The Reeb Graph Edit Distance Is Universal

Ulrich Bauer

Department of Mathematics, Technical University of Munich (TUM), Germany
ulrich.bauer@tum.de

Claudia Landi

Dipartimento di Scienze e Metodi dell'Ingegneria, Università degli Studi di Modena e Reggio Emilia, Reggio Emilia, Italy
clandi@unimore.it

Facundo Mémoli

Department of Mathematics, The Ohio State University, Columbus, OH, USA
memoli@math.osu.edu

Abstract

We consider the setting of Reeb graphs of piecewise linear functions and study distances between them that are stable, meaning that functions which are similar in the supremum norm ought to have similar Reeb graphs. We define an edit distance for Reeb graphs and prove that it is stable and universal, meaning that it provides an upper bound to any other stable distance. In contrast, via a specific construction, we show that the interleaving distance and the functional distortion distance on Reeb graphs are not universal.

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1 Introduction

The concept of a Reeb graph of a Morse function first appeared in [13] and has subsequently been applied to problems in shape analysis in [14, 10]. The literature on Reeb graphs in the computational geometry and computational topology is ever growing (see, e.g., [2, 3] for a discussion and references). The Reeb graph plays a central role in topological data analysis, not least because of the success of Mapper [15], a data analysis method providing a discretization of the Reeb graph for a function defined on a point cloud.

A recent line of work has concentrated on questions about identifying suitable notions of distance between Reeb graphs. These include the so called *functional distortion distance* [2], the *interleaving distance* [6], and various *graph edit distances* [9, 7, 1]. Naturally, there is a strong interest in understanding the connection between different existing distances. In this regard, it has been shown in [3] that the functional distortion and the interleaving distances are bi-Lipschitz equivalent. The edit distances defined in [9, 7] for Reeb graphs of curves and surfaces, respectively, are shown to be universal in their respective settings, so the functional distortion and interleaving distances restricted to the same settings are a lower bound for those distances. Moreover, an example in [7] shows that the functional distortion distance can be strictly smaller than the edit distance considered in that paper.

In this paper, we consider the setting of piecewise linear (PL) functions on compact triangulable spaces, and in this realm we study the properties of *stability* and *universality* of distances between Reeb graphs. The notion of stability has been introduced by Cohen-Steiner et al. [4] in the context of persistence diagrams, and is a key property for topological descriptors [12]. Stability means that two objects at a given distance are assigned descriptors at no more than that distance. This requires a notion of distance on both the collection of objects as well as on the collection of descriptors. The practical relevance of stability lies in the guaranteed robustness of the method with respect to bounded imprecision, caused by noise, coarse sampling, or other sources of uncertainty. However, the stability of a descriptor is not sufficient to warrant *discriminativeness*, i.e., the ability to distinguish different objects: a construction that assigns to every object the same descriptor is certainly stable, but contains no information. For that reason, given a fixed distance on the objects and a construction for a descriptor, it is desirable to assign to the descriptors a distance that is as large as possible while still satisfying the stability property. In that sense, such a distance is then the most discriminative stable distance. Following Lesnick [11], we call such a distance *universal*, noting that the concept already appears in [5] in the context of topological descriptors.

Inspired by a construction of distance between filtered spaces [12], we first construct a novel distance δ_U based on considering joint pullbacks of two given Reeb graphs and prove that this distance satisfies both stability and universality. Via analyzing a specific construction we then prove that neither the functional distortion nor the interleaving distances are universal. Finally, we define two edit-like additional distances between Reeb graphs that reinterpret those appearing in [9, 7, 1] and prove that both are stable and universal. As a consequence, both distances agree with δ_U.

2 Topological aspects of Reeb graphs

We start by exploring some topological ideas behind the definition of Reeb graphs. All maps and functions considered in this paper will be assumed to be continuous. Otherwise, we call them set maps and set functions.

2.1 Reeb graphs as quotient spaces

The classical construction of a Reeb graph [13] is given via an equivalence relation as follows:

▶ **Definition 2.1.** *For $f : X \to \mathbb{R}$ a Morse function on a compact smooth manifold, the* Reeb graph of f *is the quotient space $X/{\sim_f}$, with $x \sim_f y$ if and only if x and y belong to the same connected component of some level set $f^{-1}(t)$ (implying $t = f(x) = f(y)$).*

While this definition was originally considered in the setting of Morse theory, it does not make explicit use of the smooth structure, and so it can be applied quite broadly. However, some additional assumptions on the space X and the function f are justified in order to maintain some of the characteristic properties of Reeb graphs in a generalized setting. With this motivation in mind, we revisit the definition in terms of quotient maps and functions with discrete fibers.

A *quotient map* $p : X \to Y$ is a surjection such that a set U is open in Y if and only if $p^{-1}(U)$ is open in X. In particular, a surjection between compact Hausdorff spaces is a quotient map by the closed map lemma. A quotient map $p : X \to Y$ is characterized by the universal property that a set map $\Phi : Y \to Z$ into any topological space Z is continuous if and only if $\Phi \circ p$ is continuous.

The motivation for considering quotient maps and functions with discrete fibers is explained by the following fact.

▶ **Proposition 2.2.** *Let $f : X \to \mathbb{R}$ be a function with locally connected fibers, and let $q : X \to X/{\sim_f}$ be the canonical quotient map. Then the induced function $\tilde{f} : X/{\sim_f} \to \mathbb{R}$ with $f = \tilde{f} \circ q$ has discrete fibers.*

Proof. To see that the fibers of $\tilde{f}$ are discrete, we show that any subset S of $\tilde{f}^{-1}(t)$ is closed. Let $T = \tilde{f}^{-1}(t) \setminus S$. Then $q^{-1}(T)$ is a disjoint union of connected components of $f^{-1}(t)$. Since $f^{-1}(t)$ is locally connected, each of its connected components is open in the fiber, and so $q^{-1}(T)$ is open in $f^{-1}(t)$, implying that $q^{-1}(S)$ is closed in $f^{-1}(t)$ and hence in X. Since q is a quotient map, $q^{-1}(S)$ is closed if and only if S is closed, yielding the claim. ◀

2.2 Reeb quotient maps and Reeb graphs of piecewise linear functions

We now define a class of quotient maps that leave Reeb graphs invariant up to isomorphism. The main goal is to provide a natural construction for lifting a function $f : X \to \mathbb{R}$ to a space Y through a quotient map $Y \to X$ in a way that yields isomorphic Reeb graphs. To this end, we will define a general notion of Reeb quotient maps and Reeb graphs.

▶ **Definition 2.3.** *A* Reeb domain *is a connected compact triangulable space. A* Reeb quotient map *is a surjective piecewise linear map of Reeb domains with connected fibers.*

We remark that connectedness of Reeb domains is assumed only for the sake of simplicity (see Remark 3.4).

As shown in Corollary 2.8, Reeb domains and Reeb quotient maps constitute a subcategory of the category of triangulable spaces and piecewise linear maps.

▶ **Definition 2.4.** *A* Reeb graph *is a pair $(R_f, \tilde{f})$ where R_f is a Reeb domain endowed with a PL function $\tilde{f} : R_f \to \mathbb{R}$ with discrete fibers, called a* Reeb function*.*

In particular, the isomorphisms between Reeb graphs are PL homeomorphisms that preserve the function values of the associated Reeb functions. While the definition does not assume this explicitly, a Reeb graph is indeed a *finite topological graph* (a compact triangulable space of dimension at most 1).

▶ **Proposition 2.5.** *For any Reeb graph $(R_f, \tilde{f})$, the space R_f is a finite topological graph.*

Proof. By definition, $\tilde{f}$ is (simplexwise) linear for some triangulation of R_f. If there were a simplex σ of dimension at least 2 in the triangulation of R_f, then for any x in the interior of σ, the intersection $\sigma \cap \tilde{f}^{-1}(\tilde{f}(x))$ would have to be of dimension at least 1. But this would contradict the assumption that $\tilde{f}$ has discrete fibers. ◀

▶ **Definition 2.6.** *Generalizing the classical definition (Definition 2.1), we say that a Reeb graph $(R_f, \tilde{f})$ is a* Reeb graph of *$f : X \to \mathbb{R}$ if there is a Reeb quotient map $p : X \to R_f$ such that $f = \tilde{f} \circ p$.*

We now proceed to prove that Reeb quotient maps are closed under composition. We start by showing that not only the fibers, but more generally all preimages of closed connected sets are connected.

▶ **Proposition 2.7.** *If $p \colon X \to Y$ is a Reeb quotient map, then the preimage $p^{-1}(K)$ of a closed connected set $K \subseteq Y$ is connected.*

Proof. Assume that K is nonempty; otherwise, the claim holds trivially. Let $p^{-1}(K) = U \cup V$, with U, V nonempty and closed in $p^{-1}(K)$. To show that $p^{-1}(K)$ is connected, it suffices to show that $U \cap V$ is necessarily nonempty.

Because $p^{-1}(K)$ is closed in X, the sets U and V are also closed in X. The images $p(U)$ and $p(V)$ are closed by the closed map lemma, and their union is K. By connectedness of K, their intersection is nonempty. Let $y \in p(U) \cap p(V)$. We have

$$p^{-1}(y) = (p^{-1}(y) \cap U) \cup (p^{-1}(y) \cap V).$$

The subspaces $(p^{-1}(y) \cap U)$ and $(p^{-1}(y) \cap V)$ are closed in $p^{-1}(y)$, and by connectedness of the fiber $p^{-1}(y)$, their intersection must be nonempty. In particular, $U \cap V$ is nonempty. ◀

▶ **Corollary 2.8.** *If $p\colon X \to Y$ and $q\colon Y \to Z$ are Reeb quotient maps, then the composition $q \circ p : X \to Z$ is a Reeb quotient map too.*

As mentioned before, the main purpose of Reeb quotient maps is to lift Reeb functions to larger domains while maintaining the same Reeb graph. The following property is a consequence of the above statement:

▶ **Corollary 2.9.** *Let $(R_f, \tilde{f})$ be a Reeb graph of a function $f : X \to \mathbb{R}$, and let $q : Y \to X$ be a Reeb quotient map. Then $(R_f, \tilde{f})$ is also a Reeb graph of $f \circ q : Y \to \mathbb{R}$.*

Proof. Let $p : X \to R_f$ be the Reeb quotient map factoring $f = \tilde{f} \circ p$, as in the following diagram:

$$\begin{tikzcd}
 & & \mathbb{R} \\
Y \arrow[r, two heads, "q"] & X \arrow[ur, "f"] \arrow[r, two heads, "p"] & R_f \arrow[u, "\tilde{f}"']
\end{tikzcd}$$

Then by Corollary 2.8, $(R_f, \tilde{f})$ is also a Reeb graph for $f \circ q = \tilde{f} \circ (p \circ q) : Y \to \mathbb{R}$ via the Reeb quotient map $p \circ q : Y \to R_f$. ◀

The following lemma shows how a transformation $g = \xi \circ f$ of a function f lifts to a Reeb quotient map ζ between the corresponding Reeb graphs.

▶ **Lemma 2.10.** *Consider a commutative diagram*

$$\begin{tikzcd}
\operatorname{im} f \arrow[rr, "\chi"] & & \operatorname{im} g \\
R_f \arrow[u, "\tilde{f}"] \arrow[r, dotted, two heads, "\zeta"] & R_g \arrow[ur, "\tilde{g}"'] & \\
X \arrow[u, two heads, "p_f"] \arrow[ur, two heads, "p_g"'] \arrow[uu, bend left, "f"] \arrow[uurr, bend right, "g"'] & &
\end{tikzcd}$$

where $(R_f, \tilde{f}), (R_g, \tilde{g})$ are Reeb graphs, $p_f : X \to R_f, p_g : X \to R_g$ are Reeb quotient maps, and $\chi : \operatorname{im} f \to \operatorname{im} g$ is a PL function such that $g = \chi \circ f$. Then $\zeta = p_g \circ p_f^{-1}$ is a Reeb quotient map from R_f to R_g.

In particular, if χ is a PL homeomorphism, then so is ζ. Note that the definition of ζ does not involve the function χ; the existence of χ already ensures that ζ is a Reeb quotient map.

Proof. Let $x \in R_f$, and let $t = \tilde{f}(x)$. Then $C = p_f^{-1}(x)$ is a connected component of $f^{-1}(t)$ by the assumption that p_f is a Reeb quotient map. By commutativity, we have

$$f^{-1} \subseteq f^{-1} \circ \chi^{-1} \circ \chi = g^{-1} \circ \chi,$$

and since C is connected, there must be a single $y \in R_g$ with $p_g(C) = \{y\}$. Hence, $\zeta = p_g \circ p_f^{-1}$ is a set map. Moreover, since p_g is continuous and p_f is closed, the map ζ is continuous; since p_g and p_f are PL, the map ζ is PL as well.

Now let $y \in R_g$ and let $s = \tilde{g}(y)$. Similarly to above, $C = p_g^{-1}(y)$ is a connected component of $g^{-1}(s)$. We have $p_f(C) = p_f \circ p_g^{-1}(y) = \zeta^{-1}(y) \neq \emptyset$, so ζ is surjective, and the fiber $\zeta^{-1}(y) = p_f(C)$ is connected as the image of a connected set. ◀

▶ Remark 2.11. By Proposition 2.2 and Lemma 2.10, given a Reeb graph $(R_f, \tilde{f})$ of $f : X \to \mathbb{R}$ with Reeb quotient map $p \colon X \to R_f$, there is a canonical isomorphism $R_f \cong X/{\sim_f}$. As a consequence, the Reeb graph $(R_f, \tilde{f})$ together with the Reeb quotient map p is unique up to a unique isomorphism, defining the Reeb graph as a universal property.

We now show that Reeb quotient maps are stable under pullbacks.

▶ **Proposition 2.12.** *Consider a pullback diagram of PL maps $p_1 \colon X_1 \to Y$, $p_2 \colon X_2 \to Y$:*

$$\begin{tikzcd}
 & Y & \\
X_1 \arrow[ur, "p_1"] & & X_2 \arrow[ul, "p_2"'] \\
 & X_1 \times_Y X_2 \arrow[ul, "q_1"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

If the map p_1 (resp. p_2) is a Reeb quotient map, then so is the map q_2 (resp. q_1). Hence, the class of Reeb quotient maps is stable under pullbacks.

Proof. First note that the category of compact triangulable spaces has all pullbacks [16]. For $x_2 \in X_2$, by surjectivity of p_1 there is some $x_1 \in X_1$ such that $p_1(x_1) = p_2(x_2)$. Thus $(x_1, x_2) \in X_1 \times_Y X_2$ and $q_2(x_1, x_2) = x_2$, proving that q_2 is surjective. Moreover, for $x_2 \in X_2$, we have $q_2^{-1}(x_2) = p_1^{-1}(p_2(x_2)) \times \{x_2\}$. By assumption, $p_1^{-1}(p_2(x_2))$ is connected as a fiber of p_1, implying that $p_1^{-1}(p_2(x_2)) \times \{x_2\}$ is connected. Finally, applying Proposition 2.7 to q_2, we obtain that the pullback space $X_1 \times_Y X_2$ is connected. The proof for q_1 is analogous. ◀

3 Stable and universal distances

Throughout this paper, we will use the term *distance* to describe an extended pseudo-metric $d : X \times X \to [0, \infty]$ on some collection X. Our main goal is the introduction of a distance between Reeb graphs that is stable and universal in the following sense.

▶ **Definition 3.1.** *We say that a distance d_S between Reeb graphs is* stable *if and only if given any two Reeb graphs $(R_f, \tilde{f})$ and $(R_g, \tilde{g})$, for any Reeb domain X with Reeb quotient maps $p_f : X \to R_f$ and $p_g : X \to R_g$ we have*

$$d_S((R_f, \tilde{f}), (R_g, \tilde{g})) \leq \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty. \tag{S}$$

Note that stability implies that isomorphic Reeb graphs have distance 0*. Indeed, an isomorphism of Reeb graphs $\gamma : R_f \to R_g$ yields $d_S((R_f, \tilde{f}), (R_g, \tilde{g})) \leq \|\tilde{f} \circ \mathrm{id} - \tilde{g} \circ \gamma\|_\infty = 0$.*

Moreover, we say that a stable distance d_U between Reeb graphs is universal *if and only if for any other stable distance d_S between Reeb graphs, we have*

$$d_S((R_f,\tilde{f}),(R_g,\tilde{g})) \leq d_U((R_f,\tilde{f}),(R_g,\tilde{g})), \tag{U}$$

for all $(R_f,\tilde{f})$ and $(R_g,\tilde{g})$.

▶ Remark 3.2. By connectedness of R_f and R_g, there is at least one space X with maps p_f, p_g as needed to define the stability property: $X = R_f \times R_g$, with p_f, p_g the canonical projections. The resulting functions $f = \tilde{f} \circ p_f, g = \tilde{g} \circ p_g : R_f \times R_g \to \mathbb{R}$ then satisfy $\|f - g\|_\infty = \max(\sup \tilde{f} - \inf \tilde{g}, \sup \tilde{g} - \inf \tilde{f})$. In particular, by compactness a stable distance for Reeb graphs is always finite.

The definition of stability yields the following universal distance.

▶ **Definition 3.3.** *For any two Reeb graphs $(R_f,\tilde{f}),(R_g,\tilde{g})$, let*

$$\delta_U((R_f,\tilde{f}),(R_g,\tilde{g})) := \inf_{p_f\colon R_f \leftarrow X \rightarrow R_g \colon p_g} \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty,$$

where the infimum is taken over all possible Reeb domains X and Reeb quotient maps $p_f \colon X \to R_f$ and $p_g \colon X \to R_g$, as in the following diagram.

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R} \\
R_f \arrow[u, "\tilde{f}"] & & R_g \arrow[u, "\tilde{g}"'] \\
& X \arrow[ul, two heads, "p_f"] \arrow[ur, two heads, "p_g"'] &
\end{tikzcd}$$

▶ Remark 3.4. The connectedness assumption for Reeb domains can be dropped by adapting the definition of the universal distance as follows. If R_f and R_g have a different number of connected components, then $\delta_U(R_f, R_g) := \infty$. If both R_f and R_g have n connected components so that $R_f = \coprod_{i\in[n]} F_i$ and $R_g = \coprod_{i\in[n]} G_i$ with each F_i and G_i connected, then

$$\delta_U(R_f, R_g) := \min_{\gamma} \inf_{p:F_i \leftarrow X \rightarrow G_{\gamma(i)}:q} \|\tilde{f} \circ p - \tilde{g} \circ q\|_\infty$$

where γ varies among all permutations on n objects, $i \in [n]$, and the infimum is taken over all possible Reeb domains X and Reeb quotient maps $p \colon X \to F_i$ and $q \colon X \to G_i$.

▶ **Proposition 3.5.** *The distance δ_U is the largest stable distance on Reeb graphs. Hence, δ_U is universal.*

Proof. To see that δ_U is a distance, the only non-trivial part is showing the triangle inequality. To this end, given diagrams $p_f : R_f \leftarrow X \rightarrow R_g : p_g$ and $p'_g : R_g \leftarrow Y \rightarrow R_h : p_h$, we can form a pullback of the diagram $p_g : X \rightarrow R_g \leftarrow Y : p'_g$ to obtain the diagram

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & \mathbb{R} \\
R_f \arrow[u, "\tilde{f}"] & & R_g \arrow[u, "\tilde{g}"'] & & R_h \arrow[u, "\tilde{h}"'] \\
& X \arrow[ul, two heads, "p_f"] \arrow[ur, two heads, "p_g"'] & & Y \arrow[ul, two heads, "p'_g"] \arrow[ur, two heads, "p_h"'] & \\
& & X \times_{R_g} Y \arrow[ul, two heads, "q_X"] \arrow[ur, two heads, "q_Y"'] & &
\end{tikzcd}$$

where $X \times_{R_g} Y$ is a Reeb domain and q_X, q_Y are Reeb quotient maps by Proposition 2.12. Defining $f = \tilde{f} \circ p_f \circ q_X$, $g = \tilde{g} \circ p_g \circ q_X = \tilde{g} \circ p'_g \circ q_Y$, and $h = \tilde{h} \circ p_h \circ q_Y$, we have

$$\begin{aligned} \delta_U((R_f, \tilde{f}), (R_h, \tilde{h})) &\le \|f - h\|_\infty \le \|f - g\|_\infty + \|g - h\|_\infty \\ &= \|\tilde{f} \circ p_f - \tilde{g} \circ p_g\|_\infty + \|\tilde{g} \circ p'_g - \tilde{h} \circ p_h\|_\infty, \end{aligned}$$

where the last equality holds because q_X and q_Y are surjective. Hence

$$\delta_U((R_f, \tilde{f}), (R_h, \tilde{h})) \le \delta_U((R_f, \tilde{f}), (R_g, \tilde{g})) + \delta_U((R_g, \tilde{g}), (R_h, \tilde{h})).$$

The stability of δ_U is immediate from its definition. Moreover, for any stable distance d_S between Reeb graphs, combining the stability of d_S and the definition of δ_U, we obtain $d_S \le \delta_U$, implying that δ_U is universal. ◀

▶ **Corollary 3.6.** *The universal distance δ_U is a metric on isomorphism classes of Reeb graphs.*

Proof. According to Remark 3.2, by stability, δ_U is always finite. Moreover, we recall from [6] that there exists a stable distance d_I, the *interleaving distance*, which is a metric on isomorphism classes of Reeb graphs; in particular, $d_I((R_f, \tilde{f}), (R_g, \tilde{g})) = 0$ if and only if $(R_f, \tilde{f}) \cong (R_g, \tilde{g})$. By stability of d_I and universality of δ_U, we have $d_I((R_f, \tilde{f}), (R_g, \tilde{g})) \le \delta_U((R_f, \tilde{f}), (R_g, \tilde{g}))$. Thus, $\delta_U((R_f, \tilde{f}), (R_g, \tilde{g})) = 0$ implies $d_I(R_f, R_g) = 0$ and hence $(R_f, \tilde{f}) \cong (R_g, \tilde{g})$. ◀

▶ **Example 3.7.** Consider the one point Reeb graph $(*, c)$ endowed with the function identical to $c \in \mathbb{R}$. Then, for any Reeb graph $(R_f, \tilde{f})$, we have $\delta_U((R_f, \tilde{f}), (*, c)) = \|\tilde{f} - c\|_\infty$.

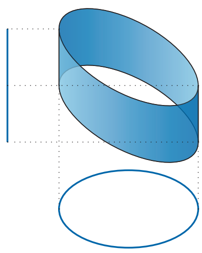
We now consider an example where we can explicitly determine the value of the distance $\delta_U((R_f, \tilde{f}), (R_g, \tilde{g}))$ between two specific simple Reeb graphs $R_f = \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with $\tilde{f}(x, y) = x$ and $R_g = [-1, 1]$ with $\tilde{g}(t) = t$. The example demonstrates the non-universality of certain distances proposed in the literature. We prove:

▶ **Proposition 3.8.** $\delta_U((R_f, \tilde{f}), (R_g, \tilde{g})) = 1$.

The proof of this proposition will be obtained from the two claims below.

▷ Claim 3.9. $\delta_U(R_f, R_g) \le 1$.

Proof. Consider the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ |2z - x| \le 1\}$ together with functions $f(x, y, z) = x$ and $g(x, y, z) = z$ defined on C. Then $(R_f, \tilde{f})$ is a Reeb graph of f



via the Reeb quotient map $(x, y, z) \mapsto (x, y)$, and $(R_g, \tilde{g})$ is a Reeb graph of g via the Reeb quotient map $(x, y, z) \mapsto z$. Since we have $|f(c) - g(c)| \le 1$ for all $c \in C$, this implies that $\delta_U((R_f, \tilde{f}), (R_g, \tilde{g})) \le 1$. ◁

▷ Claim 3.10. $\delta_U((R_f, \tilde{f}), (R_g, \tilde{g})) \ge 1$.

Proof. Assume for a contradiction that there is a diagram $p_f : R_f \leftarrow Z \rightarrow R_g : p_g$ of Reeb quotient maps such that, letting $\hat{f} = \tilde{f} \circ p_f$ and $\hat{g} = \tilde{g} \circ p_g$, we have $\|\hat{f} - \hat{g}\|_\infty = \delta < 1$. We then observe the following:

- $\hat{g}^{-1}(0) \subseteq \hat{f}^{-1}([-\delta, +\delta])$.
- $\tilde{f}^{-1}([-\delta, +\delta])$ consists of two circular arcs homeomorphic by $\tilde{f}$ to $[-\delta, +\delta]$, and thus, by Proposition 2.7, $\hat{f}^{-1}([-\delta, +\delta])$ consists of two connected components C_+ and C_- as well.
- For both components we have $\hat{f}(C_\pm) = [-\delta, \delta]$, and so $\|\hat{f} - \hat{g}\|_\infty = \delta$ implies that $0 \in \hat{g}(C_\pm)$. Thus $\hat{g}^{-1}(0) \cap C_- \neq \emptyset$ and $\hat{g}^{-1}(0) \cap C_+ \neq \emptyset$.

But since $\hat{g}^{-1}(0) \subseteq C_- \sqcup C_+$, this would contradict the assumption that the fiber $\hat{g}^{-1}(0)$ is connected. ◁

The current example illustrates that the *functional distortion distance* introduced in [2] and the *interleaving distance* introduced in [6] are both stable but fail to be universal. We first recall the definition of the former. For any Reeb graph $(R_f, \tilde{f}), (R_g, \tilde{g})$, consider the metric on R_f given by

$$d_f(x, y) = \inf\{b - a \mid x, y \text{ are in the same connected component of } \tilde{f}^{-1}([a, b])\}.$$

Given maps $\phi : R_f \rightarrow R_g$ and $\psi : R_g \rightarrow R_f$, we write

$$G(\phi, \psi) = \{(p, \phi(p)) : p \in R_f\} \cup \{(\psi(q), q) : q \in R_g\}$$

for the correspondences induced by the two maps, and

$$D(\phi, \psi) = \sup_{(p,q),(p',q') \in G(\phi,\psi)} \frac{1}{2} |d_f(p, p') - d_g(q, q')|$$

for the metric distortion induced by (ϕ, ψ). The functional distortion distance is then defined as

$$d_{FD}(R_f, R_g) = \inf_{\phi,\psi} (\max\{D(\phi, \psi), \|f - g \circ \phi\|_\infty, \|f \circ \psi - g\|_\infty\}).$$

To see that neither the functional distortion distance nor the interleaving distance are universal, we establish:

▶ **Proposition 3.11.** $d_I((R_f, \tilde{f}), (R_g, \tilde{g})) \leq d_{FD}((R_f, \tilde{f}), (R_g, \tilde{g})) \leq \frac{1}{2}$.

Proof. By [3, Lemma 8], the functional distortion distance is an upper bound on the interleaving distance on Reeb graphs [6], and so it is enough to prove that $d_{FD}((R_f, \tilde{f}), (R_g, \tilde{g})) \leq \frac{1}{2}$. To this end, consider the maps

$$\phi : R_f \rightarrow R_g,\ (x, y) \mapsto x \quad \text{and} \quad \psi : R_g \rightarrow R_f,\ t \mapsto \left(t, \sqrt{1 - t^2}\right).$$

For every pair $p, p' \in R_f$ one can verify that

$$|\tilde{f}(p) - \tilde{f}(p')| \leq d_f(p, p') \leq |\tilde{f}(p) - f(p')| + 1,$$

while for every pair $q, q' \in R_g$, we have

$$d_g(q, q') = |\tilde{g}(q) - \tilde{g}(q')|.$$

This implies that for any two corresponding pairs $(p, q), (p', q') \in G(\phi, \psi)$, we have

$$|d_f(p, p') - d_g(q, q')| \leq 1,$$

and thus $D(\phi, \psi) \leq \frac{1}{2}$. Both maps preserve function values, so $d_{FD}(R_f, R_g) \leq \frac{1}{2}$. ◀

4 Edit distances

Given a pair of Reeb graphs R_f, R_g, consider a diagram of the form

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & & & \mathbb{R} & & \mathbb{R} \\
R_f = R_1 \arrow[u, "\tilde{f}_1"] & & R_2 \arrow[u, "\tilde{f}_2"] & & \cdots & & R_{n-1} \arrow[u, "\tilde{f}_{n-1}"] & & R_n = R_g \arrow[u, "\tilde{f}_n"] \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads, dotted] & & X_{n-2} \arrow[ul, two heads, dotted] \arrow[ur, two heads] & & X_{n-1} \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd} \tag{1}$$

where for $n \in \mathbb{N}$ $\tilde{f}_1, \ldots, \tilde{f}_n$ are Reeb functions with $\tilde{f}_1 = \tilde{f}$ and $\tilde{f}_n = \tilde{g}$, and the maps $X_i \to R_i, R_{i+1}$ for $i = 1, \ldots, n-1$, are Reeb quotient maps. We call the diagram a *Reeb zigzag diagram* between R_f and R_g. Observe that, by Remark 3.2, between any two Reeb graphs R_f and R_g there exists a Reeb zigzag diagram.

A Reeb zigzag diagram can be regarded as being composed of the following elementary diagrams:

$$\begin{tikzcd}
 & \mathbb{R} & \\
 & R_i \arrow[u, "\tilde{f}_i"] & \\
X_{i-1} \arrow[ur, two heads] & & X_i \arrow[ul, two heads]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{R} & & \mathbb{R} \\
R_i \arrow[u, "\tilde{f}_i"] & & R_{i+1} \arrow[u, "\tilde{f}_{i+1}"] \\
 & X_i \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}$$

This way, we may think of a Reeb zigzag diagram as a sequence of operations transforming the R_f into R_g. The elementary diagram on the left corresponds to an *edit* operation: the space X_{i-1}, together with a function $X_{i-1} \to \mathbb{R}$ with Reeb graph R_i, is transformed to another space X_i, with a function $X_i \to \mathbb{R}$ having the same Reeb graph R_i. The elementary diagram on the right corresponds to a *relabel* operation: the function on X_i with Reeb graph R_i is transformed to another function with Reeb graph R_{i+1}. The idea of edit and relabel operations is inspired by previous work on edit distances for Reeb graphs [7, 1].

In order to define an edit distance using Reeb zigzag diagrams, we need to assign a cost to a given Reeb zigzag diagram between R_f and R_g. To that end, we can consider a cone from a space V by Reeb quotient maps $V \to R_i$:

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & & & \mathbb{R} & & \mathbb{R} \\
R_1 \arrow[u, "\tilde{f}_1"] & & R_2 \arrow[u, "\tilde{f}_2"] & & \cdots & & R_{n-1} \arrow[u, "\tilde{f}_{n-1}"] & & R_n \arrow[u, "\tilde{f}_n"] \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads, dotted] & \cdots & X_{n-2} \arrow[ul, two heads, dotted] \arrow[ur, two heads] & & X_{n-1} \arrow[ul, two heads] \arrow[ur, two heads] & \\
& & & & V \arrow[ulll, two heads] \arrow[ul, two heads] \arrow[ur, two heads] \arrow[urrr, two heads] & & & &
\end{tikzcd} \tag{2}$$

We call this diagram a *Reeb cone.* Any Reeb zigzag diagram admits such a cone. Indeed, the limit over the lower part of the diagram (1) can be constructed from iterated pullbacks, and since Reeb quotient maps are stable under pullbacks, the maps in the resulting limit diagram are Reeb quotient maps as well. In a Reeb cone, by commutativity, each of the Reeb functions $\tilde{f}_i$ induces a unique function $f_i : V \to \mathbb{R}$. By Corollary 2.9, the Reeb graph of f_i is isomorphic to R_i. This way, we pull back the individual functions $\tilde{f}_i$ to functions f_i on a common space with the same Reeb graphs, where they can be compared using the supremum norm.

Using these ideas, we can now introduce distances on Reeb graphs, and proceed to prove that they are stable and universal.

► **Definition 4.1.** *Given a Reeb cone from a space V as in* (2)*, we define the* spread *of the functions $(f_i)_{i=1,\dots,n} : V \to \mathbb{R}$, as the function*

$$s^V : V \to \mathbb{R},\ x \mapsto \max_{i=1,\dots,n} f_i(x) - \min_{j=1,\dots,n} f_j(x).$$

Moreover, for a Reeb zigzag diagram Z between R_f and R_g as in (1)*, consider the limit of Z, denoted by L. The* cost *of the Reeb zigzag diagram Z is the supremum norm of the spread s^L,*

$$c_Z := \|s^L\|_\infty = \sup_{x\in L} \left(\max_i f_i(x) - \min_j f_j(x) \right).$$

► **Definition 4.2.** *We define the* (PL) edit distance *δ_e between Reeb graphs $(R_f, \tilde{f})$ and $(R_g, \tilde{g})$ as the infimum cost of all Reeb zigzag diagrams Z between R_f and R_g:*

$$\delta_e(R_f, R_g) = \inf_Z c_Z.$$

Moreover, we define the graph edit distance *δ_{eGraph} between Reeb graphs $(R_f, \tilde{f})$ and $(R_g, \tilde{g})$ analogously by restricting the infimum to Reeb zigzag diagrams Z where all the spaces X_i and R_i are finite topological graphs.*

Thus, on Reeb graphs we have two edit distances, satisfying

$$\delta_e \leq \delta_{eGraph}\,. \tag{3}$$

The Reeb graph edit distance δ_{eGraph} is a categorical reformulation of the definition given in [1]. The main goal is to prove that these distances have the stability and universality properties (Propositions 4.4 and 4.5, Theorem 5.6, and Corollary 5.7). As a consequence, whenever applicable, they actually coincide with the canonical universal distance δ_U defined in Definition 3.3:

► **Corollary 4.3.** $\delta_U = \delta_e = \delta_{eGraph}\,.$

The proofs of stability and universality for δ_e are straightforward and are given next. The verification of stability and universality for δ_{eGraph} follows in Section 5.

► **Proposition 4.4.** *δ_e is a stable distance.*

Proof. Let $(R_f, \tilde{f}), (R_g, \tilde{g})$ be Reeb graphs. For any space X such that there exist two Reeb quotient maps $p_f : X \to R_f$ and $p_g : X \to R_g$, the diagram

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R} \\
R_f \arrow[u, "\tilde{f}"] & & R_g \arrow[u, "\tilde{g}"'] \\
& X \arrow[ul, "p_f"] \arrow[ur, "p_g"'] \arrow[uul, "f"'] \arrow[uur, "g"] &
\end{tikzcd}$$

is a Reeb zigzag diagram with limit object X. The cost of this Reeb zigzag diagram is exactly $\|f - g\|_\infty$. Hence, $\delta_e((R_f, \tilde{f}), (R_g, \tilde{g})) \leq \|f - g\|_\infty$. ◄

Our proof of universality of the edit distance is similar to previous universality proofs for the bottleneck distance [5] and for the interleaving distance [11].

▶ **Proposition 4.5.** *δ_e is a universal distance.*

Proof. Let $(R_f, \tilde f), (R_g, \tilde g)$ be Reeb graphs with $\delta_e((R_f, \tilde f), (R_g, \tilde g)) = d$. Hence, for any $\varepsilon > 0$, there is a Reeb zigzag diagram Z between $R_f = R_1$ and $R_g = R_n$, with limit L and functions f_i as in Definition 4.1, having cost

$$c_Z = \|s^L\|_\infty = \| \max_i f_i - \min_j f_j \|_\infty \le d + \varepsilon.$$

Let $p_f : L \to R_f$ and $p_g : L \to R_g$ be the induced Reeb quotient maps. If d_S is any other stable distance (cf. Definition 3.1) between R_f and R_g, we have

$$d_S((R_f, \tilde f), (R_g, \tilde g)) \le \|\tilde f \circ p_f - \tilde g \circ p_g\|_\infty \le \| \max_i f_i - \min_j f_j \|_\infty \le d + \varepsilon.$$

Since the above holds for all $\varepsilon > 0$, we have $d_S((R_f, \tilde f), (R_g, \tilde g)) \le d = \delta_e((R_f, \tilde f), (R_g, \tilde g))$. ◀

5 Stability and universality of the Reeb graph edit distance

We now turn to the proof of stability and universality for the Reeb graph edit distance. Recall that, in the case of δ_{eGraph}, the admissible Reeb zigzag diagrams are PL zigzags of finite topological graphs. As mentioned above, the distance δ_{eGraph} is applicable to Reeb graphs of compact triangulable spaces.

▶ **Lemma 5.1.** *Let X be a compact triangulable space, with PL functions $f, g : X \to \mathbb{R}$, simplexwise linear on a triangulation $|K| \cong X$ of X by some simplicial complex K. Let $\chi : \operatorname{im} f \to \operatorname{im} g$ be a weakly monotonic PL surjection such that $\chi \circ f(v) = g(v)$ for every vertex $v \in V$ of K. Then there is a Reeb quotient map $X/{\sim_f} \to X/{\sim_g}$.*

Proof. Without loss of generality, assume $X = |K|$. For simplicity, we write $R_f = X/{\sim_f}$, $R_g = X/{\sim_g}$, and $R_h = X/{\sim_h}$, where $h = \chi \circ f$. Applying Proposition 2.2, f can be factorized as $f = \tilde f \circ q_f$, where $q_f : X \to R_f$ is the canonical projection and $\tilde f : R_f \to \mathbb{R}$ is a Reeb function. Analogously, we obtain $g = \tilde g \circ q_g$ and $h = \tilde h \circ q_h$. We show that there is a Reeb quotient map $k : X \to R_h$ making the following diagram commute:

$$\begin{tikzcd}
\operatorname{im} f \arrow[rr, two heads, "\chi"] & & \operatorname{im} g \\
R_f \arrow[u, "\tilde f"] & R_h \arrow[ur, "\tilde h"'] & R_g \arrow[u, "\tilde g"'] \\
X \arrow[u, two heads, "q_f"] \arrow[ur, "q_h"'] & & X \arrow[u, two heads, "q_g"'] \arrow[ul, dotted, two heads, "k"]
\end{tikzcd}$$

The claim then follows by applying Lemma 2.10 to obtain Reeb quotient maps $R_f \to R_h$ and $R_h \to R_g$, which compose to the desired map $R_f \to R_g$.

In order to prove the existence of such a Reeb quotient map k, we define the relation

$$k = q_h \circ ((h^{-1} \circ g) \cap \operatorname{st}_K)$$

on $X \times R_h$. Here st_K denotes the open star on $X = |K|$, defined as

$$\operatorname{st}_K(x) = \{y \in X \mid \sigma \in K, y \in \sigma^\circ, x \in \sigma\},$$

where σ° is the interior of the simplex σ. Note that the converse relation to the open star is the *(closed) carrier*, $\mathrm{st}_K^{-1} = \mathrm{carr}_K$, where $\mathrm{carr}_K(A)$ is the underlying space of the smallest subcomplex of K containing $A \subseteq X$. We will also use the *open carrier* relation carr_K°, where $\mathrm{carr}_K^\circ(A)$ is the smallest union of open simplices of K covering A. Note that the open carrier relation is symmetric, i.e., $(\mathrm{carr}_K^\circ)^{-1} = \mathrm{carr}_K^\circ$. Moreover, we have $\mathrm{carr}_K^\circ \subseteq \mathrm{st}_K$.

The remainder of the proof is split into several lemmas. Lemma 5.2 describes the behaviour of the functions h and g on the simplices of K. Lemma 5.3 shows that k is a continuous surjection, and Lemma 5.4 shows that k has connected fibers. Since $\tilde{h} \circ k = g$, we conclude that k is PL. Thus, k is a Reeb quotient map, and the claim follows from Lemma 2.10. ◀

▶ **Lemma 5.2.** *For every simplex σ in K, $g(\sigma) = h(\sigma)$ and $g(\sigma^\circ) \subseteq h(\sigma^\circ)$.*

Proof. We have $h(\sigma) = g(\sigma)$ because h is equal to g on the vertices of K, and $h = \chi \circ f$ with f linear on σ and χ a weakly monotonic surjection.

To show that $g(\sigma^\circ) \subseteq h(\sigma^\circ)$, note that since g is linear on σ, either g is constant on σ and so $g(\sigma^\circ) = g(\sigma) = h(\sigma)$, or $g(\sigma^\circ) = (g(v), g(w))$ for some vertices v, w of σ. In the latter case, since h and g coincide on the vertices, we have $g(\sigma^\circ) = g(\sigma)^\circ = h(\sigma)^\circ$. Finally, since $h(\sigma^\circ) \subseteq h(\sigma) \subseteq \overline{h(\sigma^\circ)}$ are nested intervals, we have $h(\sigma)^\circ \subseteq h(\sigma^\circ)$ and the claim follows. ◀

▶ **Lemma 5.3.** *k is a continuous surjection.*

Proof. Recall that the relation $k \subseteq X \times R_h$ is a *partial set map* if for any $x \in X$ and $y, y' \in k(x)$, we have $y = y'$. Moreover, a partial set map k is a (total) set map if for every $x \in X$, $k(x) \neq \emptyset$. Finally, a set map k is a surjection if for every $y \in R_h$, there is some $x \in k^{-1}(y)$.

We first show that k is a partial set map, i.e., for any $x \in X$ and $y, y' \in k(x)$, we have $y = y'$. To see this, let $t = g(x)$ and note that $\tilde{h}(y) = \tilde{h}(y') = t$. Let $\sigma \in K$ be such that $x \in \sigma^\circ$. By Lemma 5.2 there is a point $\zeta \in \sigma^\circ$ with $h(\zeta) = g(x) = t$; in particular,

$$\zeta \in h^{-1}(t) \cap \mathrm{st}_K(x).$$

Furthermore, there are points $\xi, \xi' \in h^{-1}(t) \cap \mathrm{st}_K(x)$ with $\xi \in q_h^{-1}(y)$ and $\xi' \in q_h^{-1}(y')$. But since $h^{-1}(t) \cap \tau$ is necessarily connected for every simplex τ, we know that ζ lies in the same connected component of $h^{-1}(t) \cap \mathrm{st}_K(x)$ as both ξ and ξ', and so we have $y = q_h(\xi) = q_h(\xi') = y'$ as claimed.

To show that k is a set map, we need to show that for every $x \in X$, $k(x) \neq \emptyset$. It suffices to show that for every $x \in X$, $\mathrm{st}_K(x)$ contains a point x' with $h(x') = g(x)$. This follows by considering the simplex $\sigma \in K$ with $x \in \sigma^\circ$. Now by Lemma 5.2, there is a point $x' \in \sigma^\circ \subseteq \mathrm{st}_K(x)$ with $h(x') = g(x)$ as claimed.

To show that k is surjective, we show that for every $y \in R_h$, there is some

$$x \in k^{-1}(y) = (\mathrm{carr}_K \circ q_h^{-1})(y) \cap (g^{-1} \circ \tilde{h})(y),$$

or equivalently, there is some $x \in \mathrm{carr}_K \circ q_h^{-1}(y)$ such that $g(x) = \tilde{h}(y)$. If $q_h^{-1}(y)$ contains some vertex v of K, choose $x = v$. Otherwise, let $\xi \in q_h^{-1}(y)$, and let $\sigma \in K$ be such that $\xi \in \sigma^\circ$. Now by Lemma 5.2 there is a point $x \in \sigma \subseteq \mathrm{carr}_K \circ q_h^{-1}(y)$ with $g(x) = h(\xi) = \tilde{h}(y)$.

Finally, to show that k is continuous, we show that for every closed subset L of R_h, the preimage $k^{-1}(L)$ is closed. Since $k^{-1} = (\mathrm{carr}_K \circ q_h^{-1}) \cap (g^{-1} \circ \tilde{h})$, it is sufficient to show that both $\mathrm{carr}_K \circ q_h^{-1}(L)$ and $g^{-1} \circ \tilde{h}(L)$ are closed in X. First note that $\mathrm{carr}_K \circ q_h^{-1}(L)$ is closed as a subcomplex of K. Furthermore, the image $\tilde{h}(L)$ is closed by the closed map lemma. By continuity of g it follows that $g^{-1} \circ \tilde{h}(L)$ is closed in X. ◀

▶ **Lemma 5.4.** *The fibers of k are connected.*

Proof. Let $y \in R_h$ be a point in the Reeb graph with value $t = \tilde{h}(y)$, and $C = q_h^{-1}(y) \subseteq h^{-1}(t)$ the corresponding component of the level set of h. Let $U = \mathrm{carr}_K(C)$, and let L be the corresponding subcomplex of K. Writing $D = k^{-1}(y)$, we have $C = U \cap h^{-1}(t)$ and $D = U \cap g^{-1}(t)$. To prove that D is connected, it is sufficient to show that C and D have finite closed covers with isomorphic nerves; since C is connected, both nerves and hence also D are then connected too.

The cover of C is given by $\{\sigma \cap C \mid \sigma \in L\}$, and similarly the cover of D is $\{\sigma \cap D \mid \sigma \in L\}$. Observe that any two cover elements of C, say $\sigma \cap C$ and $\tau \cap C$, have a nonempty intersection $(\sigma \cap C) \cap (\tau \cap C) = (\sigma \cap \tau) \cap C$ if and only if $t \in h(\sigma \cap \tau)$. Similarly, $\sigma \cap D$ and $\tau \cap D$ have nonempty intersection if and only if $t \in g(\sigma \cap \tau)$. But $g(\sigma \cap \tau) = h(\sigma \cap \tau)$ by Lemma 5.2, and so the nerves of both covers are isomorphic as claimed. ◀

We thus have shown the existence of the Reeb quotient map k. This completes the proof of Lemma 5.1. We will now apply Lemma 5.1 to construct Reeb graph edit zigzags from straight line homotopies.

▶ **Lemma 5.5.** *Let X be a compact triangulable space, with PL functions $f, g : X \to \mathbb{R}$, simplexwise linear on a triangulation $|K| \cong X$. Consider the straight line homotopy $f_\lambda = (1-\lambda)f + \lambda g$, with $0 \le \lambda \le 1$. Then there exists a partition $0 = \lambda_1 < \cdots < \lambda_n = 1$ such that for every $1 \le i < n$ and $\rho \in (\lambda_i, \lambda_{i+1})$, there exist weakly monotonic PL surjections $\chi_i : \mathrm{im}\, f_\rho \to \mathrm{im}\, f_{\lambda_i}$ and $\xi_{i+1} : \mathrm{im}\, f_\rho \to \mathrm{im}\, f_{\lambda_{i+1}}$ with*

$$\chi_i \circ f_\rho(v) = f_{\lambda_i}(v) \quad \text{and} \quad \xi_{i+1} \circ f_\rho(v) = f_{\lambda_{i+1}}(v)$$

for every vertex v of K.

Proof. Consider the set of values $0 < \lambda < 1$ such that there exist vertices $v, w \in K$ with

$$f_\lambda(v) = f_\lambda(w), \quad \text{but} \quad f_\rho(v) \neq f_\rho(w) \quad \text{for every} \quad \rho \neq \lambda.$$

This set is finite because the function $\lambda \mapsto f_\lambda(v) - f_\lambda(w)$ is linear and K has a finite number of vertices. Let $\{\lambda_i\}_{1 \le i \le n}$ be this set together with 0 and 1, indexed in ascending order. By the linearity of f_λ with respect to the parameter λ, we also see that the order induced by f_ρ on the vertices is the same for every $\rho \in (\lambda_i, \lambda_{i+1})$. Indeed, if there exist two distinct vertices v, w of K such that $f_\rho(v) = f_\rho(w)$ for some $\rho \in (\lambda_i, \lambda_{i+1})$, then $f_\lambda(v) = f_\lambda(w)$ for every $\lambda \in [0,1]$. By continuity, the order is still weakly preserved along $[\lambda_i, \lambda_{i+1}]$.

Therefore, the function $f_\rho(v) \mapsto f_{\lambda_i}(v)$ is well-defined and can be extended to a piecewise linear function χ_i satisfying the claim. The function ξ_{i+1} can be defined similarly. ◀

▶ **Theorem 5.6.** *δ_{eGraph} is a stable distance.*

Proof. Let $X \cong |K|$ be a compact triangulable space with $f, g : X \to \mathbb{R}$ be PL functions, simplexwise linear on K; without loss of generality, assume $X = |K|$. Consider the straight line homotopy $f_\lambda = (1-\lambda)f + \lambda g$, with $0 \le \lambda \le 1$, and take values $\lambda_i \in [0,1]$, $1 \le i \le n$, as in Lemma 5.5. Set $\rho_i = (\lambda_i + \lambda_{i+1})/2$.

We first define a Reeb cone of the form (2), with $V = X$, $R_i = X/{\sim_{f_{\lambda_i}}}$, $i = 1, \dots, n$, and $X_i = X/{\sim_{f_{\rho_i}}}$, $i = 1, \dots, n-1$. The canonical projections $q_{\rho_i} : X \to X_i$ and $q_{\lambda_i} : X \to R_i$ are Reeb quotient maps, and the Reeb functions $R_i \to \mathbb{R}$ are induced by f_{λ_i} as in Proposition 2.2. To complete the construction, we show that there are Reeb quotient maps $p_i : X/{\sim_{f_{\rho_i}}} \to X/{\sim_{f_{\lambda_i}}}$ and $o_{i+1} : X/{\sim_{f_{\rho_i}}} \to X/{\sim_{f_{\lambda_{i+1}}}}$ that make the following diagram commute:

$$
\begin{tikzcd}
R_i = X/{\sim_{f_{\lambda_i}}} & & R_{i+1} = X/{\sim_{f_{\lambda_{i+1}}}} \\
& X_i = X/{\sim_{f_{\rho_i}}} \arrow[ul, dotted, two heads, "p_i"'] \arrow[ur, dotted, two heads, "o_{i+1}"] & \\
& X \arrow[u, two heads, "q_{\rho_i}"'] \arrow[uul, bend left, two heads, "q_{\lambda_i}"] \arrow[uur, bend right, two heads, "q_{\lambda_{i+1}}"'] &
\end{tikzcd}
$$

We prove the existence of p_i, that of o_{i+1} being analogous. By Lemma 5.5, there is a weakly monotonic PL surjection $\chi_i : \operatorname{im} f_{\rho_i} \to \operatorname{im} f_{\lambda_i}$ such that $\chi_i \circ f_{\rho_i} = f_{\lambda_i}$. Hence, Lemma 5.1 provides the desired Reeb quotient map $p_i : X/{\sim_{f_{\rho_i}}} \to X/{\sim_{f_{\lambda_i}}}$.

Now consider the limit L over the resulting Reeb zigzag diagram Z consisting of the maps p_i and o_i, with maps $r_i : L \to X_i$ and $s_i : L \to R_i$. Since the maps from X in the above Reeb cone factor through a unique map $m\colon X \to L$ by the universal property of the limit, we obtain the commutative diagram

$$
\begin{tikzcd}
& & \mathbb{R} & & \mathbb{R} & & \\
\cdots & & R_i \arrow[u, "\tilde{f}_{\lambda_i}"] & & R_{i+1} \arrow[u, "\tilde{f}_{\lambda_{i+1}}"] & & \cdots \\
& X_{i-1} \arrow[ul, dotted, two heads] \arrow[ur, two heads, "o_i"] & & X_i \arrow[ul, two heads, "p_i"'] \arrow[ur, two heads, "o_{i+1}"] & & X_{i+1} \arrow[ul, two heads, "p_{i+1}"'] \arrow[ur, dotted, two heads] & \\
& & & L \arrow[ull, two heads, "r_{i-1}"] \arrow[uul, two heads, "s_i"] \arrow[u, two heads, "r_i"'] \arrow[uur, two heads, "s_{i+1}"'] \arrow[urr, two heads, "r_{i+1}"'] & & & \\
& & & X \arrow[u, "m"] \arrow[uu, bend left, two heads, "q_{\rho_i}"] \arrow[uull, bend left, two heads, "q_{\rho_{i-1}}"] \arrow[uurr, bend right, two heads, "q_{\rho_{i+1}}"'] & & &
\end{tikzcd}
$$

We have $f_{\lambda_i} = f^L_{\lambda_i} \circ m$ for $1 \le i \le n$, with $f^L_{\lambda_i} = \tilde{f}_{\lambda_i} \circ s_i$. Hence, for every $\ell \in L$,

$$
s^L(\ell) = \max_j f^L_{\lambda_j}(\ell) - \min_k f^L_{\lambda_k}(\ell) \le \sum_{i=1}^{n-1} |f^L_{\lambda_{i+1}}(\ell) - f^L_{\lambda_i}(\ell)|.
$$

By the surjectivity of q_{ρ_i}, for every i there is $x_{\ell,i} \in X$ such that $q_{\rho_i}(x_{\ell,i}) = r_i(\ell)$. Thus,

$$
|f^L_{\lambda_{i+1}}(\ell) - f^L_{\lambda_i}(\ell)| = |f_{\lambda_{i+1}}(x_{\ell,i}) - f_{\lambda_i}(x_{\ell,i})| \le (\lambda_{i+1} - \lambda_i) \cdot \|f - g\|_\infty.
$$

Together, for every $\ell \in L$ we have

$$
s^L(\ell) \le \sum_{i=1}^{n-1} (\lambda_{i+1} - \lambda_i) \cdot \|f - g\|_\infty = \|f - g\|_\infty.
$$

We conclude that

$$
\delta_e(R_f, R_g) \le c_Z = \|s^L\|_\infty \le \|f - g\|_\infty,
$$

showing that δ_e is a stable distance. ◀

▶ **Corollary 5.7.** *$\delta_{eGraph} = \delta_U$ is the universal distance.*

Proof. The claim is a direct consequence of inequality (3) together with Theorem 5.6 and Propositions 4.4 and 4.5. ◀

6 Discussion

We believe that the following questions are of interest and could motivate further research:

- *Do minimizers in the definition of the universal distance always exist?* This would have algorithmic implications. See below.
- *Is the interleaving distance [6] bi-Lipschitz equivalent to the universal distance?* If the answer to this question is affirmative, then by results of [3], one would obtain the bi-Lipschitz equivalence between the universal distance and the functional distortion distance from [2].
- *What is the computational complexity of the universal distance?* This problem is at least graph-isomorphism hard, which can be seen as follows. First note that bipartite graphs form a graph-isomorphism complete class of graphs. Any bipartite simple graph can be interpreted as a Reeb graph with function values in $\{0, 1\}$ corresponding to the partition of the vertex set. Using Corollary 3.6, these Reeb graphs are at universal distance 0 if and only if the bipartite graphs are isomorphic, so both of these decision problems are graph-isomorphism complete. A similar observation has been made for the interleaving distance [6].

  These considerations motivate the following two ancillary questions:
  - Is the universal distance a minimum over a certain finite set, possibly of cardinality polynomial in the size of the input Reeb graphs?
  - More generally, are the possible values of the universal distance always contained in some canonical set of values, constructed from the sets of vertex function values of the two Reeb graphs? Related results in the context of manifolds endowed with Morse functions are in the work of Donatini and Frosini [8]. This work carries over to the setting of Reeb graphs by the results of [7].
- *How do the theoretical properties of the universal distance extend to more genreal settings?*
  - The definition of the universal distance also makes sense in a more general topological setting, where we consider locally compact Hausdorff spaces as Reeb domains and proper quotient maps with connected fibers as Reeb quotient maps. The distance one obtains in this larger category can still be applied to finite Reeb graphs, in which case it will be smaller or equal to the PL universal distance that we described in this paper. However, we conjecture that in this case the two distances actually coincide.
  - *Reeb spaces:* Generalizing our definitions and results up to Section 5 to Reeb spaces of piecewise linear maps $X \to \mathbb{R}^n$ is straightforward. Do our results of Section 5 generalize as well?

References

1 Ulrich Bauer, Barbara Di Fabio, and Claudia Landi. An Edit Distance for Reeb Graphs. In *Eurographics Workshop on 3D Object Retrieval*. The Eurographics Association, 2016. doi:10.2312/3dor.20161084.

2 Ulrich Bauer, Xiaoyin Ge, and Yusu Wang. Measuring distance between Reeb graphs. In *Proceedings of the Thirtieth Annual Symposium on Computational Geometry*, SoCG'14, New York, NY, USA, 2014. ACM. doi:10.1145/2582112.2582169.

3 Ulrich Bauer, Elizabeth Munch, and Yusu Wang. Strong equivalence of the interleaving and functional distortion metrics for Reeb graphs. In *31st International Symposium on Computational Geometry (SoCG 2015)*, Leibniz International Proceedings in Informatics (LIPIcs), pages 461–475, Dagstuhl, Germany, 2015. doi:10.4230/LIPIcs.SOCG.2015.461.

4 David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. *Discrete Comput. Geom.*, 37(1):103–120, 2007. doi:10.1007/s00454-006-1276-5.

5 Michele d'Amico, Patrizio Frosini, and Claudia Landi. Natural pseudo-distance and optimal matching between reduced size functions. *Acta Applicandae Mathematicae*, 109(2):527–554, 2010. doi:10.1007/s10440-008-9332-1.

6 Vin de Silva, Elizabeth Munch, and Amit Patel. Categorified Reeb graphs. *Discrete & Computational Geometry*, 55(4):854–906, 2016. doi:10.1007/s00454-016-9763-9.

7 Barbara Di Fabio and Claudia Landi. The edit distance for Reeb graphs of surfaces. *Discrete & Computational Geometry*, 55(2):423–461, 2016. doi:10.1007/s00454-016-9758-6.

8 Pietro Donatini and Patrizio Frosini. Natural pseudodistances between closed manifolds. *Forum Math.*, 16(5):695–715, 2004. doi:10.1515/form.2004.032.

9 Barbara Di Fabio and Claudia Landi. Reeb graphs of curves are stable under function perturbations. *Mathematical Methods in the Applied Sciences*, 35(12):1456–1471, 2012. doi:10.1002/mma.2533.

10 Masaki Hilaga, Yoshihisa Shinagawa, Taku Kohmura, and Tosiyasu L. Kunii. Topology matching for fully automatic similarity estimation of 3D shapes. In *Proceedings of the 28th annual conference on Computer graphics and interactive techniques*, SIGGRAPH '01, pages 203–212. ACM Press, 2001. doi:10.1145/383259.383282.

11 Michael Lesnick. The theory of the interleaving distance on multidimensional persistence modules. *Foundations of Computational Mathematics*, 15(3):613–650, 2015. doi:10.1007/s10208-015-9255-y.

12 Facundo Mémoli. A distance between filtered spaces via tripods. Preprint, 2017. arXiv:1704.03965.

13 Georges Reeb. Sur les points singuliers d'une forme de Pfaff complétement intégrable ou d'une fonction numérique. *Comptes Rendus de L'Académie des Sciences*, 222:847–849, 1946.

14 Yoshihisa Shinagawa and Tosiyasu L. Kunii. Constructing a Reeb graph automatically from cross sections. *IEEE Computer Graphics and Applications*, 11(6):44–51, 1991. doi:10.1109/38.103393.

15 Gurjeet Singh, Facundo Mémoli, and Gunnar Carlsson. Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition. In *Eurographics Symposium on Point-Based Graphics*. The Eurographics Association, 2007. doi:10.2312/SPBG/SPBG07/091-100.

16 John R. Stallings. *Brick's Quasi Simple Filtrations and 3-Manifolds*, volume 188 of *London Mathematical Society Lecture Note Series*, page 188–203. Cambridge University Press, 1993. doi:10.1017/CBO9780511661860.017.