On the Probability That a Random Digraph Is Acyclic

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Abstract

Given a positive integer n and a real number $p \in [0,1]$, let $\mathcal{D}(n,p)$ denote the random digraph defined in the following way: each of the $\binom{n}{2}$ possible edges on the vertex set $\{1,2,3,\dots,n\}$ is included with probability $2p$, where all edges are independent of each other. Thereafter, a direction is chosen independently for each edge, with probability $\frac{1}{2}$ for each possible direction. In this paper, we study the probability that a random instance of $\mathcal{D}(n,p)$ is *acyclic*, i.e., that it does not contain a directed cycle. We find precise asymptotic formulas for the probability of a random digraph being acyclic in the sparse regime, i.e., when $np = O(1)$. As an example, for each real number μ, we find an exact analytic expression for $\varphi(\mu) = \lim_{n\to\infty} n^{1/3}\mathbb{P}\left\{\mathcal{D}\left(n, \frac{1}{n}(1+\mu n^{-1/3})\right) \text{ is acyclic}\right\}$.

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1 Introduction

By a *simple digraph*, we mean a digraph (directed graph) without loops, (directed) 2-cycles or multiple edges. Such a digraph is called *acyclic* if it has no directed cycles, i.e., cycles that follow the direction of the edges. One easily observes that the only strongly connected components of an acyclic digraph are its vertices. Acyclic digraphs form an important class of digraphs that occurs naturally in many applications, such as scheduling or Bayesian networks.

The enumeration of acyclic digraphs is a classical combinatorial problem that was first considered in the 1970s, see Harary and Palmer [17], Liskovec [23, 24], Robinson [31, 32] and Stanley [34]. It is based on a recursion for the number of acyclic digraphs, which we briefly recall here. Let a_n denote the number of acyclic digraphs on n (labelled) vertices.

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Distinguishing by the number of sinks (vertices without an outgoing edge; equivalently, one can also consider sources, which are vertices without an incoming edge) and applying an elegant inclusion-exclusion argument, one finds that

$$a_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} 2^{k(n-k)} a_{n-k}$$

for $n > 1$, with initial value $a_0 = 1$. This can be rewritten as

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} 2^{-\binom{k}{2}-\binom{n-k}{2}} a_{n-k} = \begin{cases} 1 & n = 0, \\ 0 & n > 0. \end{cases}$$

Introducing the *special generating function* $A(x) = \sum_{n\geq 0} \frac{1}{n!} 2^{-\binom{n}{2}} a_n x^n$, one finds that

$$A(x) = \frac{1}{\sum_{n\geq 0} \frac{(-1)^n}{n!} 2^{-\binom{n}{2}} x^n}.$$

It can be shown that this function is meromorphic, and that the pole with minimum modulus occurs at $x \approx 1.48808$. From this, one can derive the asymptotic formula

$$\frac{a_n}{n!} 2^{-\binom{n}{2}} \sim C \cdot B^n,$$

where $C \approx 1.74106$ and $B \approx 0.67201$. These results can be found in the work of Robinson [31] (see also Liskovec [23] and Stanley [34]).

It is not difficult to include the number of edges in the count: let $a_{n,m}$ denote the number of labelled acyclic digraphs with n vertices and m edges, and set

$$A(x, y) = \sum_{n,m\geq 0} \frac{1}{n!} (1 + y)^{-\binom{n}{2}} a_{n,m} x^n y^m. \tag{1}$$

Then, we can also write this bivariate generating function in a reciprocal form:

$$A(x, y) = \frac{1}{\phi(x, y)}, \quad \text{where } \phi(x, y) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\,(1 + y)^{\binom{k}{2}}}. \tag{2}$$

This was already observed by Robinson in [31]. Bender, Richmond, Robinson and Wormald [1] exploited this generating function identity to prove asymptotic formulas for the number of acyclic digraphs with a given number of vertices and edges if the number of edges is "large" (i.e., quadratic in the number of edges). In particular, it is shown in [1] that the number of edges in a random acyclic digraph with n vertices satisfies a central limit theorem with mean $\sim \frac{n^2}{4}$ and variance $\sim \frac{n^2}{8}$.

Next, let us discuss models of random digraphs. $\mathbb{D}(n, p)$ denotes a directed digraph on n labelled vertices in which each of the $n(n - 1)$ directed edges is present with probability p, independently of the others, as described in [20, 26]. The model exhibits a phase transition that is somewhat similar to the *binomial model* $\mathbb{G}(n, p)$ of undirected graphs. This phase transition was, among others, studied by Karp [20] and Łuczak [25]. They proved the following: if np is fixed with $np < 1$ then every strong component has at most $\omega(n)$ vertices, for any sequence $\omega(n)$ tending to infinity arbitrarily slowly, and all strong components are either cycles or single vertices. If np is fixed with $np > 1$, then there exists a unique strong component of linear size, while all the other strong components are of logarithmic size (see also [15, Chapter 13]). Recently, Łuczak and Seierstad [26] obtained more precise results about the width and behaviour of the window where the phase transition occurs. They established that the scaling window is given by $np = 1 + \mu n^{-1/3}$, where μ is fixed. There, the largest strongly connected components have size of order $n^{1/3}$. Bounds on the tail probabilities of the distribution of the size of the largest component are also given by Coulson [7].

We use a slightly different model of random digraphs that has already been considered in similar contexts: first, we generate a random undirected graph according to the classical Erdős-Rényi model, where each of the possible $\binom{n}{2}$ edges between n fixed vertices is inserted with the same probability $2p$ and all edges are independent of each other. Thereafter, each edge is given a direction, where each of the two directions has probability $\frac{1}{2}$ and all choices are made independently again. Note that each possible directed edge is present in the graph with the same probability p in this model. The result is a random digraph without loops, multiple edges and 2-cycles (the latter is relevant since the presence of 2-cycles would immediately mean that the digraph is not acyclic). The random digraph generated in this way is denoted by $\mathcal{D}(n, p)$, and we ask the simple question: with what probability is $\mathcal{D}(n, p)$ acyclic? Throughout this paper, this probability will be denoted by $\mathbb{P}(n, p)$. In the case where p is of constant order, the asymptotic behaviour of $\mathbb{P}(n, p)$ can be inferred from the aforementioned results of Bender, Richmond, Robinson and Wormald. In this paper, however, we will be interested in the sparse regime, where $p = \lambda/n$ for some fixed real λ. In this case, the number of edges is only linear in n, resulting in a much higher probability of being acyclic. There is no particular reason why we chose to work with $\mathcal{D}(n, p)$ in this paper. Both models have appeared in the literature, but due to lack of space we only treat one model here. We will include $\mathbb{D}(n, p)$ in the long version of this paper.

Before we get to the statement of our main result, let us also review some related works. The model $\mathcal{D}(n, p)$ of simple random digraphs was used by Subramanian in [30], where the author studied induced acyclic subgraphs in random digraphs for fixed p. Following this work, there are also some relatively recent results on the related question of the largest acyclic subgraph in random digraphs in the stated range [9–11, 33].

The structure of the strong components of a random digraph for the $\mathbb{D}(n, p)$ model has been studied by many authors in the dense case, i.e., when $np \to \infty$ as $n \to \infty$. The largest strong components in a random digraph with a given degree sequence are studied by Cooper and Frieze [4] and the strong connectivity of an inhomogeneous random digraph was studied by Bloznelis, Göetze and Jaworski in [3]. The hamiltonicity of $\mathbb{D}(n, p)$ was investigated by Hefetz, Steger and Sudakov [18] and by Ferber, Nenadov, Noever, Peter and Škorić [13], by Cooper, Frieze and Molloy [5] and by Ferber, Kronenberg and Long [12]. Krivelevich, Lubetzky and Sudakov [21] also proved the existence of cycles of linear size with high probability (w.h.p.) when np is large enough.

Interestingly, since the enumeration of acyclic digraphs by Robinson [31] and the asymptotic results on acyclic digraphs by Bender et al. [1, 2], dense random graphs have been the focus of research in this context. However, a forthcoming independent approach of De Panafieu and Dovgal [8] gives a characterization of the probability that a digraph is acyclic inside the critical window using techniques from analytic combinatorics and the uniform model for digraphs.

Returning to the functional equation relating $A(x, y)$ and $\phi(x, y)$ in (2), it is clear that the behaviour of the zeros of $\phi(x, y)$ plays an important role in the study of acyclic digraphs. Here, by a zero of $\phi(x, y)$ we mean a function $x = x(y)$ that satisfies $\phi(x, y) = 0$. The properties of these zeros are certainly interesting in their own right. There are some known results in this direction. It is, for example, known that all zeros of ϕ are real, positive and distinct when $y > 0$, see [22, 29]. For a given $y > 0$ and $j \in \mathbb{N}$, let $\varrho_j(y)$ be the j-th smallest solution to the equation $\phi(x, y) = 0$. So, as mentioned before, we have $\varrho_1(1) \approx 1.48808$. Grabner and Steinsky [16] studied the behaviour of the other zeros of $\phi(x, 1)$, extending the work of Robinson. Our first result provides asymptotic formulas for the zeros of ϕ as $y \to 0^+$.

▶ **Theorem 1.** *Let $\phi(x, y)$ be the function defined in* (2). *For a given y, let $\varrho_j(y)$ be the solution to the equation $\phi(x, y) = 0$ that is the j-th closest to zero. If $j \in \mathbb{N}$ is fixed, then we have*

$$\varrho_j(y) = \frac{1}{e} y^{-1} - \frac{a_j}{2^{1/3} e} y^{-1/3} - \frac{1}{6e} + O(y^{1/3}), \quad \textit{as } y \to 0^+, \tag{3}$$

where a_j is the zero of the Airy function $\mathrm{Ai}(z)$ that is j-th closest to 0. Furthermore, we have the following estimate for the partial derivative of $\phi(x, y)$ at $\varrho_j(y)$:

$$\phi_x(\varrho_j(y), y) \sim -\kappa_j\, y^{1/6} \exp\left(-\tfrac{1}{2} y^{-1} + 2^{-1/3} a_j\, y^{-1/3} \right) \ \textit{as } y \to 0^+, \tag{4}$$

where

$$\kappa_j = \pi^{1/2} 2^{7/6} e^{11/12} \mathrm{Ai}'(a_j).$$

Using Theorem 1, we are able to obtain the following result on the probability that $\mathcal{D}(n, p)$ has no directed cycles.

▶ **Theorem 2.** *Let $p = \lambda/n$ with $\lambda \geq 0$ fixed. Then, the probability $\mathbb{P}(n, p)$ that a random digraph $\mathcal{D}(n, p)$ is acyclic satisfies the following asymptotic formulas as $n \to \infty$:*

$$\mathbb{P}(n, p) \sim \begin{cases} (1-\lambda) e^{\lambda + \lambda^2/2} & \textit{if } 0 \leq \lambda < 1, \\ \gamma_1 n^{-1/3} & \textit{if } \lambda = 1, \\ \gamma_2 n^{-1/3} e^{-c_1 n - c_2 n^{1/3}} & \textit{if } \lambda > 1, \end{cases} \tag{5}$$

where

$$\gamma_1 = \frac{2^{-1/3} e^{3/2}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\mathrm{Ai}(-i 2^{1/3} t)} \mathrm{d}t \approx 2.19037,$$

$$\gamma_2 = \tfrac{2^{-2/3}}{\mathrm{Ai}'(a_1)} \lambda^{5/6} e^{-\lambda^2/4 + 8\lambda/3 - 11/12},$$

$$c_1 = \tfrac{\lambda^2 - 1}{2\lambda} - \log \lambda,$$

$$c_2 = 2^{-1/3} a_1 \lambda^{-1/3} (1 - \lambda),$$

and a_1 is the zero of the Airy function $\mathrm{Ai}(z)$ with the smallest modulus.

We are also able to determine an asymptotic formula for the probability $\mathbb{P}(n, p)$ in the critical window, i.e., when $np = 1 + \mu n^{-1/3}$ and μ is bounded. This result is formulated in the next theorem.

▶ **Theorem 3.** *If $np = 1 + \mu n^{-1/3}$ such that μ is contained in a fixed bounded real interval, then*

$$\mathbb{P}(n, p) = (\varphi(\mu) + o(1)) n^{-1/3}, \quad \textit{as } n \to \infty, \tag{6}$$

where

$$\varphi(\mu) = 2^{-1/3} e^{3/2 - \mu^3/6} \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-\mu s}}{\mathrm{Ai}(-2^{1/3} s)} \mathrm{d}s. \tag{7}$$

The term that follows after "$\times$" in Equation (7) is an inverse (two-sided) Laplace transform. Hence, the function $2^{1/3}e^{\mu^3/6-3/2}\varphi(\mu)$ can be interpreted as the inverse (two-sided) Laplace transform of the function $\mathrm{Ai}(-2^{1/3}s)^{-1}$. We provide a numerical plot of $\varphi(\mu)$ in Figure 1.

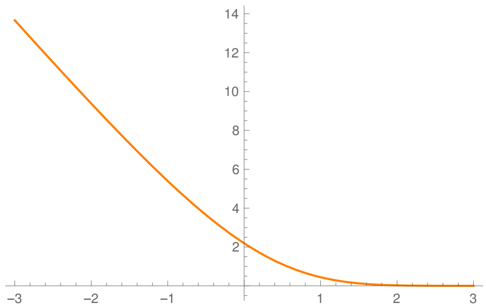

Figure 1 Numerical plot of $\varphi(\mu)$.

Throughout this paper, we use the Vinogradov notation $\ll$ interchangeably with the O-notation, i.e., as $x \to a$ (resp. $x \to \infty$), $f(x) \ll g(x)$ and $f(x) = O(g(x))$ both mean that there exists $C > 0$ independent of x such that $|f(x)| \le Cg(x)$ for all x sufficiently close to a (resp. all sufficiently large $x > 0$).

2 Estimates of $\phi(x, y)$ and its zeros

The main ingredients in the proofs of our theorems are asymptotic estimates for $\phi(x, y)$ as $y \to 0^+$, for various ranges of x, including complex values. These estimates are given in Proposition 4 and Proposition 7. The proofs of these propositions are long and rather technical, so we will not include them in this extended abstract. However, sketched proofs are provided in the Appendix. The proofs are based on the saddle-point method as it is possible to express $\phi(x, y)$ in an integral form via a formula due to Mahler [27].

2.1 Mahler's transformation

The function $\phi(x, y)$ can be expressed in terms of the function $F(z)$ in [27, Equation (6)]. In fact, they are equal if we set $z = -x$ and $q = (1 + y)^{-1}$. Thus using the integral form of $F(z)$ in [27, Equation (4)] we obtain the following formula:

$$\phi(x, y) = \sqrt{\frac{\log(1+y)}{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\log(1+y)z^2 - x(1+y)^{1/2-iz}\right) \mathrm{d}z. \tag{8}$$

It is worth noting that this equation can also derived from [14, Lemma 1]. To simplify this expression, from now on, we shall use the abbreviations

$$\alpha := \log(1+y) \quad \text{and} \quad \beta := \sqrt{1+y}. \tag{9}$$

Moreover, by making the change of variable $z \mapsto z/\alpha$, we can rewrite Equation (8) as

$$\phi(x, y) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} e^{f(z)} \mathrm{d}z, \quad \text{where } f(z) := -\frac{1}{2\alpha}z^2 - x\beta e^{-iz}. \tag{10}$$

The function f depends on the variables x and y, but we drop these dependencies in the notation for easy reading. In addition, when we say derivative of f, we always mean derivative with respect to z. In the rest of this section, we assume that x is real.

2.2 Saddle-point method

The integral in the formula for $\phi(x, y)$ in (10) is an integral over the real line. However, since the function f is entire as a function of z, we can change this path of integration without affecting the validity of the equation (Figure 2 in the appendix shows the paths that we considered). This allows us to apply the saddle-point method to the integral (10). The objective is to find a path that goes through a saddle-point, i.e., a solution of $f'(z) = 0$. Since the derivative of f is $f'(z) = -\frac{1}{\alpha}z + ix\beta e^{-iz}$, we can see that $f'(z) = 0$ if and only if

$$ize^{iz} = -x\alpha\beta. \tag{11}$$

Hence, the solutions can be expressed in terms of the branches of the Lambert-W function, which is implicitly defined by the equation $\mathrm{W}(s)e^{\mathrm{W}(s)} = s$. We choose a solution to Equation (11) that is given by the principal branch of W. So, set

$$w := W_0(-x\alpha\beta) \text{ and } z_0 := -iw, \tag{12}$$

where W_0 is the principal branch of the Lambert function. Note that z_0 still depends on the variables x and y. The fact that the Lambert function $W_0(z)$ has a singularity at $z = -1/e$ suggests that we should consider x to be a function of y such that $x\alpha\beta$ is close to $1/e$. Motivated by this, let us define x_0 and δ such that

$$x_0 = \frac{1}{e\alpha\beta} \text{ and } x = (1+\delta)x_0. \tag{13}$$

With this setting, we are now able to give asymptotic estimates of $\phi(x, y)$ when $y \to 0^+$ for several ranges of δ. This result is summarized in the following proposition.

▶ **Proposition 4.** *If x is of the form $x = (1 + \delta)x_0$, then $\phi(x, y)$ satisfies the following asymptotic formulas as $y \to 0^+$:*

(a) *If $\delta \geq -1$ and $\delta = -1 + o(1)$, then*

$$\phi(x, y) \sim e^{\frac{1}{2\alpha}(w^2+2w)}. \tag{14}$$

(b) *If $\delta < 0$ and $\alpha^{2/3} \ll |\delta| \leq 1 - \varepsilon$ for some constant $\varepsilon > 0$, then*

$$\phi(x, y) \sim 2^{5/6}\pi^{1/2}\alpha^{-1/6}|w|^{-1/3}\mathrm{Ai}(R)e^{\frac{2}{3}R^{3/2}+\frac{1}{2\alpha}(w^2+2w)} \tag{15}$$

where

$$R = 2^{-2/3}(1+w)^2 w^{-4/3}\alpha^{-2/3}.$$

(c) *If we let $\delta = \theta\alpha^{2/3}$, then*

$$\phi(x, y) = 2^{-1/2}\pi^{-1/2}\alpha^{-1/6}\left(K_1(\theta) + K_2(\theta)\,\alpha^{1/3} + O(\alpha^{2/3})\right) e^{-\frac{1}{2}\alpha^{-1}-\theta\alpha^{-1/3}}, \tag{16}$$

uniformly for θ in any fixed bounded closed interval, where

$$K_1(\theta) = \pi 2^{4/3}\mathrm{Ai}(-2^{1/3}\theta),$$
$$K_2(\theta) = \tfrac{5}{3}\pi\, 2^{1/3}\theta^2\mathrm{Ai}(-2^{1/3}\theta) - \tfrac{1}{3}\pi 2^{2/3}\mathrm{Ai}'(-2^{1/3}\theta).$$

Proof. A sketch of the proof is given in the appendix. ◀

Observe that there is an overlap in the conditions of Part (b) and Part (c), but one can show that the asymptotic formulas (15) and (16) agree in the overlap. To check this, one needs to use the classic asymptotic formula for the Airy function $\mathrm{Ai}(z)$ as well as the asymptotic formula for $W_0(z)$ near its singularity $-1/e$. These are well known facts, see for example [28, (9.7.5)] and [6, (4.22)]. Since these estimates will be referred to quite often in this paper, let us state them here. For any $\varepsilon > 0$,

$$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}} \quad \text{as } |z| \to \infty, \text{ and } |\mathrm{Arg}(z)| \le \pi - \varepsilon. \tag{17}$$

As for the Lambert function, as $z \to -1/e$, we have

$$W_0(z) = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 + \cdots \tag{18}$$

where $p = \sqrt{2(ez+1)}$ (here, $\sqrt{\cdot}$ denotes the principal branch of the square root function).

We are now ready to prove Theorem 1.

2.3 Proof of Theorem 1

Proof. We already know that the zeros of $\phi(x, y)$ are real and positive. Observe that the main terms of $\phi(x, y)$ in Part (a) and Part (b) of Proposition 4 cannot vanish (in Part (b), R is always positive, which implies $\mathrm{Ai}(R) \neq 0$). However for Part (c), the term $K_1(\theta)$ in (16) can be zero, and this happens precisely when $\theta = -2^{-1/3}a_j$, where a_j is one of the zeros of $\mathrm{Ai}(z)$.

If we let $x = (1 + \theta\alpha^{2/3})x_0$, and make θ vary in a small interval around $-2^{-1/3}a_j$, then the main term of $\phi(x, y)$ changes sign. So by the intermediate value theorem there must be a zero close to $(1 - 2^{-1/3}a_j\alpha^{2/3})x_0$. The asymptotic formula of such a zero can be obtained by a simple bootstrapping argument using (16). This eventually gives an asymptotic formula of the form

$$\frac{1}{e}y^{-1} - \frac{a_j}{2^{1/3}e}y^{-1/3} - \frac{1}{6e} + O(y^{1/3}), \quad \text{as } y \to 0^+. \tag{19}$$

To show that there is only one zero that satisfies this asymptotic formula for every a_j, we make use of the functional equation

$$\phi_x(x, y) = -\phi\left((1 + y)^{-1}x, y\right),$$

which follows easily from the definition of $\phi(x, y)$ in (2). Now, suppose that there are two different zeros ϱ' and ϱ'' that both satisfy (19) for the same j. Then by Rolle's theorem, there exists C between ϱ' and ϱ'' (which also means that C satisfies the asymptotic formula (19)) such that $(1 + y)^{-1}C$ is a zero of ϕ. This leads to a contradiction, because if C satisfies (19) then $(1+y)^{-1}C$ does not (not even if a_j is replaced by another zero of the Airy function) if y is sufficiently small. So $(1 + y)^{-1}C$ cannot be a zero of $\phi(x, y)$.

Now that we have established that there is only one zero of $\phi(x, y)$ that satisfies (19) for each fixed $j \in \mathbb{N}$ and sufficiently small y, we name it $\varrho_j(y)$. Finally, to estimate $\phi_x(\varrho_j(y), y)$ as $y \to 0^+$, we make use of the above functional equation again, which gives us

$$\phi_x(\varrho_j(y), y) = -\phi\left((1 + y)^{-1}\varrho_j(y), y\right).$$

Then, we use (19) and (16) to estimate the right-hand side. ◀

3 Proving Theorem 2 and Theorem 3

3.1 Case $0 \leq \lambda < 1$

▶ **Lemma 5.** *Consider the random digraph $\mathcal{D}(n,p)$ with $p = \lambda/n$ and $0 \leq \lambda < 1$ fixed. Let X_n be the total number of (directed) cycles in this graph. Then,*
(a) *w.h.p., all strong components of $\mathcal{D}(n,p)$ are either cycles or single vertices.*
(b) *the number of vertices on a cycle is at most ω, for any $\omega(n) \to \infty$.*
(c) *X_n converges in distribution to $\mathrm{Po}(-\log(1-\lambda) - \lambda - \frac{\lambda^2}{2})$.*

Proof. If there is a strong component that is not a cycle or a single vertex, then there are three internally disjoint paths connecting two vertices u and v such that two of them do not have the same orientation or there are two directed cycles with a common vertex. The expected number of such components is bounded above by

$$2\binom{n}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\binom{n}{i}i!p^{i+1}\binom{n}{j}j!p^{j+1}\binom{n}{k}k!p^{k+1} + \binom{n}{1}\sum_{i=2}^{n}\sum_{j=2}^{n}\binom{n}{i}i!p^{i+1}\binom{n}{j}j!p^{j+1}$$
$$\leq \frac{\lambda^3}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda^{i+j+k} + \frac{\lambda^2}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda^{i+j} = O(n^{-1}).$$

By the Markov inequality, this means that there are, w.h.p., no such components.

For (b), we can bound the expected number of cycles of length larger than ω by

$$\sum_{k=\omega}^{n}\binom{n}{k}(k-1)!p^k = \sum_{k=\omega}^{n}\frac{\prod_{i=0}^{k-1}(n-i)}{n^k}\frac{\lambda^k}{k} \leq \sum_{k=\omega}^{n}\lambda^k = O\left(\lambda^\omega\right).$$

As $0 < \lambda < 1$, (b) follows from the Markov inequality.

Now to tackle (c), we compute first the expectation of X_n. Here, we have

$$\mathbb{E}\left[X_n\right] = \sum_{k=3}^{n}\binom{n}{k}(k-1)!p^k.$$

It follows that

$$\lim_{n\to\infty}\mathbb{E}(X_n) = \lim_{n\to\infty}\sum_{k=3}^{n}\frac{\prod_{i=0}^{k-1}(n-i)}{n^k}\frac{\lambda^k}{k} \sim \sum_{k=3}^{\infty}\frac{\lambda^k}{k} = -\log(1-\lambda) - \lambda - \frac{\lambda^2}{2} = a(\lambda).$$

Since the falling factorial $(X_n)_r = X_n(X_n-1)\cdots(X_n-r+1)$ counts the number of ordered r-tuples of r disjoint cycles, the r-th factorial moment of X_n is

$$\mathbb{E}\left[(X_n)_r\right] = \sum_{k_1=3}^{n}\sum_{k_2=3}^{n-k_1}\cdots\sum_{k_r=3}^{n-\sum_{i=1}^{r-1}k_i}\binom{n}{k_1,k_2,\ldots,k_r,n-k_1-\cdots-k_r}\prod_{i=1}^{r}(k_i-1)!p^{k_i}.$$

Without going into the technical details, one can now use the statement in (b) to show that the summations can be taken to ∞. One finds that for fixed $r \geq 2$, the r-th factorial moment $\mathbb{E}\left[(X_n)_r\right]$ is asymptotically equivalent to $a(\lambda)^r$ as $n \to \infty$. So by means of [19, Corollary 6.8], we have convergence to a Poisson distribution of parameter $a(\lambda)$. ◀

Now, the case $0 \leq \lambda < 1$ of Theorem 2 is a simple consequence of Lemma 5. Indeed Part (c) of Lemma 5 implies that

$$\lim_{n\to\infty}\mathbb{P}(n,p) = \lim_{n\to\infty}\mathbb{P}\left(X_n = 0\right) = e^{-a(\lambda)} = (1-\lambda)e^{\lambda+\lambda^2/2}.$$

3.2 Case $\lambda \geq 1$

3.2.1 Preliminaries

Let us begin with a crucial lemma which relates the probability $\mathbb{P}(n,p)$ to the coefficient $[x^n]A(x,y)$.

▶ **Lemma 6.** *The probability $\mathbb{P}(n,p)$ that a random digraph $\mathcal{D}(n,p)$ is acyclic is given by*

$$\mathbb{P}(n,p) = n!(1-p)^{\binom{n}{2}}[x^n]A\left(x, \tfrac{p}{1-2p}\right). \tag{20}$$

Proof. Define $A_n(y) = \sum_{m=0}^{\binom{n}{2}} a_{n,m}y^m$ where $a_{n,m}$ is the number of acyclic digraphs with n vertices and m edges defined in (1). Therefore, we have

$$A(x,y) = \sum_{n\geq 0} A_n(y)(1+y)^{-\binom{n}{2}}\frac{x^n}{n!} \quad \text{and} \quad A_n(y) = n!(1+y)^{\binom{n}{2}}[x^n]A(x,y). \tag{21}$$

Since $\mathbb{P}(n,p)$ is defined to be the probability that $\mathcal{D}(n,p)$ is acyclic, we can express it as $\mathbb{P}(n,p) = \sum_D \mathbb{P}(\mathcal{D}(n,p) = D)$, where the sum runs over all acyclic digraphs on n fixed vertices. The probability $\mathbb{P}(\mathcal{D}(n,p) = D)$ does not depend on the structure of D but only on its number of edges. Hence, by distinguishing the number of edges, we have

$$\begin{aligned}
\mathbb{P}(n,p) &= \sum_{m=0}^{\binom{n}{2}} a_{n,m}p^m(1-2p)^{\binom{n}{2}-m} \\
&= (1-2p)^{\binom{n}{2}} \sum_{m=0}^{\binom{n}{2}} a_{n,m}\left(\frac{p}{1-2p}\right)^m \\
&= (1-2p)^{\binom{n}{2}} A_n\left(\frac{p}{1-2p}\right).
\end{aligned}$$

Applying (21) with y replaced by $p/(1-2p)$, we get after a bit of algebra that the last term is the same as the right-hand side of Equation (20). ◀

By Lemma 6, it suffices to estimate the coefficient $[x^n]A(x,y)$ when y is of order n^{-1}. To this end, we use the Cauchy integral formula

$$[x^n]A(x,y) = \frac{1}{2\pi i}\oint_{|x|=\rho}\frac{A(x,y)}{x^{n+1}}\mathrm{d}x = \frac{1}{2\pi i}\oint_{|x|=\rho}\frac{1}{\phi(x,y)x^{n+1}}\mathrm{d}x, \tag{22}$$

where $0 < \rho < \varrho_1(y)$. Notice that x here is a complex variable, so in order to estimate $[x^n]A(x,y)$ via the above integral, we need an estimate of $\phi(x,y)$ where x is complex and $y \to 0^+$. This is done in the next proposition.

▶ **Proposition 7.** *Let θ be a fixed real number which satisfies $\mathrm{Ai}(-2^{1/3}\theta) \neq 0$ and let $\delta = \theta\alpha^{2/3}$. Moreover, let*

$$x = (1+\delta)x_0e^{iu}, \ \textit{and}\ w = W_0(-(1+\delta)e^{iu-1}).$$

Then, we have the following asymptotic formulas for $\phi(x,y)$ as $y \to 0^+$:
(a) *If $\alpha^{1/2} \ll |u| \leq \pi$, then*

$$\phi(x,y) \sim \frac{e^{\frac{1}{2\alpha}(w^2+2w)}}{\sqrt{1+w}}. \tag{23}$$

(b) *If* $u = t\alpha^{2/3}$ *and* $1 \ll |t| \ll \alpha^{-1/6}$, *then*

$$\phi(x, y) \sim \pi^{1/2}\ 2^{5/6}\alpha^{-1/6}\mathrm{Ai}(-2^{1/3}(\theta + it))e^{-\frac{1}{2}\alpha^{-1}-(\theta+it)\alpha^{-1/3}-\frac{5}{6}\alpha^{1/3}t^2}. \tag{24}$$

(c) *If* $u = t\alpha^{2/3}$, *then the estimate*

$$\phi(x, y) \sim \pi^{1/2}\ 2^{5/6}\alpha^{-1/6}\mathrm{Ai}(-2^{1/3}(\theta + it))e^{-\frac{1}{2}\alpha^{-1}-(\theta+it)\alpha^{-1/3}} \tag{25}$$

holds uniformly for t *in any bounded closed interval on* $\mathbb{R}$.

Proof. A sketch of the proof is given in the appendix. ◀

▶ Remark 8. Once again, one can verify that these asymptotic formulas agree in those regions where conditions overlap.

The next lemma is a direct consequence of Proposition 7, which will be useful to estimate the integral in (22).

▶ **Lemma 9.** *Let* θ *be a fixed real number such that* $\mathrm{Ai}(-2^{1/3}\theta) \neq 0$, *and let* $\rho = (1+\theta\alpha^{2/3})x_0$. *Then, as* $y \to 0^+$,

$$\frac{1}{2\pi i}\oint_{|x|=\rho}\frac{1}{\phi(x,y)x^{n+1}}\mathrm{d}x \ll \frac{\alpha^{2/3}|\mathrm{Ai}(-2^{1/3}\theta)|}{2\pi\phi(\rho,y)\rho^n}\int_{-\infty}^{\infty}\frac{1}{|\mathrm{Ai}(-2^{1/3}(\theta+it))|}\mathrm{d}t, \tag{26}$$

where the implied constant is independent of n. *If we assume further that* n *and* α *are connected by a relation of the form* $n = \alpha^{-1} + b\alpha^{-2/3}$, *where* b *can be a function of* α *but with* $b = O(1)$, *then we have*

$$\frac{1}{2\pi i}\oint_{|x|=\rho}\frac{1}{\phi(x,y)x^{n+1}}\mathrm{d}x = \frac{\alpha^{2/3}\mathrm{Ai}(-2^{1/3}\theta)}{2\pi\phi(\rho,y)\rho^n}\left(\int_{-\infty}^{\infty}\frac{e^{-ibt}}{\mathrm{Ai}(-2^{1/3}(\theta+it))}\mathrm{d}t + o(1)\right). \tag{27}$$

Proof. We will only present the proof of the second estimate, which is the harder one, the idea of the proof of the first estimate will be very similar but simpler since it is only an upper bound. First, we have

$$\frac{1}{2\pi i}\oint_{|x|=\rho}\frac{1}{\phi(x,y)x^{n+1}}\mathrm{d}x = \frac{1}{2\pi\phi(\rho,y)\rho^n}\int_{-\pi}^{\pi}\frac{\phi(\rho,y)e^{-iun}}{\phi(\rho e^{iu},y)}\mathrm{d}u.$$

Next, we choose a fixed constant $c \in (1/2, 2/3)$, and we split the integral on the right-hand side into three pieces corresponding to each of the following ranges of u: $|u| \le \alpha^c$, $\alpha^c < |u| \le \alpha^{1/2}$, and $\alpha^{1/2} < |u| \le \pi$. Let us now treat these cases separately.

- If $\alpha^{1/2} < |u| \le \pi$, then we can use Part (a) of Proposition 7 to estimate $|\phi(\rho e^{iu}, y)|$ and Part (c) of Proposition 4 to estimate $|\phi(\rho, y)|$. We get

  $$\left|\frac{\phi(\rho,y)e^{-iun}}{\phi(\rho e^{iu},y)}\right| \ll \alpha^{-1/6}\sqrt{|1+w|}\ e^{-\frac{1}{2}\alpha^{-1}\mathrm{Re}((1+w)^2)-\theta\alpha^{-1/3}},$$

  with w as defined in (12). Note that w is bounded in this case. Moreover, one can show (see Lemma 10 in the appendix) that $\mathrm{Re}((1 + w)^2)$ remains positive if u is bounded away from zero, and by means of (18) (with $p = \sqrt{2(1 - (1 + \theta\alpha^{2/3})e^{iu})}$), one gets

  $$\mathrm{Re}((1+w)^2) = -2\theta\alpha^{2/3} + (1+o(1))\frac{4}{3}|u|^{3/2}, \tag{28}$$

  if $u \to 0$ and $|u| \ge \alpha^{1/2}$. Hence, we have

  $$-\frac{1}{2}\alpha^{-1}\mathrm{Re}((1+w)^2) - \theta\alpha^{-1/3} = -(\tfrac{2}{3} + o(1))\alpha^{-1}|u|^{3/2} \gg \alpha^{-1/4}$$

  uniformly for $\alpha^{1/2} < |u| \le \pi$. This implies that the contribution from $\alpha^{1/2} < |u| \le \pi$ to the above integral tends to zero exponentially quickly.

- If $\alpha^c < |u| \le \alpha^{1/2}$, then we use Part (b) of Proposition 7 to estimate $|\phi(\rho e^{iu}, y)|$. We obtain

  $$\left| \frac{\phi(\rho, y)e^{-iun}}{\phi(\rho e^{iu}, y)} \right| \sim \frac{e^{\frac{5}{6}\alpha^{1/3}t^2}}{|\mathrm{Ai}(-2^{1/3}(\theta + it))|},$$

  where $t = u\alpha^{-2/3}$. So, in particular $|t| > \alpha^{c-2/3}$. But since we chose $c < 2/3$, we have $|t| \to \infty$. Since θ is fixed, we have $|\mathrm{Arg}(-2^{1/3}(\theta + it))| \to \pi/2$. Therefore, by the asymptotic formula (17) for the Airy function, the contribution from $\alpha^c < |u| \le \alpha^{1/2}$ to the integral above is also tending to zero exponentially fast in α.
- If $|u| \le \alpha^c$, then we let $u = t\alpha^{2/3}$. So $\alpha^{1/3}t^2 = O(\alpha^{2c-1})$, which tends to zero. Also by the assumption on n in the statement of the lemma, we have

  $$-iun = -it\alpha^{-1/3} - ibt.$$

  Using Part (c) of Proposition 7 and $n = \alpha^{-1} + b\alpha^{-2/3}$, we get

  $$\frac{\phi(\rho, y)e^{-iun}}{\phi(\rho e^{iu}, y)} \sim \frac{\mathrm{Ai}(-2^{1/3}\theta)e^{-ibt}}{\mathrm{Ai}(-2^{1/3}(\theta + it))}.$$

  By making the change of variable $u = t\alpha^{2/3}$, we obtain

  $$\int_{-\alpha^c}^{\alpha^c} \frac{\phi(\rho, y)e^{-iun}}{\phi(\rho e^{iu}, y)}\mathrm{d}u \sim \alpha^{2/3}\mathrm{Ai}(-2^{1/3}\theta) \int_{-\alpha^{c-2/3}}^{\alpha^{c-2/3}} \frac{e^{-ibt}}{\mathrm{Ai}(-2^{1/3}(\theta + it))}\mathrm{d}t.$$

  Once again, by the asymptotic formula (17) for the Airy function, the integral on the right-hand side can be extended to infinity with an exponentially small error term, uniformly in b.

The proof of the asymptotic formula (27) is complete. ◀

We now have everything we need to complete the proof of Theorem 2 and to prove Theorem 3. Since the case $\lambda = 1$ in Theorem 2 is a particular case of Theorem 3, it does not need to be treated separately. Let us begin with a proof of Theorem 3.

3.2.2 Case $\lambda = 1 + \mu n^{-1/3}$

Let $np = 1 + \mu n^{-1/3}$, where μ is contained in a fixed bounded interval. In view of Lemma 6 we set $y = p/(1 - 2p)$, and with $\alpha = \log(1 + y)$, we can show that n satisfies

$$n = \alpha^{-1} + \mu\alpha^{-2/3} + O(\alpha^{-1/3}).$$

We choose $\rho = x_0 = \frac{1}{e\alpha\beta}$ (which is smaller than $\varrho_1(y)$). Hence, by Equation (27) of Lemma 9 (with $\theta = 0$), we have

$$[x^n]A(x, y) = \frac{\alpha^{2/3}\mathrm{Ai}(0)}{2\pi\phi(\rho, y)\rho^n} \left(\int_{-\infty}^{\infty} \frac{e^{-i\mu t}}{\mathrm{Ai}(-2^{1/3}it)}\mathrm{d}t + o(1) \right).$$

Thus Lemma 6 yields

$$\mathbb{P}(n, p) = n!(1 - p)^{\binom{n}{2}} \frac{\alpha^{2/3}\mathrm{Ai}(0)}{2\pi\phi(\rho, y)\rho^n} \left(\int_{-\infty}^{\infty} \frac{e^{-i\mu t}}{\mathrm{Ai}(-2^{1/3}it)}\mathrm{d}t + o(1) \right).$$

We apply Part (c) of Proposition 4 (or Part (c) of Proposition 7) to estimate $\phi(\rho, y)$, then write everything in terms of n. With the help of a computer algebra system (we used asymptotic expansions in SageMath [35]) we obtain the asymptotic formula

$$\mathbb{P}(n, p) = 2^{-1/3}e^{3/2-\mu^3/6}\, n^{-1/3} \times \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{e^{-i\mu t}}{\mathrm{Ai}(-2^{1/3}it)}\mathrm{d}t + o(1) \right)$$

as $n \to \infty$, which is equivalent to the estimate of $\mathbb{P}(n, p)$ in Theorem 3.

3.2.3 Case $\lambda > 1$

Let $\lambda > 1$ be a fixed real number, and let $np = \lambda$. We follow the same argument as above, but we choose $\rho = (1 + \theta\alpha^{2/3})x_0$, where θ is fixed and satisfies $-a_1 < 2^{1/3}\theta < -a_2$. This implies that $\mathrm{Ai}(-2^{1/3}\theta) \neq 0$ and that $\varrho_1(y) < \rho < \varrho_2(y)$. Hence, by the residue theorem, we have

$$[x^n]A(x,y) = -\frac{1}{\varrho_1(y)^{n+1}\phi_x(\varrho_1(y),y)} + \frac{1}{2\pi i}\oint_{|x|=\rho} \frac{1}{\phi(x,y)x^{n+1}}\mathrm{d}x. \tag{29}$$

Then, we use (26) of Lemma 9 to estimate the integral on the right-hand side, so we get

$$[x^n]A(x,y) = -\frac{1}{\varrho_1(y)^{n+1}\phi_x(\varrho_1(y),y)} + O\left(\frac{\alpha^{2/3}}{|\phi(\rho,y)|\rho^n}\right).$$

In view of the formula $\mathbb{P}(n,p) = n!(1-p)^{\binom{n}{2}}[x^n]A(x,y)$, we can express the contribution to $\mathbb{P}(n,p)$ from each of the terms above in terms of n: we use Theorem 1 to estimate $\varrho_1(y)$ and $\phi_x(\varrho_1(y),y)$ and Part (c) of Proposition 4 to estimate $\phi(\rho,y)$. Then, with the help of a computer algebra system, we obtain

$$-\frac{n!(1-p)^{\binom{n}{2}}}{\varrho_1(y)^{n+1}\phi_x(\varrho_1(y),y)} \sim \gamma_2 n^{-1/3} e^{-c_1 n + 2^{1/3}a_1\lambda^{-1/3}(\lambda-1)n^{1/3}}, \tag{30}$$

$$\frac{n!(1-p)^{\binom{n}{2}}\alpha^{2/3}}{|\phi(\rho,y)|\rho^n} \ll n^{-1/3} e^{-c_1 n - \theta\lambda^{-1/3}(\lambda-1)n^{1/3}}, \tag{31}$$

where the constants γ_2 and c_1 are precisely as defined in Theorem 2. Since we chose θ in such a way that $\theta > -2^{-1/3}a_1$, the left-hand side of (31) is exponentially (in $n^{1/3}$) smaller than that of (30) as $n \to \infty$. Therefore, the right-hand side of (30) is indeed the main term of $\mathbb{P}(n,p)$.

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A Sketched Proof of Proposition 4

The proof is based on the saddle-point method applied to the integral in the formula for $\phi(x,y)$ in (10). Recall that we set $x = (1+\delta)x_0$, where $x_0 = (e\alpha\beta)^{-1}$. Note that the case $\delta = -1$ corresponds to $x = 0$, and so it immediately follows from the definition of $\phi(x,y)$ in (2) that $\phi(0,y) = 1$. Hence, we may assume that $\delta > -1$, but it can be a function of α. Recall the saddle-point z_0 defined in (12), $z_0 = -iw$, where $w = \mathrm{W}_0(-(1+\delta)e^{-1})$. One can see that if δ goes from -1 to 0, then w goes from 0 to -1. So if $\delta \in (-1, 0]$, then the point z_0 lies on the segment $[0,\ i]$. The Taylor series around z_0 is

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + f''(z_0)\frac{(z-z_0)^2}{2!} + f'''(z_0)\frac{(z-z_0)^3}{3!} + \cdots$$

where

$$f(z_0) = -\frac{1}{2\alpha}z_0^2 + \frac{iz_0}{\alpha} = \frac{1}{2\alpha}(w^2+2w) \tag{32}$$

$$f''(z_0) = -\frac{1}{\alpha}(1+iz_0) = -\frac{1}{\alpha}(1+w) \tag{33}$$

$$f'''(z_0) = -\frac{z_0}{\alpha} = \frac{i}{\alpha}w \tag{34}$$

$$|f^{(k)}(\eta+z_0)| = \frac{|w|}{\alpha} \le \frac{1}{\alpha} \text{ for every } \eta \in \mathbb{R} \text{ and } k \ge 4. \tag{35}$$

Looking at the first few terms in the Taylor series above, note that the quadratic term $f''(z_0)(z-z_0)^2$ also vanishes when $z_0 = i$. So we proceed as follows: if z_0 is sufficiently far from i (for our case it means $|z_0 - i| \gg \alpha^{2/3}$), then we shift the path of integration to the horizontal line passing through z_0, and if $|z_0 - i| \ll \alpha^{2/3}$, then we take the path Γ on the right in Figure 2.

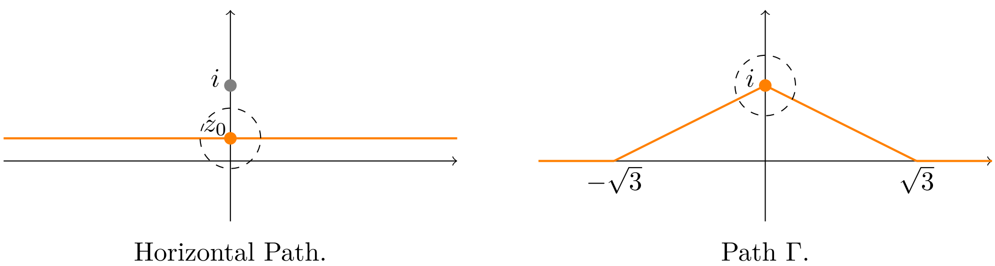

Figure 2 Two Paths.

We can use the path Γ in the integral (10) since $f(z)$ is an entire function. But we can also shift the path of integration to any horizontal line. To see this, for any real numbers a, b, (assuming that x is real for now) we have

$$\mathrm{Re}(f(a+ib)) = -\frac{1}{2\alpha}(a^2 - b^2) - x\beta e^b \cos(a).$$

This implies that $|e^{f(z)}| = e^{\mathrm{Re}(f(z))}$ tends to zero exponentially fast as $|\mathrm{Re}(z)| \to \infty$ on any fixed horizontal strip for $\alpha > 0$. Hence we can shift the path of integration within this strip.

Before we begin proving each part of Proposition 4, let us adopt some terminology. The dashed circles in the two graphs in Figure 2 represent circles of radius α^c, where c is a positive constant. But c will be chosen separately for each range of x. The part of the integral from the path within the circle will be called the *local integral* and the rest will be called the *tail*.

A.1 Part (a)

Here, we assume that $\delta > -1$, but that the asymptotic formula $\delta = -1 + o(1)$ is satisfied as $\alpha \to 0^+$. Then, we choose $c \in (1/3, 1/2)$. To estimate the local integral, we have

$$f(t+z_0) = f(z_0) + f''(z_0)\frac{t^2}{2} + O(\alpha^{3c-1}),$$

for $|t| \le \alpha^c$, where the implicit constant in the O term is independent of t. This implies that

$$\int_{-\alpha^c}^{\alpha^c} e^{f(t+z_0)}\mathrm{d}t = (1 + O(\alpha^{3c-1}))e^{f(z_0)} \int_{-\alpha^c}^{\alpha^c} e^{f''(z_0)t^2/2}\mathrm{d}t.$$

The condition $\delta = -1 + o(1)$ implies that $w = o(1)$, hence $f''(z_0) \sim -\alpha^{-1}$. This allows us to extend the range of integration in the integral above to infinity at the expense of an exponentially small error term. Thus, we have

$$\int_{-\alpha^c}^{\alpha^c} e^{f(t+z_0)}\mathrm{d}t \sim e^{f(z_0)} \int_{-\infty}^{\infty} e^{f''(z_0)t^2/2}\mathrm{d}t \sim \sqrt{\frac{2\pi}{-f''(z_0)}}e^{f(z_0)}.$$

Now for the estimate of the tail. From the definition of the function $f(z)$, we can show that

$$\mathrm{Re}(f(t+z_0)) - f(z_0) = -\frac{1}{2\alpha}\left(t^2 + 2(1-\cos t)w\right) \le -\frac{1}{2\alpha}(1+w)t^2. \tag{36}$$

The last line follows using the well known inequality $1 - \cos t \le t^2/2$ for all $t \in \mathbb{R}$ and the fact that $w < 0$. Thus, we have

$$\int_{|t|>\alpha^c} e^{f(t+z_0)}\mathrm{d}t \le e^{f(z_0)} \int_{|t|\ge\alpha^c} e^{\mathrm{Re}(f(t+z_0))-f(z_0)}\mathrm{d}t \le 2e^{f(z_0)} \int_{\alpha^c}^{\infty} e^{-(1+w)\alpha^{-1}t^2/2}\mathrm{d}t.$$

Again, since $w = o(1)$ the rightmost integral tends to zero faster than any power of α. Thus, we deduce that

$$\int_{-\infty}^{\infty} e^{f(t+z_0)}\mathrm{d}t \sim \sqrt{\frac{2\pi}{-f''(z_0)}}e^{f(z_0)}.$$

Therefore, from (10), we get $\phi(x,y) \sim e^{f(z_0)}$, which is equivalent to Part (a) of Proposition 4.

A.2 Part (b)

In this case, $\delta \in (-1, 0)$ is assumed to satisfy the condition $\alpha^{2/3} \ll |\delta| \ll 1 - \varepsilon$ for some fixed $\varepsilon > 0$. We proceed in the same manner as in the previous case, but since w can be very close to -1, $f''(z_0)$ can be small. We can still use the horizontal line passing through z_0 as our path of integration. However, this time, we do not ignore the term with $f'''(z_0)$ in the local integral. We choose $c \in (1/4, 1/3)$, and the Taylor approximation gives

$$f(t+z_0) = f(z_0) + f''(z_0)\frac{t^2}{2} + f'''(z_0)\frac{t^3}{6} + O(\alpha^{4c-1}),$$

uniformly for $|t| \le \alpha^c$. Thus,

$$\int_{-\alpha^c}^{\alpha^c} e^{f(t+z_0)}\mathrm{d}t = (1 + O(\alpha^{4c-1}))e^{f(z_0)} \int_{-\alpha^c}^{\alpha^c} e^{f''(z_0)t^2/2+f'''(z_0)t^3/6}\mathrm{d}t.$$

Let us denote the integral on the right-hand side by J. Making use of the values of $f''(z_0)$ and $f'''(z_0)$ in (33) and (34) respectively, and with the change of variable $t \mapsto \alpha^{1/3}t$, we can rewrite J in the following way:

$$J = 2\alpha^{1/3} \int_0^{\alpha^{c-1/3}} e^{-(1+w)\alpha^{-1/3}t^2/2} \cos\left(w\frac{t^3}{6}\right) \mathrm{d}t.$$

The term $1+w$ is always positive, and since $\delta \gg \alpha^{2/3}$, we can show from the asymptotic estimate (18) of the Lambert W function that $1+w \gg \alpha^{1/3}$. This implies that $(1+w)\alpha^{-1/3} \gg 1$. Adding the fact that $\alpha^{c-1/3} \to \infty$, we can extend the range of integration of J to infinity at the expense of a negligible error term. So we have

$$J \sim 2\alpha^{1/3} \int_0^{\infty} e^{-(1+w)\alpha^{-1/3}t^2/2} \cos\left(w\frac{t^3}{6}\right) \mathrm{d}t.$$

With an appropriate change of variable, the right-hand side can be written in terms of the Airy function $\mathrm{Ai}(z)$ (here we can use the integral representation of $\mathrm{Ai}(z)$ in [28, (9.5.7)]). Skipping the calculations, we have

$$J \sim 2^{4/3}\pi|w|^{-1/3}\alpha^{1/3}e^{2R^{3/2}/3}\mathrm{Ai}(R),$$

where $R = 2^{-2/3}(1+w)^2 w^{-4/3}\alpha^{-2/3}$.

Now for the estimates of the tail, we use the same argument as in the previous case. Inequality (36) is valid in this case as well, so we still have

$$\int_{|t|\geq\alpha^c} e^{f(t+z_0)}\mathrm{d}t \leq 2e^{f(z_0)} \int_{\alpha^c}^{\infty} e^{-(1+w)\alpha^{-1}t^2/2}\mathrm{d}t.$$

With $1+w \gg \alpha^{1/3}$ and $c > 1/3$ the integral on the right-hand side tends to zero faster than any power of α. Hence, the main contribution would come from the local integral if we can show that $e^{2R^{3/2}/3}\mathrm{Ai}(R)$ is bounded below by some power of α. But the definition of R given above guarantees that R is always positive, and it is bounded above by a function that is $O(\alpha^{-2/3})$. Hence, by the asymptotic formula of the Airy function $e^{2R^{3/2}/3}\mathrm{Ai}(R) \gg R^{-1/4} \gg \alpha^{-1/6}$. Therefore, we deduce that

$$\phi(x,y) \sim 2^{5/6}\pi^{1/2}|w|^{-1/3}\mathrm{Ai}(R)\alpha^{-1/6}e^{2R^{3/2}/3+f(z_0)}.$$

The latter gives the formula in Part (b) of Proposition 4.

A.3 Part (c)

This case is slightly different from the previous two. The saddle-point z_0 is too close to i so we choose the path of integration Γ shown in Figure 2. For the local integral, we consider up to the 5-th term in the Taylor approximation of the function $f(z)$ around i. We have $f'(i) = i\delta\alpha^{-1}$, $f''(i) = \delta\alpha^{-1}$, $f'''(i) = -i(1+\delta)\alpha^{-1}$ and $f^{(4)}(i) = -(1+\delta)\alpha^{-1}$. We choose $c \in (1/5, 1/3)$. For $|z-i| \leq \alpha^c$ and $z \in \Gamma$, we can write

$$e^{f(z)} = e^{f(i)+\mu_1(z-i)+\mu_3(z-i)^3}\left(1 + \zeta_2(z-i)^2 + \zeta_4(z-i)^4 + O(\alpha^{5c-1})\right),$$

where $\mu_1 = i\theta\alpha^{-1/3}$, $\mu_3 = -\frac{1}{6}i\alpha^{-1}$, $\zeta_2 = \frac{1}{2}\theta\alpha^{-1/3}$, $\zeta_4 = -\frac{1}{24}(1+\theta\alpha^{2/3})\alpha^{-1}$ (we used the fact that $\delta = \theta\alpha^{2/3}$). Let us denote by Γ_c the part of Γ that lies in the disk $|z-i| \leq \alpha^c$. For each integer $k \geq 0$, let

$$\mathcal{I}_k := \int_{\Gamma_c} (z-i)^k e^{\mu_1(z-i)+\mu_3(z-i)^3}\mathrm{d}z.$$

We parametrize Γ_c and show that the the two half-segments of Γ_c can be extended to ∞ with an error smaller than any power of α. Then, with some suitable change of variable, (skipping all details) we can express $\mathcal{I}_k$ in terms of the k-th derivative of the Airy function as follows

$$\mathcal{I}_k = -i^{k+2}\,\pi\, 2^{(k+4)/3}(1+\delta)^{-(k+1)/3}\alpha^{(k+1)/3}\mathrm{Ai}^{(k)}(R) + \cdots$$

where by "$\cdots$" we mean an exponentially small term, and

$$R = i\mu_1(3i\mu_3)^{-1/3} = -2^{1/3}\delta(1+\delta)^{-1/3}\delta\alpha^{-2/3} = -2^{1/3}\theta\,(1+\delta)^{-1/3}.$$

Thus, we obtain

$$\begin{aligned}
\zeta_0\,\mathcal{I}_0 &= \pi\, 2^{4/3}(1+\delta)^{-1/3}\alpha^{1/3}\mathrm{Ai}(R) + \cdots\\
\zeta_2\,\mathcal{I}_2 &= -2\pi\theta\,(1+\delta)^{-1}\alpha^{2/3}\mathrm{Ai}^{(2)}(R) + \cdots\\
\zeta_4\,\mathcal{I}_4 &= -\tfrac{1}{3}\pi\, 2^{-1/3}(1+\delta)^{-2/3}\alpha^{2/3}\mathrm{Ai}^{(4)}(R) + \cdots
\end{aligned}$$

The higher derivatives of the Airy function can be written in terms of $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ using the well known Airy differential equation as we can easily show by induction on k that

$$\mathrm{Ai}^{(k+3)}(z) = (k+1)\,\mathrm{Ai}^{(k)}(z) + z\,\mathrm{Ai}^{(k+1)}(z) \;\text{ for }\; k \geq 0.$$

Reducing all higher derivatives of the Airy function and using Taylor approximation to estimate each of the $\mathrm{Ai}^{(k)}(R)$'s, we get

$$\int_{\Gamma_c} e^{f(z)}\mathrm{d}z \sim e^{f(i)}\left(K_1(\theta)\,\alpha^{1/3} + K_2(\theta)\,\alpha^{2/3} + O(\alpha)\right),$$

where $K_1(\theta)$ and $K_2(\theta)$ are precisely as defined in Proposition 4.

Now we estimate the contribution of the tail. First, if t is real and $|t| \geq \sqrt{3}$, then we have

$$\mathrm{Re}(f(t)) - f(i) = -\frac{1}{2\alpha}(t^2-1) - \frac{\cos t}{e\alpha} + \frac{\delta}{\alpha}\left(1 - \frac{\cos t}{e}\right) \leq -\frac{|t|}{2\alpha}\left(1 + O(\alpha^{2/3})\right),$$

where the constant in the O term is independent of t. The inequality follows from the fact that if $|t| \geq \sqrt{3}$ then we have $((t^2-1)/2 - \cos(t)/e) \leq 0.5|t|$, which is not too difficult to verify. The above estimate is enough to show that the contribution from the real half-lines $|t| \geq \sqrt{3}$ is negligible.

If $z = i + te^{i\pi/6}$ and $t \in [-2, -\alpha^c]\}$, then we let $\xi(t) = 2e^{t/2}\cos(\sqrt{3}t/2)$ so that we have

$$\mathrm{Re}(f(i+te^{\pi i/6})) - f(i) = -\frac{1}{2\alpha}\left(\xi(t) - 2 - t + \frac{t^2}{2} + \delta\;(\xi(t)-2)\right).$$

We can show (e.g., using a Taylor approximation of $\xi(t)$) that there exist positive absolute constants C_1 and C_2 such that $|\xi(t) - 2| \leq C_1|t|$ and $\xi(t) - 2 - t + \frac{t^2}{2} \geq C_2|t|^3$, for any $t \in [-2, 0]$. Hence, uniformly for $t \in [-2, -\alpha^c]$,

$$\mathrm{Re}(f(i+te^{\pi i/6})) - f(i) \leq -\frac{1}{2\alpha}\left(C_2|t|^3 + O(\alpha^{2/3}|t|)\right) \leq -C_2\frac{|t|^3}{2\alpha}(1 + O(\alpha^{2/3-2c})).$$

In particular, $\mathrm{Re}(f(i+te^{\pi i/6})) - f(i)$ is negative for sufficiently small α and $|\mathrm{Re}(f(i+te^{\pi i/6})) - f(i)| \gg \alpha^{3c-1}$ uniformly for $t \in [-2, -\alpha^c]$. This is enough to show that the contribution from the segment $\{i + te^{i\pi/6} \;:\; t \in [-2, -\alpha^c]\}$ is also negligible. The contribution from the segment $\{i + te^{-i\pi/6} \;:\; t \in [\alpha^c,\; 2]\}$ can also be dealt with in the same way.

B Idea of the Proof of Proposition 7

We are unable to give much detail here due to lack of space. But the proof of Proposition 7 is essentially the same as the one above. The main difference is, of course, that x is complex. Let $x = (1+\delta)x_0 e^{iu}$ where $u \in (-\pi, \pi]$, $\delta = \theta\alpha^{2/3}$ and θ is a fixed number that satisfies $\mathrm{Ai}(-2^{1/3}\theta) \neq 0$. This implies that the number w, which is now defined as $W_0(-(1+\delta)e^{iu-1})$, is no longer real.

The proof of Part (c) above proceeds in essentially the same way as δ and u are both $O(\alpha^{2/3})$. But for Part (a) and Part (b) to work, we need the following result about w.

▶ **Lemma 10.** *The functions* $\mathrm{Re}((1+w)^2)$*,* $\mathrm{Re}(1+w)$ *and* $1-|w|$ *are all positive and nonzero if* u *is bounded away from 0. Moreover, if* $|u| \to 0$ *as* $\alpha \to 0^+$*, but* $|u| \gg \alpha^{2/3}$*, then*

$$w = -1 + \alpha^{1/3}\sqrt{-2(\theta + it)} + \frac{2}{3}\alpha^{2/3}(\theta + it) + O(|u|^{3/2}), \tag{37}$$

where $t = u\alpha^{-2/3}$*, and the implied constant does not depend on* u*. In particular, we have* $\mathrm{Re}(1+w) \gg \alpha^{1/3}$ *uniformly for* $\alpha^{2/3} \ll |u| \leq \pi$*.*

Proof. The asymptotic formula (37) follows easily from (18). Now to show that $\mathrm{Re}(1+w)$ and $1-|w|$ stay positive, we just need to show that this is the case for $1-|w|$ since $\mathrm{Re}(1+w) \geq 1-|w|$.

If $|w| = 1$, then from the definition of w we have

$$|w|e^{\mathrm{Re}(1+w)} = e^{\mathrm{Re}(1+w)} = 1 + O(\alpha^{2/3}).$$

This implies that $\mathrm{Re}(1+w) \to 0$, which means $w = -1 + o(1)$. The latter forces u to tend to zero.

Similarly if $\mathrm{Re}((1+w)^2) = 0$, then w must be of the form $w = -1 + a(1 \pm i)$, where $a \in \mathbb{R}$. This implies that

$$|w|e^{\mathrm{Re}(1+w)} = \sqrt{a^2 + (a-1)^2}\; e^a = 1 + O(\alpha^{2/3}).$$

This is only possible if $a \to 0$ as $\alpha \to 0$. ◀

This lemma will be very useful in proving Part (a) and Part (b). It shows, for example, that $\mathrm{Re}(-f''(z_0))$ is positive and it satisfies $\mathrm{Re}(-f''(z_0)) = \mathrm{Re}((1+w)\alpha^{-1}) \gg \alpha^{-2/3}$ uniformly for $\alpha^{2/3} \ll |u| \leq \pi$. This will make the proofs of Part (a) and Part (b) above work in this case as well.