Removing Connected Obstacles in the Plane Is FPT

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Abstract

Given two points in the plane, a set of obstacles defined by closed curves, and an integer k, does there exist a path between the two designated points intersecting at most k of the obstacles? This is a fundamental and well-studied problem arising naturally in computational geometry, graph theory, wireless computing, and motion planning. It remains NP-hard even when the obstacles are very simple geometric shapes (e.g., unit-length line segments). In this paper, we show that the problem is fixed-parameter tractable (FPT) parameterized by k, by giving an algorithm with running time $k^{O(k^3)}n^{O(1)}$. Here n is the number connected areas in the plane drawing of all the obstacles.

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1 Introduction

In the Connected Obstacle Removal problem we are given as input a source point s and a target point t in the plane, and our goal is to move from the source to the target along a continuous curve. The catch is that the plane is also littered with obstacles – each obstacle is represented by a bounded closed connected subset of the plane, and the goal is to get from the source to the target while intersecting as few of the obstacles as possible. Equivalently we can ask for the minimum number of obstacles that have to be removed so that one can move from s to t without touching any of the remaining one. The problem has a wealth of applications, and has been studied under different names, such as Barrier Coverage or Barrier Resilience in networking and wirless computing [1, 3, 15, 16, 17, 18], or Minimum Constraint Removal in planning [7, 10, 13, 14]. The problem is NP-hard even when the obstacles are restricted to simple geometric shapes, such as line segments (e.g., see [1, 17, 18]). On the other hand, for unit-disk obstacles in a restricted setting, the problem can be solved in polynomial time [16]. Whether Connected Obstacle Removal can be solved in polynomial time for unit-disk obstacles remains open. The problem is known to be APX-hard [2], and also no factor $o(n)$-approximation is known. For restricted inputs (such as unit disc or rectangle obstacles) better approximation algorithms are known [2, 3].

In this paper we approach the general Connected Obstacle Removal problem from the perspective of parameterized algorithms (see [4] for an introduction). In particular it is easy to see that the problem is solvable in time $n^{k+O(1)}$ if the solution curve is to intersect at most k obstacles. Here n is the number of connected regions in the plane defined by

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the simultaneous drawing of all the obstacles. If k is considered a constant then this is polynomial time, however the exponent of the polynomial grows with the parameter k. A natural problem is whether the algorithm can be improved to a *Fixed Parameter Tractable* (FPT) one, that is an algorithm with running time $f(k)n^{O(1)}$. In this paper we give the first FPT algorithm for the problem. Our algorithm substantially generalizes previous work by Kumar et al. [16] as well as the first author and Kanj [8].

▶ **Theorem 1.1.** *There is an algorithm for* Connected Obstacle Removal *with running time* $k^{O(k^3)}n^{O(1)}$.

Our arguments and the relation between our results and previous work are more conveniently stated in terms of an equivalent graph problem, which we now discuss. Given a graph G, a set $C \subset \mathbb{N}$ (interpreted as a set of *colors*), and a function $\chi : V(G) \to 2^C$ that assigns a set of colors to every vertex of v, a vertex set S *uses* the color set $\bigcup_{v \in S} \chi(v)$. In the Colored Path problem input consists of G, s, t, χ and k, and the goal is to find an $s - t$ path P that uses at most k colors. Note, that to obtain computational results for the problem, we assume that the regions and intersections formed by the obstacles can be computed and enumerated in polynomial time. We do not assume that the obstacles are simply-connected, however we assume that the boundary of each obstacle is union of finite number of disjoint simple closed curves. We may also assume that s and t are not on a boundary of any obstacle. It is easy to see that Connected Obstacle Removal reduces to Colored Path (see also Figure 1). In particular, we let the vertices of G be the connected components, called regions, of the plane minus the union of the boundaries of the obstacles and we put an edge between two vertices if their boundaries have a curve of positive length in common. The color set is exactly the set of obstacles and the color set of a vertex is the set of obstacles containing the region associated with the vertex. The equivalence of the instances is rather straightforward. One way, the sequence of (closures of) regions a path in the plane intersects when traversing it from s to t, determines an s-t walk in G. On the other hand, we can easily define an s-t path in plane from a path in the graph that intersects precisely the regions associated with the vertices of the path and crosses between consecutive regions in the common boundary. Of course, reducing from Connected Obstacle Removal in this way can not produce all possible instances of Colored Path: the graph G is always a planar graph, and for every color $c \in C$ the set $\chi^{-1}(c) = \{v \in V(G) : c \in \chi(v)\}$ induces a connected subgraph of G. We shall denote the Colored Path problem restricted to instances that satisfy the two properties above by Colored Path*. With these additional restrictions it is easy to reduce back (we can just take the dual of G and let each obstacle be the closure of the union of the faces containing the associated color), and therefore Connected Obstacle Removal and Colored Path* are, for all practical purposes, different formulations of the same problem.

Related Work in Parameterized Algorithms, and Barriers to Generalization. Korman et al. [15] initiated the study of Connected Obstacle Removal from the perspective of parameterized complexity. They show that Connected Obstacle Removal is FPT parameterized by k for unit-disk obstacles, and extended this result to similar-size fat-region obstacles with a constant *overlapping number*, which is the maximum number of obstacles having nonempty intersection. Eiben and Kanj [8] generalize the results of Korman et al. [15] by giving algorithms for Colored Path* with running time $f(k,t)n^{O(1)}$ and $g(k,\ell)n^{O(1)}$ where t is the treewidth of the input graph G, and ℓ is an upper bound on the number of *vertices* on the shortest solution path P.

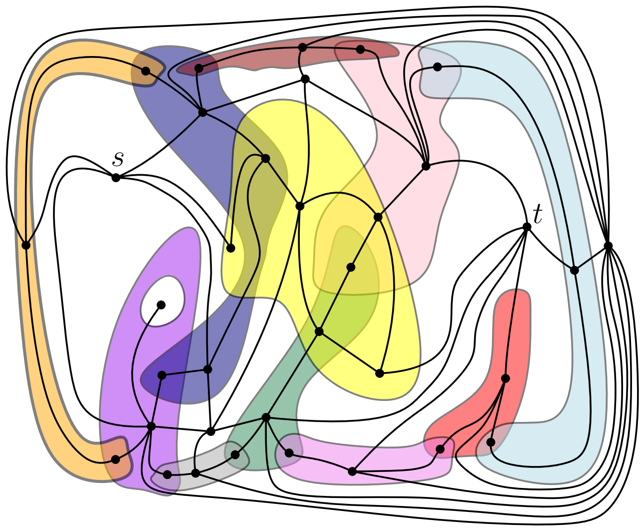

Figure 1 The figure shows an instance of Connected Obstacle Removal and the graph G of an equivalent instance of Colored Path. Every obstacle corresponds to a color, and the color set of a vertex are the obstacles that contain the vertex in their interior.

Eiben and Kanj [8] leave open the existence of an FPT algorithm for Colored Path* - Theorem 1.1 provides such an algorithm. Interestingly, Eiben and Kanj [8] also show that if an FPT algorithm for Colored Path* were to exist, then in many ways it would be the best one can hope for. More concretely, for each of the most natural ways to generalize Thoerem 1.1, Eiben and Kanj [8] provide evidence of hardness. Specifically, the Colored Path* problem imposes two constraints on the input – the graph G has to be planar and the color sets need to be connected. Eiben and Kanj [8] show that lifting *either one* of these constraints results in a W[1]-hard problem (i.e. one that is not FPT assuming plausible complexity theoretic hypotheses) *even* if the treewidth of the input graph G is a small constant, *and* the length of the a solution path (if one exists) is promised to be a function of k.

Algorithms that determine the existence of a path can often be adapted to algorithms that find the *shortest* such path. Eiben and Kanj [8] show that for Colored Path*, this can not be the case. Indeed, they show that an algorithm with running time $f(k)n^{O(1)}$ that given a graph G, color function χ and integers k and ℓ determines whether there exists an $s-t$ path of length at most ℓ using at most k colors, would imply that $\mathsf{FPT} = \mathrm{W}[1]$. Thus, unless $\mathsf{FPT} = \mathrm{W}[1]$ the algorithm of Theorem 1.1 can not be adapted to an FPT algorithm that finds a *shortest* path through k obstacles.

1.1 Overview of the Algorithm

The naive $n^{k+O(1)}$ time algorithm enumerates all choices of a set S on at most k colors in the graph, and then decides in polynomial time whether S is a feasible color set, in other words whether there exists a solution path that only uses colors from S. At a very high level our algorithm does the same thing, but it only computes sets S that can be obtained as a union of colors of at most k vertices and additionally it performs a pruning step so that not all n^k choices for S are enumerated.

In FPT algorithms such a pruning step is often done by clever *branching*: when choosing the i'th vertex defining S one would show that there are only $f(k)$ viable choices that could possibly lead to a solution. We are not able to implement a pruning step in this way. Instead, our pruning step is inspired by algorithms based on representative sets [12].

In particular, our algorithm proceeds in k rounds. In each round we make a family $\mathcal{P}_i$ of color sets of size at most i, with the following properties. First, $|\mathcal{P}_i| \leq k^{O(k^3)} n^{O(1)}$. Second, if there exists a solution path, then there exists a solution such that the set containing the *first i visited* colors is in $\mathcal{P}_i$.

In each round i the algorithm does two things: first it *extends* the already computed families $\mathcal{P}_0, \ldots \mathcal{P}_{i-1}$ by going over every set $S \in \bigcup_{j=0}^{i-1} \mathcal{P}_j$ and every vertex $v \in V(G)$ and inserting $S \cup \chi(v)$ into the new family $\hat{\mathcal{P}}_i$ if $|S \cup \chi(v)| = i$. It is quite easy to see that $\hat{\mathcal{P}}_i$ satisfies the second property - however it is a factor of n larger than the union of previous $\mathcal{P}_j$'s. If we keep extending $\hat{\mathcal{P}}_i$ in this way then after a super-constant number of steps we will break the first requirement that the family size should be at most $k^{O(k^3)} n^{O(1)}$. For this reason the algorithm also performs an *irrelevant set* step: as long as $\hat{\mathcal{P}}_i$ is "too large" we show that one can identify a set $S \in \hat{\mathcal{P}}_i$ that can be removed from $\hat{\mathcal{P}}_i$ without breaking the second property. We repeat this irrelevant set step until $\hat{\mathcal{P}}_i$ is sufficiently small. At this point we declare that this is our i'th family $\mathcal{P}_i$ and proceed to step $i+1$.

The most technically involved part of our argument is the proof of correctness for the irrelevant set step, see Section 3.3. This argument crucially exploits the structure of a large set of paths in a planar graph that start and end in the same vertex.

2 Preliminaries

For integers n, m with $n \leq m$, we let $[n, m] := \{n, n+1, \ldots, m\}$ and $[n] := [1, n]$. Let $\mathcal{F}$ be a family of subsets of a universe U. A *sunflower* in $\mathcal{F}$ is a subset $\mathcal{F}' \subseteq \mathcal{F}$ such that all pairs of elements in $\mathcal{F}'$ have the same intersection.

▶ **Lemma 2.1** ([9, 11]). *Let $\mathcal{F}$ be a family of subsets of a universe U, each of cardinality exactly b, and let $a \in \mathbb{N}$. If $|\mathcal{F}| \geq b!(a-1)^b$, then $\mathcal{F}$ contains a sunflower $\mathcal{F}'$ of cardinality at least a. Moreover, $\mathcal{F}'$ can be computed in time polynomial in $|\mathcal{F}|$.*

We assume familiarity with the basic notations and terminologies in graph theory and parameterized complexity. We refer the reader to the standard books [4, 5, 6] for more information on these subjects.

Graphs. All graphs in this paper are simple (i.e., loop-less and with no multiple edges). Let G be an undirected graph. For an edge $e = uv$ in G, *contracting* e means removing the two vertices u and v from G, replacing them with a new vertex w, and for every vertex y in the neighborhood of v or u in G, adding an edge wy in the new graph, not allowing multiple edges. Given a connected vertex-set $S \subseteq V(G)$, *contracting* S means contracting the edges between the vertices in S to obtain a single vertex at the end. For a set of edges $E' \subseteq E(G)$, the subgraph of G induced by E' is the graph whose vertex-set is the set of endpoints of the edges in E', and whose edge-set is E'.

A graph is *planar* if it can be drawn in the plane without edge intersections (except at the endpoints). A *plane graph* is a planar graph together with a fixed drawing. Each maximal connected region of the plane minus the drawing is an open set; these are the *faces*.

Let $W_1 = (u_1, \ldots, u_p)$ and $W_2 = (v_1, \ldots, v_q)$, $p, q \in \mathbb{N}$, be two walks such that $u_p = v_1$. Define the *gluing* operation $\circ$ that when applied to W_1 and W_2 produces that walk $W_1 \circ W_2 = (u_1, \ldots, u_p, v_2, \ldots, v_q)$. For a path $P = (v_1, \ldots, v_q)$, $q \in \mathbb{N}$ and $i \in [q]$, we let $\mathbf{pre}(P, v_i)$ be the *prefix* of the P ending at v_i, that is the path $(v_1, v_2, \ldots v_i)$. Similarly, we let $\mathbf{suf}(P, v_i)$ be the *suffix* of the P starting at v_i, that is the path $(v_i, v_{i+1}, \ldots v_q)$.

For a graph G and two vertices $u, v \in V(G)$, we denote by $d_G(u, v)$ the *distance* between u and v in G, which is the length (number of edges) of a shortest path between u and v in G.

Parameterized Complexity. A *parameterized problem* Q is a subset of $\Omega^* \times \mathbb{N}$, where Ω is a fixed alphabet. Each instance of the parameterized problem Q is a pair (x, k), where $k \in \mathbb{N}$ is called the *parameter*. We say that the parameterized problem Q is *fixed-parameter tractable* (FPT) [6], if there is a (parameterized) algorithm, also called an *FPT-algorithm*, that decides whether an input (x, k) is a member of Q in time $f(k) \cdot |x|^{O(1)}$, where f is a computable function. Let FPT denote the class of all fixed-parameter tractable parameterized problems. By *FPT-time* we denote time of the form $f(k) \cdot |x|^{O(1)}$, where f is a computable function and $|x|$ is the input instance size.

Colored Path and Colored Path*. For a set S, we denote by 2^S the power set of S. Let $G = (V, E)$ be a graph, let $C \subset \mathbb{N}$ be a finite set of colors, and let $\chi : V \longrightarrow 2^C$. A vertex v in V is *empty* if $\chi(v) = \emptyset$. A color c *appears on*, or is *contained in*, a subset S of vertices if $c \in \bigcup_{v \in S} \chi(v)$. For $u, v \in V(G)$, $\ell \in \mathbb{N}$, a u-v walk $W = (u = v_0, \ldots, v_r = v)$ in G is *ℓ-valid* if $|\bigcup_{i=0}^{r} \chi(v_i)| \le \ell$; i.e., if the total number of colors appearing on the vertices of W is at most ℓ. A color c is *connected* in G, or simply *connected*, if $\bigcup_{c \in \chi(v)} \{v\}$ induces a connected subgraph of G. The graph G is *color-connected*, if for every $c \in C$, c is connected in G.

For an instance (G, C, χ, s, t, k) of Colored Path*, if s and t are nonempty vertices, we can remove their colors and decrement k by $|\chi(s) \cup \chi(t)|$ because their colors appear on every s-t path. If afterwards k becomes negative, then there is no k-valid s-t path in G. Moreover, if s and t are adjacent, then the path (s, t) is a path with the minimum number of colors among all s-t paths in G. Therefore, we will assume:

▶ **Assumption 2.2.** *For an instance (G, C, χ, s, t, k) of Colored Path or Colored Path*, we can assume that s and t are nonadjacent empty vertices.*

▶ **Definition 2.3.** Let s, t be two designated vertices in G, and let x, y be two adjacent vertices in G such that $\chi(x) = \chi(y)$. We define the following operation to x and y, referred to as a *color contraction* operation, that results in a graph G', a color function χ', and two designated vertices s', t' in G', obtained as follows:

- G' is the graph obtained from G by contracting the edge xy, which results in a new vertex z;
- $s' = s$ (resp. $t' = t$) if $s \notin \{x, y\}$ (resp. $t \notin \{x, y\}$), and $s' = z$ (resp. $t' = z$) otherwise;
- $\chi' : V(G') \longrightarrow 2^C$ is defined as $\chi'(w) = \chi(w)$ if $w \neq z$, and $\chi'(z) = \chi(x) = \chi(y)$.

G is *irreducible* if there does not exist two vertices in G to which the color contraction operation is applicable.

▶ **Observation 2.4.** *Let G be a color-connected plane graph, C a color set, $\chi : V \longrightarrow 2^C$, $s, t \in V(G)$, and $k \in \mathbb{N}$. Suppose that the color contraction operation is applied to two vertices x, y in G to obtain G', χ', s', t', as described in Definition 2.3. For any two vertices $u, v \in V(G)$ and $p \subseteq C$ there is a u-v walk W with $\chi(W) = p$ in G if and only if there is a u'-v' walk W' with $\chi(W') = p$, where $u' = u$ (resp. $v' = v$) if $u \notin \{x, y\}$ (resp. $v \notin \{x, y\}$), and $u' = z$ (resp. $v' = z$) otherwise.*

3 FPT algorithm for Colored Path*

Given an instance (G, C, χ, s, t, k) and a vertex $v \in V(G)$, we say that a vertex u is *reachable* from a vertex v by a color set $p \subseteq C$ if there exists a v-u path p with $\chi(P) \subseteq p$. Furthermore, we say that a color set $p \subseteq C$ is *v-opening* if there is a vertex $u \in V(G)$ such that u is reachable from v by p, but not by any proper subset of p. Note that necessarily $\chi(v) \subseteq p$. A set of colors p *completes* a v-t walk Q if there is an s-v path P with $\chi(P) = p$, $|p \cup \chi(Q)| \le k$, and v is the only vertex on Q reachable from s by p. We say p *minimally completes* a v-t

walk Q, if p completes Q and there is no s-v path P' with $\chi(P') \subsetneq p$. We say that an s-t path P is *nice*, if for every prefix $\mathbf{pre}(P,u)$ of P ending at the vertex $u \in V(G)$ there is no s-u path P' with $\chi(P') \subsetneq \chi(\mathbf{pre}(P,u))$.

▶ **Observation 3.1.** *There is a k-valid s-t path if and only if there is a nice k-valid s-t path.*

▶ **Definition 3.2** (k-representation)**.** *Given an instance (G, C, χ, s, t, k) of* Colored Path**, a vertex $v \in V(G)$, and two families $\mathcal{P}$ and $\mathcal{P}'$ of s-opening subsets of C of size $\ell \leq k$, we say that $\mathcal{P}'$ k-represents $\mathcal{P}$ w.r.t. v if for every $p \in \mathcal{P}$ and every v-t walk Q such that p minimally completes Q, there is a set $p' \in \mathcal{P}'$ such that $|p' \cup \chi(Q)| \leq k$, $p' \cap \chi(Q) \supseteq p \cap \chi(Q)$, and there is an s-v path P' with $\chi(P') = p'$.*

The main technical result of this paper is then the following theorem stating that if a family $\mathcal{P}$ of color sets is large, then we can find an irrelevant color set in $\mathcal{P}$.

▶ **Lemma 3.3.** *Let (G, C, χ, s, t, k) be an instance of* Colored Path**. Given a family $\mathcal{P}$ of s-opening color sets of set of size $\ell \leq k$ and a vertex $v \in V(G)$, if $|\mathcal{P}| > f(k)$, $f(k) = k^{\mathcal{O}(k^3)}$, then we can in time polynomial in $|\mathcal{P}| + |V(G)|$ find a set $p \in \mathcal{P}$ such that $\mathcal{P} \setminus \{p\}$ k-represents $\mathcal{P}$ w.r.t. v.*

3.1 Algorithm assuming Lemma 3.3

In this subsection, we show how to get an FPT-algorithm for Colored Path* assuming Lemma 3.3 is true. The whole algorithm is relatively simple and is given in Algorithm 1. The main goal of the subsection is to show that the algorithm is correct and runs in FPT-time.

While the definition of k-representation is not the most intuitive definition of representation (for example it is not transitive), we show that it is sufficient to preserve a path of some specific form. Let P be a k-valid s-t path. For $i \in [0,k]$ let $v_i(P)$ be the last vertex on P such that $|\chi(\mathbf{pre}(P, v_i(P)))| \leq i$ and let $\ell_i(P)$ be the length, i.e., number of edges, of $\mathbf{suf}(P, v_i(P))$. If the path P is clear from the context, we write v_i and ℓ_i instead of $v_i(P)$ and $\ell_i(P)$. For example, we write $\mathbf{pre}(P, v_i)$ instead of $\mathbf{pre}(P, v_i(P))$. Note that for a k-valid s-t path P, $\ell_k(P) = 0$ and since G is irreducible w.r.t. color contraction, $\ell_0(P)$ is precisely the length of P. For two vectors $(a_0, a_1, a_2, \ldots, a_k), (b_0, b_1, b_2, \ldots, b_k)$ we say $(a_0, \ldots, a_k) < (b_0, \ldots, b_k)$ if there exists $i \in [0,k]$ such that $a_i < b_i$ and for all $j > i$ $a_j = b_j$. For a k-valid s-t path, we call the vector $\vec{\ell}(P) = (\ell_0(P), \ldots, \ell_k(P))$ the *characteristic* vector of P (see also Figure 2).

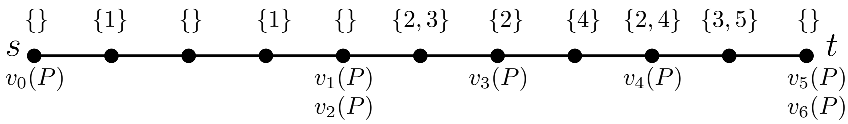

Figure 2 Figure depicting the definition of $v_i(P)$ for $k = 6$ and a path using 5 colors. The characteristic vector $\vec{\ell}(P) = (\ell_0(P), \ldots, \ell_6(P))$ is $(10, 6, 6, 4, 2, 0, 0)$.

▶ **Lemma 3.4.** *Let P be a k-valid s-t path with characteristic vector $\vec{\ell}(P)$, then there exists a nice k-valid s-t path P' with characteristic vector $\vec{\ell}(P')$ such that $\vec{\ell}(P') \leq \vec{\ell}(P)$.*

The following technical lemma will help us later show that replacing a prefix of a path P with $\chi(\mathbf{pre}(P, v_i)) \in \mathcal{P}$ by its representative will always lead to a path P' with $\vec{\ell}(P') \leq \vec{\ell}(P)$.

▶ **Lemma 3.5.** *Let P be an s-t path, $w \in V(P)$, let $\mathrm{pre} = \mathbf{pre}(P, w)$, $\mathrm{suf} = \mathbf{suf}(P, w)$, and let pre' be an s-w path such that $|\chi(\mathrm{pre}') \cup (\chi(\mathrm{pre}) \cap \chi(\mathrm{suf}))| \leq |\chi(\mathrm{pre})|$ and $|\chi(\mathrm{pre}')| < |\chi(\mathrm{pre})|$. Then $\vec{\ell}(\mathrm{pre}' \circ \mathrm{suf}) < \vec{\ell}(P)$.*

Algorithm 1 The algorithm for COLORED PATH⋆.

```
Data: An instance (G, C, χ, s, t, k) of COLORED PATH⋆
Result: A k-valid s-t path or NO, if such a path does not exists
P_0 = {∅};
for i ∈ [k] do
    P̂_i = ∅
    for v ∈ V(G) do
        for p ∈ ⋃_{j∈[0,i−1]} P_j do
            if |χ(v) ∪ p| = i then
                if there is a k-valid s-t path P with χ(P) ⊆ χ(v) ∪ p then
                    Output P and stop
                end
                P̂_i = P̂_i ∪ {χ(v) ∪ p}
            end
        end
    end
    for v ∈ V(G) do
        P_i^v = P̂_i
        while |P_i^v| > f(k) do
            Compute p ∈ P_i^v such that P_i^v \ {p} k-represents P_i^v w.r.t. v (by
             Lemma 3.3)
            P_i^v = P_i^v \ {p}
        end
    end
    P_i = ⋃_{v∈V(G)} P_i^v
end
Output NO
```

Next, we show that k-representativity preserve in a sense a representation of a k-valid paths with minimal characteristic vector. Before we state the next lemma we introduce the following notation. We say that a set of colors p *i-captures* a s-t path P if $|\chi(\mathbf{pre}(P, v_i)| = |p|$, p completes $\mathbf{suf}(P, v_i)$, and p contains $\chi(\mathbf{pre}(P, v_i)) \cap \chi(\mathbf{suf}(P, v_i))$.

▶ **Lemma 3.6.** *Let (G, C, χ, s, t, k) be a* YES*-instance, P a nice k-valid path minimizing $\vec{\ell}(P)$, and $\mathcal{P}'$ and $\mathcal{P}$ two families of s-opening subsets of C of size $i \le k$. If $|\chi(\mathbf{pre}(P, v_i))| = i$, $\mathcal{P}'$ k-represents $\mathcal{P}$ w.r.t. $v_i = v_i(P)$, and there is $p \in \mathcal{P}$ such that p i-captures P. Then there is $p' \in \mathcal{P}'$ such that p' i-captures P.*

Proof. Since $|p| = |\mathbf{pre}(P, v_i)| = i$ and p completes $\mathrm{suf}\, Pv_i$, it follows from the choice of P and Lemma 3.5 that p minimally completes P. Because, $\mathcal{P}'$ k-represents $\mathcal{P}$ w.r.t. v_i, it follows that there exists $p' \in \mathcal{P}'$ such that $|p' \cup \chi(\mathrm{suf}\, Pv_i)|$, there is a s-v_i path P' with $\chi(P') = p'$ and $p' \cap \chi(\mathbf{suf}(P, v_i)) \supseteq p \cap \chi(\mathbf{suf}(P, v_i)) \supseteq \chi(\mathbf{pre}(P, v_i)) \cap \chi(\mathbf{suf}(P, v_i))$. Where the second containment follows, because p i-captures P. Therefore p' contains $\chi(\mathbf{pre}(P, v_i)) \cap \chi(\mathbf{suf}(P, v_i))$. To finish the proof it only remains to show that no vertex on $\mathbf{suf}(P, v_i)$ other than v_i is reachable from s by p'. Assume otherwise and let $w \in V(\mathbf{suf}(P, v_i)) \setminus \{v_i\}$ be the last vertex that is reachable by p'. We show that $|p' \cup (\chi(\mathbf{pre}(P, w)) \cap \chi(\mathbf{suf}(P, w)))| \le |\chi(\mathbf{pre}(P, w))|$. Since clearly $|p'| = i < |\chi(\mathbf{pre}(P, w))|$, the lemma then follows by applying Lemma 3.5 and from the choice of P. ◀

▶ **Lemma 3.7.** *Let (G, C, χ, s, t, k) be a* YES*-instance, P a nice k-valid s-t path minimizing the vector $\vec{\ell}(P)$. Moreover, let $\mathcal{P}_0 = \emptyset$ and $\mathcal{P}_1, \ldots, \mathcal{P}_k$ the color sets created in the step on line 21 of Algorithm 1. Then for all $i \in [0, k]$ such that $|\chi(\mathbf{pre}(P, v_i))| = i$, there is $p_i \in \mathcal{P}_i$ such that p_i i-captures P.*

Proof. We will prove the lemma by induction. Since $\mathcal{P}_0$ contains $\emptyset$ and $\chi(s) = \emptyset$, it is easy to see that the lemma is true for $i = 0$ and that $\chi(\mathbf{pre}(P, v_0)) = 0$. Let us assume that the lemma is true for all $j < i$. If $v_i = v_{i-1}$,[1] then the statement is true for i, because $|\chi(\mathbf{pre}(P, v_i))| \leq i - 1$. Hence, we assume for the rest of the proof that $v_i \neq v_{i-1}$. Let $j \in [0, i-1]$ be such that $v_{j-1} \neq v_{i-1}$ but $v_j = v_{i-1}$ and let u be the vertex on P just after v_j. It follows from definition of v_{j-1}, v_j, and v_{i-1} that $|\chi(\mathbf{pre}(P, v_j))| = j$ and $|\chi(\mathbf{pre}(P, u))| = i$. By the induction hypothesis there is $p_j \in \mathcal{P}_j$ such that p_j i-captures P. In particular v_j is the last vertex on $\mathbf{suf}(P, v_j)$ reachable from s by p_j and $p_j \supseteq \chi(\mathbf{pre}(P, v_j)) \cap \chi(\mathbf{suf}(P, v_j))$.

▷ Claim 3.8. $|p_j \cup \chi(u)| = i$ and $p_j \cup \chi(u)$ minimally completes $\mathbf{suf}(P, v_i)$.

From the above claim, it follows that $\hat{\mathcal{P}}_i$ contains a color set $\hat{p} = p_j \cup \chi(u)$ such that $|\hat{p}| = i$ minimally completes $\mathbf{suf}(P, v_i)$. Moreover, $\hat{p} \supseteq \chi(\mathbf{pre}(P, v_i)) \cap \chi(\mathbf{suf}(P, v_i))$ and $\hat{p}$ i-captures P. The rest of the proof follows by applying Lemma 3.6 in every loop between the steps on lines 16 and 19 for $v = v_i$. ◀

Now, if the nice k-valid s-t path P minimizing the vector $\vec{\ell}(P)$ contains $i \leq k$ colors, then $v_i(P)$ is a singleton path (t). Since by Lemma 3.7 there is $p \in \mathcal{P}_i$ that i-captures P, it means that t is reachable from s by p and Algorithm 1 outputs a s-t path using only the colors in p. Moreover, whenever it outputs a path it check whether it is k-valid. Therefore after analyzing the running time of Algorithm 1 we obtain the following theorem.

▶ **Theorem 3.9.** *There is an algorithm that given an instance (G, C, χ, s, t, k) of* COLORED PATH* *either outputs k-valid s-t path or decides that no such path exists, in time $\mathcal{O}(k^{\mathcal{O}(k^3)} \cdot |V(G)|^{\mathcal{O}(1)})$.*

Note that by the reduction from CONNECTED OBSTACLE REMOVAL to COLORED PATH* discussed in the introduction, Theorem 3.9 implies also an algorithm for CONNECTED OBSTACLE REMOVAL with the asymptotically same running time and hence Theorem 1.1.

3.2 Proof of Lemma 3.3

▶ **Observation 3.10.** *Let $\mathcal{P}$ be a family of s-opening subsets of C of size $\ell \leq k$, $v \in V(G)$, and $p \in \mathcal{P}$. If there is an s-v path P with $\chi(P) \subsetneq p$, then $\mathcal{P} \setminus \{p\}$ k-represents $\mathcal{P}$.*

For the rest of the section we will fix $v \in V(G)$, $\ell \in [k]$, and we let $\mathcal{P}$ be a family of s-opening color sets of size ℓ such that, for every $p \in \mathcal{P}$, v is reachable from s by p but is not reachable from s by any proper subset of p. Our goal in the remainder of the section is to show that if $|\mathcal{P}| > f(k)$, $f(k) = k^{\mathcal{O}(k^3)}$, then we can find in FPT-time a color set $p \in \mathcal{P}$ such that $\mathcal{P} \setminus \{p\}$ k-represents $\mathcal{P}$ w.r.t. v. We refer to such p also as an *irrelevant* color set.

[1] Throughout the proof, to improve readability we write v_i instead of $v_i(P)$.

3.2.1 Sketch of the Proof

The main idea is to show that if the family $\mathcal{P}$ is large, in our case of size at least $k^{\mathcal{O}(k^3)}$, then we can find a subfamily of $\mathcal{P}$ that is structured and this structure makes it easier to find an irrelevant color set that can be always represented within the structured subfamily. We can first apply sunflower lemma and restrict our search to a subfamily of size at least $k^{\mathcal{O}(k^2)}$ whose color sets pairwise intersect in the same color sets c, but are otherwise pairwise color-disjoint. Now we can remove colors in c from the graph and apply the color contraction operation to newly created neighbors with the same color (see Subsection 3.2.3).

In the rest of the proof, we can restrict our search for an irrelevant color set to a family $\mathcal{P}$ whose color sets are pairwise color disjoint. Moreover, we assume the graph is irreducible w.r.t. color contraction. Now for each $p_i \in \mathcal{P}$ we compute an s-v path P_i such that $\chi(P_i) = p_i$, by Observation 3.10 this is simply done by finding an s-v path in the subgraph induced on vertices with colors in p_i. The goal is to further restrict the search for an irrelevant path to a set of paths $\mathbf{P}$ such that there is a small set of vertices U, $|U| \leq 2k$, such that all the paths in $\mathbf{P}$ visit all vertices of U in the same order, but every vertex in $V(G) \setminus (U \cup \{s, v\})$ appears on at most $\frac{|\mathbf{P}|}{f(k)}$ paths. This is simply done by finding a vertex that appear on the most paths in $\mathbf{P}$, including the vertex in U if the vertex appears on at least $\frac{|\mathbf{P}|}{|U|!\cdot f(k)}$ paths, and restricting $\mathbf{P}$ to the paths containing the vertex. Otherwise, we stop. We show in Lemma 3.13 that because each path in $\mathbf{P}$ has at most k colors, we stop after including at most $2k$ vertices into U. To get the paths that visit U in the same order, we just go through all $|U|!$ orderings of U and pick the one most paths adhere to. To finish the proof, we show that thanks to the structure of paths in $\mathbf{P}$, for any two consecutive vertices in U, there is a large set of paths that are pairwise vertex disjoint between the two consecutive vertices of U (Lemma 3.16). Hence, we get into the situation similar to the one in Figure 3. Any v-t path (walk) that contains at most k colors and does not contain vertices in U can only interact with a few of these paths between the two consecutive vertices. Hence, because $\mathcal{P}$ was large and because of the structure of paths in $\mathbf{P}$, we find a path that cannot share a color with any v-t walk with at most k colors (Lemma 3.17). But the color set of such a path is then represented by any other color set in $\mathcal{P}$, as they have the same size.

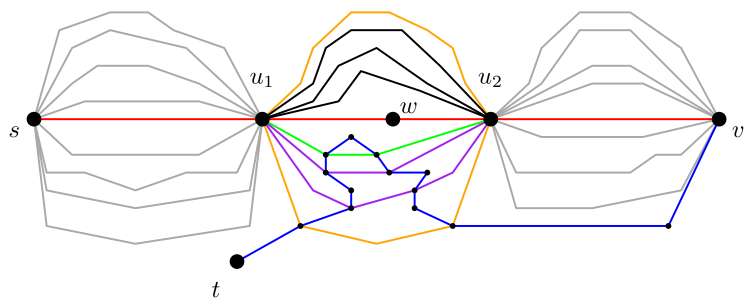

Figure 3 A set of pairwise color-disjoint paths that intersects exactly in u_1 and u_2 in the same order. If a path P from v to t do not contain s, u_1, nor u_2 but it shares a color with some vertex w on the part of the red. Then P has to cross at least 4 of the color-disjoint path and hence it has to contain at least 3 colors. For example for the blue path are vertices outside of the orange region, inside the purple region, and the region between red and green path pairwise color-disjoint. In each of these regions the blue path contains at least 2 consecutive vertices, hence at least one is not empty.

3.2.2 The Color-Disjoint Case

The goal of this subsection is to show that Lemma 3.3 is true for a special case when the color sets in $\mathcal{P}$ are pairwise color-disjoint and the input graph is irreducible w.r.t. color contraction. This is the most difficult and technical part of the proof. For the rest of the subsection we will have the following assumption:

▶ **Assumption 3.11.** *For an instance (G, C, χ, s, t, k) of* Colored Path* *and family $\mathcal{P}$ of color sets each of size $\ell \leq k$, we assume that G is irreducible w.r.t. color contraction and the sets in $\mathcal{P}$ are pairwise color-disjoint.*

In this subsection, it will be more convenient to work with a set of paths instead of a set of color sets. Given a set $\mathcal{P} = \{p_1, \ldots, p_{|\mathcal{P}|}\}$ of color-disjoint color sets such that v is reachable by each $p \in \mathcal{P}$ from s but not by any proper subset of p, we will construct a set of paths $\mathbf{P} = \{P_1, \ldots, P_{|\mathcal{P}|}\}$ such that $\chi(P_i) = p_i$ for all $i \in [|\mathcal{P}|]$. Note that, since v is not reachable from s by any proper subset of p_i, this can be simply done by finding a shortest s-v path in the graph obtained from G by removing all vertices containing a color not in p_i.

Now we restrict our attention to a subset of paths $\mathbf{Q}$ constructed by Algorithm 2.

Algorithm 2 Refining the set of important s-v paths.

Data: A set of pairwise color-disjoint paths $\mathbf{P}$ in a graph G
Result: A subset $\mathbf{Q}$ of $\mathbf{P}$ and $U \subseteq V(G)$ such that $|\mathbf{Q}| > \frac{|\mathbf{P}|}{((|U|+1)! \cdot (8k^2+8k+2))^{|U|}}$, all paths in $\mathbf{Q}$ contains all the vertices in U, and for every vertex $w \in V(G) \setminus U$ at most $\frac{|\mathbf{Q}|}{(|U|+1)! \cdot (8k^2+8k+2)}$ paths in $\mathbf{Q}$ contains w.

1 $U = \emptyset$ and $\mathbf{Q} = \mathbf{P}$
2 let u be a vertex in $V(G) \setminus U$ contained by the highest number of paths in $\mathbf{Q}$
3 **if** *u is contained in more than $\frac{|\mathbf{Q}|}{(|U|+1)! \cdot (8k^2+8k+3)}$ paths* **then**
4 $\quad U = U \cup \{u\}$
5 $\quad$ restrict $\mathbf{Q}$ to contain only the paths containing u
6 $\quad$ go to the step on line 2
7 **end**

▶ **Lemma 3.12.** *Let G a color-connected plane graph that is irreducible w.r.t. color contraction, s, u_1, u_2, u_3, v be vertices in G and let $\mathbf{P} = \{P_1, \ldots, P_{|\mathbf{P}|}\}$ be pairwise color-disjoint s-v paths all going through the vertices u_1, u_2, and u_3 in the same order. Then there are at most two paths $P_i \in \mathcal{P}$ such that if w^i_j, $j \in [3]$, denotes the vertex on P_i immediately after u_j then $\chi(w^i_1) \cap \chi(w^i_3) \neq \emptyset$.*

Now we can show that if $|U| \geq 2k+1$, then at the point when Algorithm 2 adds $2k+1$-st element to U, we can find k^2+k+1 paths in $\mathbf{Q}$ that visit the first $2k+1$ vertices of U in the same order. Lemma 3.12 then implies that there is a path $P_i \in \mathbf{P}$ such that $\chi(w^i_j) \cap \chi(w^i_{j'}) = \emptyset$ for all $j \neq j'$, $j, j' \in \{1, 3, 5, \ldots, 2k+1\}$, where w^i_j denotes the vertex on P_i immediately after u_j. Then $|\chi(P_i)| \geq k+1$ which contradicts definition of $\mathbf{P}$.

▶ **Lemma 3.13.** *If $|\mathbf{P}| \geq f(k)$, $f(k) = k^{\mathcal{O}(k^2)}$, then when Algorithm 2 terminates, it holds that $|U| < 2k+1$.*

We can now fix an ordering $\tau = (u_1, u_2, \ldots, u_{|U|})$ of vertices in U which maximizes the number of paths in $\mathbf{Q}$ that visit U in the same order as τ and let $\mathbf{Q}'$ be the restriction of $\mathbf{Q}$

to the paths that are consistent with this ordering. Clearly $|\mathbf{Q}| \le |\mathbf{Q}'| \cdot (2k)!$ and it suffice to show that we can find an irrelevant path in $\mathbf{Q}'$ if $|\mathbf{Q}'|$ is large. The agenda for the rest of the proof is as follows. Because $|U| \le 2k$ and intersection number of each vertex outside $|U|$ is small compared to the size of $\mathbf{Q}'$, only "few" paths can share a color with any k-valid v-t walk that do not contain a vertex in U hence we can find an irrelevant path. The color set of this irrelevant path is then the irrelevant color set in $\mathcal{P}$.

Recall that due to Assumption 3.11, we assume that the graph G is color contracted and no two neighbors have the same color set. Moreover, the paths in $\mathbf{Q}'$ are color-disjoint, so the vertices in $U \cup \{s, v\}$ are all empty and each neighbor of these vertices belongs to at most one path in $\mathbf{Q}'$. The goal in the following few technical lemmas is to show that for any two consecutive vertices u_i and u_{i+1} in U we can find a large (of size at least $4k+1$) subsets of paths in $\mathbf{Q}'$ that pairwise do not intersect between u_i and u_{i+1}.

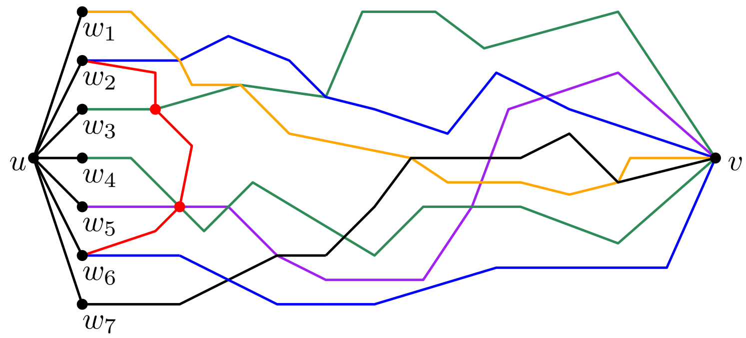

Figure 4 Situation in Lemma 3.14. On the picture are seven u-v paths, no 3 of them intersecting in the same vertex. The red w_2-w_6 path on the picture intersects the three paths containing w_3, w_4, and w_5, respectively. Any such path has to contain at least 2 vertices, else the only vertex on the path would be the intersection of 3 u-v paths.

▶ **Lemma 3.14.** *Given an instance (G, C, χ, s, t, k) which is irreducible w.r.t. color contraction, two vertices u, v, $b \in \mathbb{N}$ and a set $\mathbf{P}$ of k-valid u-v paths such that no b paths intersect in the same vertex. Let $w_1, \ldots, w_r$ be the neighbors of u, each the second vertex of a different path in $\mathbf{P}$, in counterclockwise order. For $i \in [r]$ let P_i denote the path in $\mathbf{P}$ containing w_i. Let $1 \le i < j \le r$, then the shortest curve σ from w_i to w_j that intersects G only in vertices of $V(G) \setminus \{u, v\}$ contains at least $\frac{\min\{j-i, r+i-j\}-1}{b}$ vertices on paths in $\mathcal{P} \setminus \{P_i, P_j\}$.*

Proof. See an example of the situation in Figure 4. Given a curve σ, we can easily find a closed curve σ' that intersect G in u, w_i, w_j and the vertices that are intersected by σ. The vertices on σ' are then the vertex separator separating v from either $w_{i+1}, \ldots, w_{j-1}$ or from $w_1, \ldots, w_{i-1}$ and $w_{j+1}, \ldots, w_r$. If the vertices on σ' are the vertex separator separating v from $w_{i+1}, \ldots, w_{j-1}$, then all the paths $P_{i+1}, \ldots, P_{j-1}$ has to pass a vertex on σ different than w_i or w_j. Since no b paths intersect in the same vertex, we get that σ contains at least $\frac{j-i-1}{b}$ vertices in this case. The case when the vertices on σ' are the vertex separator separating v from $w_1, \ldots, w_{i-1}$ and $w_{j+1}, \ldots, w_r$ is symmetric and the lemma follows. ◀

▶ **Lemma 3.15.** *Let (G, C, χ, s, t, k) be an instance of* Colored Path★ *such that G is irreducible w.r.t. color contraction, H a subgraph of G, and P a k-valid u-v path with $u, v \in V(H)$ and $\chi(P) \cap \chi(H) = \emptyset$. Then P intersects at most k faces of H.*

The combination of the two above lemmas immediately yields the following:

▶ **Lemma 3.16.** *Given an instance (G, C, χ, s, t, k) which is irreducible w.r.t. color contraction, two vertices u, v, an integer $b \in \mathbb{N}$ and a set $\mathbf{P}$ of k-valid pairwise color-disjoint u-v paths such that no b paths intersect in the same vertex. Let $w_1, \ldots, w_r$ be the neighbors of u, each the second vertex of a different path in $\mathcal{P}$, in counterclockwise order. Let $1 \leq i < j \leq r$ and let P_i and P_j be the two paths in $\mathcal{P}$ containing w_i and w_j, respectively. If $\min\{j - i, r + i - j\} > 2k \cdot b$, then P_i and P_j do not intersect.*

▶ **Lemma 3.17.** *If no b paths in $\mathbf{Q}'$ intersect in the same vertex in $V(G) \setminus (U \cup \{s, v\})$ and $|\mathbf{Q}'| > (8k^2 + 8k + 2) \cdot (|U| + 1) \cdot b$, then we can in polynomial time find a path $P \in \mathbf{Q}'$ such that for every k-valid v-t walk Q that does not contain a vertex in U holds $\chi(P) \cap \chi(Q) = \emptyset$.*

Proof. For the convenience let us denote s by u_0 and v by $u_{|U|+1}$. We will show that for every $i \in \{0, \ldots, |U|\}$, every k-valid v-t walk can intersect at most $(8k^2 + 8k + 2) \cdot b$ paths in a vertex on the path between u_i and u_{i+1}. For a path $P \in \mathbf{Q}'$ let P^i denote the subpath between u_i and u_{i+1} and let $\mathbf{Q}^i = \{P^i \mid P \in \mathbf{Q}'\}$. Clearly, the paths in $\mathbf{Q}^i$ are color-disjoint u_i-u_{i+1} each containing at most $\ell \leq k$ colors and no b paths in $\mathbf{Q}^i$ intersect in the same vertex beside u_i and u_{i+1}. Now let H^i be the subgraph of G induced by the edges on paths in $\mathbf{Q}^i$. Since G is color contracted, u_i is an empty vertex, and the paths in $\mathbf{Q}^i$ are colored disjoint, each neighbor of u_i appears on a unique path in $\mathbf{Q}^i$. Let $w_1, w_2, \ldots, w_{|\mathbf{Q}^i|}$ be the neighbors of u_i in H^i in counterclockwise order and let P^i_j be the path in $\mathbf{Q}^i$ that contains w_j. Clearly, t is in the interior of some face f of H^i and there is at least one path that contains an edge incident on f in H^i. Without loss of generality let P^i_1 be such path (note that we can always choose a counterclockwise order around u_i for which this is true).

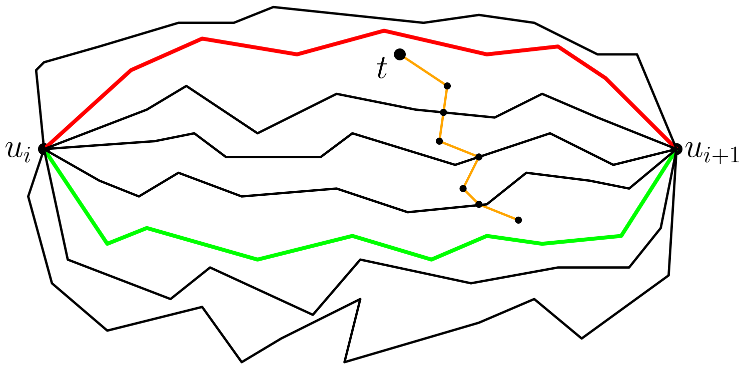

Figure 5 Any path that starts in a face incident on the red path and finishes in a face incident on the green path that does not contain u_i nor u_{i+1} has to appear in at least 4 different faces. Since the paths are color-disjoint, only the consecutive faces can share colors and hence any such path contains at least 2 colors.

▷ Claim 3.18. Let $j \in [|\mathbf{Q}^i|]$. If $(2k+1)(2k+1) \cdot b < j < |\mathbf{Q}^i| - (2k+1)(2k+1) \cdot b$, k-valid v-t walk Q that does not contain u_i nor u_{i+1} in the interior holds $\chi(P^i_j) \cap \chi(Q) = \emptyset$.

Proof. Consider the following set of paths: $P^i_1, P^i_{2k+2}, P^i_{4k+3}, \ldots, P_{4k^2+4k+1}, P^i_j$, P^i_{j+2k+1}, $P^i_{j+4k+2}, \ldots, P^i_{j+4k^2+4k}$. By Lemma 3.16, these paths are pairwise non-intersecting. Hence, we are in the situation as depicted in Figure 5. Since the paths in $\mathbf{Q}^i$ are pairwise color-disjoint, the colors of P^i_j are only on vertices of G inside the region bounded by P_{2k^2+k+1} and P^i_{j+2k+1}. Therefore, if $\chi(Q) \cap P^i_j \neq \emptyset$ for some v-t walk Q, then Q contains a vertex w inside the region bounded by P_{2k^2+k+1} and P^i_{j+2k+1}. Moreover, Q does not contain u_i nor u_{i+1} as an inner vertex then it either crosses all the paths in $\mathbf{P}_1 = \{P^i_{2k+2}, P^i_{4k+3}, \ldots, P_{4k^2+4k+1}\}$ or all the paths in $\mathbf{P}_2 = \{P^i_{j+2k+1}, P^i_{j+4k+2}, \ldots, P^i_{j+4k^2+2k}\}$. Without loss of generality, let us assume that Q crosses all the paths in $\mathbf{P}_1$. The other case is symmetric. As G is color contracted, no two consecutive vertices of P are empty. Hence, Q either crosses a path in $\mathbf{P}_1$ in a colored vertex or there is a colored vertex on Q between two consecutive paths in $\mathbf{P}_1$ (resp. $\mathbf{P}_2$). Let us partition the paths in $\mathbf{P}_1 \cup \{P_1, P_j\}$ into $k+1$ group of two consecutive pairs. that is we partition $\mathbf{P}_1$ into groups $\{P_1, P_{2k+2}\}$, $\{P_{4k+3}, P_{6k+4}\}$,. . .,$\{P_{4k^2-1}, P_{4k^2+2k}\}$,$\{P_{4k^2+4k+1}, P_j\}$. If the walk Q crosses all paths in $\mathbf{P}_1$, it has to contains a colored vertex in each of the $k+1$ groups. However, each two groups are separated by color-disjoint paths. Therefore, two colored vertices in two different groups have to be color-disjoint. But then $\chi(Q)$ contains at least $k+1$ colors, this is however not possible, because Q is k-valid. ◁

The lemma then follows by marking for each of $|U|+1$ consecutive pairs $2(2k+1)^2 \cdot b$ paths that can share a color with some Q and outputting any non-marked path. ◀

Since $\chi(P) \cap \chi(Q) = \emptyset$, $\chi(P)$ can be replaced by any other color set of $|\chi(P)|$ colors and we can safely remove it from $\mathcal{P}$. Since we chose $\mathbf{Q}'$ such that no $\frac{|\mathbf{Q}|}{(|U|+1)!\cdot(8k^2+8k+3)} = \frac{|\mathbf{Q}'|}{(|U|+1)\cdot(8k^2+8k+3)}$ paths intersect in $\mathbf{Q}'$, we get the following main result of this subsection.

▶ **Lemma 3.19.** *Let (G, C, χ, s, t, k) be an instance of* Colored Path⋆ *such that G is irreducible w.r.t. color contraction. Given a family $\mathcal{P}$ of pairwise color-disjoint s-reachable color sets of set of size $\ell \le k$ and a vertex $v \in V(G)$, if $|\mathcal{P}| > 2^{\mathcal{O}(k^2 \log(k))}$, then we can in time polynomial in $|\mathcal{P}| + |V(G)|$find a set $p \in \mathcal{P}$ such that $\mathcal{P} \setminus \{p\}$ k-represents $\mathcal{P}$ w.r.t. v.*

3.2.3 Finishing the Proof

Proof of Lemma 3.3. Since each set in $\mathcal{P}$ has precisely $\ell \le k$ colors, if $|\mathcal{P}| > \ell! \cdot (g(k))^{\ell+1}$, $g(k) = k^{\mathcal{O}(k^2)}$ then, by Lemma 2.1 we can, in time polynomial in $|\mathcal{P}|$, find a set $\mathcal{Q}$ of $g(k)+1$ sets in $\mathcal{P}$ such that there is a color set $c \subseteq C$ and for any two distinct sets p_1, p_2 in $\mathcal{Q}$ it holds $p_1 \cap p_2 = c$. Now let $(G, C', \chi', s, t, k-|c|)$ be the instance of Colored Path⋆ such that $C' = C \setminus c$ and for every $v \in V(G)$, $\chi'(v) = \chi(v) \setminus c$ and let $\mathcal{Q}' = \{p \setminus c \mid p \in \mathcal{Q}\}$.

▷ Claim 3.20. For all $p \in \mathcal{Q}$, $\mathcal{Q}' \setminus \{p \setminus c\}$ $(k-|c|)$-represents $\mathcal{Q}'$ w.r.t. v in $(G, C', \chi', s, t, k-|c|)$ if and only if $\mathcal{Q}^v \setminus \{p\}$ k-represents $\mathcal{Q}^v$ w.r.t. v in (G, C, χ, s, t, k).

Removing the colors in c from G can result in an instance that is not irreducible w.r.t. color contraction. However, in our algorithm for color-disjoint case, we crucially rely on the fact that G is irreducible w.r.t. color contraction. Now let $G_0 = G$, $\chi_0 = \chi'$, $s_0 = s$, $t_0 = t$, $v_0 = v$ and for $i \ge 1$ let $(G_i, C, \chi_i, s_i, t_i, k-|c|)$ be an instance we obtain from $(G_{i-1}, C, \chi_{i-1}, s_{i-1}, t_{i-1}, k-|c|)$ by a single color contraction of vertices x_i and y_i into a vertex z_i and let $v_i = z_i$ if $v_{i-1} \in \{x_i, y_i\}$ and $v_i = v_{i-1}$ otherwise.

▷ Claim 3.21. For all $p \in \mathcal{P}$, if the set $\mathcal{P} \setminus p$ $(k-|c|)$-represents $\mathcal{P}$ w.r.t. v_i in $(G_i, C, \chi_i, s_i, t_i, k-|c|)$, then $\mathcal{P} \setminus p$ $(k-|c|)$-represents $\mathcal{P}$ w.r.t. v in $(G_{i+1}, C, \chi_{i+1}, s_{i+1}, t_{i+1}, k-|c|)$.

Let $(G_i, C, \chi_i, s_i, t_i, k-|c|)$ be the instance obtained from $(G, C', \chi', s, t, k-|c|)$ by repeating color contraction operation until G_i is irreducible w.r.t. color contraction and let v_i be the image of v. Since G_i is irreducible w.r.t. color contraction, the sets in $\mathcal{Q}'$ are pairwise color-disjoint, and $|\mathcal{Q}'| = g(k)+1 > g(k-|c|)$, we can use Lemma 3.19 to find in time polynomial in $|\mathcal{Q}'|+|V(G)|$ a set $p \in \mathcal{Q}'$ such that $\mathcal{Q}' \setminus \{p\}$ $(k-|c|)$-represents $\mathcal{Q}'$ w.r.t. v_i in $(G_i, C, \chi_i, s_i, t_i, k-|c|)$. By Claim 3.21, it follows that $\mathcal{Q}' \setminus \{p\}$ $(k-|c|)$-represents $\mathcal{Q}'$ w.r.t. v in $(G, C', \chi', s, t, k-|c|)$ and by Claim 3.20 $\mathcal{Q} \setminus \{p \cup c\}$ k-represents $\mathcal{Q}$ in (G, C, χ, s, t, k). Finally, since for all $p' \in \mathcal{P} \setminus \mathcal{Q}$ is $p' \in \mathcal{P} \setminus \{p \cup c\}$ it follows that $\mathcal{P} \setminus \{p \cup c\}$ k-represents $\mathcal{P}$. ◀

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