T\bar{x} \to \bigvee_{s \in X} \chi^{\ell}_{\mathfrak{A},s}(\bar{x})\big) \end{array} \right) \Big\}.$$

▶ **Corollary 19** (Second Characterisation Theorem for Guarded Team Semantics)**.**

$$\mathsf{FO}^T/{\sim_{\mathsf{g}}} \equiv \mathsf{GF}^T/{\sim_{\mathsf{g}}} \equiv \mathsf{FO}^{\mathsf{hg}}(\mathsf{non}) \equiv \mathsf{GF}(\mathsf{non}) \equiv \mathsf{FO}(\mathsf{non})/{\sim_{\mathsf{g}}}.$$

It is tempting to assume that this is also equivalent to GF^T, i.e. to sentences $\psi(T) \in \mathsf{GF}(\tau \,\dot\cup\, \{T\})$ (with Tarski semantics). But this is not the case. As we shall see below, GF^T is only $\approx_{\mathsf{g}}$-invariant, and not $\sim_{\mathsf{g}}$-invariant.

7 Invariance under strong guarded team bisimulation

Recall that the strong guarded bisimulation equivalence $\mathfrak{A}, X \approx^{\ell}_{\mathsf{g}} \mathfrak{B}, Y$ is based, by definition, on ordinary guarded bisimulation equivalence between the expansions of the underlying τ-structures by the relational encodings of the teams, $(\mathfrak{A}, [\![X]\!]) \sim_{\mathsf{g}} (\mathfrak{B}, [\![Y]\!])$. It is therefore clear that $\mathsf{GF}^T(\tau)$, the classical guarded fragment applied to these same expansions, is preserved under $\approx_{\mathsf{g}}$. The classical characterisation theorem of Andréka, van Benthem and Németi as well as its finite model theory analogue from [24] can thus be phrased as follows.

▶ **Proposition 20.** $\mathsf{FO}^T/{\approx_{\mathsf{g}}} \equiv \mathsf{GF}^T$ *(classically and in fmt).*

Note that these formalisms are more expressive than $\mathsf{FO}/{\approx_{\mathrm{g}}}$ and GF, since they can obviously express invariant team properties that are neither flat nor downward- or union-closed.

Unlike its plain counterpart, strong bisimulation is a priori not a flat notion, and it is in fact strictly stronger than $\sim_g$. This is also witnessed by some important and familiar atomic team properties.

▶ **Proposition 21.** *Inclusion and exclusion dependencies (as well as their strong negations) are* GF^T*-definable, and thus* $\approx_{\mathrm{g}}$*-invariant. However, they are not* $\sim_{\mathrm{g}}$*-invariant.*

Proof. For a team X with variables $x, y, \bar{z}$, we have

$$\mathfrak{A} \models_X (x \subseteq y) \iff (\mathfrak{A}, X) \models \forall xy\bar{z}(Txy\bar{z} \to \exists u\bar{v}Tux\bar{v})$$
$$\mathfrak{A} \models_X (x \mid y) \iff (\mathfrak{A}, X) \models \forall xy\bar{z}(Txy\bar{z} \to \neg\exists u\bar{v}Tux\bar{v}).$$

This extends in the obvious way to general inclusion and exclusion atoms between arbitrary tuples of variables to show that these are definable in GF^T.

To prove that exclusion atoms are not $\sim_{\mathrm{g}}$-invariant, consider a graph $\mathfrak{A}$ with vertices a, b, c and edges (a, b) and (b, c), and the assignments $s : (x, y) \mapsto (a, b)$ and $s' : (x, y) \mapsto (b, c)$. On the other side let $\mathfrak{B}$ be the graph with vertices u, v, u', v', w and edges $(u, v), (u', v'), (v, w), (v', w)$ with assignments $t : (x, y) \mapsto (u, v)$ and $t' : (x, y) \mapsto (v', w)$. Clearly $\mathfrak{A}, s \sim_{\mathrm{g}} \mathfrak{B}, t$ and $\mathfrak{A}, s' \sim_{\mathrm{g}} \mathfrak{B}, t'$. For the teams $X = \{s, s'\}$ and $Y = \{t, t'\}$ we thus have that $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$. However, $\mathfrak{B} \models_Y (x \mid y)$ but $\mathfrak{A} \not\models (x \mid y)$. Notice, however, that $\mathfrak{A}, X \not\approx_{\mathrm{g}} \mathfrak{B}, Y$, and indeed, even $\mathfrak{A}, X \approx^2_{\mathrm{g}} \mathfrak{B}, Y$ fails as the second player has no valid response if the first player makes a move in $(\mathfrak{A}, [\![X]\!])$ from the assignment $s : (x, y) \mapsto (a, b) \in X$ to $s'' : (y, z) \mapsto (b, c)$. An almost identical argument applies to inclusion atoms. ◀

On the other side, the two notions of bisimulation invariance coincide as far as flat team properties are concerned. To prove this, we use the notion of guarded tree decompositions cf. [14, 23, 24], which are available in the tree unravellings induced by guarded bisimulations.

▶ **Proposition 22.** *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are guarded tree-decomposable, then, for any two guarded assignments* s *and* t*, we have* $\mathfrak{A}, s \sim_{\mathrm{g}} \mathfrak{B}, t \;\Rightarrow\; \mathfrak{A}, \{s\} \approx_{\mathrm{g}} \mathfrak{B}, \{t\}$.

Proof. The second player can use a strategy whose trace in the underlying tree-like transition system of guarded configurations (induced by the guarded tree decompositions, which are themselves linked by a bisimulation), respects distances from the roots $s(\bar{x})$ and $t(\bar{x})$. ◀

▶ **Proposition 23.** *Properties of guarded teams that are flat and* $\approx_{\mathrm{g}}$*-invariant are in fact also* $\sim_{\mathrm{g}}$*-invariant.*

Proof. Assume that φ defines a flat property of guarded teams that is preserved under $\approx_{\mathrm{g}}$, and let $\mathfrak{A}, X \sim_{\mathrm{g}} \mathfrak{B}, Y$. We need to show that $\mathfrak{A} \models_X \varphi$ if, and only if, $\mathfrak{B} \models_Y \varphi$. Assume towards a contradiction that $\mathfrak{A} \models_X \varphi$ while $\mathfrak{B} \not\models_Y \varphi$. Consider the guarded tree unfoldings $(\mathfrak{A}^*, [\![X]\!]^*)$ and $(\mathfrak{B}^*, [\![Y]\!]^*)$ of $(\mathfrak{A}, [\![X]\!])$ and $(\mathfrak{B}, [\![Y]\!])$, respectively. Clearly

$$(\mathfrak{A}^*, [\![X]\!]^*) \sim_{\mathrm{g}} (\mathfrak{A}, [\![X]\!]) \;\text{ and }\; (\mathfrak{B}^*, [\![Y]\!]^*) \sim_{\mathrm{g}} (\mathfrak{B}, [\![Y]\!])$$

imply that $\mathfrak{A}^*, X^* \approx_{\mathrm{g}} \mathfrak{A}, X$ and $\mathfrak{B}^*, Y^* \approx_{\mathrm{g}} \mathfrak{B}, Y$ for the associated 'unfolded teams' X^* and Y^*. By $\approx_{\mathrm{g}}$-invariance, therefore, $\mathfrak{A}^* \models_{X^*} \varphi$ and $\mathfrak{B}^* \not\models_{Y^*} \varphi$. By flatness of φ this implies on one hand that $\mathfrak{B}^* \not\models_{\{t\}} \varphi$ for some $t \in Y^*$. On the other hand there is some $s \in X^*$ such that $\mathfrak{A}^*, s \sim_{\mathrm{g}} \mathfrak{B}^*, t$ for which, also by flatness, $\mathfrak{A}^* \models_{\{s\}} \varphi$. So $\mathfrak{A}^*, s \sim_{\mathrm{g}} \mathfrak{B}^*, t$ while $\mathfrak{A}^* \models_{\{s\}} \varphi$ but $\mathfrak{B}^* \not\models_{\{t\}} \varphi$. In view of Proposition 22, this contradicts $\approx_{\mathrm{g}}$-invariance of φ. ◀

▶ **Corollary 24.** $\mathsf{FO}/{\sim_\mathsf{g}} \equiv \mathsf{FO}/{\approx_\mathsf{g}}$ *(classically, open in fmt).*

On the other hand, the interplay of $\approx_\mathsf{g}$- or $\approx^\ell_\mathsf{g}$-invariance with team semantic constructs in stronger logics than FO is far less clear-cut. Clearly (team) conjunction and strong negation preserve $\approx_\mathsf{g}$-invariance. However, the following example shows that $\approx_\mathsf{g}$-invariance is not compatible with team disjunction, not even for atomic team properties.

▶ **Proposition 25.** *The formula $(x \mid y) \vee (x \mid y)$ is not $\approx_\mathsf{g}$-invariant.*

Proof. Let C_n be the directed cycle of length n and let X_n be the team of all its edges. The formula $(x \mid y) \vee (x \mid y)$ says that the team can be split in a bipartite manner. We thus have that $C_n \models_{X_n} (x \mid y) \vee (x \mid y)$ if, and only if, n is even. On the other side, the guarded assignments on graphs are just singletons, edges, and inverse edges, so obviously, $(C_n, X_n) \approx_\mathsf{g} (C_m, X_m)$ for all $m, n > 2$. ◀

Together with Proposition 21, this example shows that GF^T is in particular not closed under team disjunction.

From the characterisation theorem for guarded fixed-point logic $\mu\mathsf{GF}$ in [14] it follows that $\mu\mathsf{GF}^T \equiv \mathrm{GSO}/{\approx_\mathsf{g}}$, for guarded second-order logic GSO. The question arises whether we can also obtain a characterisation theorem that relates guarded inclusion logic with (a fragment of) guarded fixed-point logic. From Proposition 7 we conclude that $\mathsf{FO}^{\mathsf{hg}}(\subseteq) \equiv \mathsf{GF}(\subseteq)$, and this can be translated, by [10], into sentences in $\mu\mathsf{GF}^T$ of the form $\forall\bar{x}(T\bar{x} \rightarrow \psi(T, \bar{x}))$ where $\psi(T, \bar{x})$ has only greatest fixed points and is positive in T.

Question: Is this fragment equivalent with $\mathsf{GF}(\subseteq)$ and/or $\mathsf{FO}(\subseteq)/\approx_\mathsf{g}$?

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