# The $\epsilon$-$t$-Net Problem


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## Abstract

We study a natural generalization of the classical $\epsilon$-net problem (Haussler–Welzl 1987), which we call *the $\epsilon$-$t$-net problem*: Given a hypergraph on $n$ vertices and parameters $t$ and $\epsilon \geq \frac{t}{n}$, find a minimum-sized family $S$ of $t$-element subsets of vertices such that each hyperedge of size at least $\epsilon n$ contains a set in $S$. When $t = 1$, this corresponds to the $\epsilon$-net problem.

We prove that any sufficiently large hypergraph with VC-dimension $d$ admits an $\epsilon$-$t$-net of size $O(\frac{(1+\log t)d}{\epsilon} \log \frac{1}{\epsilon})$. For some families of geometrically-defined hypergraphs (such as the dual hypergraph of regions with linear union complexity), we prove the existence of $O(\frac{1}{\epsilon})$-sized $\epsilon$-$t$-nets.

We also present an explicit construction of $\epsilon$-$t$-nets (including $\epsilon$-nets) for hypergraphs with bounded VC-dimension. In comparison to previous constructions for the special case of $\epsilon$-nets (i.e., for $t = 1$), it does not rely on advanced derandomization techniques. To this end we introduce a variant of the notion of VC-dimension which is of independent interest.



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# 1 Introduction

## 1.1 Preliminaries

### Hypergraphs and VC-dimension

A *hypergraph* is a pair $H = (V, \mathcal{E})$ where $V$ is a set of *vertices* and $\mathcal{E} \subseteq 2^V$ is the set of *hyperedges* of $H$. When $V$ is finite, $H$ is a *finite* hypergraph.

A subset $V' \subseteq V$ is *shattered* if all its subsets are realized by $\mathcal{E}$, meaning $\{V' \cap e \colon e \in \mathcal{E}\} = 2^{V'}$. The *VC-dimension* of $H$, denoted by $\dim H$, is the cardinality of a largest shattered subset of $V$ or $+\infty$ if arbitrarily large subsets are shattered (which does not happen in finite hypergraphs). This parameter plays a central role in statistical learning, computational geometry, and other areas of computer science and combinatorics [36, 26, 28].

### $\epsilon$-nets, Mnets

Let $\epsilon \in (0, 1)$. An *$\epsilon$-net* for a finite hypergraph $(V, \mathcal{E})$ is a subset of vertices $S \subseteq V$ such that $S \cap e \neq \emptyset$ for every hyperedge $e \in \mathcal{E}$ such that $|e| \geq \epsilon|V|$.

Haussler and Welzl [18] proved that finite hypergraphs with VC-dimension $d$ admit $\epsilon$-nets of size $O(\frac{d}{\epsilon} \log \frac{d}{\epsilon})$, later improved to $O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ [22]. In the last three decades, $\epsilon$-nets have found applications in diverse areas of computer science, including machine learning [9], algorithms [12], computational geometry [6] and social choice [2].

Mustafa and Ray introduced the notion of *Mnets* [27]. For a hypergraph $(V, \mathcal{E})$ and for a fixed $\epsilon \in (0, 1)$, an *$\epsilon$-Mnet* is a family $\{V_1, V_2, \ldots, V_\ell\}$ such that each $V_i \subseteq V$, each $V_i$ is of size $\Theta(\epsilon|V|)$, and, for each $e \in \mathcal{E}$ such that $|e| \geq \epsilon|V|$, $V_i \subseteq e$ for some $V_i$. They constructed small $\epsilon$-Mnets (i.e., such families with small $\ell$) for several classes of geometric hypergraphs. These results were extended by Dutta et al. [16] using polynomial partitioning.

### Explicit constructions

Although Hausssler and Welzl's proof of the $\epsilon$-net theorem is probabilistic, several deterministic constructions of $\epsilon$-nets for hypergraphs with finite VC-dimension have been devised [10, 24, 13]. The best result of this kind is Brönniman, Chazelle and Matoušek's $O(\epsilon^{-d} \log^d \frac{1}{\epsilon}|V|)$-time algorithm for computing an $\epsilon$-net of size $O(\frac{d}{\epsilon} \log \frac{d}{\epsilon})$ [10]. These constructions are used to derandomize applications of $\epsilon$-nets, such as low-dimensional linear programming [12].

In scenarios where the VC-dimension is $\Omega(\log|V|)$, the running time of these constructions becomes exponential in $|V|$. For one such scenario – the hypergraph induced by half-spaces on the discrete cube $V = \{-1, 1\}^d$ – Rabani and Shpilka [31] presented an efficient explicit construction of an $\epsilon$-net, alas of sub-optimal size: $O(\epsilon^{-b}|V|^a)$ for some universal constants $a, b > 0$, whereas $O(|V|/\epsilon)$ can be obtained by random sampling. Like the aforementioned explicit constructions, the construction of [31] is based on derandomization.

## 1.2 Our problem

We denote by $\binom{X}{k}$ the set of all subsets of cardinality $k$ (or "$k$-subsets") of the set $X$.

▶ **Definition 1.** *Let $H = (V, \mathcal{E})$ be a finite hypergraph, $t$ a positive integer and $\epsilon \in (t/|V|, 1)$. A family $S \subseteq \binom{V}{t}$ of $t$-subsets of $V$ is an $\epsilon$-$t$-*net *for $H$ if for every $e \in \mathcal{E}$ with $|e| \geq \epsilon|V|$ there is an $s \in S$ such that $s \subseteq e$.*

As mentioned already, for $t = 1$ this is equivalent to the $\epsilon$-net notion, and for $t = \Theta(\epsilon|V|)$ this corresponds to the notion of $\epsilon$-Mnets. In this paper we study the following problem.

▶ **Problem.** *How small are the smallest $\epsilon$-$t$-nets for $H$? Can we compute them efficiently?*

## Motivation

Instances of the $\epsilon$-$t$-net problem appear naturally in various contexts in computer science and combinatorics. For example, the following is a basic motivating example for *secret sharing* [23, 34]: "Eleven scientists are working on a secret project. They wish to lock up the documents in a cabinet so that the cabinet can be opened if and only if six or more of the scientists are present. What is the smallest number of locks needed?". Consider a variant of this question in which the number of scientists is large. We still insist on the basic security condition – that no less than six scientists can open the cabinet. On the other hand, due to the large number of scientists, we do not require that any six should be able to do so, but rather any sufficiently large group of a certain kind, e.g., at least one tenth of all scientists including a representative of each university involved.

The classical secret sharing methods (see, e.g., [8]) distribute "keys" to subsets of 6 scientists so that any six scientists will be able to open the cabinet but no five will be able to do that. But as we require only certain groups of scientists to be able to open it, it is possible to distribute shared keys to only some of the 6-subsets. The questions: "What is the minimal number of 6-subsets we can achieve? and how can we choose the 6-subsets of scientists we distribute keys to?" are an instance of the $\epsilon$-$t$-net problem – with $t = 6$, $\epsilon = 1/10$, and the hyperedges of the hypergraph being all groups of scientists that are required to be able to open the cabinet.

Other contexts in which the $\epsilon$-$t$-net problem appears (described in the full version of this paper [3]) include the Turán numbers of hypergraphs, $\chi$-boundedness of graphs, edge-coloring of hypergraphs and more.

## Related work: $\epsilon$-Nets and Mnets

For any $t$, the minimum size of an $\epsilon$-$t$-net is sandwiched between the corresponding minimum sizes of $\epsilon$-nets and of Mnets. Indeed, given an Mnet, one obtains an $\epsilon$-$t$-net by picking one $t$-subset from each subset, and given an $\epsilon$-$t$-net, one obtains an $\epsilon$-net by taking one vertex from each $t$-subset. The survey [28] has most known bounds on these objects.

## 1.3 Results

**Notation:** we write $O_{x,y}(\cdot)$ when the implicit constants depend on parameters $x$ and $y$.

## Hypergraphs of finite VC-dimension have small $\epsilon$-$t$-nets

Our main result is an existence result for small $\epsilon$-$t$-nets.

▶ **Theorem 2.** *For every $\epsilon \in (0, 1)$ and $t \in \mathbb{N} \setminus \{0\}$, every hypergraph on $\geq C_1 \left(\frac{t-1}{\epsilon}\right)^{d^*}$ vertices with VC-dimension $d$ and dual shatter function $\pi^*_H(m) \leq Cm^{d^*}$ admits an $\epsilon$-$t$-net of size $O(\frac{d(1+\log t)}{\epsilon} \log \frac{1}{\epsilon})$, all elements of which are pairwise disjoint. Here $C_1 = C_1(d^*, C)$.*

(The dual shatter function, described in Section 2, is a property of the hypergraph such that we may always take $d^* < 2^{d+1}$.)

This bound is asymptotically tight when $t = O(1)$, in the sense that there exist hypergraphs for which any $\epsilon$-net, and consequently also any $\epsilon$-$t$-net, is of size $\Omega(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$ [22]. The proof of Theorem 2 involves a surprising relation between the $\epsilon$-$t$-net problem and the existence of *spanning trees with a low crossing number*, proved by Welzl in 1988 [37].

Hypergraphs with VC-dimension 1 admit $O(\frac{1}{\epsilon})$-sized $\epsilon$-nets [22] and $\epsilon$-Mnets [16]. The latter fact yields the following result, albeit with worse constants. We offer a simple proof.

▶ **Theorem 3.** *For every positive integer $t$ and $\epsilon \leq \frac{1}{2}$, every finite hypergraph on $\geq t\lceil\frac{1}{\epsilon}\rceil$ vertices with VC-dimension 1 admits an $\epsilon$-$t$-net of size at most $t\lceil\frac{1}{\epsilon}\rceil + 1$.*

## An efficient explicit construction of $\epsilon$-$t$-nets

Our second result is a new explicit construction of $\epsilon$-$t$-nets, for all $t \geq 1$. The case of $t = 1$ (i.e., $\epsilon$-nets) is of independent interest, as in this case our construction does not follow the proof strategy of Haussler and Welzl and does not use derandomization (unlike all previously known explicit constructions of $\epsilon$-nets). On the other hand, it has a sub-optimal size of $O_d(\frac{1}{\epsilon^d})$, where $d$ is the VC-dimension of the underlying hypergraph.

For a higher $t$, we introduce a new parameter of the hypergraph, which we call the *$t$-VC-dimension*. For hypergraphs of $t$-VC-dimension $d$, we construct $\epsilon$-$t$-nets of size $O_d(\frac{1}{\epsilon^{d+t-1}})$. We give some first results on the relation between this new parameter and the standard VC-dimension.

## Small $\epsilon$-2-nets for geometric hypergraphs

In view of Theorem 2, which shows that for hypergraphs with a constant VC dimension one can obtain an $\epsilon$-$t$-net of roughly the same size as the smallest $\epsilon$-net, it is natural to ask whether a similar result can be achieved for geometrically-defined hypergraphs that admit an $\epsilon$-net of size $O(\frac{1}{\epsilon})$. We obtain such results for several geometrically-defined hypergraphs in $\mathbb{R}^2$, including the intersection hypergraph of two families of pseudo-disks and the dual hypergraph of a family of regions with linear union complexity. Namely, we show that these hypergraphs have $O(\frac{1}{\epsilon})$-sized $\epsilon$-2-nets provided they have $\Omega(\frac{1}{\epsilon})$ vertices. Interestingly, in some scenarios the minimum size of an $\epsilon$-2-net is sensitive to the exact multiplicative constant: there are subhypergraphs (of the same hypergraph which is described in the appendix) on $\Theta(\frac{1}{\epsilon})$ vertices for which any $\epsilon$-2-net is of size $\Omega(\frac{1}{\epsilon^2})$.

# 2 Construction of auxiliary hypergraphs

## 2.1 Some preparatory results

### Sauer's lemma

Given a hypergraph $H = (V, \mathcal{E})$ the *trace (also known as projection or restriction) of $H$ on* $A \subseteq V$ is $\Pi_H(A) = \{A \cap e\colon e \in \mathcal{E}\}$; shattered subsets are those for which $\Pi_H(A) = 2^A$. The shatter function of $H$ is

$$\pi_H\colon n \in \mathbb{N} \mapsto \max\{|\Pi_H(A)| : A \subseteq V,\ |A| \leq n\}.$$

It is bounded by the *Sauer–Shelah lemma*:

▶ **Lemma 4** ([36, 33, 35]). *If $\dim H = d$ then $\pi_H(n) \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}$. In particular, for $1 \leq d \leq n$ one has $\pi_H(n) \leq (\frac{e}{d})^d \cdot n^d$, where $e$ is Euler's number.*

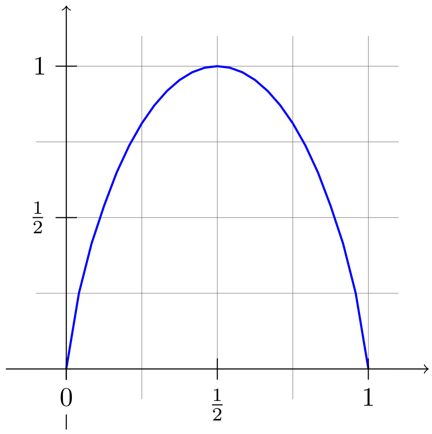


**Figure 1** The binary entropy function.

## Binary entropy function

This is $h\colon x \in (0,1) \mapsto -x\log x - (1-x)\log(1-x)$. (All logarithms are binary. See Figure 1.) We will use the following inequality:

$$\forall \alpha \in \left(0, \frac{1}{2}\right], \ \forall n \in \mathbb{N}, \quad \log \sum_{i=0}^{\lfloor \alpha n \rfloor} \binom{n}{i} \leq nh(\alpha). \tag{1}$$

The binary entropy function restricted to $(0, \frac{1}{2}]$ is invertible, and [11, Th. 2.2]:

$$\forall x \in (0,1), \quad \frac{x}{2\log\frac{6}{x}} \leq h^{-1}(x) \leq \frac{x}{\log\frac{1}{x}}. \tag{2}$$

## 2.2 A first hypergraph on $t$-subsets

▶ **Definition 5.** *Given a hypergraph $H = (V, \mathcal{E})$ and a positive integer $t$, let $H^t$ be the hypergraph $(V^t, \mathcal{E}^t)$ where $V^t = \binom{V}{t}$ and $\mathcal{E}^t = \left\{\binom{e}{t} : e \in \mathcal{E}\right\}$. That is, its vertices are all $t$-element subsets of $V$ and each hyperedge of $H^t$ consists of all such subsets contained in a given hyperedge of $H$.*

For $t \in \mathbb{N} \setminus \{0,1\}$, let $\gamma_t = (th^{-1}(1/t))^{-1}$. Note that $\log t \leq \gamma_t \leq 2\log 6t$.

▶ **Proposition 6.** *If $H$ is a hypergraph with $\dim H = d$ then $d - t + 1 \leq \dim H^t \leq \gamma_t d$.*

**Proof.** We assume that $t \geq 2$, as for $t = 1$, $\dim H^t = \dim H^1 = \dim H = d$.

To prove the left inequality, let $\{v_1, \dots, v_d\}$ be a shattered subset of vertices in $H$, with $d \geq t - 1$. There are $d - t + 1$ sets containing all vertices in $\{v_1, v_2, \dots, v_{t-1}\}$ and exactly one in $\{v_t, v_{t+1}, \dots v_d\}$. It is easy to see that they form a shattered subset in $H^t$.

For the right inequality, suppose to the contrary that $P$ is a shattered set in $H^t$ with $d' = |P| > \gamma_t d$. Let $S = \cup_{p \in P} p$; clearly $|S| \leq td'$. Observe also that $d' + t - 1 \leq |S|$. If this were not the case there would exist some $p_1 \in P$ such that $p_1 \subseteq \cup_{p \in P \setminus \{p_1\}} p$, which would contradict the fact that $P$ is shattered.

We denote $|S| = \beta d'$; we have $1 < \beta \leq t$.

Since $P$ is shattered in $H^t$, each $P_1 \subseteq P$ is of the form $P \cap \binom{e}{t} = \{p \in P : p \subseteq (S \cap e)\}$ for some $e \in \mathcal{E}$. Thus $|\Pi_H(S)| \geq |2^P| = 2^{d'}$.

On the other hand, $\dim H = d$, and so by Lemma 4, $|\Pi_H(S)| \le \binom{\beta d'}{0} + \binom{\beta d'}{1} + \cdots + \binom{\beta d'}{d}$. It follows from Equation (1) (with $\beta d' \ge \beta\gamma_t d > 2d$) that $d' \le \log|\Pi_H(S)| \le \beta d' h(\frac{d}{\beta d'})$. We show that $1 > \beta h(\frac{d}{\beta d'})$, a contradiction.

Note that $\frac{1}{t\gamma_t} \le \frac{1}{\gamma_t \beta} < \frac{1}{2}$. Since $t \mapsto \frac{h(t)}{t}$ is monotone decreasing in the range $(0,1)$, we have $\gamma_t\beta \cdot h(\frac{1}{\gamma_t\beta}) \le t\gamma_t \cdot h(\frac{1}{t\gamma_t}) = \gamma_t$. As $h$ is increasing on $(0, \frac{1}{2})$, it follows that $\beta h(\frac{d}{\beta d'}) < \beta h(\frac{1}{\beta\gamma_t}) \le 1$. ◀

Proposition 6 allows us to slightly improve the "trivial" upper bound of $O(\frac{d^t}{\epsilon^t}(\log\frac{1}{\epsilon})^t)$ on the minimum size of an $\epsilon$-$t$-net for any hypergraph with constant VC-dimension.

▶ **Corollary 7.** *Let $H$ be a hypergraph on $n$ vertices with VC-dimension $d$. For any $t, \epsilon$ such that $n \ge \frac{t}{\epsilon}$, $H$ admits an $\epsilon$-$t$-net of size $O(\frac{dt(1+\log t)}{\epsilon^t}\log\frac{1}{\epsilon})$.*

Indeed, observe that an $\epsilon^t$-net for $H^t$ is an $\epsilon$-$t$-net for $H$, and apply the classical $\epsilon$-net theorem to $H^t$.

## 2.3 A smaller, well-behaved hypergraph on $t$-subsets

A *spanning cycle* $P$ for $H = (V, \mathcal{E})$ is a cycle graph on $V$ that visits all vertices (exactly once). For $e \in \mathcal{E}$, let $\mathrm{cr}(P, e)$ be the number of edges of $P$ with one endpoint in $e$ and the other in $V \setminus e$. The *crossing number* of $P$ with respect to $H$ is $\sup\{\mathrm{cr}(P, e) : e \in \mathcal{E}\}$.

The *dual hypergraph* of $H$ is $H^* = (\mathcal{E}, \mathcal{E}^*)$, where $\mathcal{E}^*$ consists of all hyperedges $v^* = \{e \in \mathcal{E} : v \in e\}$ for $v \in V$. Its shatter function is the *dual shatter function* of $H$, and is denoted by $\pi^*_H$.

If $\dim H = d$ then $\dim H^* \le 2^{d+1}$ [7], and hence $\pi^*_H(m) \le C_d m^{2^{d+1}}$ for every positive $m$, where $C_d$ is a constant depending on $d$. In particular, any hypergraph with finite VC-dimension satisfies the hypotheses of the following theorem.

▶ **Theorem 8** ([37, Lemma 3.3 and Theorem 4.2])**.** *Let $H$ be a hypergraph on $n$ vertices such that $\pi^*_H(m) \le Cm^d$ for some constants $C > 0$ and $d > 1$. Then there exists another constant $C_1$ (depending on $C$ and $d$) and a spanning cycle for $H$ with crossing number $\le C_1 n^{1-\frac{1}{d}}$.*

(An additional $\log n$ factor in Welzl's original result was later removed [25, Sec. 5.4]. Up to constant factors, this theorem is equivalent to the same result for paths or trees.)

▶ **Definition 9.** *Let $H = (V, \mathcal{E})$ be a finite hypergraph with $\pi^*_H(m) \le Cm^d$. Let $P$ be a spanning cycle for $H$ whose crossing number is minimal (and thus $\le C_1|V|^{1-\frac{1}{d}}$). Fix an arbitrary starting point $v_0 \in P$ and orientation of $P$. For $0 \le i < |V|$, let $v_i \in V$ be the $i$-th vertex along $P$. Let $V^t_{lc} = \{\{v_{kt}, v_{kt+1}, \ldots, v_{kt+t-1}\} : 0 \le k < \lfloor\frac{|V|}{t}\rfloor\} \subsetneq \binom{V}{t}$ (where the subscript lc stands for low crossing). Observe that its elements are pairwise disjoint. Let $H^t_{lc}$ be the hypergraph on $V^t_{lc}$ whose hyperedges are of the form $\{v \in V^t_{lc} : v \subseteq e\}$ for each $e \in \mathcal{E}$.*

▶ Remark 10. In order to make $H^t_{lc}$ uniquely defined, $P$ is chosen arbitrarily from all suitable spanning cycles. As $H^t_{lc}$ is a subhypergraph of $H^t$, $\dim H^t_{lc} \le \dim H^t$, and thus we also have $\dim H^t_{lc} \le \gamma_t \dim H$.

# 3 Existence of small $\epsilon$-$t$-nets

▶ **Theorem 2.** *For every $\epsilon \in (0,1)$ and $t \in \mathbb{N} \setminus \{0\}$, every hypergraph on $\ge C_1\left(\frac{t-1}{\epsilon}\right)^{d^*}$ vertices with VC-dimension $d$ and dual shatter function $\pi^*_H(m) \le Cm^{d^*}$ admits an $\epsilon$-$t$-net of size $O(\frac{d(1+\log t)}{\epsilon}\log\frac{1}{\epsilon})$, all elements of which are pairwise disjoint. Here $C_1 = C_1(d^*, C)$.*

**Proof.** For $t = 1$, this is simply the $\epsilon$-net theorem. For higher $t$, let $H = (V, \mathcal{E})$ be such a hypergraph and $n = |V|$. Consider the hypergraph $H^t_{lc}$ defined in Section 2. It has $\lfloor \frac{n}{t} \rfloor$ vertices and VC-dimension $\leq \gamma_t d$ (by Remark 10), and thus admits an $\frac{\epsilon}{2}$-net of size $O(\frac{\gamma_t d}{\epsilon} \log \frac{1}{\epsilon})$. We claim that any such $\frac{\epsilon}{2}$-net $N \subseteq \binom{V}{t}$ is also an $\epsilon$-$t$-net for $H$.

Indeed, the crossing number of the associated spanning cycle is $O_{C,d^*}(n^{1-1/d^*})$. Every hyperedge $e$ of $H$ with $|e| \geq \epsilon n$ fully contains at least $\lfloor \frac{\epsilon n}{t} \rfloor - O_{C,d^*}(n^{1-1/d^*})$ elements of $V^t_{lc}$, which is $\geq \frac{\epsilon n}{2t}$ as soon as $n = \Omega_{C,d^*}(\frac{t}{\epsilon} n^{1-1/d^*})$, or equivalenty (noting also that $2(t-1) \geq t$ for $t \geq 2$) when $n = \Omega_{C,d^*}(\frac{t-1}{\epsilon}^{d^*})$. One of these $t$-subsets is in $N$. ◀

▶ Remark 11. In general, some fast growth of $n = |V|$ as a function of $\frac{1}{\epsilon}$ is necessary. For example, given any $\epsilon$ such that $\frac{t}{\epsilon} \in \mathbb{N}$, the complete $t$-uniform hypergraph on $\frac{t}{\epsilon}$ vertices does not have any $\epsilon$-$t$-net with fewer than $\binom{t/\epsilon}{t}$ elements. Moreover, there exist geometrically-defined hypergraphs that do not admit $\epsilon$-2-nets of size $o(\frac{1}{\epsilon^2})$ (see the full version of the paper [3]). On the other hand, in Section 5 we show that certain classes of geometrically-defined hypergraphs have "small" $\epsilon$-$t$-nets even for "small" values of $n$.

## Small $\epsilon$-nets, small $\epsilon$-2-nets

A natural question arising from Theorem 2 is whether any hypergraph that admits small $\epsilon$-nets must also admit $\epsilon$-$t$-nets of approximately same size. In general, the answer is negative. Take for example a hypergraph whose smallest $\epsilon$-net is of size $\Omega(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$ (see [22], [29]), and augment it by adding a vertex that belongs to all hyperedges. Clearly, this second hypergraph has the same VC-dimension and a one-element $\epsilon$-net, but any $\epsilon$-2-net is of size $\Omega(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$.

However, this example is quite artificial. In "natural" scenarios (and for sufficiently large vertex sets) the smallest $\epsilon$-nets and $\epsilon$-2-nets might still have approximately same size. In Section 5 we show that this is the case for some geometrically-defined hypergraphs.

Another scenario in which there exist both an $\epsilon$-net and an $\epsilon$-2-net of size $O(\frac{1}{\epsilon})$ is when the VC-dimension of the hypergraph is 1. In this case, the existence of an $\epsilon$-net of size $O(\frac{1}{\epsilon})$ was proved in [22]. The next theorem could be derived from results on Mnets [16], at the cost of poor multiplicative constants. Here we give a simpler proof for it.

▶ **Theorem 3.** *For every positive integer $t$ and $\epsilon \leq \frac{1}{2}$, every finite hypergraph on $\geq t\lceil \frac{1}{\epsilon} \rceil$ vertices with VC-dimension 1 admits an $\epsilon$-$t$-net of size at most $t\lceil \frac{1}{\epsilon} \rceil + 1$.*

**Proof.** Let $(V, \mathcal{E})$ be such a hypergraph and $n = |V|$. Without loss of generality, $\min\{|e| \colon e \in \mathcal{E}\} \geq \epsilon n$. For $1 \leq i \leq t$, there exists an $\epsilon$-net $N_i$ that hits each $e \in \mathcal{E}$ at least $i$ times, and $|N_i| = i\lceil \frac{1}{\epsilon} \rceil$. To see this let $N_1$ be an $\epsilon$-net of size $\lceil \frac{1}{\epsilon} \rceil$ [22]. In the hypergraph induced on $V \setminus N_i$ the hyperedges hit only $i$ times by $N_i$ have cardinality $\geq \epsilon n - i$, while the number of vertices is $n - i\lceil \frac{1}{\epsilon} \rceil$, for a ratio $\frac{\epsilon n - i}{n - i\lceil \frac{1}{\epsilon} \rceil} \geq \epsilon$. Take an $\epsilon$-net $N$ of size $\lceil \frac{1}{\epsilon} \rceil$ for this hypergraph and let $N_{i+1} = N_i \cup N$. Finally, let the desired $\epsilon$-$t$-net consist of one $t$-subset from each element of $\Pi_H(N_t)$ with $\geq t$ vertices, of which there are at most $|N_t| + 1$ by Lemma 4. ◀

## 4 Deterministic construction of $\epsilon$-$t$-nets

Let $H = (V, \mathcal{E})$ be a finite hypergraph with VC-dimension $d$, and fix $\epsilon \in (0, 1)$. In this section we provide an explicit polynomial-time construction of $\epsilon$-nets that immediately implies an explicit construction of $\epsilon$-$t$-nets. The size is far from optimal, but the construction is simpler than previous explicit constructions, as it does not rely on packing numbers nor on pseudo-random choices.

## 4.1 Deterministic construction of $\epsilon$-nets

We start with the following definition:

▶ **Definition 12.** *Let $A, B$ be two subsets of $V$. We say that $A$* stabs *$B$ if for every hyperedge $S \in \mathcal{E}$ with $B \subseteq S$ we have $S \cap A \neq \emptyset$.*

Let $S \in \mathcal{E}$ be a hyperedge, $|S| \geq d+1$, and let $X \in \binom{S}{d+1}$. Since the VC-dimension is $d$ the set $X$ is not shattered. Notice that $X = X \cap S \in \Pi_H(X)$. We can also assume that $\emptyset \in \Pi_H(X)$, for otherwise $X$ is a transversal for $H$ of size $d+1$. Hence there exists at least one non-trivial, proper subset $A \subsetneq X$ such that $(X \setminus A) \notin \Pi_H(X)$. Equivalently, there is a non-trivial partition of $X$ into $A$ and $X \setminus A$ such that $A$ stabs $X \setminus A$. We say that $X$ is of type $|A| \in \{1, \ldots, d\}$. Note that $X$ could have several types. By the pigeonhole principle, there is a type $i$ and a subset $A \in \binom{S}{i}$ such that a fraction $d^{-1}\binom{|S|}{i}^{-1}$ of the elements of $\binom{S}{d+1}$ are stabbed by $A$, hence the following lemma holds:

▶ **Lemma 13.** *Let $S$ be a hyperedge containing $\geq d+1$ vertices of $V$. Then there exists an integer $i \in \{1, \ldots, d\}$ and a subset $A \in \binom{S}{i}$ that stabs $\binom{|S|}{d+1} d^{-1} \binom{|S|}{i}^{-1}$ subsets of cardinality $d+1-i$.*

### Constructing $\epsilon$-nets

Put $n := |V|$. We construct an $\epsilon$-net of size $O_d(\frac{1}{\epsilon^d})$ as follows. Start with $N = \emptyset$. As long as there is a hyperedge $S \in \mathcal{E}$ with $|S| \geq \epsilon n$ and $S \cap N = \emptyset$, Lemma 13 asserts that some $i$-subset from $S$ stabs $\Omega_d((\epsilon n)^{d+1-i})$ subsets of $S$ with cardinality $d+1-i$ for an appropriate $i \in \{1, \ldots, d\}$. Add all elements of this subset to $N$; we call this a type $i$ iteration.

The resulting set is an $\epsilon$-net by construction. It is left to show that $|N| = O_d(\frac{1}{\epsilon^d})$. As each step of the construction adds at most $d$ vertices to $N$ it is enough to bound the number of iterations $T$. By the pigeonhole principle, at least $\frac{T}{d}$ of the iterations have the same type, say $i$. After a type-$i$ iteration $N$ stabs an additional $\Omega_d((\epsilon n)^{d+1-i})$ subsets of cardinality $d+1-i$ none of which were previously stabbed. Since there are $\binom{n}{d+1-i}$ subsets of cardinality $d+1-i$ we have $\frac{T}{d} = O_d(\binom{n}{d+1-i}(\epsilon n)^{-(d+1-i)}) = O_d(\frac{1}{\epsilon^d})$.

### Complexity analysis

We analyze the running time of the above algorithm. We assume that for the algorithm we have a data structure which is the incidence matrix of the hypergraph $H$. Without loss of generality, each hyperedge of $\mathcal{E}$ may be replaced with a subset of cardinality $\lceil \epsilon n \rceil$. This can be done in time $O(\epsilon n^{d+1})$ due to the fact that $|\mathcal{E}| = O(n^d)$.

We consider each $X \in \binom{V}{d+1}$. Firstly we check if there is a hyperedge $S \in \mathcal{E}$ which contains $X$, if not, we continue to the next subset. If yes, we consider each of the $2^{d+1} - 2$ proper subsets of $X$. Let $A \subset X$ be such a subset. We check if $X \setminus A$ is stabbed by $A$. We can do it by going over all $O(n^d)$ hyperedges of $H$. Hence, in total this pre-processing step takes $O(n^{d+1} \cdot 2^{d+1} \cdot n^d) = O_d(n^{2d+1})$ running time. While determining the type of any $(d+1)$-subset of $X$ and scanning all the hyperedges of the hypergraph, we maintain for any $i$-subset $A \subset X$ $(1 \leq i \leq d)$, a list of all the $(d+1-i)$-subsets of $X$ that $A$ stabs and their number.

Consider some iteration of the algorithm and let $S \in \mathcal{E}$ be such that $|S| \geq \epsilon n$ and $S \cap N = \emptyset$ where $N$ is the collection of elements found until this iteration. We find a subset $A \subset S$ of size at most $d$ which stabs the most subsets of size $(d+1) - |A|$.

The running time of each iteration is $O(|S|^d \cdot n^d) = O(\epsilon^d n^{2d})$. Hence in total the running time of the algorithm after the pre-processing step is $O_d(\frac{\epsilon^d n^{2d}}{\epsilon^d}) = O_d(n^{2d})$. Hence the total running of the algorithm described in the previous section is $O_d(n^{2d})$.

### Immediate applications to $\epsilon$-$t$-nets

The construction of $\epsilon$-nets in Section 4.1 gives two straightforward constructions of $\epsilon$-$t$-nets.

1. *Trivial construction.* Use the above algorithm to explicitly construct $t$ disjoint $\epsilon$-nets of size $O_d(1/\epsilon^d)$, and take all $t$-subsets of elements in their union that contain one element from each net. The resulting $\epsilon$-$t$-net is of size $O_d(1/\epsilon^{td})$.
2. *Construction via $H^t_{lc}$.* Use the above algorithm to explicitly construct an $\frac{\epsilon}{2}$-net for the hypergraph $H^t_{lc}$, which is an $\epsilon$-$t$-net for $H$ (as was shown in the proof of Theorem 2). The resulting $\epsilon$-$t$-net is of size $O_{d,t}(1/\epsilon^{\dim H^t_{lc}})$. (The cycle with a low crossing number required for constructing the hypergraph $H^t_{lc}$ can be found in polynomial time [37, 25]).

## 4.2 Deterministic construction of $\epsilon$-$t$-nets

We present a direct construction of $\epsilon$-$t$-nets without passing through $\epsilon$-nets. For the sake of convenience, we present the method for $t = 2$, and extend it in the full version [3] for $t > 2$.

The following definition extends the classical notion of VC-dimension.

▶ **Definition 14.** *Let $t$ be a positive integer. Also let $H = (V, \mathcal{E})$ be a hypergraph, and $T', T$ such that $T' \subseteq T \subseteq V$. We say that $T'$ is* $t$-realized *by $H$ (with respect to $T$) if $T' \cup S \in \Pi_H(T)$ for some $S \subseteq T$ such that $|S| < t$. We say that $T$ is* $t$-shattered *by $H$ if every $T' \subseteq T$ is $t$-realized by $H$ (with respect to $T$). The $t$-VC-dimension of $H$, denoted by $\dim_t H$, is the maximal size of a vertex set that is $t$-shattered by $H$.*

Note that the 1-VC-dimension is the standard VC-dimension. Moreover, the $t$-VC-dimension is at most the $(t+1)$-VC-dimension for any psitive integer $t$. We use the following adaptation of Definition 12:

▶ **Definition 15.** *Let $H = (V, \mathcal{E})$ be a hypergraph. Given two vertex sets $A, B \subseteq V$, we say that $A$* 2-stabs *$B$ if each hyperedge of $\mathcal{E}$ that contains $B$ also contains at least two vertices from $A$.*

▶ **Theorem 16.** *For a hypergraph $H = (V, \mathcal{E})$ with 2-VC-dimension $d$, one can construct explicitly an $\epsilon$-2-net of size $O_d(1/\epsilon^{d-1})$.*

**Proof.** Let $S \in \mathcal{E}$ be a hyperedge and let $X \in \binom{S}{d+1}$. Since the 2-VC-dimension is $d$ the set $X$ is not 2-shattered. Notice that $X = X \cap S$ and so $X$ and all elements of $\binom{X}{d}$ are 2-realized by $H$ with respect to $X$. For our purpose, we can also assume that $\emptyset$ is 2-realized by $H$ (with respect to $X$), for otherwise $\binom{X}{2}$ is a transversal for $H$ of size $\binom{d+1}{2}$. This means that there is a partition, say $X = A \cup (X \setminus A)$, such that $A$ 2-stabs $X \setminus A$. Let $i = |A|$. Note that $i \in \{2, \ldots, d\}$. We say that $X = A \cup (X \setminus A)$ is a type $i$ partition. We need the following lemma, whose proof is similar to that of Lemma 13.

▶ **Lemma 17.** *Let $S$ be a hyperedge containing $\geq d+1$ vertices of $V$. Then there exists an integer $i \in \{2, \ldots, d\}$ and a subset $A \subset S$ with cardinality $i$ that 2-stabs $\frac{\binom{|S|}{d+1}}{(d-1)\binom{|S|}{i}}$ subsets $B$ of cardinality $d + 1 - i$.*

## Constructing $\epsilon$-2-nets

Let $H = (V, \mathcal{E})$ be as above and let $\epsilon > 0$ be fixed. Put $n = |V|$. We construct an $\epsilon$-2-net of size $O_d(\frac{1}{\epsilon^{d-1}})$ as follows. We start with a set $N = \emptyset$. As long as there is a hyperedge $S \in \mathcal{E}$ with $|S| \geq \epsilon n$ that does not contain any pair $\{v, w\} \in N$, for an appropriate $i \in \{2, \ldots, d\}$ we take an $i$-subset $A \subset S$ 2-stabbing $\Omega_d((\epsilon n)^{d+1-i})$ subsets of $S$ with cardinality $d+1-i$, and add to $N$ all $\binom{i}{2}$ elements of $A$. We call this a type $i$ iteration. This is possible by Lemma 17.

The resulting set is an $\epsilon$-2-net by construction. It is left to show that $|N| = O_d(\frac{1}{\epsilon^{d-1}})$. In each step of the construction we add at most $\binom{d}{2}$ pairs to $N$ so it is enough to bound the number of iterations $T$. By the pigeonhole principle, at least $\frac{T}{d-1}$ of the iterations have the same type, say $i$. There are $\binom{n}{d+1-i}$ subsets of cardinality $d+1-i$, and in each of the at least $\frac{T}{d-1}$ type $i$ iterations we 2-stab at least $\Omega_d((\epsilon n)^{d+1-i})$ additional subsets of cardinality $d+1-i$, so we have $\frac{T}{d-1} = O_d(\frac{\binom{n}{d+1-i}}{(\epsilon n)^{d+1-i}}) = O_d(\frac{1}{\epsilon^{d+1-i}})$ so $t = O_d(\frac{1}{\epsilon^{d-1}})$ (since $i \geq 2$). This completes the proof of Theorem 16. ◀

## Complexity analysis

The only significant difference between the constructions of Section 4.1 and of Section 4.2 is the factor that depends on the size of the resulting net. Hence, the complexity of the algorithm in this section is bounded by $O_d(n^{2d})$, where $d$ is the 2-VC-dimension of $H$.

## 4.3 $t$-VC-dimension versus classical VC-dimension

What can be said about the relation between VC-dimension and our newly introduced $t$-VC-dimension, for $t \geq 2$? By definition, $\dim H \leq \dim_2 H$. Ideas from Dudley's unpublished lecture notes [15, Th. 4.37] (see also the full version [3]) yield $\dim_2 H \leq 2\dim H + 1$. This is sharp for some small hypergraphs, such as that with vertex set $\{a, b, c\}$ and hyperedges $\{a\}$, $\{b, c\}$, $\{a, c\}$, and $\{a, b, c\}$, which has VC-dimension 1 but 2-VC-dimension 3. For general $t$, we conjecture that $\dim_t H \leq 2\dim H + 2t - 1$. The reasoning below gives roughly $\dim_t H \leq 9.09 \max\{\dim H, t-1\}$.

Let $H$ be a hypergraph of finite VC-dimension with a largest $t$-shattered subset of vertices $T$. As $T$ is $t$-shattered, we have $2^T = \{e \setminus S \colon e \in \Pi_H(T),\ S \subseteq T,\ |S| < t\}$. This yields

$$2^{\dim_t H} \leq |\Pi_H(T)| \cdot \sum_{i=0}^{t-1} \binom{\dim_t H}{i} \leq \sum_{i=0}^{\dim H} \binom{\dim_t H}{i} \cdot \sum_{i=0}^{t-1} \binom{\dim_t H}{i},$$

with the last inequality following from Lemma 4. When $\dim_t H \geq 2\max\{t-1, \dim H\}$, applying Equation (1) gives

$$1 \leq h\left(\frac{\dim H}{\dim_t H}\right) + h\left(\frac{t-1}{\dim_t H}\right).$$

From this inequality we obtain:

▶ **Proposition 18.** *For $t \in \mathbb{N} \setminus \{0\}$, the $t$-VC-dimension of a hypergraph of VC-dimension $d$ is at least $d$, at most $2\gamma_2 \max\{d, t-1\}$ (where $\gamma_2 \simeq 4.54$), and, as $d \to \infty$, at most $2d + o(d)$.*

An interesting geometric example is the hypergraph $H$ whose vertex set is a finite subset of $\mathbb{R}^{d-1}$ and whose hyperedges are induced by half-spaces. It is well-known that $\dim H = d$.

More generally, we have $\dim_t H \le td$ for all $t$. Indeed, by Tverberg's theorem (see, e.g., [26]), every set $T$ of $td+1$ points in $\mathbb{R}^{d-1}$ admits a partition into $t+1$ pairwise disjoint and non-empty sets $T = X \cup Y_1 \cup \cdots \cup Y_t$ such that the intersection of their convex hulls is non-empty. No half-space can $t$-realize $X$ since any half-space that contains $X$ must contain at least one point from each $Y_i$, that is, at least $t$ points of $T \setminus X$.

Therefore, for this hypergraph and $t = 2$, the direct construction yields an $\epsilon$-2-net of size $O_d(1/\epsilon^{2d-1})$, while the trivial construction (described at the end of Section 4.1) yields only a weaker upper bound of $O_d(1/\epsilon^{2d})$. With good bounds on $\dim H^2_{lc}$, the construction via $H^2_{lc}$ (see again Section 4.1) might provide even smaller $\epsilon$-2-nets. In the plane (namely, where $d = 3$), it follows from [17] that $\dim H^2_{lc} \le 5$, and so the upper bounds obtained using the direct construction and using $H^2_{lc}$ are the same – $O(1/\epsilon^5)$.

## 5 Geometric $\epsilon$-2-nets

For a fixed $\epsilon > 0$, any hypergraph with VC-dimension $d$ and $n \ge \frac{C_d}{\epsilon^{2^{d+1}}}$ vertices admits, by Theorem 2, an $\epsilon$-2-net of size $O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$. This leaves open two interesting questions:

1. In cases where the hypergraph admits an $\epsilon$-net of small size, say $O(\frac{1}{\epsilon})$, does it also admit an $O(\frac{1}{\epsilon})$-sized $\epsilon$-2-net (or, more generally, $\epsilon$-$t$-nets)?
2. Does this extend to smaller values of $n$?

In this section we answer both in the affirmative for several classes of geometrically-defined hypergraphs.

▶ **Definition 19.** *Given two families $B$ and $R$ of sets, the intersection hypergraph $H(B, R)$ is the hypergraph on vertex set $B$, where any $r \in R$ defines a hyperedge $\{b \in B : b \cap r \ne \emptyset\}$.*

Note that $H(B, R)$ and $H(R, B)$ are (in general) not isomorphic but dual to each other. Intersection hypergraphs are ubiquitous in discrete and computational geometry. Particular attention is given to the case where either $B$ or $R$ is a set of points, with $H(B, R)$ respectively known as a *primal hypergraph* defined by $R$ or a *dual hypergraph* defined by $B$. See the survey [28] and the references therein.

We present below and in the full version of the paper [3] several intersection hypergraphs that admit $O(\frac{1}{\epsilon})$-sized $\epsilon$-nets, and prove that each of them has $\epsilon$-2-nets of the same size. Furthermore, while Theorem 2 applies only to hypergraphs with a very large number of vertices, the geometric hypergraphs discussed do not have to contain "many" vertices in order to guarantee the existence of "small" $\epsilon$-2-nets. In some cases (see, e.g., the full version of the paper [3]), the behavior is sharp: we can point out two constants $c_1 < c_2$ s.t. if the number of vertices satisfies $|V| \ge \frac{c_2}{\epsilon}$ the hypergraph admits an $O(\frac{1}{\epsilon})$-sized $\epsilon$-2-net, while for $|V| \le \frac{c_1}{\epsilon}$, there exist hypergraphs from the same family that admit only $\epsilon$-2-nets of size $\Omega(\frac{1}{\epsilon^2})$.

### 5.1 Non-piercing regions

For our first example we consider a large class of geometric objects introduced by Raman and Ray [32]. A *family of non-piercing regions* is a family of regions of $\mathbb{R}^2$ such that for any two regions $\gamma_1$ and $\gamma_2$ the difference $\gamma_1 \setminus \gamma_2$ is connected. (Each region may contain holes. See [32] for the exact definitions.)

This extends the more familiar notion of pseudo-disks.

▶ **Theorem 20.** *The intersection hypergraph of two families $B$ and $R$ of non-piercing regions with $B$ finite admits an $\epsilon$-net of size $O(\frac{1}{\epsilon})$ and, if $\epsilon|B| \ge 2$, an $\epsilon$-2-net of size $O(\frac{1}{\epsilon})$.*

The proof relies on several intermediary results. The first one is about an analogue of the Delaunay graph for non-piercing regions [32]. The important specific case where the regions are pseudo-disks had already been studied [4, 20, 21].

▶ **Definition 21.** *A* planar support *for the hypergraph* $(V, \mathcal{E})$ *is a planar graph* $G$ *on the same vertex set* $V$ *such that any hyperedge in* $\mathcal{E}$ *induces a connected subgraph of* $G$.

▶ **Theorem 22** ([32])**.** *Given two families* $B$ *and* $R$ *of non-piercing regions,* $B$ *finite, their intersection hypergraph* $H(B, R)$ *admits a planar support.*

The following corollary has already been noted for families of pseudo-discs [4].

▶ **Corollary 23.** *Given two families* $B$ *and* $R$ *of non-piercing regions,* $\dim H(B, R) \leq 4$.

**Proof.** Let $B' \subseteq B$ be a shattered subset of vertices in $H(B, R)$. As the non-piercing property is clearly hereditary, the hypergraph $H(B', R)$ also admits a planar support. For every pair of vertices in $B'$ there exists a hyperedge of $H(B', R)$ that contains these two vertices and no other. Following Definition 21 these two vertices must share an edge in any planar support of $H(B', R)$. Thus said *planar* support is a complete graph on $B'$, forcing $|B'| \leq 4$. ◀

**Proof of Theorem 20.** First we observe that $H(B, R)$ has $\epsilon$-nets of size $O(\frac{1}{\epsilon})$. Since $H(B, R)$ is finite, we may assume that $R$ is finite as well. To paraphrase from Pyrga and Ray [30, Theorem 4], the following properties suffice:

- For any $0 < \epsilon < 1$ and any $B' \subseteq B$, $H(B', R)$ admits an $\epsilon$-net whose size depends only on $\epsilon$.
- There exist constants $\alpha > 0$, $\beta \geq 0$ and $\tau > 0$ s.t. for any $R' \subseteq R$ there is a graph $G_{R'} = (R', E_{R'})$ with $|E_{R'}| \leq \beta |R'|$ so that for any element $b \in B$ we have $m_b \geq \alpha n_b - \tau$, where $n_b$ is the number of regions of $R'$ intersecting $b$ and $m_b$ is the number of edges in $E_{R'}$ whose both endpoints (which are regions of $R'$) intersect $b$.

The first condition is verified because $\dim H(B', R) \leq 4$ for every $B'$. For the second one, let $\alpha = \tau = 1$ and $\beta = 3$, and let $G_{R'}$ be a planar support of $H(R', B)$. (Note the use of duality!) The inequalities follow from its planarity and the connectedness of the subgraph “cut out” by each $b \in B$.

Finally, to obtain an $\epsilon$-2-net, let $K_1 \subseteq B$ be an $\epsilon$-net for $H(B, R)$ of size $O(\frac{1}{\epsilon})$. Let $R'$ consist of the regions of $R$, if any, that intersect $\geq \epsilon |B|$ regions of $B$ but only one of $K_1$, and let $K_2$ be an $\frac{\epsilon}{2}$-net for $H(B \setminus K_1, R')$ also of size $O(\frac{1}{\epsilon})$. Then the desired $\epsilon$-2-net consists of all edges in a planar support of $H(K_1 \cup K_2, R)$. ◀

## 5.2 Small union complexity

Next, we prove the existence of a small $\epsilon$-2-net for the intersection hypergraph of regions in the plane with linear union complexity and points (i.e. the dual hypergraph defined by the regions).

The union complexity of a family of objects is the function $\kappa : \mathbb{N} \to \mathbb{N}$ that sends each $n \in \mathbb{N}$ to the number of faces of all dimensions in the boundary of the union of $\leq n$ objects, maximized over all subsets of $\leq n$ objects. If $\kappa(n) = O(n)$, we say that the family has *linear union complexity.* Families with linear union complexity include, e.g., families of pseudo-discs: the boundary of the union of $n \geq 3$ pseudo-discs consists of at most $6n - 12$ arcs and as many vertices [19].

The $(\leq k)$-level complexity of the family is defined by counting all faces included in at most $k$ objects (not just on the boundary). To make these definitions precise, one needs to define faces and their dimension; see the survey by Agarwal, Pach and Sharir [1].

A specific case of the following result could also be derived from previous results on Mnets [16], if one adds the additional assumption that the regions have bounded "semi-algebraic description complexity". (The proof of [16] is involved and uses algebraic arguments).

▶ **Theorem 24.** *Let $L$ be a finite family of regions in $\mathbb{R}^2$ with linear union complexity and let $P \subseteq \mathbb{R}^2$ be a set of points. If $|L| \geq \frac{2}{\epsilon}$ then $H(L,P)$ admits an $\epsilon$-2-net of size $O(\frac{1}{\epsilon})$.*

**Proof.** Let $n := |L|$. First, construct a set $K \subseteq L$ of size $O(\frac{1}{\epsilon})$ such that every "heavy" point of $P$ is included in at least two elements of $K$, as in the proofs of Theorem 3 or Theorem 20. This relies on the existence of $\epsilon$-nets of size $O(\frac{1}{\epsilon})$ for $H(L,P)$, a result of Aronov, Ezra and Sharir [5].

Since linear union complexity is a hereditary property, $K$ as a subset of $L$ also has linear union complexity. By a standard argument using the Clarkson–Shor theorem [14], the $(\leq 2)$-level complexity of $K$ is linear as well. Hence, by Euler's formula, the number of hyperedges of size 2 in $H(K,P)$ (whose order of magnitude is equal to the number of $(\leq 2)$-level faces in the arrangement of $K$) is at most $c|K|$ for some constant $c$. By the pigeonhole principle, some region $d \in K$ participates in at most $c$ such hyperedges (i.e., pairs of regions). We pick these at most $c$ pairs of regions to be elements of the $\epsilon$-2-net we construct, and repeat the process for $K \setminus \{d\}$.

We continue in this fashion until all elements of $K$ are removed, and set the $\epsilon$-2-net $N$ to be the set of pairs we picked. Clearly, $|N| = O(|K|) = O(\frac{1}{\epsilon})$. To see that $N$ is indeed an $\epsilon$-2-net, let $p$ be a point that belongs to at least $\epsilon n$ regions of $L$. By construction, $p$ belongs to at least two regions of $K$. Consider the process in which the elements of $K$ are gradually removed, until none of them are left. As a single region is removed at every step, we can look at the step in which the number of remaining regions that contain $p$ is reduced from 2 to 1. Since at that step $p$ is included in exactly two regions of the arrangement, the corresponding pair of regions is added to the $\epsilon$-2-net. Hence, $p$ is covered by both elements of a pair in the $\epsilon$-2-net, as asserted. This completes the proof. ◀

▶ Remark 25. By essentially the same argument, the hypergraph $H(L,P)$ admits an $\epsilon$-$t$-net of size $O_t(\frac{1}{\epsilon})$ for any constant $t \leq \epsilon|L|$.

We can extend Theorem 24 to a family $L$ with union complexity $\kappa(n) = n \cdot f(n)$. In this case, the size of the $\epsilon$-2-net is $O(\frac{1}{\epsilon} \cdot \log f(\frac{1}{\epsilon}) \cdot f(\frac{1}{\epsilon} \cdot \log f(\frac{1}{\epsilon})))$. For example, if $\kappa(n) = n \log n$, then one obtains an $\epsilon$-2-net of size $O(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon} \cdot \log\log \frac{1}{\epsilon})$.

Indeed, by [5], the hypergraph $H(L,P)$ admits an $\epsilon$-net of size $O(\frac{1}{\epsilon} \cdot \log f(\frac{1}{\epsilon}))$. Let $n' = \frac{1}{\epsilon} \cdot \log f(\frac{1}{\epsilon})$. By the Clarkson–Shor theorem [14], the $(\leq 2)$-level complexity is bounded by $O(n' \cdot f(n'))$, hence there exists a region that participates in at most $f(n')$ hyperedges of order 2. This means that the size of the obtained $\epsilon$-2-net is bounded by $O(n' \cdot f(n'))$.

## 6 Discussion and open problems

A hypergraph $H$ with finite VC-dimension $d$ has $\epsilon$-2-nets of size $O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ when $n$ is very large as a function of $\frac{1}{\epsilon}$. This upper bound is the best possible in general, and as we saw in Section 3 may also be best possible even if $H$ admits smaller $\epsilon$-nets. However, we conjecture that in any "reasonable" setting, (including, e.g., all the geometric scenarios discussed in Section 5, and all hypergraphs with hereditarily small $\epsilon$-nets), the existence of an $\epsilon$-net of some order of magnitude, implies the existence of an $\epsilon$-2-net of roughly the same order of magnitude.

Furthermore, we are not aware of any hypergraph in which the dependence of $n$ in $\frac{1}{\epsilon}$ has to be as large as in the assumption of Theorem 2. It may be interesting to extend our results to smaller values of $n$ (as a function of $\frac{1}{\epsilon}$), and to understand whether (as in some of the geometric cases discussed above), there exists a sharp threshold (as a function of $\frac{1}{\epsilon}$) such that if $n$ is above this threshold, then the hypergraph admits an $\epsilon$-2-net of size $\tilde{O}(\frac{1}{\epsilon})$, but if $n$ is below it, then any $\epsilon$-2-net for the hypergraph contains at least $\Omega(\frac{1}{\epsilon^2})$ pairs.

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