Book Embeddings of Nonplanar Graphs with Small Faces in Few Pages

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Abstract

An embedding of a graph in a book, called book embedding, consists of a linear ordering of its vertices along the spine of the book and an assignment of its edges to the pages of the book, so that no two edges on the same page cross. The book thickness of a graph is the minimum number of pages over all its book embeddings. For planar graphs, a fundamental result is due to Yannakakis, who proposed an algorithm to compute embeddings of planar graphs in books with four pages. Our main contribution is a technique that generalizes this result to a much wider family of nonplanar graphs, which is characterized by a biconnected skeleton of crossing-free edges whose faces have bounded degree. Notably, this family includes all 1-planar and all optimal 2-planar graphs as subgraphs. We prove that this family of graphs has bounded book thickness, and as a corollary, we obtain the first constant upper bound for the book thickness of optimal 2-planar graphs.

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1 Introduction

Book embeddings of graphs form a well-known topic in topological graph theory that has been a fruitful subject of intense research over the years, with seminal results dating back to the 70s [38]. In a book embedding of a graph $G$, the vertices of $G$ are restricted to a line, called the spine of the book, and the edges of $G$ are assigned to different half-planes delimited by the spine, called pages of the book. From a combinatorial point of view, computing a book embedding of a graph corresponds to finding a linear ordering of its vertices and a partition of its edges, such that no two edges in the same part cross; see Fig. 1. The book thickness (also known as stack number or page number) of a graph is the minimum number of pages required by any of its book embeddings, while the book thickness of a family of graphs $G$ is the maximum book thickness of any graph $G \in G$.

Book embeddings were originally motivated by the design of VLSI circuits [14, 42], but they also find applications, among others, in sorting permutations [39, 43], compact graph encodings [30, 35], graph drawing [9, 10, 45], and computational origami [1]; for a more complete list, we point the reader to [22]. Unfortunately, determining the book thickness of a graph turns out to be an NP-complete problem even for maximal planar graphs [44]. This negative result has motivated a large body of research devoted to the study of upper bounds on the book thickness of meaningful graph families.

In this direction, there is a very rich literature concerning planar graphs. The most notable result is due to Yannakakis, who back in 1986 exploited a peeling-into-levels technique (a flavor of it is given in Section 2) to prove that the book thickness of any planar graph is at most 4 [46, 47], improving upon a series of previous results [13, 27, 29]. Even though it is not yet known whether the book thickness of planar graphs is 3 or 4, there exist several improved bounds for particular subfamilies of planar graphs. Bernhart and Kainen [8] showed that the book thickness of a graph $G$ is 1 if and only if $G$ is outerplanar, while its book thickness is at most 2 if and only if $G$ is subhamiltonian, that is, $G$ is a subgraph of a Hamiltonian planar graph. In particular, several subfamilies of planar graphs are known to be subhamiltonian, e.g., 4-connected planar graphs [37], planar graphs without separating triangles [31], Halin graphs [15], series-parallel graphs [40], bipartite planar graphs [17], planar graphs of maximum degree 4 [6], triconnected planar graphs of maximum degree 5 [28], and maximal planar graphs of maximum degree 6 [24]. In this plethora of results, we should also mention that planar 3-trees have book thickness 3 [27] and that general (i.e., not necessarily triconnected) planar graphs of maximum degree 5 have book thickness at most 3 [26].

In contrast to the planar case, there exist far fewer results for non-planar graphs. Bernhart and Kainen first observed that the book thickness of a graph can be linear in the number of its vertices; for instance, the book thickness of the complete graph $K_n$ is $\lceil n/2 \rceil$ [8]. Improved bounds are usually obtained by meta-theorems exploiting standard parameters of
the graph. In particular, Malitz proved that if a graph has $m$ edges, then its book thickness is $O(\sqrt{m})$ [34], while if its genus is $g$, then its book thickness is $O(\sqrt{g})$ [33]. Also, Dujmovic and Wood [23] showed that if a graph has treewidth $w$, then its book thickness is at most $w + 1$, improving an earlier linear bound by Ganley and Heath [25]. It is also known that all graphs belonging to a minor-closed family have bounded book thickness [11], while the other direction is not necessarily true. As a matter of fact, the family of 1-planar graphs is not closed under taking minors [36], but it has bounded book thickness [3, 4]. We recall that a graph is $h$-planar (with $h \geq 0$), if it can be drawn in the plane such that each edge is crossed at most $h$ times; see, e.g., [19, 32] for recent surveys.

Notably, the approaches presented in [3, 4] form the first non-trivial extensions of the above mentioned peeling-into-levels technique by Yannakakis [46, 47] to graphs that are not planar. Both approaches exploit an important property of 3-connected 1-planar graphs, namely, they can be augmented and drawn so that all pairs of crossing edges are “caged” in the interior of degree-4 faces of a planar skeleton, i.e., the graph consisting of all vertices and of all crossing-free edges of the drawing [41]. A similar property also holds for the optimal 2-planar graphs. Each graph in this family admits a drawing whose planar skeleton is simple, biconnected, and has only degree 5 faces, each containing five crossing edges [7]. The book thickness of these graphs, however, has not been studied yet; the best-known upper bound of $O(\log n)$ is derived from the corresponding one for general $h$-planar graphs [21].

**Our contribution.** We present a technique that further generalizes the result by Yannakakis to a much wider family of non-planar graphs, called partial $k$-framed graphs, which is general enough to include all 1-planar graphs and all optimal 2-planar graphs. A graph is $k$-framed, if it admits a drawing having a simple biconnected planar skeleton, whose faces have degree at most $k \geq 3$, and whose crossing edges are in the interiors of these faces. A partial $k$-framed graph is a subgraph of a $k$-framed graph. Clearly, the book thickness of partial $k$-framed graphs is lower bounded by $\left\lceil k/2 \right\rceil$, as they may contain cliques of size $k$ [8]. In this work, we present an upper bound on the book thickness of partial $k$-framed graphs that depends linearly only on $k$ (but not on $n$). Our main result is as follows.

**Theorem 1.** The book thickness of a partial $k$-framed graph is at most $6\left\lceil \frac{k}{2} \right\rceil + 5$.

Note that the partial 3-framed graphs are exactly the (simple) planar graphs. Also, it is known that 3-connected 1-planar graphs are partial 4-framed [2], while general 1-planar graphs can be augmented to 8-framed. Hence, Theorem 1 implies constant upper bounds for the book thickness of these families of graphs. Since optimal 2-planar graphs are 5-framed, the next corollary guarantees the first constant upper bound on the book thickness of this family.

**Corollary 2.** The book thickness of an optimal 2-planar graph is at most 23.

More in general, each partial $k$-framed graph is $h$-planar for $h = (\frac{k-2}{2})^2$, and hence for this family of $h$-planar graphs we prove that the book thickness is $O(\sqrt{h})$, while the best-known upper bound for general $h$-planar graphs is $O(h \log n)$ [21].

**Preliminaries.** We assume familiarity with basic graph-theoretic [20] and graph-drawing [18] concepts. Let $\Gamma$ be a drawing of a graph $G$. The planar skeleton $\sigma(G)$ of $G$ in $\Gamma$ is the plane subgraph of $G$ induced by the crossing-free edges of $G$ in $\Gamma$ (where the embedding of $\sigma(G)$ is the one induced by $\Gamma$). The edges of $\sigma(G)$ are crossing-free, while the edges that belong to $G$ but not to $\sigma(G)$ are crossing edges. A $k$-framed drawing of a graph is one such that its crossing-free edges determine a planar skeleton, which is simple, biconnected, spans all
the vertices, and has faces of degree at most \( k \geq 3 \). A graph is \( k \)-framed, if it admits a \( k \)-framed drawing; see Fig. 2. A partial \( k \)-framed graph is a subgraph of a \( k \)-framed graph. Clearly, if a \( k \)-framed graph has book thickness at most \( b \), then the book thickness of any of its subgraphs is at most \( b \). Thus, we will only consider \( k \)-framed graphs. Further, w.l.o.g., we will also assume that each pair of vertices that belongs to a face \( f \) of \( \sigma(G) \) is connected either by a crossing-free edge (on the boundary of \( f \)) or by a crossing edge (drawn inside \( f \)). In other words, the vertices on the boundary of \( f \) induce a clique of size at most \( k \). Under this assumption, graph \( G \) may contain parallel crossing edges connecting the same pair of vertices, but drawn in the interior of different faces of \( \sigma(G) \); see, e.g., the dashed edges of Fig. 2.

2 Proof of Theorem 1

Our approach adopts some ideas from the seminal work by Yannakakis on embeddings of planar graphs in books with five pages [47], not four. The main challenges of our generalization are posed by the crossing edges and by the fact that we cannot augment the input graph so that its underlying planar skeleton is internally-triangulated. Our technique is based on the so-called peeling-into-levels decomposition. Let \( G \) be an \( n \)-vertex \( k \)-framed graph with a \( k \)-framed drawing \( \Gamma \). We classify the vertices of \( G \) as follows: (i) vertices on the unbounded face of \( \sigma(G) \) are at level 0, and (ii) vertices that are on the unbounded face of the subgraph of \( \sigma(G) \) obtained by deleting all vertices of levels \( i \) are at level \( i \) \((0 < i < n)\); see, e.g., Fig. 3. Denote by \( \sigma_i(G) \) the subgraph of \( \sigma(G) \) induced by the vertices of \( L_i \). Observe that \( \sigma_i(G) \) is outerplane, but not necessarily connected. Next, we consider \( \sigma_i(G) \) and delete any edge that is not incident to the unbounded face. The resulting spanning subgraph of \( \sigma_i(G) \) is denoted by \( C_i(G) \). By definition, each connected component of \( C_i(G) \) is a cactus. Also, the only edges that belong to \( \sigma_i(G) \) but not to \( C_i(G) \) are the chords of \( \sigma_i(G) \). Finally, we denote by \( G_i \) the subgraph of \( G \) induced by the vertices of \( L_0 \cup \ldots \cup L_i \) containing neither chords of \( \sigma_i(G) \) nor the crossing edges that are in the interior of the unbounded face of \( \sigma(G) \).

Consider an edge \( e \) of \( \sigma(G) \). If the endpoints of \( e \) are assigned to the same level, \( e \) is a level edge; otherwise, \( e \) connects vertices of consecutive levels and is called a binding edge; see Fig. 3. By the definition of the level-partition, there is no edge \( e \in E \), that connects two vertices of levels \( i \) and \( j \), such that \(|i - j| > 1\). Another consequence of the level-partition is that any vertex of level \( i + 1 \) lies in the interior of a cycle of level \( i \). Next, we give a characterization for bounded faces of \( \sigma(G) \). A bounded face of \( \sigma(G) \) is an intra-level face of \( \sigma_i(G) \) if it is incident to at least one vertex of \( L_{i-1} \) but to no vertex of \( L_{i-2} \). We denote by \( \mathcal{F}_i \) the set of all the intra-level faces of \( \sigma_i(G) \). By definition, the unbounded face of \( \sigma_i(G) \) is not an intra-level face. Each intra-level face of \( \sigma_i(G) \) has either at least one binding edge between \( L_{i-1} \) and \( L_i \) on its boundary, or it consists exclusively of \( L_{i-1} \)-level edges.
At a high level, we will inductively compute a book embedding of \( G_{i+1} \), assuming that we have already computed a book embedding of \( G_i \). For this inductive strategy to work, the computed book embeddings satisfy particular invariants, which we define subsequently. We first focus on the base case, in which \( G \) consists of only two levels \( L_0 \) and \( L_1 \) under some additional assumptions (see Section 2.1). Afterwards, we consider the inductive case, in which \( G \) consists of more than two levels (see Section 2.2).

### 2.1 Base case: two-level instances

A two-level instance is a \( k \)-framed graph \( G \) consisting of two levels \( L_0 \) and \( L_1 \), such that there is no crossing edge in the unbounded face of \( \sigma_0(G) \), and either \( L_1 = \emptyset \) or \( \sigma_1(G) = C_1(G) \), i.e., \( \sigma_1(G) \) is chordless; refer to Fig. 4 of a two-level instance (see Fig. 4). Since \( \sigma(G) \) is biconnected, \( C_0(G) \) is a simple cycle. Let \( u_0, u_1, \ldots, u_{s-1} \) with \( s \geq 3 \) be the vertices of \( L_0 \) in the order they appear in a clockwise traversal of \( C_0(G) \) starting from \( u_0 \). An edge \((u_i, u_j)\) of \( \sigma_0(G) \) is short if \( i - j = \pm 1 \); otherwise it is long. By definition, \((u_0, u_{s-1})\) is long. In the following we will refer to the intra-level faces of \( \sigma_1(G) \) simply as intra-level faces, and we will further denote \( F_1 \) as \( F \). Consider now the graph \( C_1(G) \). Each of its connected components is a cactus; thus, its biconnected components, called blocks, are either single edges or simple cycles (that are chordless, as \( \sigma_1(G) = C_1(G) \)). A connected component of \( C_1(G) \) may degenerate into a single vertex, and this vertex itself is a degenerate block. A block that consists of more than one vertex is called non-degenerate. We equip \( F \) with a linear ordering \( \lambda(F) \) as follows. For \( i = 0, \ldots, s - 1 \), the intra-level faces incident to vertex \( u_i \) are appended to \( \lambda(F) \) as they appear in clockwise order around \( u_i \) starting from the one incident to \((u_{i-1}, u_i)\) and ending at the one incident to \((u_i, u_{i+1})\) (indices taken modulo \( s \)), unless already present. For a pair of intra-level faces \( f \) and \( f' \), we write \( f \prec_{\lambda} f' \) if \( f \) precedes \( f' \) in \( \lambda(F) \); similarly, we write \( f \preceq_{\lambda} f' \) if \( f = f' \) or \( f \prec_{\lambda} f' \).

Let \( C_1, \ldots, C_\gamma \) be the connected components of \( C_1(G) \) and let \( C \in \{C_1, \ldots, C_\gamma\} \). In general, several intra-level faces in \( F \) may contain vertices of \( C \) on their boundary. Let \( f_C \) be the first face in the ordering \( \lambda(F) \) that contains a vertex of \( C \). Consider now a clockwise traversal of the boundary of \( f_C \) starting from the vertex of \( L_0 \) with the smallest subscript that belongs to \( f_C \). We refer to the vertex, say \( v_C \), of \( C \) that is encountered first in this traversal as the first vertex of \( C \). Observe that, by definition, \( v_C \) is incident
to a binding edge that is on the boundary of $f_C$. We will further assume that $v_C$ forms a degenerate block $r_C$ of $C$. The leader of a block $B$ of $C$, denoted by $\ell(B)$, is the first vertex of $B$ that is encountered in any path of $C$ from $v_C$ to $B$; note that $\ell(B)$ is uniquely defined.

Consider a vertex $v$ of $C$. If $v$ belongs to only one block of $C$, then $v$ is assigned to that block. Otherwise $v$ is assigned to the block of $C$ such that $v$ belongs to $B$ and the graph-theoretic distance in $C$ between $\ell(B)$ and $v_C$ is the smallest. It follows that $v_C$ is assigned to the degenerate block $r_C$, and that for any non-degenerate block $B$ the leader $\ell(B)$ is not assigned to $B$. We denote by $B(v)$ the block of $C$ that a vertex $v$ is assigned to. Let $B$ be a block of $C$. Assume first that $B$ is non-degenerate. We refer to the first face in the ordering $\lambda(\mathcal{F})$ containing an edge of $B$ as the face that discovers $B$. Assume now that $B$ is degenerate, i.e., it consists of a single vertex $v$. We refer to the first face in the ordering $\lambda(\mathcal{F})$ that has $v$ on its boundary as the face that discovers $B$. In both cases, we denote by $d(B)$ the face in $\mathcal{F}$ that discovers block $B$. We extend the notion of discovery to the vertices of $G$. To this end, let $v$ be a vertex of $G$ (which can be incident to several intra-level faces in $\mathcal{F}$). We distinguish whether $v$ belongs to $L_0$ or $L_1$. In the former case, face $f$ of $\mathcal{F}$ discovers vertex $v$ if $f$ is the first intra-level face in the ordering $\lambda(\mathcal{F})$ that contains $v$ on its boundary. In the latter case, face $f$ in $\mathcal{F}$ discovers vertex $v$ if $f$ is the face that discovers the block vertex $v$ is assigned to. In both cases we denote by $d(f)$ the face in $\mathcal{F}$ that discovers vertex $v$.

This yields $d(v) = d(B(v))$ for any $v \in L_1$. The dominator $\text{dom}(B)$ of block $B$ is the vertex of $L_0$ with the smallest subscript that is on the boundary of $d(B)$. Several blocks of $C$ can be discovered by the same face, and by definition, these blocks have the same dominator. Analogously, we define the dominator $\text{dom}(f)$ of an intra-level face $f$ as the vertex of $L_0$ with the smallest subscript that is on the boundary of $f$. This yields $\text{dom}(B) = \text{dom}(d(B))$.

**Property 3.** The face $d(B)$ that discovers block $B$ is the first face in $\lambda(\mathcal{F})$ that has a vertex assigned to block $B$ on its boundary.
Proof. If $B$ is a degenerate block, the property follows by definition. Otherwise, $B$ contains at least one edge on its boundary. The face $d(B)$ is the first intra-level face in $\lambda(F)$ that contains an edge $(v, w)$ of $B$ on its boundary. Since only $\ell(B)$ is not assigned to $B$ and since $(v, w)$ is a boundary edge of $B$, at least one of $v$ and $w$ is assigned to $B$. The property follows from the fact that at most one of the endpoints of $(v, w)$ is not assigned to $B$. ▶

Let $B$ and $B'$ be two blocks of $C_1(G)$. We say that $B$ precedes $B'$ and write $B \prec B'$ if (i) $d(B) \succcurlyeq_{\lambda} d(B')$, or (ii) $d(B) = d(B')$ and in a counterclockwise traversal of $d(B)$ starting from $\text{dom}(d(B))$ block $B$ is encountered before block $B'$. Since $\lambda(F)$ is a well-defined ordering, the relationship “precedes” defines a total ordering of the blocks of $C_1(G)$.

► Property 4. Let $v$ be a vertex of $G$ and let $f_v \in F$ be an intra-level face that contains $v$ on its boundary. Then, $d(v) \preceq_{\lambda} f_v$ holds.

Proof. If $v$ belongs to $L_0$, then the property follows by definition. Otherwise, $v$ belongs to $L_1$, and $d(v)$ is the intra-level face that discovers the block $B(v)$, that is, $d(v) = d(B(v))$. If $B(v)$ is degenerate, then $d(v)$ is the first intra-level face in $\lambda(F)$ that has $v$ on its boundary. Hence, $d(v) \preceq_{\lambda} f_v$. Otherwise, by Property 3, $d(B(v))$ is the first intra-level face in $\lambda(F)$ that contains a vertex assigned to block $B$ on its boundary. Since $d(v) = d(B(v))$ and since $v$ is assigned to block $B$, it follows that $d(v) \preceq_{\lambda} f_v$. ▶

A vertex $v$ of $L_0$ belonging to the boundary of an intra-level face $f$ is prime with respect to $f$ if no vertex of $L_1$ and no long level edge is encountered in the clockwise traversal of $f$ from $\text{dom}(f)$ to $v$. By definition, $\text{dom}(f)$ is prime with respect to $f$. We say that a vertex $v$ is $f$-prime if either $v$ is prime with respect to face $f$ or $v$ belongs to $L_1$. By definition, any vertex of $L_1$ is $g$-prime with respect to any intra-level face $g$. Let $u_j$ be a vertex on $L_0$ that is not $d(u_j)$-prime with $j \in \{1, \ldots, s - 1\}$. Let $f_0^{u_j}, \ldots, f_s^{u_j}$ be the faces that have $u_j$ on their boundary in a counterclockwise traversal of $u_j$ starting from $(u_{j-1}, u_j)$ and ending at $(u_j, u_{j+1})$ (indices taken modulo $s$). Let $d$ be smallest index such that $f_d^{u_j} = d(u_j)$. The faces $f_0^{u_j}, \ldots, f_d^{u_j}$ that have $u_j$ as their dominator are called small.

### 2.1.1 Linear ordering

We compute the linear ordering $\rho$ of the vertices by first embedding the vertices of $L_0$ in the order $u_0, u_1, \ldots, u_{s-1}$, and by embedding the remaining vertices of $L_1$ based on the blocks that they have been assigned to and according to the following rules:

- **R.1** For $j = 0, \ldots, s-1$, let $B_0^j, \ldots, B_{d-1}^j$ be the blocks with $u_j$ as dominator such that the faces that discover them are not small (are small, resp.), and $B_i^j \prec B_i^{j+1}$ for $i = 0, 1, \ldots, t-2$.
- The vertices assigned to these blocks are placed right after (before, resp.) $u_j$ in $\rho$.

- **R.2** The vertices assigned to $B_t^j$ are right before those assigned to $B_{t+1}^i$, for each $i = 0, \ldots, t-2$.

- **R.3** The vertices assigned to the same block $B_t^j$ are in the order they appear in a counterclockwise traversal of the boundary of $B_t^j$ starting from the leader of $B_t^j$, for $i = 0, \ldots, t-1$.

For a pair of distinct vertices $v$ and $w$, we write $v \prec_{\rho} w$ if $v$ precedes $w$ in $\rho$. By Rule R.1, the vertices of $L_1$ discovered by $f$ and the $f$-prime vertices of $L_0$ are right next to each other in $\rho$. The next property is consequence of Rules R.1–R.3.

► Property 5. The vertices assigned to a block $B$ of $L_1$ appear consecutively in $\rho$.

Properties 6 to 8 will be useful in Section 2.2; for the proofs of Properties 7 and 8 refer to [5].
Let $C_1$ and $C_2$ be two connected components of $C_1(G)$ rooted at their first vertices, and let $B_1$ and $B_2$ be two non-degenerate blocks of $C_1$ and $C_2$, respectively. If there exists a vertex $v$ assigned to $B_2$ between $\ell(B_1)$ and the vertices assigned to $B_1$ in $\rho$, then all vertices assigned to $B_2$ appear in $\rho$ between $\ell(B_1)$ and the vertices assigned to $B_1$.

**Proof.** Let $B'_1$ be the block that $\ell(B_1)$ is assigned to. Then $B'_1$ is a block of $C_1$ and $B'_1 \neq B_1$. Let $w$ be a vertex assigned to block $B_1$. Then we have $\ell(B_1) \prec w$ and $\ell(B_1)$ is assigned to $B'_1$, $v$ assigned to $B_2$, and $w$ assigned to $B_1$. By Property 5, all vertices assigned to the same block are consecutive in $\rho$, and the claim follows.

**Property 6.** Let $C$ be a connected component of $C_1(G)$ rooted at its first vertex, and let $B$ be a non-degenerate block of $C$ with two children $B_1$ and $B_2$. If $\ell(B_1) \preceq \ell(B_2)$ and $B_2 \prec B_1$, then all vertices assigned to descendant blocks of $B_2$ (including $B_2$) precede in $\rho$ all vertices assigned to descendant blocks of $B_1$ (including $B_1$).

**Property 7.** Let $C$ be a connected component of $C_1(G)$, and let $B_1$ and $B_2$ be two distinct non-degenerate blocks of $C$. If there is a vertex $v$ assigned to a block $B_1$ between $\ell(B_2)$ and the remaining vertices of $B_2$ such that $\ell(B_1) \prec \ell(B_2)$, then $\ell(B_2)$ is assigned to $B_1$.

### 2.1.2 Edge-to-page assignment

An edge $(v, w)$ is a dominator edge if $v$ is the dominator of an intra-level face $f_w$ containing $w$ on its boundary. A dominator edge $(v, w)$ is backward if $v \prec w$ or forward otherwise. Next, we prove that all backward edges of $G$ can be assigned to a single page. The proof is reminiscent of a corresponding one by Yannakakis [47] for similarly-defined backward edges.

**Lemma 9.** Let $(v, w)$ and $(v', w')$ be two backward edges of $G$, such that $v, w, v', w'$ are four distinct vertices of $G$ with $v \prec w$, $v' \prec w'$ and $v \prec w'$. Then, $v \prec w \prec v' \prec w'$ or $v \prec w \prec v' \prec w \prec w'$ holds.

**Proof.** By definition, $v$ and $v'$ are the dominators of two intra-level faces $f_w$ and $f_{w'}$ containing $w$ and $w'$ on their boundaries. If $w \prec v'$, we have $v \prec w \prec v' \prec w$. Thus, assume $v' \prec w$. If $w$ belongs to $L_0$, then $w$ is not $f_w$-prime; see Fig. 5a. Since $v \prec v'$, and $v$ and $v'$ are the dominators of $f_w$ and $f_{w'}$, respectively, it follows that $f_w \prec f_{w'}$. Since vertex $w$ is not $f_w$-prime, we have $w' \prec w$. Hence, $v \prec v' \prec w \prec w'$. Assume now that $w$ belongs to $L_1$; see Fig. 5b. Since $v$ is the dominator of $f_w$, and $v \prec w$, the vertex $w$ belongs to a block $B(w)$ discovered by $v$. By Rule R.1, there is no vertex of $L_0$ between $v$ and the vertices assigned to $B(w)$ in $\rho$. Hence, $v'$ cannot appear between $v$ and $w$ in $\rho$. □

Similarly, we can prove that all forward edges can be assigned to a single page.
Lemma 10. Let \((v, w)\) and \((v', w')\) be two forward edges of \(G\), such that \(v, w, v'\) and \(w'\) are four distinct vertices of \(G\) with \(w \prec_{\rho} v, w' \prec_{\rho} v'\) and \(v' \prec_{\rho} v\). Then, \(w' \prec_{\rho} v' \prec_{\rho} w \prec_{\rho} v\) or \(w \prec_{\rho} w' \prec_{\rho} v' \prec_{\rho} v\) holds.

We now present properties helpful for the page assignment of the non-dominator edges.

Property 11. Let \(v\) be a \(d(v)\)-prime vertex of \(L_0\). Then \(v\) is \(f\)-prime for any intra-level face \(f\) that has \(v\) on its boundary. Also, \(v = \text{dom}(f)\), except possibly for \(f = d(v)\).

Proof. Let \(f\) be an intra-level face that is different from \(d(v)\) such that \(f\) has \(v\) on its boundary. By planarity, vertex \(v\) is the dominator of face \(f\). Thus, \(v\) is \(f\)-prime.

Property 12. Let \(w\) be a \(d(w)\)-prime vertex. For any vertex \(v\) with \(v \prec_{\rho} w\), \(d(v) \preceq_{\lambda} d(w)\).

Proof. Since \(w\) is \(d(w)\)-prime, \(w\) precedes any vertex discovered by a face \(f\) with \(d(w) \preceq_{\lambda} f\). Assuming to the contrary that \(d(w) \prec_{\lambda} d(v)\), we get \(w \prec_{\rho} v\); a contradiction.

Lemma 13. Let \(v\) and \(w\) be two vertices of \(G\), such that \(v \prec_{\rho} w\). Also, let \(f_v\) and \(f_w\) be two intra-level faces containing \(v\) and \(w\) on their boundaries, respectively, such that \(f_v \prec_{\lambda} f_w\). If the following conditions hold (i) \(v\) is \(d(v)\)-prime, (ii) \(w\) is \(d(w)\)-prime, and (iii) \(v\) and \(w\) are not the dominators of \(f_v\) and \(f_w\), respectively, then \(f_v \preceq_{\lambda} d(w)\) holds.

Proof. First, observe that by Property 12, we have \(d(v) \preceq_{\lambda} d(w)\). We split the proof into four cases based on whether \(v\) and \(w\) belong to \(L_0\) or to \(L_1\). 

(a) \(v\) and \(w\) belong to \(L_0\). Since \(v\) is \(d(v)\)-prime, it follows by Property 11 that \(v\) is also \(f_v\)-prime. However, since \(v\) is not the dominator of \(f_v\), it follows that \(d(v) = f_v\). The same holds for vertex \(w\) and the faces \(d(w)\) and \(f_w\). Now the claim \(f_v \preceq_{\lambda} d(w)\) is an immediate consequence of the assumption \(f_v \prec_{\lambda} f_w\).

(b) \(v\) belongs to \(L_0\) and \(w\) belongs to \(L_1\). By Property 11 and Condition (i), we know that \(v\) is \(f_v\)-prime. By Property 4, we have \(d(v) \preceq_{\lambda} f_v\). If \(d(v) \prec_{\lambda} f_v\), Property 11 implies \(v = \text{dom}(f_v)\) which contradicts Condition (iii). However, if \(d(v) = f_v\), the claim follows from \(d(v) \preceq_{\lambda} d(w)\).

(c) \(v\) belongs to \(L_1\) and \(w\) belongs to \(L_0\). Consider vertex \(w\). As above, by Property 11 and Condition (ii), it follows that \(w\) is \(f_w\)-prime and therefore, by Condition (iii), \(d(w) = f_w\) holds. Recalling the assumption \(f_v \prec_{\lambda} f_w\), the claim \(f_v \preceq_{\lambda} d(w)\)
is a direct consequence of \( f_v \prec \lambda f_w \). (d) \( v \) and \( w \) belong to \( L_1 \). Assume to the contrary that \( d(w) \prec \lambda f_v \). This implies \( d(v) \preceq \lambda d(w) \prec \lambda f_v \prec \lambda f_w \). We consider the two cases, namely, \( d(v) \prec \lambda d(w) \) and \( d(v) = d(w) \). In the former, since \( v \) belongs to \( L_1 \), vertex \( v \) belongs to the boundary of block \( B(v) \) discovered by \( d(v) \). Similarly, vertex \( w \) belongs to the boundary of block \( B(w) \) discovered by \( d(w) \). Hence, we have \( B(v) \neq B(w) \), as \( d(v) \prec \lambda d(w) \); see Fig. 6a. The order \( f_v \prec \lambda f_w \) violates the planarity of \( \sigma(G) \); a contradiction. In the latter, since \( v \) belongs to \( L_1 \), vertex \( v \) belongs to the boundary of block \( B(v) \) discovered by \( d(v) = d(w) \). Similarly, vertex \( w \) belongs to the boundary of block \( B(w) \) discovered by \( d(v) = d(w) \). For the two blocks \( B(v) \) and \( B(w) \) either \( B(v) \neq B(w) \) or \( B(v) = B(w) \) holds. First, assume that \( B(v) \neq B(w) \); see Fig. 6b. \( B(v) \) and \( B(w) \) are discovered by the same face, and \( v \prec \rho w \). By Rule R.2 it follows \( B(v) \) precedes \( B(w) \) in the counterclockwise traversal of \( d(v) = d(w) \). With \( f_v \prec \lambda f_w \), the planarity of \( \sigma(G) \) is violated; a contradiction. Next, assume \( B(v) = B(w) \). Since \( v \preceq \rho w \), by Rule R.3, in the counterclockwise traversal of \( B(v) = B(w) \) starting from its leader, vertex \( v \) precedes \( w \); see Fig. 6c. The order \( f_v \prec \lambda f_w \) violates the planarity of \( \sigma(G) \); a contradiction.

Lemma 14. Let \( v, w, x \), and \( z \) be four vertices of \( G \), such that \( (v, w) \) and \( (x, z) \) are two non-dominator edges of \( G \), and \( v \preceq_{\rho} x \preceq_{\rho} w \preceq_{\rho} z \). Let \( f_{vw} \) be a face with \( v \) and \( w \) on its boundary, and let \( f_{xz} \) be a face with \( x \) and \( z \) on its boundary such that \( f_{vw} \) and \( f_{xz} \) are two distinct faces. Moreover, \( v \) and \( w \) are \( f_{vw} \)-prime, whereas \( x \) and \( z \) are \( f_{xz} \)-prime. Then \( d(x) = f_{vw} \) or \( d(w) = f_{xz} \).

Observe that in Lemma 14 the edges \( (v, w) \) and \( (x, z) \) form two non-dominator edges that cannot be assigned to the same page. Lemma 14 translates this conflict into a relationship between the two faces \( f_{vw} \) and \( f_{xz} \) containing these edges. In the following, we model these conflicts as edges of an auxiliary graph which we call the conflict graph and denote by \( C(G) \).

Definition 15. The conflict graph \( C(G) \) of \( G \) is an undirected graph whose vertices are the faces of \( \mathcal{F} \). There exists an edge \( (f, g) \) with \( f \neq g \) in \( C(G) \) if and only if there exists a vertex \( w \) of level \( L_1 \) on the boundary of \( g \) such that \( f = df(w) \); see Fig. 7.

With this definition, we are able to restate Lemma 14 as follows.
Lemma 16. Let \((v, w)\) and \((x, z)\) be two non-dominator edges of \(G\) belonging to two distinct faces \(f_{vw}\) and \(f_{xz}\) such that \(v\) and \(w\) are \(f_{vw}\)-prime, \(x\) and \(z\) are \(f_{xz}\)-prime, \(v \prec_{\rho} w\), and \(x \prec_{\rho} z\). If \((v, w)\) and \((x, z)\) cross in \(\rho\), then there is an edge \((f_{vw}, f_{xz})\) in \(C(G)\).

Proof. W.l.o.g. assume \(v \prec_{\rho} x \prec_{\rho} w \prec_{\rho} z\). As in Lemma 14, we show \(v, x \in L_1\). By Lemma 14, \(f_{vw} = d(x)\) or \(f_{xz} = d(w)\) holds. Since \(x \in L_1\), \((f_{vw}, f_{xz}) \in C(G)\) if \(f_{vw} = d(x)\) holds. Thus, consider \(f_{xz} = d(w)\). If \(w \in L_1\), \((f_{vw}, f_{xz}) \in C(G)\). Assume \(w \in L_0\). If \(f_{vw} \prec_{\lambda} f_{xz}\), we get \(d(w) \preceq_{\lambda} f_{xz} \prec_{\lambda} f_{vw} = d(w)\) by Property 4; a contradiction. Otherwise, if \(w\) is \(d(w)\)-prime, we have \(d(w) = f_{xz} \prec_{\lambda} f_{vw}\) and thus, \(w = \dom(f_{vw})\) by Property 11; a contradiction. So, \(w\) is not \(d(w)\)-prime. Since \(w\) is \(f_{vw}\)-prime with \(w \neq \dom(f_{vw})\), at least one vertex of \(L_0\) on \(f_{vw}\) is right before \(w\) in a clockwise traversal of \(L_0\); see Fig. 8. By Property 12 and \(v, x \in L_1\), we have \(d(v) \preceq_{\lambda} d(x)\). By Property 4, we get \(d(v) \preceq_{\lambda} d(x) \preceq_{\lambda} f_{xz}\).

In fact, \(d(v) = d(x) = f_{xz}\) holds as otherwise \(d(v)\) and \(f_{vw}\) cannot bound \(B(v)\) without violating planarity. Thus, \(d(v) = f_{xz}\) and \(v \in L_1\) imply \((f_{vw}, f_{xz}) \in C(G)\).

In the following lemma, we prove an important property of the conflict graph.

Lemma 17. Graph \(C(G)\) is 1-page book embeddable.

Proof. We order the vertices of \(C(G)\) as in \(\lambda(F)\). For a contradiction, suppose \(C(G)\) contains two crossing edges \((f, g)\) and \((f', g')\) such that, w.l.o.g., \(f \prec_{\lambda} f' \prec_{\lambda} g \prec_{\lambda} g'\). Then, there is either \(v \in L_1\) on \(f\) with \(g = d(v)\), or \(w \in L_1\) on \(g\) with \(f = d(w)\). In the former, by Property 4, we have \(d(v) \preceq_{\lambda} f\), contradicting \(g = d(v) \preceq_{\lambda} f \prec_{\lambda} g\). In the latter, we argue analogously for \((f', g')\). Hence, there exist \(w, w' \in L_1\) on \(g\) and \(g'\), respectively, with \(f = d(w)\) and \(f' = d(w')\). This yields \(d(w) \prec_{\lambda} d(w') \prec_{\lambda} g \prec_{\lambda} g'\). Since \(w, w' \in L_1\), they are \(d(w)\)- and \(d(w')\)-prime. By Property 12 and since \(w \neq w'\), we have \(w \prec_{\rho} w'\). We apply Lemma 13 on \(w\) and \(w'\) with \(f_v = g\) and \(f_w = g'\), and obtain \(g \preceq_{\lambda} d(w)\), contradicting \(d(w) \prec_{\lambda} g\).

Since \(C(G)\) is 1-page book embeddable, it is outerplanar [8]. Hence, we have the following.

Corollary 18. Graph \(C(G)\) admits a vertex coloring with three colors.

We are now ready to give the main result of the section.

Theorem 19. The book thickness of a two-level \(k\)-framed graph \(G\) is at most \(3\lceil\frac{k}{2}\rceil + 2\).

Sketch. By Lemma 9, we embed all backward edges in page \(p_0\), and all forward edges in page \(p_1\). We next assign the remaining edges of \(G\) to three sets \(R_1\), \(B_1\) and \(G_1\), each containing \(\lceil\frac{k}{2}\rceil\) pages. We process the intra-level faces of \(F\) according to \(\lambda(F)\). Let \(f\) be the next face to process. By Corollary 18, face \(f\) has a color in \(\{r, b, g\}\). The vertices of \(f\) induce at most a \(k\)-clique \(C_f\) in \(G\). We assign the non-dominator edges of \(C_f\) to the pages of one of the sets \(R_1\), \(B_1\) and \(G_1\) depending on whether the color of \(f\) is \(r\), \(b\), or \(g\), respectively. This is possible since \(C_f\) is at most a \(k\)-clique [8]. Let \((v, w)\) and \((x, z)\) be two non-dominator edges, and let \(f_{vw}\) and \(f_{xz}\) be the faces of \(F\) responsible for assigning \((v, w)\) and \((x, z)\) to one
of the pages of $R_1 \cup B_1 \cup G_1$. If $v$ and $w$ are $f_{vw}$-prime, and $x$ and $z$ are $f_{xz}$-prime, then by Lemma 16, $(v, w)$ and $(x, z)$ cannot cross. In the full version [5], we prove that no two edges in the same page can cross, even if their endpoints are non-prime vertices of $L_0$. ◀

### 2.2 Inductive step: multi-level instances

In this section, we consider the general instances, which we call multi-level instances, in which the input $k$-framed graph $G$ consists of $q \geq 3$ levels $L_0, L_1, \ldots, L_{q-1}$. We refer to Fig. 9 for a schematic representation of a multi-level instance. Initially, we assume that the unbounded face of $\sigma(G)$ contains no crossing edges in its interior; we will eventually drop this assumption. Recall that $G_i$ denotes the subgraph of $G$ induced by the vertices of $L_0 \cup \ldots \cup L_i$ containing neither chords of $\sigma_i(G)$ nor the crossing edges that are in the interior of the unbounded face of $\sigma(G)$. We will further denote by $\hat{G}_i$ the subgraph of $G_i$ that is induced by the vertices of $L_{i-1} \cup L_i$ without the chords of $\sigma_{i+1}(G)$. Observe that $\hat{G}_i$ is not necessarily connected; however, its maximal biconnected components, referred to as bicomponents in the following, form two-level instances. To ease the description, we refer to the blocks of all bicomponents of $\hat{G}_i$ simply as the blocks of $\hat{G}_i$. In a book embedding of $G_i$, we say that two vertices of the level $L_j$ (with $j \leq i$) are sequential if there is no other vertex of level $L_j$ between them along the spine. We say that a set $U$ of vertices of level $L_{j'}$ is $j'$-delimited, with $j' \neq j$, if either: (a) there exist two sequential vertices of level $L_j$ such that all vertices of $U$ appear between them along the spine, or (b) all vertices of $U$ are preceded or followed along the spine by all vertices of $L_j$.
A book embedding $E_i$ of $G_i$ is good if it satisfies the following properties:\footnote{We stress at this point that even though Properties P.7c, P.7d and P.7e might be a bit difficult to be parsed, they formalize the main idea of Yannakakis’ algorithm for reusing the same set of pages in a book embedding. Notably, this formalization in the original seminal paper [47] is not present.}

**P.1** The left-to-right order of the vertices on the boundary of each non-degenerate block $B$ of $\hat{G}_i$ in $E_i$ complies with the order of these vertices in a counterclockwise (clockwise) traversal of the boundary of $B$, if $i$ is odd (even).

**P.2** All vertices of each block $B$ of $\hat{G}_i$, except possibly for its leftmost vertex, are consecutive and $(i - 1)$-delimited.

**P.3** If between the leftmost vertex $\ell(B)$ of a block $B$ of $\hat{G}_i$ and the remaining vertices of $B$ there is a vertex $v$ of $L_i$ that belongs to a block $B'$ of $\hat{G}_i$ in the same connected component as $B$, such that the leftmost vertex $\ell(B')$ of $B'$ is to the left of $\ell(B)$, then $B$ and $B'$ share $\ell(B)$.

**P.4** Let $B$ and $B'$ be two blocks of $\hat{G}_i$ for which P.3 does not apply, and let $\ell(B)$ and $\ell(B')$ be their leftmost vertices. If $\ell(B)$ precedes $\ell(B')$, then either $\ell(B')$ precedes all remaining vertices of $B$ or all remaining vertices of $B'$ precede all remaining vertices of $B$.

**P.5** For any $j \leq i - 2$, all the vertices of each block of $\hat{G}_i$ are $j$-delimited.

**P.6** The edges of $G_i$ are assigned to $6[\lceil k/2 \rceil] + 5$ pages partitioned as (i) $P = \{p_0, \ldots, p_4\}$, and (ii) $R^j = \{r_1^j, \ldots, r_{\lceil k/2 \rceil}^j\}$, $B^j = \{b_1^j, \ldots, b_{\lceil k/2 \rceil}^j\}$, $G^j = \{g_1^j, \ldots, g_{\lceil k/2 \rceil}^j\}$, $j \in \{0, 1\}$.

**P.7** The edges of $G_i$ are classified as backward, forward, or non-dominator such that:

a. For $\zeta \leq i$, the non-dominator edges of $\hat{G}_\zeta$ belong to $R^j \cup B^j \cup G^j$ with $j = \zeta \mod 2$.

b. The edges that are incident to the leftmost vertex of a bicomponent of $\hat{G}_i$ and that are in its interior are backward.

c. Let $B_1$ be a bicomponent of $\hat{G}_i$. The backward edges of $\hat{G}_i$ in the interior of $B_1$ are assigned to a single page $b(B_1)$, while the forward edges are assigned to two pages $f_1(B_1)$ and $f_2(B_1)$ of $P$ different from $b(B_1)$; refer to Fig. 9.

d. Let $B_{i-1}$ be a bicomponent of $\hat{G}_{i-1}$. The blocks $B_{i-1}^1, \ldots, B_{i-1}^\mu$ of $B_{i-1}$ are the boundaries of several bicomponents of $\hat{G}_{i-1}$. Then, the forward edges of $\hat{G}_{i-1}$ incident to $B_{i-1}^j$, with $j = 1, \ldots, \mu$, are either all assigned to $f_1(B_{i-1})$ or to $f_2(B_{i-1})$.

e. Let $\langle p_0', \ldots, p_4' \rangle$ be a permutation of $P$. Assume that the backward edges of $\hat{G}_{i-2}$ that are in the interior of a bicomponent $B_{i-2}$ of $\hat{G}_{i-2}$ have been assigned to $p_0'$ (in accordance with P.7c), while the forward edges of $\hat{G}_{i-2}$ that are in the interior of $B_{i-2}$ have been assigned to $p_1'$ and $p_2'$ (in accordance to P.7c and P.7d). The blocks of $B_{i-2}$ are the boundaries of several bicomponents $B_{i-2}^1, \ldots, B_{i-2}^\mu$ of $\hat{G}_{i-1}$. Consider now a bicomponent $B_{i-1}^j$ with $1 \leq j \leq \mu$ of $\hat{G}_{i-1}$. Assume w.l.o.g. that the forward edges of $B_{i-2}$ incident to $B_{i-1}^j$ are assigned to $p_1'$. Then, the backward edges of $B_{i-1}^j$ (which are incident to its blocks, and thus to the bicomponents of $\hat{G}_i$) are assigned to $p_2'$, while its forward edges to $p_3'$.

The book embeddings computed in Section 2.1 can be easily adjusted to become good.

▶ **Lemma 20.** Any two-level instance admits a good book embedding.

Finally, the next lemma deals with good book embeddings of multi-level instances.

▶ **Lemma 21.** Any multi-level instance admits a good book embedding.
Sketch. Assume to have recursively computed a good book embedding $E_i$ of $G_i$. We show how to extend $E_i$ to a good book embedding $E_{i+1}$ of $G_{i+1}$. Consider the set $H$ of bicomponents $B_1, \ldots, B_s$ of $G_{i+1}$, each of which forms a two-level instance. Hence, the vertices delimiting the unbounded faces of $B_1, \ldots, B_s$ form blocks $B_1, \ldots, B_s$ of $G_i$, which form a set of cacti in $\sigma(G)$. By rooting each connected component in this set at one of its blocks, we associate each bicomponent in $H$ with a root bicomponent denoted by $r(B_i), i = 1, \ldots, \chi$, and with a parity bit $e(B_i)$ that expresses whether the distance between $B_i$ and $r(B_i)$ is even or odd.

Assume to have processed the first $x - 1 < \chi$ bicomponents $B_1, \ldots, B_{x-1}$ of $H$ and to have extended $E_i$ to a good book embedding $E_i^{x-1}$ of $G_i$, together with $B_1, \ldots, B_{x-1}$. Consider the next bicomponent $B_x$ of $G_{i+1}$ in $H$. The boundary of $B_x$ is a simple cycle of vertices of $L_i$. Therefore, the vertices and the edges of this cycle are present in $G_i$, have been embedded in $E_i$ and thus in $E_i^{x-1}$. We show how to extend $E_i^{x-1}$ to a good book embedding $E_i^x$ of $G_i$ together with $B_1, \ldots, B_x$. Once all blocks in $H$ have been processed, the obtained book embedding $E_i^x$ is the desired good book embedding $E_{i+1}$ of $G_{i+1}$. The vertices that delimit the unbounded face of $B_x$ form a block $B_x$ of $G_i$. Let $u_0, \ldots, u_s = 1$ be the order of these vertices by Property $P.1$. We proceed by computing a good book embedding $E_x$ of $B_x$ which exists by Lemma 20, such that the left-to-right order of the vertices of $B_x$ is $u_0, \ldots, u_s = 1$ in $E_x$. If $i$ is even, this can be achieved by flipping $B_x$. Further, note that $E_x$ is good by Lemma 20. We extend $E_i^{x-1}$ to a good book embedding $E_i^x$ in two steps as follows.

In the first step, for $j = 0, 1, \ldots, s - 1$, the vertices of $B_x$ that appear between $u_j$ and $u_{j+1}$ in $E_x$, if any, are embedded right before $u_{j+1}$ in $E_i^{x-1}$ in the same left-to-right order as in $E_x$: also, the vertices of $B_x$ that appear after $u_{s-1} = 1$ in $E_x$, if any, are embedded right after $u_{s-1}$ in $E_i^{x-1}$ in the same left-to-right order as in $E_x$. In the second step, we assign the internal edges of $B_x$ to the already existing pages of $E_i^x$. This step will complete the extension of $E_i^{x-1}$ to $E_i^x$. The backward, forward, and non-dominator edges of $E_i$ that are internal in $B_x$ will be classified as backward, forward, and non-dominator, respectively, also in $E_i^x$. The non-dominator edges of $E_i$ that are internal in $B_x$ and are assigned to $r_1, \ldots, r_{[k/2]}$, $b_1, \ldots, b_{[k/2]}, g_1, \ldots, g_{[k/2]}$ in $E_i$, are assigned to $r_{1+1}, \ldots, r_{[k/2]+1}, b_1, \ldots, b_{[k/2]}, g_1, \ldots, g_{[k/2]}$ in $E_i^x$, respectively, where $j = i + 1 \mod 2$. All backward edges of $E_i$ have been assigned to page $p_0$ in $E_x$, while its forward edges have been assigned to $p_1$ and $p_3$, respectively. Denote by $p_{0,1}$ a common page $b$ of $P$, respectively, not necessarily to $p_0$, while the corresponding forward edges assigned to $p_1$ and $p_3$ in $E_x$ will be reassigned to two pages $p_1$ and $p_3$. We determine pages $p_1, p_2$, and $p_3$ as follows. Assuming $i \geq 3$, there is a bicomponent $B_{i-2}$ of $G_{i-2}$, whose boundary vertices form a cycle that, in $G_i$, contains the bicomponent $B_x$ in its interior. Assume w.l.o.g. that the backward edges of $B_{i-2}$ are assigned to page $p_0' \in P$, in accordance to $P.7c$. It follows by $P.7e$ that we may further assume w.l.o.g. that all the backward edges of the bicomponents of $G_{i-1}$, whose boundaries are blocks of $B_{i-2}$, have been assigned to pages $p_1'$ and $p_2'$ different from $p_0'$. Assume also, w.l.o.g., that the forwards edges of $B_{i-2}$ incident to $B_x$ have been assigned to $p_1'$. By Property $P.7e$, this implies that the backward (forward) edges of bicomponent $B_x$ must be assigned to page $p_2'$ (to $p_1'$ and $p_3'$, respectively). Note that also of all the previously processed bicomponents of $G_{i+1}$ in $H$ make use of these three pages plus the page $p_1'$. The choice between the two pages $p_1'$ and $p_3'$ is done based on the parity bit of $e(B_x)$, so that all forward edges of all bicomponents in $H$ having the same parity bit will be assigned to the same page in $\{p_1', p_3', p_0'\}$.

We initially assumed that the unbounded face of $\sigma(G)$ contains no crossing edges in its interior, to support the recursive strategy. We drop this assumption as follows. We assign these edges to the pages of $R^0 \cup B^0 \cup G^0$, which results in a good book embedding of $G$, since the endvertices of the edges already assigned to these pages are 0-delimited.

Altogether, Lemma 21 in conjunction with Lemma 20 completes the proof of Theorem 1.
3 Conclusions and open problems

Our research generalizes a fundamental result by Yannakakis in the area of book embeddings. To achieve $O(k)$ pages for partial $k$-framed graphs, we exploit the special structure of these graphs which allows us to model the conflicts of the crossing edges by means of a graph with bounded chromatic number (thus keeping the unavoidable relationship with $k$ low).

Even though our result only applies to a subclass of $h$-planar graphs, it provides useful insights towards a positive answer to the intriguing question of determining whether the book thickness of (general) $h$-planar graphs is bounded by a function of $h$ only. Another direction for extending our result is to drop the biconnectivity requirement of partial $k$-framed graphs.

We conclude that the time complexity of our algorithm is $O(k^2 n)$, assuming that a $k$-framed drawing of the considered graph is also provided. It is of interest to investigate whether (partial) $k$-framed graphs can be recognized in polynomial time. The question remains valid even for the class of optimal 2-planar graphs, which exhibit a quite regular structure. Brandenburg [12] provided a corresponding linear-time recognition algorithm for the class of optimal 1-planar graphs, while Da Lozzo et al. [16] showed that the related question of determining whether a graph admits a planar embedding whose faces have all degree at most $k$ is polynomial-time solvable for $k \leq 4$ and NP-complete for $k \geq 5$.

References


