On a Theorem of Lovász that $\hom(\cdot, H)$ Determines the Isomorphism Type of H

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Abstract

Graph homomorphism has been an important research topic since its introduction [14]. Stated in the language of binary relational structures in that paper [14], Lovász proved a fundamental theorem that the graph homomorphism function $G \mapsto \hom(G, H)$ for 0-1 valued H (as the adjacency matrix of a graph) determines the isomorphism type of H. In the past 50 years various extensions have been proved by Lovász and others [15, 9, 1, 19, 17]. These extend the basic 0-1 case to admit vertex and edge weights; but always with some restrictions such as all vertex weights must be positive. In this paper we prove a general form of this theorem where H can have arbitrary vertex and edge weights. An innovative aspect is that we prove this by a surprisingly simple and unified argument. This bypasses various technical obstacles and unifies and extends all previous known versions of this theorem on graphs. The constructive proof of our theorem can be used to make various complexity dichotomy theorems for graph homomorphism *effective*, i.e., it provides an algorithm that for any H either outputs a P-time algorithm solving $\hom(\cdot, H)$ or a P-time reduction from a canonical #P-hard problem to $\hom(\cdot, H)$.

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1 Introduction

More than 50 years ago the concept of graph homomorphism was introduced [14, 13]. Given two graphs G and H, a mapping from $V(G)$ to $V(H)$ is called a homomorphism if every edge of G is mapped to an edge of H. The graphs G and H can be either both directed or undirected. Presented in the language of binary relational structures, Lovász proved in that paper [14] the following fundamental theorem about graph homomorphism: If H and H' are two graphs, then they are isomorphic iff they define the same counting graph homomorphism

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function, namely, for every G, the number of homomorphisms from G to H is the same as that from G to H'. This number is denoted by $\hom(G, H)$. (Formal definitions are in Section 2.)

In [14] the graph H is a 0-1 adjacency matrix; there are no vertex and edge weights. In [9] Freedman, Lovász and Schrijver define a weighted version of the homomorphism function $\hom(\cdot, H)$, where H has *positive* vertex weights and *real* edge weights. The paper [9] investigates what graph properties can be expressed as such graph homomorphism functions. They gave a necessary and sufficient condition for this expressibility. This work has been extended to the case with arbitrary vertex and edge weights in a field [5], and to "edge models", e.g., [20, 18]. A main technical tool introduced in [9] is the so-called graph algebras. In [15] Lovász further investigated these graph algebras and proved precise bounds for their dimensions. These dimensions are a quantitative account of the space of all isomorphisms from H to H'. They are expressed in a theory of labeled graphs. Schrijver [19] studied the function $\hom(\cdot, H)$ where H is an undirected graph with complex edge weights (but all vertex weights are restricted to 1). He also gave a characterization of a graph property expressible in this form, and proved that $\hom(\cdot, H) = \hom(\cdot, H')$ implies that $H \cong H'$ for undirected graphs with complex edge weights (but unit vertex weights). Regts in [18], in addition to finding interesting connections between edge-coloring models and invariants of the orthogonal group, also proved multiple theorems in the framework of graph homomophisms (corresponding to "vertex models") requiring that all (nonempty) sums of vertex weights be nonzero. The possibility that vertex weights may sum to zero has been a difficult point. Our main result is to extend this isomorphism theorem to both directed and undirected graphs with arbitrary vertex and edge weights. We also determine the precise values of the dimensions of the corresponding graph algebras. A variant of our result, in terms of dimensions of associated algebras was proved by Goodall, Regts and Vena [11]; please see Remark 12 in Section 5.

To prove our theorem, we introduce a surprisingly simple and completely elementary argument, which we call the *Vandermonde Argument*. All of our results are proved by this one technique.

Two vertices i and j in an unweighted graph H are called *twins* iff the neighbor sets of i and j are identical. For weighted graphs, i and j are called twins iff the edge weights $\beta(i, k) = \beta(j, k)$ (and for directed graphs also $\beta(k, i) = \beta(k, j)$) for all k. In order to identify the isomorphism class of H, a natural step is to combine twin vertices. This creates a super vertex with a combined vertex weight (even when originally all vertices are unweighted, i.e., have weight 1). After this "twin reduction" step, our isomorphism theorem can be stated. The following is a simplified form:

▶ **Theorem 1.** *Let $\mathbb{F}$ be a field of characteristic 0. Let H and H' be (directed or undirected) weighted graphs with arbitrary vertex and edge weights from $\mathbb{F}$. Without loss of generality all individual vertex weights are nonzero. Suppose H and H' are twin-free. If for all simple graphs G (i.e., without loops and multiedges),*

$$\hom(G, H) = \hom(G, H'), \tag{1}$$

then the graphs H and H' are isomorphic as weighted graphs, i.e., there is a bijective map from H to H' that preserves all vertex and edge weights.

Theorem 1 is the special case of $k = 0$ of the more general Theorem 2 which deals with k-labeled graphs. In Section 8 we also determine the dimensions of the corresponding graph algebras in terms of the rank of the so-called connection tensors, introduced in [5]. These improve the corresponding theorems in [15, 19, 18] as follows.

From the main theorem (Theorem 2.2) of [15] we generalize from positive vertex weights and real edge weights to arbitrary weights. The main technique in [15] is algebraic. The proof relies on the notion of quantum graphs and structures built from them, and uses idempotent elements in the graph algebras. Similarly, from the isomorphism theorem in [19] we generalize from unit vertex weights and complex edge weights to arbitrary weights. Also we allow directed and undirected weighted graphs H. Theorem 2 also weakens the condition (1) on G to simple graphs (i.e., no multiedges or loops). Schrijver's proof technique is different from that of Lovász [15], but is also algebraic and built on quantum graphs. He uses a Reynolds operator and the Möbius transform (of a graph). The results of Lovász [15] and Schrijver [19] are incomparable. While requiring all vertex weights to be positive is not unreasonable, it is nonetheless a severe restriction, and has been a technical obstacle to all existing proofs. In Regts' thesis [18], multiple theorems were proved with the explicit requirement that all (nonempty) sums of vertex weights be nonzero, which circumvented this issue. In this paper, we allow arbitrary vertex weights with no assumptions. In particular, H can have arbitrary complex vertex and edge weights.

However, more than the explicit strengthening of the theorems, we believe the most innovative aspect of this work is that we found a direct elementary argument that bypassed various technical obstacles and unified all previously known versions. We can also show that the only restriction – $\mathbb{F}$ has characteristic 0 – cannot be removed, and thus our results are the most general extensions on graphs. We give counterexamples for fields of finite characteristic in Section 7.

This line of work has already led to significant applications in the graph limit literature, such as on quasi-random graphs [16]. In [17] Lovász and B. Szegedy also studied these graph algebras where "contractors" and "connectors" are used. In our treatment these "contractors" and "connectors" can also be constructed with simple graphs.

In terms of applications to complexity theory, there has been a series of significant complexity dichotomy theorems on counting graph homomorphisms which show that the function $\hom(\cdot, H)$ is either P-time computable or #P-hard, *depending* on H [7, 8, 2, 12, 21, 10, 4, 6, 3]. These theorems differ in the scope of what types of H are allowed, from 0-1 valued to complex valued, from undirected to directed. In all these theorems a P-time tractability condition on H is given, such that if H satisfies the condition then $\hom(\cdot, H)$ is P-time computable, otherwise $\hom(\cdot, H)$ is #P-hard. In the latter case, the theorem asserts that there is a P-time reduction from a canonical #P-hard problem to $\hom(\cdot, H)$. *However*, various *pinning lemmas* are proved nonconstructively; for undirected complex weighted graphs [4] it was unknown how to make this constructive. Consequently, there was no known algorithm to produce a #P-hardness reduction from H. Because the proof in this paper is constructive, it can be applied as a crucial step in making all these dichotomy theorems *effective*, i.e., we can obtain an algorithm that for any H either outputs a P-time algorithm solving $\hom(\cdot, H)$ or a P-time reduction from a canonical #P-hard problem to $\hom(\cdot, H)$.

2 Preliminaries

We first recap the notion of weighted graph homomorphisms [9], but state it for an arbitrary field $\mathbb{F}$. We let $[k] = \{1, \ldots, k\}$ for integer $k \geq 0$. In particular, $[0] = \emptyset$. By covention $\mathbb{F}^0 = \{\emptyset\}$, and $0^0 = 1$ in $\mathbb{Z}$, $\mathbb{F}$, etc. Often we discuss both directed and undirected graphs together.

An ($\mathbb{F}$-)*weighted graph* H is a finite (di)graph with a weight $\alpha_H(i) \in \mathbb{F} \setminus \{0\}$ associated with each vertex i (0-weighted vertices can be deleted) and a weight $\beta_H(i, j) \in \mathbb{F}$ associated with each edge ij (or loop if $i = j$). For undirected graphs, $\beta_H(i, j) = \beta_H(j, i)$. It is convenient to assume that H is a complete graph with a loop at all nodes by adding all missing edges and loops with weight 0. Then H is described by an integer $q = |V(H)| \geq 0$ (H can be the empty graph), a nowhere zero vector $\alpha = (\alpha_H(1), \dots, \alpha_H(q)) \in \mathbb{F}^q$ and a matrix $B = (\beta_H(i,j)) \in \mathbb{F}^{q \times q}$. An isomorphism from H to H' is a bijective map from $V(H)$ to $V(H')$ that preserves vertex and edge weights.

According to [9], let G be an unweighted graph (with possible multiple edges, but no loops) and H a weighted graph given by (α, B), we define

$$\begin{aligned} \hom(G,H) &= \sum_{\phi\colon V(G)\to V(H)} \alpha_\phi \hom_\phi(G,H) \\ &= \sum_{\phi\colon V(G)\to V(H)} \prod_{u\in V(G)} \alpha_H(\phi(u)) \prod_{uv\in E(G)} \beta_H(\phi(u),\phi(v)). \end{aligned} \tag{2}$$

The unweighted case is when all node weights are 1 and all edge weights are 0-1 in H, and $\hom(G,H)$ is the number of homomorphisms from G into H.

A k-labeled graph ($k \geq 0$) is a finite graph in which k nodes are labeled by $1, 2, \dots, k$ (the graph can have any number of unlabeled nodes). Two k-labeled graphs are isomorphic if there is a label-preserving isomorphism between them. U_k denotes the k-labeled graph on k nodes with no edges. In particular, U_0 is the empty graph with no nodes and no edges. The *product* of two k-labeled graphs G_1 and G_2 is defined as follows: take their disjoint union, and then identify nodes with the same label. Hence for two 0-labeled graphs, $G_1G_2 = G_1 \sqcup G_2$ (disjoint union). Clearly, the graph product is associative and commutative with the identity U_k, so the set of all (isomorphism classes) of k-labeled graphs togegher with the product operation forms a commutative monoid which we denote by $\mathcal{PLG}[k]$. We denote by $\mathcal{PLG}^{\text{simp}}[k]$ the submonoid of simple graphs in $\mathcal{PLG}[k]$; these are graphs with no loops, at most one edge between any two vertices i and j, and no edge between labeled vertices. A directed labeled graph is simple if its underlying undirected one is simple; in particluar, for any i and j, we require that if $i \to j$ is an edge then $j \to i$ is not an edge. Clearly, $\mathcal{PLG}^{\text{simp}}[k]$ is closed under the product operation (for both directed and undirected types).

Fix a weighted graph $H = (\alpha, B)$. For any k-labeled graph G and mapping $\psi\colon [k] \to V(H)$, let

$$\hom_\psi(G,H) = \sum_{\substack{\phi\colon V(G)\to V(H)\\ \phi \text{ extends } \psi}} \frac{\alpha_\phi}{\alpha_\psi} \hom_\phi(G,H), \tag{3}$$

where ϕ extends ψ means that if $u_i \in V(G)$ is labeled by $i \in [k]$ then $\phi(u_i) = \psi(i)$, and $\alpha_\psi = \prod_{i=1}^k \alpha_H(\psi(i))$, $\alpha_\phi = \prod_{v\in V(G)} \alpha_H(\phi(v))$, so $\frac{\alpha_\phi}{\alpha_\psi}$ is the product of vertex weights of α_ϕ *not* in α_ψ. Then

$$\hom(G,H) = \sum_{\psi\colon [k]\to V(H)} \alpha_\psi \hom_\psi(G,H). \tag{4}$$

When $k = 0$, we only have the empty map $\emptyset$ with the domain $\emptyset$. Then $\hom(G,H) = \hom_\emptyset(G,H)$ for every $G \in \mathcal{PLG}[k]$. The functions $\hom_\psi(\cdot, H)$ where $\psi\colon [k] \to V(H)$ and $k \geq 0$ satisfy

$$\begin{cases} \hom_\psi(G_1G_2,H) = \hom_\psi(G_1,H)\hom_\psi(G_2,H), & G_1, G_2 \in \mathcal{PLG}[k], \\ \hom_\psi(U_k,H) = 1. \end{cases} \tag{5}$$

Given a directed or undirected $\mathbb{F}$-weighted graph H, we call two vertices $i, j \in V(H)$ *twins* if for every vertex $\ell \in V(H)$, $\beta_H(i,\ell) = \beta_H(j,\ell)$ and $\beta_H(\ell,i) = \beta_H(\ell,j)$. Note that the vertex weights $\alpha_H(w)$ do not participate in this definition. If H has no twins, we call it twin-free.

The twin relation partitions $V(H)$ into nonempty equivalence classes, $I_1, \dots, I_s$ where $s \geq 0$. We can define a *twin contraction* graph $\widetilde{H}$, having $I_1, \dots, I_s$ as vertices, with vertex weight $\sum_{t \in I_r} \alpha_H(t)$ for I_r, and edge weight from I_r to I_q to be $\beta_H(u, v)$ for some arbitrary $u \in I_r$ and $v \in I_q$. After that, we remove all vertices in $\widetilde{H}$ with zero vertex weights together with all incident edges (still called $\widetilde{H}$). This defines a twin-free $\widetilde{H}$. Clearly, $\hom(G, H) = \hom(G, \widetilde{H})$ for all G.

We denote by $\mathrm{Isom}(H, H')$ the set of $\mathbb{F}$-weighted graph isomorphisms from H to H' and by $\mathrm{Aut}(H)$ the group of ($\mathbb{F}$-weighted) graph automorphisms of H.

It is obvious that for directed (or undirected) $\mathbb{F}$-weighted graphs H and H', and the maps $\varphi\colon [k] \to V(H)$ and $\psi\colon [k] \to V(H')$ such that $\psi = \sigma \circ \varphi$ for some isomorphism $\sigma\colon V(H) \to V(H')$ from H to H', we have $\hom_\varphi(G, H) = \hom_\psi(G, H')$ for every $G \in \mathcal{PLG}[k]$.

3 Our results

Theorem 1 is a direct consequence of the case $k = 0$ of the following Main Theorem.

▶ **Theorem 2.** *Let $\mathbb{F}$ be a field of characteristic 0. Let H, H' be (directed or undirected) $\mathbb{F}$-weighted graphs such that H is twin-free and $m = |V(H)| \geq m' = |V(H')|$. Suppose $\varphi\colon [k] \to V(H)$ and $\psi\colon [k] \to V(H')$ where $k \geq 0$. If $\hom_\varphi(G, H) = \hom_\psi(G, H')$ for every $G \in \mathcal{PLG}^{\mathrm{simp}}[k]$ then there exists an isomorphism of $\mathbb{F}$-weighted graphs $\sigma\colon V(H) \to V(H')$ from H to H' such that $\psi = \sigma \circ \varphi$ (a fortiori, H' is twin-free and $m = m'$).*

In Section 8 we will give our results about the space of such isomorphisms, expressed in terms of the dimensions of the corresponding graph algebras.

In Corollaries 3 to 6, $\operatorname{char} \mathbb{F} = 0$. The following two corollaries extend Lovász's theorem (and lemmas that are of independent interest) in [15] from real edge weight and positive vertex weight. Furthermore, it holds for both directed and undirected graphs, and the condition on G is weakened so that it is sufficient to assume it for simple graphs only. The fact that the theorem holds under the condition $\hom(G, H) = \hom(G, H')$ for loopless graphs G is important in making the complexity dichotomies *effective* in the sense defined in Section 1.

Theorem 2 is stated in a technical way where we only assume H is twin-free and $|V(H)| \geq |V(H')|$. Corollary 3 is a symmetric statement.

▶ **Corollary 3.** *Let H, H' be (directed or undirected) $\mathbb{F}$-weighted twin-free graphs. Let $\varphi\colon [k] \to V(H)$ and $\psi\colon [k] \to V(H')$ where $k \geq 0$. If $\hom_\varphi(G, H) = \hom_\psi(G, H')$ for every $G \in \mathcal{PLG}^{\mathrm{simp}}[k]$, then there exists an isomorphism σ from H to H' such that $\psi = \sigma \circ \varphi$.*

▶ **Corollary 4.** *Let H, H' be $\mathbb{F}$-weighted twin-free graphs, either both directed or both undirected. If $\hom(G, H) = \hom(G, H')$ for every* simple *graph G, then H and H' are isomorphic as $\mathbb{F}$-weighted graphs.*

For edge weighted graphs with unit vertex weight, the requirement of twin-freeness can be dropped. The following two corollaries directly generalize Schrijver's theorem (Theorem 2) in [19]. Corollary 6 is a restatement of Corollary 5 using the terminology in [19]. Here we strengthen his theorem by requiring the condition $\hom(G, H) = \hom(G, H')$ for only simple graphs G. Also our result holds for fields $\mathbb{F}$ of characteristic 0 generalizing from $\mathbb{C}$, and for directed as well as undirected graphs.

▶ **Corollary 5.** *Let H, H' be (directed or undirected) $\mathbb{F}$-edge-weighted graphs. If $\hom(G, H) = \hom(G, H')$ for every* simple *graph G, then H and H' are isomorphic as $\mathbb{F}$-weighted graphs.*

▶ **Corollary 6.** *Let $A \in \mathbb{F}^{m\times m}$ and $A' \in \mathbb{F}^{m'\times m'}$. Then $\hom(G, A) = \hom(G, A')$ for every* simple *graph G iff $m = m'$ and there is a permutation matrix $P \in \mathbb{F}^{m\times m}$ such that $A' = P^T AP$.*

Our proof of Theorem 2 will show that for any given H, H', there is an explicitly constructed finite family of graphs in $\mathcal{PLG}^{\text{simp}}[k]$ such that the condition for all $G \in \mathcal{PLG}^{\text{simp}}[k]$ can be replaced with for all G in this family, thus we have an explicit finitary family of test graphs. Moreover, this provides an explicit set of "witnesses" that can be used to make various complexity dichotomy theorems for graph homomorphism *effective*, in particular, making the pinning steps in [4] computable, which was an open problem.

4 Technical statements

We start with an exceedingly simple lemma, based on which all of our results will be derived. We will call this lemma and its corollary the *Vandermonde Argument*.

▶ **Lemma 7.** *Let $n \geq 0$, and $a_i, x_i \in \mathbb{F}$ for $1 \leq i \leq n$. Suppose*

$$\sum_{i=1}^{n} a_i x_i^j = 0, \qquad \textit{for all} \;\; 0 \leq j < n. \tag{6}$$

Then for any function $f : \mathbb{F} \to \mathbb{F}$, we have $\sum_{i=1}^{n} a_i f(x_i) = 0$.

▶ Remark 8. The statement is vacuously true if $n = 0$, since an empty sum is 0. If (6) is true for $1 \leq j \leq n$, then the same conclusion holds for any function f satisfying $f(0) = 0$.

Proof. We may assume $n \geq 1$. We partition $[n]$ into $\bigsqcup_{\ell=1}^{p} I_\ell$ such that i, i' belong to the same I_ℓ iff $x_i = x_{i'}$. Then (6) is a Vandermonde system of rank p with a solution $(\sum_{i\in I_\ell} a_i)_{\ell\in[p]}$. Thus $\sum_{i\in I_\ell} a_i = 0$ for all $1 \leq \ell \leq p$. It follows that $\sum_{i=1}^{n} a_i f(x_i) = 0$ for any function $f : \mathbb{F} \to \mathbb{F}$. We also note that if (6) is true for $1 \leq j \leq n$, then the same proof works except when some $x_i = 0$. In that case, we can separate out the term $\sum_{i\in I_{\ell_0}} a_i$ for the unique I_{ℓ_0} that contains this i, and we get a Vandermonde system of rank $p - 1$ on the other terms $(\sum_{i\in I_\ell} a_i)_{\ell\in[p],\ell\neq\ell_0}$, which must be all zero. ◀

By iteratively applying Lemma 7 we get the following Corollary.

▶ **Corollary 9.** *Let I be a finite (index) set, $s \geq 1$, and $a_i, b_{ij} \in \mathbb{F}$ for all $i \in I, j \in [s]$. Further, let $I = \bigsqcup_{\ell\in[p]} I_\ell$ be the partition of I into equivalence classes, where i, i' are equivalent iff $b_{ij} = b_{i'j}$ for all $j \in [s]$. If $\sum_{i\in I} a_i \prod_{j\in[s]} b_{ij}^{\ell_j} = 0$, for all choices of $(\ell_1, \dots, \ell_s)$ where each $0 \leq \ell_j < |I|$, then $\sum_{i\in I_\ell} a_i = 0$ for every $\ell \in [p]$.*

Proof. We iteratively apply Lemma 7. First, we define an equivalence relation where i, i' belong to the same equivalence class $\widetilde{I}$ iff $b_{is} = b_{i's}$. For any $\widetilde{I}$, choose f with $f(x) = 1$ for $x = b_{is}$ where $i \in \widetilde{I}$, and $f(x) = 0$ otherwise. After the first application we get $\sum_{i\in\widetilde{I}} a_i \prod_{j\in[s-1]} b_{ij}^{\ell_j} = 0$, for an arbitrary $\widetilde{I}$, and all $0 \leq \ell_j < |I|, j \in [s-1]$. The Corollary follows after applying Lemma 7 s times. ◀

5 Proof of Main Theorem

In this conference version for the sake of simplicity of presentation we prove for undirected graphs; the full version has proofs for directed graphs as well as other results.

We may assume that H is on the vertex set $V(H) = [m]$, given by vertex and edge weights $(\alpha_i)_{i\in[m]} \in (\mathbb{F} \setminus \{0\})^m$, $(\beta_{ij})_{i,j\in[m]} \in \mathbb{F}^{m\times m}$. Similarly, H' is on $V(H') = [m']$, given by $(\alpha'_i)_{i\in[m']} \in (\mathbb{F} \setminus \{0\})^{m'}$ and $(\beta'_{ij})_{i,j\in[m']} \in \mathbb{F}^{m'\times m'}$.

We first make a technical condition of "super surjectivity"; it will be removed in Theorem 2.

▶ **Lemma 10.** *Let H, H' be undirected $\mathbb{F}$-weighted graphs such that H is twin-free and $m \geq m'$. Suppose $\varphi\colon [k] \to V(H)$ and $\psi\colon [k] \to V(H')$ where $k \geq 0$. Assume φ is "super surjective", namely: $|\varphi^{-1}(u)| \geq 2m^2$ for every $u \in V(H)$. If $\hom_\varphi(G, H) = \hom_\psi(G, H')$ for every $G \in \mathcal{PLG}^{\mathrm{simp}}[k]$ then there exists an isomorphism σ from H to H' such that $\psi = \sigma \circ \varphi$.*

Proof. Assume $m \geq 1$ (the case $m = 0$ is trivial). Taking any $u \in V(H)$, we get $k \geq |\varphi^{-1}(u)| \geq 2m^2 > 0$, thus $m' \geq |\psi([k])| \geq 1$. For each $\kappa = (b_i)_{i\in[k]} \in \{0, 1\}^k$, we can define a graph $G_\kappa \in \mathcal{PLG}^{\mathrm{simp}}[k]$:

$V(G_\kappa) = \{u_1, \ldots, u_k, v\}$, with each u_i labeled i. For each $i \in [k]$, there is an edge (v, u_i) iff $b_i = 1$. There are no other edges.

We now define a specific set of G_κ, given φ and ψ. We can partition $[k] = \bigsqcup_{i=1}^m I_i$ where each $I_i = \varphi^{-1}(i)$ and $|I_i| \geq 2m^2$. For every $i \in [m]$, since $|I_i| \geq 2mm'$, there exists $J_i \subseteq I_i$ such that $|J_i| \geq 2m > 0$ and the restriction $\psi_{|J_i}$ takes a constant value $s(i)$, for some function $s\colon [m] \to [m']$. Next, for each $i \in [m]$ and for every $0 \leq k_i < 2m$, we can fix $K_i \subset J_i$, with $|K_i| = k_i$. Then we let the tuple $\chi = \chi(K_1, \ldots, K_m) \in \{0, 1\}^k$ take $\chi_{|K_i} = 1$, $i \in [m]$, and all other entries are 0. Let R be the set of all such tuples χ. Then $\hom_\varphi(G_\chi, H) = \hom_\psi(G_\chi, H')$ for every G_χ with $\chi \in R$ is expressed by: For all $0 \leq k_j < 2m$, $j \in [m]$,

$$\sum_{i=1}^{m} \alpha_i \prod_{j=1}^{m} \beta_{ij}^{k_j} = \sum_{i=1}^{m'} \alpha'_i \prod_{j=1}^{m} (\beta'_{is(j)})^{k_j}. \tag{7}$$

In (7) the sums on i come from assigning $v \in V(G_\chi)$ to $i \in V(H)$ or to $i \in V(H')$, respectively.

Because H is twin-free the m-tuples $(\beta_{ij})_{j\in[m]} \in \mathbb{F}^m$ for $1 \leq i \leq m$ are pairwise distinct. In (7) the sum in the LHS has m terms, while the sum in the RHS has $m' \leq m$ terms. Transferring the RHS to the LHS we get at most $2m$ terms. Now we apply Corollary 9 to the sum obtained by moving all terms of the RHS to the LHS in (7). By the pairwise distinctness of the m-tuples $(\beta_{ij})_{j\in[m]} \in \mathbb{F}^m$ for $1 \leq i \leq m$, and since there are only $m' \leq m$ terms from the RHS and every $\alpha_i \neq 0$, we see that each term from the LHS of (7) must be canceled by exactly one term from the RHS. And this can only occur if $m = m'$, and there is a bijective map $\sigma\colon [m] \to [m]$:

$$\alpha_i = \alpha'_{\sigma(i)} \quad \text{for } i \in [m], \qquad (\beta_{ij})_{j\in[m]} = (\beta'_{\sigma(i)s(j)})_{j\in[m]} \quad \text{for } i \in [m]. \tag{8}$$

Since H, H' are undirected graphs, we also have $\beta_{ij} = \beta_{ji} = \beta'_{\sigma(j)s(i)} = \beta'_{s(i)\sigma(j)}$ for $i, j \in [m]$.

Next we show that s is bijective. If for some $x, y \in [m]$ we have $s(x) = s(y)$, then

$$(\beta_{xj})_{j\in[m]} = (\beta'_{s(x)\sigma(j)})_{j\in[m]} = (\beta'_{s(y)\sigma(j)})_{j\in[m]} = (\beta_{yj})_{j\in[m]}.$$

Since H is twin-free we conclude that $x = y$. Thus the map $s\colon [m] \to [m]$ is injective and (since $[m]$ is finite) s is bijective. However, $\sigma\colon [m] \to [m]$ is also bijective, so it follows from (8) that the tuples $(\beta'_{ij})_{j\in[m]}$ for $i \in [m]$ are pairwise distinct, which means that H' is twin-free as well.

Next we show $\psi_{|I_i} = s(i)$ for all $i \in [m]$. If for all $i \in [m]$, we have $J_i = I_i$, then we are done. Otherwise, take any $w \in [m]$ such that J_w is a proper subset of I_w and we take any $t \in I_w \setminus J_w$. Observe that $t \notin K_i$ for all $i \in [m]$, in particular, $\chi(t) = 0$ for each $\chi \in R$.

For each $\chi \in R$, let χ_+ be the tuple obtained from χ by reassigning $\chi(t)$ (changing its t-th entry) from 0 to 1 and let R_+ be the set of all such χ_+. Then $\hom_\varphi(G_\kappa, H) = \hom_\psi(G_\kappa, H')$ for every G_κ with $\kappa \in R_+$ is expressed as (recall that we have already proved that $m' = m$)

$$\sum_{i=1}^{m} \alpha_i \beta_{iw} \prod_{j=1}^{m} \beta_{ij}^{k_j} = \sum_{i=1}^{m} \alpha'_i \beta'_{i\psi(t)} \prod_{j=1}^{m} (\beta'_{is(j)})^{k_j},$$

which can be compared to (7), and here for $\kappa \in R_+$ we have one extra edge (v, u_t) in G_κ, and $\varphi(t) = w$ since $t \in I_w$. So this holds for every $0 \le k_j < 2m$ where $j \in [m]$. Transferring the RHS to the LHS and using (8), we get

$$\sum_{i=1}^{m} (\alpha_i \beta_{iw} - \alpha'_{\sigma(i)} \beta'_{\sigma(i)\psi(t)}) \prod_{j=1}^{m} \beta_{ij}^{k_j} = 0,$$

for every $0 \le k_j < 2m$ where $j \in [m]$. Since $\alpha_i = \alpha'_{\sigma(i)} \neq 0$, and the tuples $(\beta_{ij})_{j \in [m]}$ for $1 \le i \le m$ are pairwise distinct, by Corollary 9, we get $\beta_{iw} - \beta'_{\sigma(i)\psi(t)} = 0$ for $i \in [m]$. On the other hand by (8), $\beta_{iw} = \beta'_{\sigma(i)s(w)}$ for $i \in [m]$. It follows that $\beta'_{\sigma(i)\psi(t)} = \beta'_{\sigma(i)s(w)}$ for $i \in [m]$. However, since $\sigma\colon [m] \to [m]$ is a bijection and, as shown before, H' is twin-free, this implies that $\psi(t) = s(w)$. Recall that $\psi_{|J_w} = s(w)$. This proves that on $I_w \setminus J_w$, ψ also takes the constant value $s(w)$. Thus $\psi_{|I_i} = s(i)$ for all $i \in [m]$.

Next we prove that $\beta_{ij} = \beta'_{\sigma(i)\sigma(j)}$ for all $i, j \in [m]$, i.e., σ preserves the edge weights.

For each $\lambda = (b_i)_{i \in [k]}$, $\tau = (c_i)_{i \in [k]} \in \{0,1\}^k$, we define a graph $G_{\lambda,\tau} \in \mathcal{PLG}^{\mathrm{simp}}[k]$:

$V(G_{\lambda,\tau}) = \{u_1, \ldots, u_k, v, v'\}$, with each u_i labeled i. There is an edge (v, v') and, for each $i \in [k]$, there is an edge (v, u_i) if $b_i = 1$, and an edge (v', u_i) if $c_i = 1$. There are no other edges.

Let $R^2 = R \times R$. By the definition of R, every $(\lambda, \tau) \in R^2$ corresponds to a sequence of pairs (K_i, L_i), $i \in [m]$, where $K_i \subset J_i \subseteq I_i$ and $L_i \subset J_i \subseteq I_i$ such that $|K_i| = k_i$, $|L_i| = \ell_i$, where $0 \le k_i, \ell_i < 2m$, and $\lambda_{|K_i} = 1$, $\tau_{|L_i} = 1$, for $i \in [m]$, and all other entries are 0.

Then $\hom_\varphi(G_{\lambda,\tau}, H) = \hom_\psi(G_{\lambda,\tau}, H')$ for every $G_{\lambda,\tau}$ with $(\lambda, \tau) \in R^2$ is expressed as

$$\sum_{i,j=1}^{m} \alpha_i \alpha_j \beta_{ij} \prod_{r=1}^{m} (\beta_{ir}^{k_r} \beta_{jr}^{\ell_r}) = \sum_{i,j=1}^{m} \alpha'_i \alpha'_j \beta'_{ij} \prod_{r=1}^{m} \left((\beta'_{is(r)})^{k_r} (\beta'_{js(r)})^{\ell_r} \right).$$

This holds for all $0 \le k_r, \ell_r < 2m$ where $r \in [m]$. Transferring the RHS to the LHS and using (8), we get

$$\sum_{i,j=1}^{m} (\alpha_i \alpha_j \beta_{ij} - \alpha_i \alpha_j \beta'_{\sigma(i)\sigma(j)}) \prod_{r=1}^{m} (\beta_{ir}^{k_r} \beta_{jr}^{\ell_r}) = 0,$$

for every $0 \le k_r, \ell_r < 2m$, where $r \in [m]$. Since the tuples $(\beta_{ir}, \beta_{jr})_{r \in [m]}$ for $1 \le i, j \le m$ are pairwise distinct, and $\alpha_i \alpha_j \neq 0$, by Corollary 9,

$$\beta_{ij} = \beta'_{\sigma(i)\sigma(j)}, \quad \text{for } i, j \in [m]. \tag{9}$$

This means that the bijection $\sigma\colon [m] \to [m]$ preserves the edge weights in addition to the vertex weights by (8). Hence $\sigma\colon [m] \to [m]$ is an isomorophism of $\mathbb{F}$-weighted graphs from H to H'.

Finally, we show that $\psi = \sigma \circ \varphi$. From (8) and (9), we have $\beta'_{\sigma(i)s(j)} = \beta_{ij} = \beta'_{\sigma(i)\sigma(j)}$ for $i, j \in [m]$. As H' is twin-free and σ is bijective we get $\sigma(j) = s(j)$ for $j \in [m]$. Now let $x \in [k]$, then $x \in I_i$ for some $i \in [m]$. Therefore $\varphi(x) = i$ and so $\psi(x) = s(i) = \sigma(i) = \sigma(\varphi(x))$ confirming $\psi = \sigma \circ \varphi$. ◀

Proof of Theorem 2. Consider an arbitrary $\ell \geq k$ and let $G \in \mathcal{PLG}^{\text{simp}}[\ell]$ be any ℓ-labeled graph. Let $G^* = \pi_{[k]}(G)$ be the graph obtained by unlabeling the labels not in $[k]$ from G (if $k = \ell$, then $G^* = G$). Clearly, $G^* \in \mathcal{PLG}^{\text{simp}}[k]$ so $\hom_\varphi(G^*, H) = \hom_\psi(G^*, H')$. Expanding the sums on the LHS and the RHS of this equality representing the maps φ, ψ along the vertices formerly labeled by $[\ell] \setminus [k]$, and then regrouping the terms corresponding to the same extension maps of φ from $[\ell]$ to $[m]$ and of ψ from $[\ell]$ to $[m']$, respectively, and then bringing back the labels from $[\ell] \setminus [k]$, we get

$$\sum_{\substack{\mu\colon [\ell]\to[m] \\ \mu \text{ extends } \varphi}} \left(\prod_{k<i\leq\ell} \alpha_{\mu(i)} \right) \hom_\mu(G, H) = \sum_{\substack{\nu\colon [\ell]\to[m'] \\ \nu \text{ extends } \psi}} \left(\prod_{k<i\leq\ell} \alpha'_{\nu(i)} \right) \hom_\nu(G, H') \tag{10}$$

for every $G \in \mathcal{PLG}^{\text{simp}}[\ell]$. (Here if $\ell = k$, the empty product $\prod_{k<i\leq\ell}$ is 1.)

Now choose $\ell \geq k$ so that we can extend φ to a map $\eta\colon [\ell] \to [m]$ such that $|\eta^{-1}(u)| \geq 2m^2$ for every $u \in V(H)$. Clearly $\ell \leq k + 2m^3$ suffices. (If φ already satisfies the property, we can take $\ell = k$ and $\eta = \varphi$.) We fix ℓ and η to be such. Define

$$I = \{\mu\colon [\ell] \to [m] \mid \big(\mu_{|[k]} = \varphi\big) \wedge \big((\exists \sigma \in \operatorname{Aut}(H))\ \mu = \sigma \circ \eta\big)\},$$
$$J = \{\nu\colon [\ell] \to [m'] \mid \big(\nu_{|[k]} = \psi\big) \wedge \big((\exists \sigma \in \operatorname{Isom}(H, H'))\ \nu = \sigma \circ \eta\big)\}.$$

Obviously, $\eta \in I$ so $I \neq \emptyset$. For now, we do not exclude the possibility $J = \emptyset$ but our goal is to show that this is not the case. If $\mu\colon [\ell] \to V(H)$ extends φ but $\mu \notin I$, then by Lemma 10, there exists a graph $G_{\eta,\mu} \in \mathcal{PLG}^{\text{simp}}[\ell]$ such that $\hom_\eta(G_{\eta,\mu}, H) \neq \hom_\mu(G_{\eta,\mu}, H)$. Similarly, if $\nu\colon [\ell] \to V(H')$ extends ψ but $\nu \notin J$, then by Lemma 10, there exists a graph $G'_{\eta,\nu} \in \mathcal{PLG}^{\text{simp}}[\ell]$ such that $\hom_\eta(G'_{\eta,\nu}, H) \neq \hom_\nu(G'_{\eta,\nu}, H')$. Let S be the set consisting of these graphs $G_{\eta,\mu}$ and $G'_{\eta,\nu}$ (we remove any repetitions). Note that if $\mu' \in I$, then $\hom_\eta(G, H) = \hom_{\mu'}(G, H)$ for any $G \in \mathcal{PLG}[\ell]$, and if $\nu' \in J$, then $\hom_\eta(G, H) = \hom_{\nu'}(G, H')$ for any $G \in \mathcal{PLG}[\ell]$. In particular, both equalities hold for each $G \in S$. We impose a linear order on S and regard any set indexed by S as a tuple. For any tuple $\bar{h} = (h_G)_{G\in S}$ of integers, where each $0 \leq h_G < 2m^\ell$, consider the graph $G_{\bar{h}} = \prod_{G\in S} G^{h_G} \in \mathcal{PLG}^{\text{simp}}[\ell]$. Substituting $G = G_{\bar{h}}$ in (10) and using the multiplicativity of partial graph homomorphisms (5), we obtain

$$\sum_{\substack{\mu\colon [\ell]\to[m] \\ \mu \text{ extends } \varphi}} \left(\prod_{k<i\leq\ell} \alpha_{\mu(i)} \right) \prod_{G\in S} (\hom_\mu(G, H))^{h_G} = \sum_{\substack{\nu\colon [\ell]\to[m'] \\ \nu \text{ extends } \psi}} \left(\prod_{k<i\leq\ell} \alpha'_{\nu(i)} \right) \prod_{G\in S} (\hom_\nu(G, H'))^{h_G}$$

for every $0 \leq h_G < 2m^\ell$. By the previous observations and the fact that S contains each $G_{\eta,\mu}$ and $G'_{\eta,\nu}$, the tuple $(\hom_\eta(G, H))_{G\in S}$ coincides with the tuple $(\hom_\mu(G, H))_{G\in S}$ for $\mu \in I$ and it also coincides with the tuple $(\hom_\nu(G, H'))_{G\in S}$ for $\nu \in J$; on the other hand, this tuple $(\hom_\eta(G, H))_{G\in S}$ is different from the tuple $(\hom_\mu(G, H))_{G\in S}$ for each $\mu\colon [k] \to V(H)$ extending φ and not in I and it is also different from the tuple $(\hom_\nu(G, H'))_{G\in S}$ for each $\nu\colon [k] \to V(H')$ extending ψ and not in J. Transferring the RHS to the LHS and then applying Corollary 9, we conclude that

$$\sum_{\mu\in I} \left(\prod_{k<i\leq\ell} \alpha_{\mu(i)} \right) = \sum_{\nu\in J} \left(\prod_{k<i\leq\ell} \alpha'_{\nu(i)} \right). \tag{11}$$

If $\mu \in I$, then $\mu = \sigma \circ \eta$ for some $\sigma \in \mathrm{Aut}(H)$, and therefore $\alpha_{\mu(i)} = \alpha_{\sigma(\eta(i))} = \alpha_{\eta(i)}$ for $1 \le i \le \ell$. Hence $\prod_{k<i\le\ell} \alpha_{\mu(i)} = \prod_{k<i\le\ell} \alpha_{\eta(i)}$ (if $k = \ell$, then both sides are 1), so (11) transforms to

$$|I|_{\mathbb{F}} \cdot \prod_{k<i\le\ell} \alpha_{\eta(i)} = \sum_{\nu\in J} \left(\prod_{k<i\le\ell} \alpha'_{\nu(i)} \right). \tag{12}$$

Here we let$|I|_{\mathbb{F}} = |I| \cdot 1_{\mathbb{F}} = 1_{\mathbb{F}} + \ldots + 1_{\mathbb{F}} \in \mathbb{F}$ ($1_{\mathbb{F}}$ occurs $|I|$ times). Since all $\alpha_i \neq 0$, we have $\prod_{k<i\le\ell} \alpha_{\eta(i)} \neq 0$ (if $k = \ell$, this product is $1_{\mathbb{F}} \neq 0$). Because $I \neq \emptyset$ (as $\eta \in I$) we have $|I| \ge 1$; but $\operatorname{char} \mathbb{F} = 0$ so $|I|_{\mathbb{F}} \neq 0$ and therefore $|I|_{\mathbb{F}} \cdot \prod_{k<i\le\ell} \alpha_{\eta(i)} \neq 0$. This implies that the RHS of (12) is nonzero as well which can only occur when $J \neq \emptyset$. Take $\xi \in J$. Then $\xi = \sigma \circ \eta$ for some isomorphism of $\mathbb{F}$-weighted graphs $\sigma\colon V(H) \to V(H')$ from H to H'. Restricting to $[k]$, we obtain $\psi = \sigma \circ \varphi$ which completes the proof. ◀

▶ Remark 11. Theorem 2 shows that the condition $\hom_\varphi(G, H) = \hom_\psi(G, H')$ for all $G \in \mathcal{PLG}^{\mathrm{simp}}[k]$ is equivalent to the existence of an isomorphism from H to H' such that $\psi = \sigma \circ \varphi$. This condition is effectively checkable. However, for the purpose of effectively producing a #P-hardness reduction in the dichotomy theorems, e.g., in [4], we need a witness G such that $\hom_\varphi(G, H) \neq \hom_\psi(G, H')$.

The proof of Theorem 2 gives an *explicit finite* list to check. For $\ell = k + 2m^3$, let $\mathcal{P}_{\ell,m} = \{\prod_{G\in S} G^{h_G} \mid 0 \le h_G < 2m^\ell\}$, where S is from the proof of Theorem 2 using the construction of Lemma 10. We then define $\mathcal{Q}_{k,m} = \pi_{[k]}(\mathcal{P}_{\ell,m}) \subseteq \mathcal{PLG}^{\mathrm{simp}}[k]$. Then the proof of Theorem 2 shows that: the existence of an isomorphism $\sigma\colon V(H) \to V(H')$ from H to H' such that $\varphi = \sigma \circ \psi$, is equivalent to $\hom_\varphi(G, H) = \hom_\psi(G, H')$ for every $G \in \mathcal{Q}_{k,m}$.

▶ Remark 12. In their proof of Theorem 2.1 in [11], Goodall, Regts and Vena noted in a footnote on p. 268 that "Even though Lovász [9] (this is [15]) works over $\mathbb{R}$ and assumes $a_i > 0$ for all i, it is easy to check that the proof of his Lemma 2.4 remains valid in our setting." The following is a sketch of this approach due to Goodall, Regts and Vena; we thank Guus Regts for enlightening discussions. Notations below can be found in [15].

Following the proof of Lemma 2.4 by Lovász in [15] one comes to Claim 4.2. On p. 968 the set Ψ is defined as consisting of all maps equivalent to μ. This set appears in the sum on line 2 of p. 969. Let Ψ_0 be the subset of all maps $\eta \in \Psi$ that restrict to ϕ. Since $\mu \in \Psi$ and μ restricts to ϕ, the subset Ψ_0 is nonempty; however it could have cardinality > 1. Then in the sum on line 2 of p. 969 when one collects all terms with the same restriction, ϕ appears with the coefficient $\sum_{\eta\in\Psi_0} \alpha(\eta(k+1))$. The crucial step in the proof of Claim 4.2 is that this sum is nonzero. In [15] this is nonzero since all vertex weights are positive. Without this positivity, it is possible that such a sum is 0. Thus one cannot directly follow the proof in [15] in the general case when a nonempty subset of vertex weights can sum to 0.

However, instead of Claim 4.2 in [15] on line 6 of p. 970 after the proof of Claim 4.4, one can extend ϕ to a surjective map $\mu : [\ell] \to [m]$ for some $\ell \ge k$. Then apply the trace operator of [15] $\ell - k$ times to an analogous sum defined on line -1 of p. 968, which is in $\mathcal{A}''_\ell$. This gives a sum $\Sigma = \sum_{\eta\in\Psi} \prod_{i\in[k+1:\ell]} \alpha(\eta(i))\eta_{\mathrm{res}}$, where η_{res} is the restriction of η to $[k]$.

Then use Claim 4.4, all $\alpha(\eta(i))$ are the same for $\eta \in \Psi$ and thus $P = \prod_{i\in[k+1:\ell]} \alpha(\eta(i))$ is a common factor. Since each $\alpha(\eta(i)) \neq 0$, this $P \neq 0$. Since $\mu \in \Psi$, $\phi = \mu_{\mathrm{res}}$ appears as a term in Σ with a coefficient $|S|P$, where S is the subset of Ψ consisting of maps that restrict to ϕ. As $\mathbb{F}$ has characteristic 0, this $|S|P \neq 0$.

Now since ψ is given as equivalent to ϕ, and the sum Σ is in $\mathcal{A}''_k$, ψ must have the same coefficient as that of ϕ, which has the form $|S'|P$, where S' is the subset of Ψ that restrict to ψ. So $|S'| = |S|$, in particular S' is nonempty. Thus there is some $\eta \in \Psi$ such that η extends ψ, and η is equivalent to μ.

This leads to the following theorem:

▶ Theorem 13. *Let $\mathbb{F}$ be a field of characterisitic 0. Let H be an undirected $\mathbb{F}$-weighted twin-free graph. For any $k \geq 0$, if $\varphi, \psi \colon [k] \to V(H)$ and $\hom_\phi(G,H) = \hom_\psi(G,H')$ for all undirected graphs G, where G may have multiloops and multiedges, then there is an automorphism σ of H such that $\phi = \sigma \circ \psi$.*

6 Effective GH Dichotomies

We briefly discuss how to use Theorem 2 to make complexity dichotomies for graph homomorphism effective. A long and fruitful sequence of work [7, 8, 2, 12, 21, 10] led to the following complexity dichotomy for weighted graph homomorphisms [4] which unifies these previous ones: There is a tractability condition $\mathcal{P}$ such thatfor any (algebraic) complex symmetric matrix H, if H satisfies $\mathcal{P}$ then $\hom(\cdot, H)$ is P-time computable, otherwise there is a P-time reduction from a canonical #P-hard problem to $\hom(\cdot, H)$. However, in the long sequence of reductions in [4] there are nonconstructive steps, a prominent example is the first pinning lemma (Lemma 4.1, p. 937). This involves condensing H by collapsing "equivalent" vertices, while introducing vertex weights. Consider all 1-labeled graphs G. We say two vertices $u, v \in V(H)$ are "equivalent" if $\hom_u(G,H) = \hom_v(G,H)$ where the notation $\hom_u(G,H)$ denotes the partial sum of $\hom(G,H)$ where we restrict to all mappings which map the labeled vertex of G to u, and similarly for $\hom_v(G,H)$. This is just the special case $k = 1$ in Theorem 2 (note that we first apply the twin compression step to H). Previously the P-time reduction was proved existentially. Using Theorem 2 (see Remark 11 after the proof), this step can be made effective.

There is a finer distinction between making the dichotomy effective in the sense discussed here versus the decidability of the dichotomy. The dichotomy criterion in [4] is decidable in P-time (measured in the size of the description of H); however, according to the proof in [4] which involves nonconstructive steps, when the decision algorithm decides $\hom(\cdot, H)$ is #P-hard it does not produce a reduction. The results in this paper can constructively produce such a reduction when $\hom(\cdot, H)$ is #P-hard.

Previous versions of Theorem 2 (e.g., [14, 15]) show that the above equivalence on u, v for suitably restricted classes of H can be decided by testing for graph isomorphism (with pinning). However, to actually obtain the promised P-time reduction one has to search for "witness" graphs G to $\hom_u(G,H) \neq \hom_v(G,H)$. Having no graph isomorphism mapping u to v does not readily yield such a "witness" graph G, although an open ended search is guaranteed to find one. Thus Theorem 2 gives a double exponential time (in the size of H) algorithm to find a reduction algorithm, while directly applying previous versions of the theorem gives a computable process with no definite time bound. (But we emphasize that no previous versions of Theorem 2 apply to the dichotomy in [4].)

7 Counterexample for fields of finite characteristic

In Lemma 10, the field $\mathbb{F}$ is arbitrary. By contrast, for Theorem 2 the proof uses the assumption that $\operatorname{char} \mathbb{F} = 0$. We show that this assumption cannot be removed, for any fixed k, by an explicit counterexample. The counterexample also applies to Corollaries 3 to 6.

Let $\operatorname{char} \mathbb{F} = p > 0$. For $n \geq 2$ and $\ell_1 > \ldots > \ell_n > 0$, define an (undirected) $\mathbb{F}$-weighted graph $H = H_{n,\ell_1,\ldots,\ell_n}$ with the vertex set $U \cup \bigcup_{i=1}^n V_i$ where $U = \{u_1, \ldots, u_n\}$ and $V_i = \{v_{i,j} \mid 1 \leq j \leq \ell_i p\}$, for $i \in [n]$, and the edge set being the union of the edge sets that form a copy of the complete graph K_n on U and $K_{1+\ell_i p}$ on $\{u_i\} \cup V_i$ for $i \in [n]$. H

is a simple graph with no loops. To make H an $\mathbb{F}$-weighted graph, we assign each vertex and edge weight 1. (So H is really unweighted.) It is easy to see that H is twin-free: First, any two distinct vertices from U or from the same V_i are not twins because H is loopless. (Note that for vertices i, j to be twin in an undirected graph, if (i, j) is an edge, then the loops (i, i), (j, j) must also exist.) Second, for any $i \in [n]$, $u_i \in U$ and any $v \in V_i$ are not twins by $\deg(u_i) > \deg(v)$. Third, $u_i \in U$ (or any $v \in V_i$) and any $w \in V_j$, for $j \neq i$, are not twins because w has some neighbor in V_j while u_i (or v) do not. Let $\sigma \in \mathrm{Aut}(H)$ be an automorphism of H. Each vertex $u \in U$ has the property that u has two neighbors (one in U and one not in U) such that they are not neighbors to each other. This property separates U from the rest. Furthermore $\deg(u_1) > \ldots > \deg(u_n)$. Therefore σ must fix U pointwise. Then it is easy to see that σ must permute each V_i.

For any $\varphi\colon [k] \to U \subset V(H)$ where $k \geq 0$, we claim that $\hom_\varphi(G, H) = \hom_\varphi(G, K_U)$ for every $G \in \mathcal{PLG}[k]$, where K_U is the complete graph with the vertex set U.

Let $\mathfrak{S}_N$ denote the symmetry group on N letters. We define a group action of $\prod_{i=1}^n \mathfrak{S}_{\ell_i p}$ on $\{\xi \mid \xi\colon V(G) \to V(H)\}$ which permutes the images of ξ within each of $V_1, \ldots, V_n$, and fixes U pointwise. Thus for $g = (g_1, \ldots, g_n) \in \prod_{i=1}^n \mathfrak{S}_{\ell_i p}$, if $\xi(w) \in V_i$, then $\xi^g(w) = g_i(\xi(w))$. This group action partitions all ξ into orbits. Consider any $\xi\colon V(G) \to V(H)$ extending φ, such that the image $\xi(V(G)) \not\subseteq U$. Let η be in the same orbit of ξ. The nonzero contributions to $\hom_\xi(G, H)$ and $\hom_\eta(G, H)$ come from either edge weights within U, where they are identical, or within each $\{u_i\} \cup V_i$. Hence by the definition of the group action, $\hom_\xi(G, H) = \hom_\eta(G, H)$. The stabilizer of ξ consists of those g such that each g_i fixes the image set $\xi(V(G)) \cap V_i$ pointwise. Since $\xi(V(G)) \not\subseteq U$, the orbit has cardinality, which is the index of the stablizer, divisible by some $\ell_i p$. In particular it is 0 mod p. Thus the total contribution from each orbit is zero in $\mathbb{F}$, except for those ξ with $\xi(V(G)) \subseteq U$. The claim follows.

For $k \geq 1$, we take $H' = H$. We say that maps $\varphi, \psi\colon [k] \to U$ have the *same type* if for every $i, j \in [k]$, $\varphi(i) = \varphi(j)$ iff $\psi(i) = \psi(j)$. Thus the inverse image sets $\varphi^{-1}(\varphi(i))$ and $\psi^{-1}(\psi(i))$ have the same cardinality for every $i \in [k]$. It follows that the image sets $\varphi([k])$ and $\psi([k])$, consisting of the elements of U having a nonempty inverse image sets under φ^{-1} and ψ^{-1} respectively, are of the same cardinality. So there is a bijection σ of U, mapping $U \setminus \varphi([k])$ to $U \setminus \psi([k])$, and also for every $i \in [k]$, $(\sigma \circ \varphi)(i) = \psi(i)$. Thus $\sigma \in \mathrm{Aut}(K_U)$ and $\sigma \circ \varphi = \psi$. Take $\varphi, \psi\colon [k] \to U$ such that $\varphi \neq \psi$, and they have the same type. For example, we can take $\varphi(i) = u_1$ and $\psi(i) = u_2$ for every $i \in [k]$. Since φ and ψ have the same type, clearly $\hom_\varphi(G, K_U) = \hom_\psi(G, K_U)$. For every $G \in \mathcal{PLG}[k]$, we have already shown that $\hom_\varphi(G, K_U) = \hom_\varphi(G, H)$, and since $H' = H$, $\hom_\psi(G, K_U) = \hom_\psi(G, H')$, implying that $\hom_\varphi(G, H) = \hom_\psi(G, H')$. If Theorem 2 were to hold for the field $\mathbb{F}$ of $\operatorname{char} \mathbb{F} = p > 0$, there would be an automorphism $\widehat{\sigma} \in \mathrm{Aut}(H)$ such that $\widehat{\sigma} \circ \varphi = \psi$. But every automorphism of H must fix U pointwise, and thus it restricts to the identity map on U. And since $\varphi([k]) \subseteq U$, we have $\widehat{\sigma} \circ \varphi = \varphi \neq \psi$, a contradiction.

When $k = 0$, in addition to $H = H_{n, \ell_1, \ldots, \ell_n}$ we also take $H' = H_{n, \ell'_1, \ldots, \ell'_n}$ on the vertex set $U \cup \bigcup_{i=1}^n V'_i$, where $n \geq 2$, $\ell'_1 > \ldots > \ell'_n > 0$ and $(\ell'_1, \ldots, \ell'_n) \neq (\ell_1, \ldots, \ell_n)$. As $k = 0$, the only possible choices are the empty maps $\varphi = \emptyset$ and $\psi = \emptyset$, and $\hom(G, H) = \hom(G, K_U) = \hom(G, H')$ still holds for every $G \in \mathcal{PLG}[0]$. However, the same property that every vertex $u \in U$ has two neighbors such that they are not neighbors to each other separates U from the rest in both H and H'. Then the monotonicity $\deg(u_1) > \ldots > \deg(u_n)$ within both H and H' shows that any isomorphism from H to H', if it exists, must fix U pointwise. Then it is easy to see that σ must be a bijection from V_i of H to the corresponding copy V'_i in H'. This forces $(\ell_1, \ldots, \ell_n) = (\ell'_1, \ldots, \ell'_n)$, a contradiction.

8 Rank of Connection Tensors and Dimension of Graph Algebras

The purpose of this section is to summarize the extensions of the main results from [15]. These are stated as Theorems 14 and 15.

An $\mathbb{F}$-valued *graph parameter* is a function from finite graph ismorphism classes to $\mathbb{F}$. For convinience, we think of a graph parameter as a function defined on finite graphs and invariant under graph isomorphism. We allow multiple edges in our graphs, but no loops, as input to a graph parameter. A graph parameter f is called *multiplicative*, if for any disjoint union $G_1 \sqcup G_2$ of graphs G_1 and G_2 we have $f(G_1 \sqcup G_2) = f(G_1)f(G_2)$. A graph parameter on a labeled graph ignores its labels. Every weighted graph homomorphism $f_H = \hom(\cdot, H)$ is a multiplicative graph parameter.

A (k-labeled, $\mathbb{F}$-)*quantum graph* is a finite formal $\mathbb{F}$-linear combination of finite k-labeled graphs. $\mathcal{G}[k] = \mathbb{F}\mathcal{PLG}[k]$ is the monoid algebra of k-labeled $\mathbb{F}$-quantum graphs. We denote by $\mathcal{G}^{\text{simp}}[k]$ the monoid algebra of *simple* k-labeled $\mathbb{F}$-quantum graphs; it is a subalgebra of $\mathcal{G}[k]$. U_k is the multiplicative identity and the empty sum is the additive identity in both $\mathcal{G}[k]$ and $\mathcal{G}^{\text{simp}}[k]$.

Let f be any graph parameter. For all integers $k, n \geq 0$, we define the following n-dimensional array $T(f, k, n) \in \mathbb{F}^{(\mathcal{PLG}[k])^n}$, which can be identified with $(V^{\otimes n})^*$, the dual space of $V^{\otimes n}$, where $V = \bigoplus_{\mathcal{PLG}[k]} \mathbb{F}$ is the infinite dimensional vector space with coordinates indexed by $\mathcal{PLG}[k]$. The entry of $T(f, k, n)$ at coordinate $(G_1, \ldots, G_n)$ is $f(G_1 \cdots G_n)$; when $n = 0$, we define $T(f, k, n)$ to be the scalar $f(U_k)$. The arrays $T(f, k, n)$ are symmetric with respect to its coordinates, i.e., $T(f, k, n) \in \operatorname{Sym}(\mathbb{F}^{(\mathcal{PLG}[k])^n})$. Fix f, k and n, we call the n-dimensional array $T(f, k, n)$ the (k-th, n-dimensional) *connection tensor* of the graph parameter f. When $n = 2$, a connection tensor is exactly a *connection matrix* of the graph parameter f studied in [9], i.e., $T(f, k, 2) = M(f, k)$.

For graph parameters of the form $f_H = \hom(\cdot, H)$, where H has positive vertex weights and real edge weights, the main results of [15] are to compute the rank of the corresponding connection matrices, and the dimension of graph algebras, etc. We will prove these results for arbitrary $\mathbb{F}$-weighted graphs (without vertex or edge weight restrictions). Moreover we will prove these for connection tensors (see [5]). Below we let H be a (directed or undirected) $\mathbb{F}$-weighted graph.

For $k \geq 0$, let $N(k, H)$ be the matrix whose rows are indexed by maps $\varphi \colon [k] \to V(H)$ and columns are indexed by $\mathcal{PLG}[k]$, and the row indexed by φ is $\hom_\varphi(\cdot, H)$. We have a group action of $\operatorname{Aut}(H)$ on the k-tuples from $V(H)^k = \{\varphi \colon [k] \to V(H)\}$ by $\varphi \mapsto \sigma \circ \varphi$ for $\sigma \in \operatorname{Aut}(H)$ and $\varphi \colon [k] \to V(H)$. We use $\operatorname{orb}_k(H)$ to denote the number of its orbits.

As mentioned before, $f_H = \hom(\cdot, H)$ ignores labels on a labeled graph, so we can think of f_H as defined on $\mathcal{PLG}[k]$ and then by linearity as defined on $\mathcal{G}[k]$. Then we can define the following bilinear symmetric form on $\mathcal{G}[k]$:

$$\langle x, y \rangle = f_H(xy), \quad x, y \in \mathcal{G}[k].$$

Let

$$\mathcal{K}_{[k]} = \{x \in \mathcal{G}[k] \colon f_H(xy) = \langle x, y \rangle = 0 \ \forall y \in \mathcal{G}[k]\}.$$

Clearly, $\mathcal{K}_{[k]}$ is an ideal in $\mathcal{G}[k]$, so we can form a quotient algebra $\mathcal{G}'[k] = \mathcal{G}[k]/\mathcal{K}_{[k]}$. It is easy to see that $h \in \mathcal{K}_{[k]}$ iff $M(f_H, k)h = 0$.

In order to be consistent with the notation in [15], when $f = f_H = \hom(\cdot, H)$ for an $\mathbb{F}$-weighted (directed or undirected) graph H, we let $T(k, n, H) = T(f_H, k, n)$ where $k, n \geq 0$ and $M(k, H) = M(f_H, k)$ where $k \geq 0$.

The following theorems extend Theorem 2.2, Corollary 2.3 and the results of Section 3 in [15].

▶ **Theorem 14.** *Let $\mathbb{F}$ be a field of characteristic* 0*. Let H be a (directed or undirected) $\mathbb{F}$-weighted twin-free graph. Then $\mathcal{G}'[k] \cong \mathbb{F}^{r_k}$ as isomorphic algebras, where*

$$r_k = \dim \mathcal{G}'[k] = \operatorname{rk}_{\mathsf{S}} T(k, n, H) = \operatorname{rk} T(k, n, H) = \operatorname{rk} M(k, H) = \operatorname{rk} N(k, H) = \operatorname{orb}_k(H)$$

for $k \geq 0$ and $n \geq 2$. Here $\operatorname{rk}_{\mathsf{S}}$ denotes symmetric tensor rank, and rk *on T and on M, N denote tensor and matrix rank, respectively. In particular, if H has no nontrivial $\mathbb{F}$-weighted automorphisms, then the above quantities are all equal to $|V(H)|^k$.*

The following theorem generalizes Lemma 2.5 in [15]. Let $k \geq 0$ be an integer and H be an $\mathbb{F}$-weighted graph. We say that a vector $f\colon V(H)^k \to \mathbb{F}$ is *invariant under the automorphisms* of H if $f(\sigma \circ \varphi) = f(\varphi)$ for every $\sigma \in \operatorname{Aut}(H)$ and $\varphi \in V(H)^k$.

▶ **Theorem 15.** *Let $\mathbb{F}$ be a field of characteristic* 0*. Let H be a (directed or undirected) $\mathbb{F}$-weighted twin-free graph. Then for $k \geq 0$, the column space of $N(k, H)$ consists of precisely those vectors $f\colon V(H)^k \to \mathbb{F}$ that are invariant under the authomorphisms of H. Moreover, every such vector can be obtained as a finite linear combination of the columns of $N(k, H)$ indexed by $\mathcal{PLG}^{\mathrm{simp}}[k]$.*

From this theorem, we can immediately conclude the existence of simple contractors and connectors from $\mathcal{PLG}^{\mathrm{simp}}[k]$ when $\operatorname{char} \mathbb{F} = 0$ for GH functions (see [17] for the definitions).

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