A Toroidal Maxwell-Cremona-Delaunay Correspondence

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Abstract

We consider three classes of geodesic embeddings of graphs on Euclidean flat tori:

- A torus graph G is *equilibrium* if it is possible to place positive weights on the edges, such that the weighted edge vectors incident to each vertex of G sum to zero.
- A torus graph G is *reciprocal* if there is a geodesic embedding of the dual graph G^* on the same flat torus, where each edge of G is orthogonal to the corresponding dual edge in G^*.
- A torus graph G is *coherent* if it is possible to assign weights to the vertices, so that G is the (intrinsic) weighted Delaunay graph of its vertices.

The classical Maxwell-Cremona correspondence and the well-known correspondence between convex hulls and weighted Delaunay triangulations imply that the analogous concepts for plane graphs (with convex outer faces) are equivalent. Indeed, all three conditions are equivalent to G being the projection of the 1-skeleton of the lower convex hull of points in $\mathbb{R}^3$. However, this three-way equivalence does not extend directly to geodesic graphs on flat tori. On any flat torus, reciprocal and coherent graphs are equivalent, and every reciprocal graph is equilibrium, but not every equilibrium graph is reciprocal. We establish a weaker correspondence: Every equilibrium graph on any flat torus is affinely equivalent to a reciprocal/coherent graph on *some* flat torus.

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1 Introduction

The Maxwell-Cremona correspondence is a fundamental theorem establishing an equivalence between three different structures on straight-line graphs G in the plane:

- An *equilibrium stress* on G is an assignment of non-zero weights to the edges of G, such that the weighted edge vectors around every interior vertex p sum to zero:

$$\sum_{p\colon pq\in E} \omega_{pq}(p-q) = \begin{pmatrix}0\\0\end{pmatrix}$$

- A *reciprocal diagram* for G is a straight-line drawing of the dual graph G^*, in which every edge e^* is orthogonal to the corresponding primal edge e.

- A *polyhedral lifting* of G assigns z-coordinates to the vertices of G, so that the resulting lifted vertices in $\mathbb{R}^3$ are not all coplanar, but the lifted vertices of each face of G are coplanar.

Building on earlier seminal work of Varignon [76], Rankine [62, 61], and others, Maxwell [52, 51, 50] proved that any straight-line planar graph G with an equilibrium stress has both a reciprocal diagram and a polyhedral lifting. In particular, positive and negative stresses correspond to convex and concave edges in the polyhedral lifting, respectively. Moreover, for any equilibrium stress ω on G, the vector $1/\omega$ is an equilibrium stress for the reciprocal diagram G^*. Finally, for any polyhedral liftings of G, one can obtain a polyhedral lifting of the reciprocal diagram G^* via projective duality. Maxwell's analysis was later extended and popularized by Cremona [25, 26] and others; the correspondence has since been rediscovered several times in other contexts [3, 39]. More recently, Whiteley [77] proved the converse of Maxwell's theorem: every reciprocal diagram and every polyhedral lift corresponds to an equilibrium stress; see also Crapo and Whiteley [24]. For modern expositions of the Maxwell-Cremona correspondence aimed at computational geometers, see Hopcroft and Kahn [38], Richter-Gebert [64, Chapter 13], or Rote, Santos, and Streinu [66].

If the outer face of G is convex, the Maxwell-Cremona correspondence implies an equivalence between equilibrium stresses in G that are *positive* on every interior edge, *convex* polyhedral liftings of G, and reciprocal *embeddings* of G^*. Moreover, as Whiteley *et al.* [78] and Aurenhammer [3] observed, the well-known equivalence between convex liftings and weighted Delaunay complexes [5, 4, 13, 32] implies that all three of these structures are equivalent to a fourth:

- A *Delaunay weighting* of G is an assignment of weights to the vertices of G, so that G is the (power-)weighted Delaunay graph [4, 7] of its vertices.

Among many other consequences, combining the Maxwell-Cremona correspondence [77] with Tutte's spring-embedding theorem [75] yields an elegant geometric proof of Steinitz's theorem [70, 69] that every 3-connected planar graph is the 1-skeleton of a 3-dimensional convex polytope. The Maxwell-Cremona correspondence has been used for scene analysis of planar drawings [24, 74, 3, 5, 39], finding small grid embeddings of planar graphs and polyhedra [31, 15, 59, 64, 63, 67, 30, 40], and several linkage reconfiguration problems [22, 29, 73, 72, 60].

It is natural to ask how or whether these correspondences extend to graphs on surfaces other than the Euclidean plane. Lovász [47, Lemma 4] describes a spherical analogue of Maxwell's polyhedral lifting in terms of Colin de Verdière matrices [17, 20]; see also [44]. Izmestiev [42] provides a self-contained proof of the correspondence for planar frameworks, along with natural extensions to frameworks in the sphere and the hyperbolic plane. Finally, and most closely related to the present work, Borcea and Streinu [11], building on their earlier study of rigidity in infinite periodic frameworks [10, 9], develop an extension of the Maxwell-Cremona correspondence to infinite periodic graphs in the plane, or equivalently, to geodesic graphs on the Euclidean flat torus. Specifically, Borcea and Streinu prove that *periodic* polyhedral liftings correspond to *periodic* stresses satisfying an additional homological constraint.[1]

[1] Phrased in terms of toroidal frameworks, Borcea and Streinu consider only equilibrium stresses for which the corresponding reciprocal toroidal framework contains no essential cycles.

1.1 Our Results

In this paper, we develop a different generalization of the Maxwell-Cremona-Delaunay correspondence to geodesic embeddings of graphs on Euclidean flat tori. Our work is inspired by and uses Borcea and Streinu's recent results [11], but considers a different aim. Stated in terms of infinite periodic planar graphs, Borcea and Streinu study periodic equilibrium stresses, which necessarily include both positive and negative stress coefficients, that include periodic *polyhedral lifts*; whereas, we are interested in periodic *positive* equilibrium stresses that induce periodic reciprocal *embeddings* and periodic *Delaunay weights*. This distinction is aptly illustrated in Figures 8–10 of Borcea and Streinu's paper [11].

Recall that a Euclidean flat torus $\mathbb{T}$ is the metric space obtained by identifying opposite sides of an arbitrary parallelogram in the Euclidean plane. A *geodesic* graph G in the flat torus $\mathbb{T}$ is an embedded graph where each edge is represented by a "line segment". Equilibrium stresses, reciprocal embeddings, and weighted Delaunay graphs are all well-defined in the intrinsic metric of the flat torus. We prove the following correspondences for any geodesic graph G on any flat torus $\mathbb{T}$.

- Any equilibrium stress for G is also an equilibrium stress for the affine image of G on any other flat torus $\mathbb{T}'$ (Lemma 2.2). Equilibrium depends only on the common *affine* structure of all flat tori.
- Any reciprocal embedding G^* on $\mathbb{T}$ – that is, any geodesic embedding of the dual graph such that corresponding edges are orthogonal – defines unique equilibrium stresses in both G and G^* (Lemma 3.1).
- G has a reciprocal embedding if and only if G is coherent. Specifically, each reciprocal diagram for G induces an essentially unique set of Delaunay weights for the vertices of G (Theorem 4.5). Conversely, each set of Delaunay weights for G induces a *unique* reciprocal diagram G^*, namely the corresponding weighted Voronoi diagram (Lemma 4.1). Thus, a reciprocal diagram G^* may not be a weighted Voronoi diagram of the vertices of G, but some unique translation of G^* is.
- Unlike in the plane, G may have equilibrium stresses that are not induced by reciprocal embeddings; more generally, not every equilibrium graph on $\mathbb{T}$ is reciprocal (Theorem 3.2). Unlike equilibrium, reciprocality depends on the *conformal* structure of $\mathbb{T}$, which is determined by the shape of its fundamental parallelogram. We derive a simple geometric condition that characterizes which equilibrium stresses are reciprocal on $\mathbb{T}$ (Lemma 5.4).
- More generally, we show that for any equilibrium stress on G, there is a flat torus $\mathbb{T}'$, unique up to rotation and scaling of its fundamental parallelogram, such that the same equilibrium stress is reciprocal for the affine image of G on $\mathbb{T}'$ (Theorem 5.7). In short, every equilibrium stress for G is reciprocal on *some* flat torus. This result implies a natural toroidal analogue of Steinitz's theorem (Theorem 6.1): Every essentially 3-connected torus graph G is homotopic to a weighted Delaunay graph on some flat torus.

Due to space limitations, we defer several proofs to the full version of the paper [33].

1.2 Other Related Results

Our results rely on a natural generalization (Theorem 2.3) of Tutte's spring-embedding theorem to the torus, first proved (in much greater generality) by Colin de Verdière [18], and later proved again, in different forms, by Delgado-Friedrichs [28], Lovász [48, Theorem 7.1][49, Theorem 7.4], and Gortler, Gotsman, and Thurston [36]. Steiner and Fischer [68] and Gortler *et al.* [36] observed that this toroidal spring embedding can be computed by solving the Laplacian linear system defining the equilibrium conditions. We describe this result and the necessary calculation in more detail in Section 2. Equilibrium and reciprocal graph embeddings can also be viewed as discrete analogues of harmonic and holomorphic functions [49, 48].

Our weighted Delaunay graphs are (the duals of) *power diagrams* [4, 6] in the intrinsic metric of the flat torus. Toroidal Delaunay triangulations are commonly used to generate finite-element meshes for simulations with periodic boundary conditions, and several efficient algorithms for constructing these triangulations are known [53, 37, 14, 8]. Building on earlier work of Rivin [65] and Indermitte *et al.* [41], Bobenko and Springborn [7] proved that on any piecewise-linear surface, intrinsic Delaunay triangulations can be constructed by an intrinsic incremental flipping algorithm, mirroring the classical planar algorithm of Lawson [46]; their analysis extends easily to intrinsic weighted Delaunay graphs. Weighted Delaunay complexes are also known as *regular* or *coherent* subdivisions [79, 27].

Finally, equilibrium and reciprocal embeddings are closely related to the celebrated Koebe-Andreev circle-packing theorem: Every planar graph is the contact graph of a set of interior-disjoint circular disks [43, 1, 2]; see Felsner and Rote [34] for a simple proof, based in part on earlier work of Brightwell and Scheinerman [12] and Mohar [54]. The circle-packing theorem has been generalized to higher-genus surfaces by Colin de Verdière [16, 19] and Mohar [55, 56]. In particular, Mohar proves that any well-connected graph G on the torus is *homotopic* to an essentially unique circle packing *for a unique Euclidean metric* on the torus. This disk-packing representation immediately yields a weighted Delaunay graph, where the areas of the disks are the vertex weights. We revisit this result in Section 6.

Discrete harmonic and holomorphic functions, circle packings, and intrinsic Delaunay triangulations have numerous applications in discrete differential geometry; we refer the reader to monographs by Crane [23], Lovász [49], and Stephenson [71].

2 Background and Definitions

2.1 Flat Tori

A **flat torus** is the metric surface obtained by identifying opposite sides of a parallelogram in the Euclidean plane. Specifically, for any nonsingular 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\mathbb{T}_M$ denote the flat torus obtained by identifying opposite edges of the **fundamental parallelogram** $\lozenge_M$ with vertex coordinates $\binom{0}{0}$, $\binom{a}{c}$, $\binom{b}{d}$, and $\binom{a+b}{c+d}$. In particular, the *square* flat torus $\mathbb{T}_\square = \mathbb{T}_I$ is obtained by identifying opposite sides of the Euclidean unit square $\square = \lozenge_I = [0,1]^2$. The linear map $M : \mathbb{R}^2 \to \mathbb{R}^2$ naturally induces a homeomorphism from $\mathbb{T}_\square$ to $\mathbb{T}_M$.

Equivalently, $\mathbb{T}_M$ is the quotient space of the plane $\mathbb{R}^2$ with respect to the lattice Γ_M of translations generated by the columns of M; in particular, the square flat torus is the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. The quotient map $\pi_M : \mathbb{R}^2 \to \mathbb{T}_M$ is called a *covering* map or **projection**. A **lift** of a point $p \in \mathbb{T}_M$ is any point in the preimage $\pi_M^{-1}(p) \subset \mathbb{R}^2$. A **geodesic** in $\mathbb{T}_M$ is the projection of any line segment in $\mathbb{R}^2$; we emphasize that geodesics are *not* necessarily shortest paths.

2.2 Graphs and Embeddings

We regard each edge of an undirected graph G as a pair of opposing **darts**, each directed from one endpoint, called the **tail** of the dart, to the other endpoint, called its **head**. For each edge e, we arbitrarily label the darts e^+ and e^-; we call e^+ the **reference dart** of e. We explicitly allow graphs with loops and parallel edges. At the risk of confusing the reader, we often write $p{\to}q$ to denote an arbitrary dart with tail p and head q, and $q{\to}p$ for the reversal of $p{\to}q$.

A **drawing** of a graph G on a torus $\mathbb{T}$ is any continuous function from G (as a topological space) to $\mathbb{T}$. An **embedding** is an injective drawing, which maps vertices of G to distinct points and edges to interior-disjoint simple paths between their endpoints. The **faces** of an embedding are the components of the complement of the image of the graph; we consider only *cellular* embeddings, in which all faces are open disks. (Cellular graph embeddings are also called *maps*.) We typically do not distinguish between vertices and edges of G and their images in any embedding; we will informally refer to any embedded graph on any flat torus as a **torus graph**.

In any embedded graph, $\mathit{left}(d)$ and $\mathit{right}(d)$ denote the faces immediately to the left and right of any dart d. (These are possibly the same face.)

The **universal cover** $\widetilde{G}$ of an embedded graph G on any flat torus $\mathbb{T}_M$ is the unique infinite periodic graph in $\mathbb{R}^2$ such that $\pi_M(\widetilde{G}) = G$; in particular, each vertex, edge, or face of $\widetilde{G}$ projects to a vertex, edge, or face of G, respectively. A torus graph G is **essentially simple** if its universal cover $\widetilde{G}$ is simple, and **essentially 3-connected** if $\widetilde{G}$ is 3-connected [55, 56, 57, 58, 35]. We emphasize that essential simplicity and essential 3-connectedness are features of *embeddings*; see Figure 1.

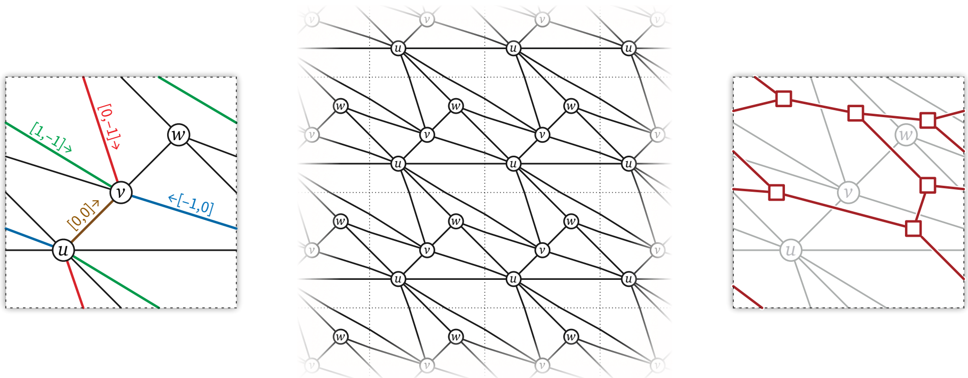

Figure 1 An essentially simple, essentially 3-connected geodesic graph on the square flat torus (showing the homology vectors of all four darts from u to v), a small portion of its universal cover, and its dual graph.

2.3 Homology, Homotopy, and Circulations

For any embedding of a graph G on the square flat torus $\mathbb{T}_\square$, we associate a **homology vector** $[d] \in \mathbb{Z}^2$ with each dart d, which records how the dart crosses the boundary edges of the unit square. Specifically, the first coordinate of $[d]$ is the number of times d crosses the vertical boundary rightward, minus the number of times d crosses the vertical boundary leftward; and the second coordinate of $[d]$ is the number of times d crosses the horizontal boundary upward, minus the number of times d crosses the horizontal boundary downward. In particular, reversing a dart negates its homology vector: $[e^+] = -[e^-]$. Again, see Figure 1. For graphs on any other flat torus $\mathbb{T}_M$, homology vectors of darts are similarly defined by how they crosses the edges of the fundamental parallelogram $\Diamond_M$.

The **(integer) homology class** $[\gamma]$ of a directed cycle γ in G is the sum of the homology vectors of its forward darts. A cycle is *contractible* if its homology class is $\binom{0}{0}$ and *essential* otherwise. In particular, the boundary cycle of each face of G is contractible.

Two cycles on a torus $\mathbb{T}$ are **homotopic** if one can be continuously deformed into the other, or equivalently, if they have the same integer homology class. Similarly, two drawings of the same graph G on the same flat torus $\mathbb{T}$ are homotopic if one can be continuously deformed into the other. Two drawings of the same graph G on the same flat torus $\mathbb{T}$ are homotopic if and only if every cycle has the same homology class in both embeddings [45, 21].

A **circulation** ϕ in G is a function from the darts of G to the reals, such that $\phi(p{\to}q) = -\phi(q{\to}p)$ for every dart $p{\to}q$ and $\sum_{p\to q}\phi(p{\to}q) = 0$ for every vertex p. We represent circulations by column vectors in $\mathbb{R}^E$, indexed by the edges of G, where $\phi_e = \phi(e^+)$. Let Λ denote the $2 \times E$ matrix whose columns are the homology vectors of the reference darts in G. The homology class of a circulation is the matrix-vector product

$$[\phi] \;=\; \Lambda\phi \;=\; \sum_{e\in E}\phi(e^+)\cdot[e^+].$$

(This identity directly generalizes our earlier definition of the homology class $[\gamma]$ of a cycle γ.)

2.4 Geodesic Drawings and Embeddings

A **geodesic drawing** of G on any flat torus $\mathbb{T}_M$ is a drawing that maps edges to geodesics; similarly, a **geodesic embedding** is an embedding that maps edges to geodesics. Equivalently, an embedding is geodesic if its universal cover $\widetilde{G}$ is a straight-line plane graph.

A geodesic drawing of G in $\mathbb{T}_M$ is uniquely determined by its **coordinate representation**, which consists of a coordinate vector $\langle p\rangle \in \lozenge_M$ for each vertex p, together with the homology vector $[e^+] \in \mathbb{Z}^2$ of each edge e.

The **displacement vector** Δ_d of any dart d is the difference between the head and tail coordinates of any lift of d in the universal cover $\widetilde{G}$. Displacement vectors can be equivalently defined in terms of vertex coordinates, homology vectors, and the shape matrix M as follows:

$$\Delta_{p\to q} := \langle q\rangle - \langle p\rangle + M\,[p{\to}q].$$

Reversing a dart negates its displacement: $\Delta_{q\to p} = -\Delta_{p\to q}$. We sometimes write Δx_d and Δy_d to denote the first and second coordinates of Δ_d. The **displacement matrix** Δ of a geodesic drawing is the $2 \times E$ matrix whose columns are the displacement vectors of the reference darts of G. Every geodesic drawing on $\mathbb{T}_M$ is determined *up to translation* by its displacement matrix.

On the *square* flat torus, the integer homology class of any directed cycle is also equal to the sum of the *displacement* vectors of its darts:

$$[\gamma] \;=\; \sum_{p\to q\in\gamma}[p{\to}q] \;=\; \sum_{p\to q\in\gamma}\Delta_{p\to q}.$$

In particular, the total displacement of any contractible cycle is zero, as expected. Extending this identity to circulations by linearity gives us the following useful lemma:

▶ **Lemma 2.1.** *Fix a geodesic drawing of a graph G on $\mathbb{T}_\square$ with displacement matrix Δ. For any circulation ϕ in G, we have $\Delta\phi = \Lambda\phi = [\phi]$.*

2.5 Equilibrium Stresses and Spring Embeddings

A **stress** in a geodesic torus graph G is a real vector $\omega \in \mathbb{R}^E$ indexed by the edges of G. Unlike circulations, homology vectors, and displacement vectors, stresses can be viewed as *symmetric* functions on the *darts* of G. An **equilibrium stress** in G is a stress ω that satisfies the following identity at every vertex p:

$$\sum_{p\to q}\omega_{pq}\Delta_{p\to q} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Unlike Borcea and Streinu [11, 10, 9], we consider only **positive** equilibrium stresses, where $\omega_e > 0$ for every edge e. It may be helpful to imagine each stress coefficient ω_e as a linear spring constant; intuitively, each edge pulls its endpoints inward, with a force equal to the length of e times the stress coefficient ω_e.

Recall that the linear map $M\colon \mathbb{R}^2 \times \mathbb{R}^2$ associated with any nonsingular 2×2 matrix induces a homeomorphism $\underline{M}\colon \mathbb{T}_\square \to \mathbb{T}_M$. In particular, applying this homeomorphism to a geodesic graph in $\mathbb{T}_\square$ with displacement matrix Δ yields a geodesic graph on $\mathbb{T}_M$ with displacement matrix $M\Delta$. Routine definition-chasing now implies the following lemma.

▶ **Lemma 2.2.** *Let G be a geodesic graph on the square flat torus $\mathbb{T}_\square$. If ω is an equilibrium stress for G, then ω is also an equilibrium stress for the image of G on any other flat torus $\mathbb{T}_M$.*

Our results rely on the following natural generalization of Tutte's spring embedding theorem to flat torus graphs.

▶ **Theorem 2.3** (Colin de Verdiére [18]; see also [28, 48, 36])**.** *Let G be any essentially simple, essentially 3-connected embedded graph on any flat torus $\mathbb{T}$, and let ω be any positive stress on the edges of G. Then G is homotopic to a geodesic embedding in $\mathbb{T}$ that is in equilibrium with respect to ω; moreover, this equilibrium embedding is unique up to translation.*

Theorem 2.3 implies the following sufficient condition for a displacement matrix to describe a geodesic embedding on the square torus.

▶ **Lemma 2.4.** *Fix an essentially simple, essentially 3-connected graph G on $\mathbb{T}_\square$, a $2 \times E$ matrix Δ, and a positive stress vector ω. Suppose for every directed cycle (and therefore any circulation) ϕ in G, we have $\Delta\phi = \Lambda\phi = [\phi]$. Then Δ is the displacement matrix of a geodesic **drawing** on $\mathbb{T}_\square$ that is homotopic to G. If in addition ω is an equilibrium stress for that drawing, the drawing is an embedding.*

Proof. A result of Ladegaillerie [45] implies that two embeddings of a graph on the same surface are homotopic if the images of each directed cycle are homotopic. Since homology and homotopy coincide on the torus, the assumption $\Delta\phi = \Lambda\phi = [\phi]$ for every directed cycle immediately implies that Δ is the displacement matrix of a geodesic drawing that is homotopic to G.

If ω is an equilibrium stress for that drawing, then the uniqueness clause in Theorem 2.3 implies that the drawing is in fact an embedding. ◀

Following Steiner and Fischer [68] and Gortler, Gotsman, and Thurston [36], given the coordinate representation of any geodesic graph G on the square flat torus, with any positive stress vector $\omega > 0$, we can compute an isotopic equilibrium embedding of G by solving the linear system

$$\sum_{p \to q} \omega_{pq}\big(\langle q\rangle - \langle p\rangle + [p{\to}q]\big) = \binom{0}{0} \qquad \text{for every vertex } q$$

for the vertex locations $\langle p\rangle$, treating the homology vectors $[p{\to}q]$ as constants. Alternatively, Lemma 2.4 implies that we can compute the displacement vectors of every isotopic equilibrium embedding directly, by solving the linear system

$$\sum_{p \to q} \omega_{pq} \Delta_{p \to q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{for every vertex } q$$
$$\sum_{left(d)=f} \Delta_d = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{for every face } f$$
$$\sum_{d \in \gamma_1} \Delta_d = [\gamma_1]$$
$$\sum_{d \in \gamma_2} \Delta_d = [\gamma_2]$$

where γ_1 and γ_2 are any two directed cycles with independent non-zero homology classes.

2.6 Duality and Reciprocality

Every embedded torus graph G defines a **dual graph** G^* whose vertices correspond to the faces of G, where two vertices in G are connected by an edge for each edge separating the corresponding pair of faces in G. This dual graph G^* has a natural embedding in which each vertex f^* of G^* lies in the interior of the corresponding face f of G, each edge e^* of G^* crosses only the corresponding edge e of G, and each face p^* of G^* contains exactly one vertex p of G in its interior. We regard any embedding of G^* to be *dual* to G if and only if it is homotopic to this natural embedding. Each dart d in G has a corresponding dart d^* in G^*, defined by setting $head(d^*) = left(d)^*$ and $tail(d^*) = right(d^*)$; intuitively, the dual of a dart in G is obtained by rotating the dart counterclockwise.

It will prove convenient to treat vertex coordinates, displacement vectors, homology vectors, and circulations in any dual graph G^* as *row* vectors. For any vector $v \in \mathbb{R}^2$ we define $v^\perp := (Jv)^T$, where $\mathbf{J} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the matrix for a 90° counterclockwise rotation. Similarly, for any $2 \times n$ matrix A, we define $A^\perp := (JA)^T = -A^T J$.

Two dual geodesic graphs G and G^* on the same flat torus $\mathbb{T}$ are **reciprocal** if every edge e in G is orthogonal to its dual edge e^* in G^*.

A **cocirculation** in G a *row* vector $\theta \in \mathbb{R}^E$ whose transpose describes a circulation in G^*. The *cohomology* class $[\theta]^*$ of any cocirculation is the transpose of the homology class of the circulation θ^T in G^*. Recall that Λ is the $2 \times E$ matrix whose columns are homology vectors of edges in G. Let λ_1 and λ_2 denote the first and second rows of Λ. The following lemma is illustrated in Figure 2; we defer the proof to the full version of the paper [33].

▶ **Lemma 2.5.** *The row vectors λ_1 and λ_2 describe cocirculations in G with cohomology classes $[\lambda_1]^* = (0\ 1)$ and $[\lambda_2]^* = (-1\ 0)$.*

2.7 Coherent Subdivisions

Let G be a geodesic graph in $\mathbb{T}_M$, and fix arbitrary real weights π_p for every vertex p of G. Let $p{\to}q$, $p{\to}r$, and $p{\to}s$ be three consecutive darts around a common tail p in clockwise order. Thus, $left(p{\to}q) = right(p{\to}r)$ and $left(p{\to}r) = right(p{\to}s)$. We call the edge pr **locally Delaunay** if the following determinant is positive:

$$\begin{vmatrix} \Delta x_{p\to q} & \Delta y_{p\to q} & \frac{1}{2}|\Delta_{p\to q}|^2 + \pi_p - \pi_q \\ \Delta x_{p\to r} & \Delta y_{p\to r} & \frac{1}{2}|\Delta_{p\to r}|^2 + \pi_p - \pi_r \\ \Delta x_{p\to s} & \Delta y_{p\to s} & \frac{1}{2}|\Delta_{p\to s}|^2 + \pi_p - \pi_s \end{vmatrix} > 0. \tag{2.1}$$

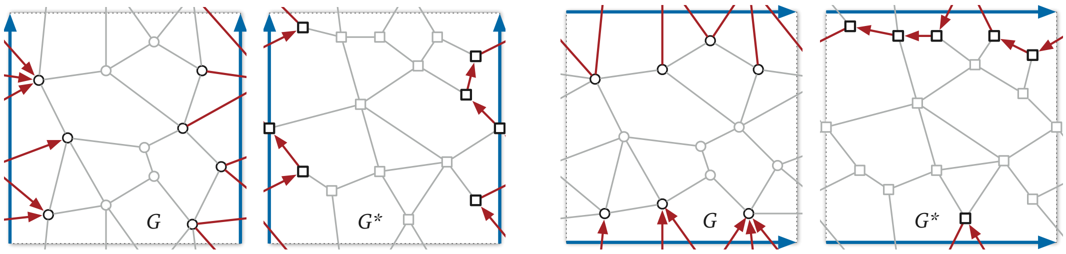

Figure 2 Proof of Lemma 2.5: The darts in G crossing either boundary edge of the fundamental square dualize to a closed walk in G^* parallel to that boundary edge.

This inequality follows by elementary row operations and cofactor expansion from the standard determinant test for appropriate lifts of the vertices p, q, r, s to the universal cover:

$$\begin{vmatrix} 1 & x_p & y_p & \frac{1}{2}(x_p^2 + y_p^2) - \pi_p \\ 1 & x_q & y_q & \frac{1}{2}(x_q^2 + y_q^2) - \pi_q \\ 1 & x_r & y_r & \frac{1}{2}(x_r^2 + y_r^2) - \pi_r \\ 1 & x_s & y_s & \frac{1}{2}(x_s^2 + y_s^2) - \pi_s \end{vmatrix} > 0. \tag{2.2}$$

(The factor $1/2$ simplifies our later calculations, and is consistent with Maxwell's construction of polyhedral liftings and reciprocal diagrams.) Similarly, we say that an edge is **locally flat** if the corresponding determinant is zero. Finally, G is the **weighted Delaunay graph** of its vertices if every edge of G is locally Delaunay and every diagonal of every non-triangular face is locally flat.

One can easily verify that this condition is equivalent to G being the projection of the weighted Delaunay graph of the lift $\pi_M^{-1}(V)$ of its vertices V to the universal cover. Results of Bobenko and Springborn [7] imply that any finite set of weighted points on any flat torus has a unique weighted Delaunay graph. We emphasize that weighted Delaunay graphs are *not* necessarily either simple or triangulations; however, every weighted Delaunay graphs on any flat torus is both essentially simple and essentially 3-connected. The dual **weighted Voronoi graph** of P, also known as its *power diagram* [4, 6], can be defined similarly by projection from the universal cover.

Finally, a geodesic torus graph is **coherent** if it is the weighted Delaunay graph of its vertices, with respect to some vector of weights.

3 Reciprocal Implies Equilibrium

▶ **Lemma 3.1.** *Let G and G^* be reciprocal geodesic graphs on some flat torus $\mathbb{T}_M$. The vector ω defined by $\omega_e = |e^*|/|e|$ is an equilibrium stress for G; symmetrically, the vector ω^* defined by $\omega^*_{e^*} = 1/\omega_e = |e|/|e^*|$ is an equilibrium stress for G^*.*

Proof. Let $\omega_e = |e^*|/|e|$ and $\omega^*_{e^*} = 1/\omega_e = |e|/|e^*|$ for each edge e. Let Δ denote the displacement matrix of G, and let Δ^* denote the (transposed) displacement matrix of G^*. We immediately have $\Delta^*_{e^*} = \omega_e \Delta_e^\perp$ for every edge e of G. The darts leaving each vertex p of G dualize to a facial cycle around the corresponding face p^* of G^*, and thus

$$\left(\sum_{q\colon pq\in E} \omega_{pq} \Delta_{p\to q} \right)^\perp = \sum_{q\colon pq\in E} \omega_{pq} \Delta_{p\to q}^\perp = \sum_{q\colon pq\in E} \Delta^*_{(p\to q)^*} = (0\ 0)\,.$$

We conclude that ω is an equilibrium stress for G, and thus (by swapping the roles of G and G^*) that ω^* is an equilibrium stress for G^*. ◀

A stress vector ω is a **reciprocal stress** for G if there is a reciprocal graph G^* on the same flat torus such that $\omega_e = |e^*|/|e|$ for each edge e. Thus, a geodesic torus graph is reciprocal if and only if it has a reciprocal stress.

▶ **Theorem 3.2.** *Not every positive equilibrium stress for G is a reciprocal stress. More generally, not every equilibrium graph on $\mathbb{T}$ is reciprocal/coherent on $\mathbb{T}$.*

Proof. Let G_1 be the geodesic triangulation in the flat square torus $\mathbb{T}_\square$ with a single vertex p and three edges, whose reference darts have displacement vectors $\binom{1}{0}$, $\binom{1}{1}$, and $\binom{2}{1}$. Every stress ω in G is an equilibrium stress, because the forces applied by each edge cancel out. The weighted Delaunay graph of a single point is identical for all weights, so it suffices to verify that G_1 is not an intrinsic Delaunay triangulation. We easily observe that the longest edge of G_1 is not Delaunay. See Figure 3.

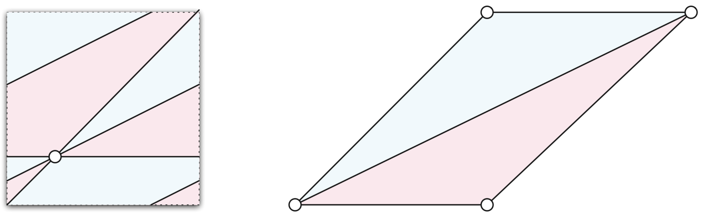

Figure 3 A one-vertex triangulation G_1 on the square flat torus, and a lift of its faces to the universal cover. Every stress in G_1 is an equilibrium stress, but G_1 is not a (weighted) intrinsic Delaunay triangulation.

More generally, for any positive integer k, let G_k denote the $k \times k$ covering of G_1. The vertices of G_k form a regular $k \times k$ square toroidal lattice, and the edges of G_k fall into three parallel families, with displacement vectors $\binom{1/k}{1/k}$, $\binom{2/k}{1/k}$, and $\binom{1/k}{0}$. Every positive stress vector where all parallel edges have equal stress coefficients is an equilibrium stress.

For the sake of argument, suppose G_k is coherent. Let $p{\rightarrow}r$ be any dart with displacement vector $\binom{2/k}{1/k}$, and let q and s be the vertices before and after r in clockwise order around p. The local Delaunay determinant test implies that the weights of these four vertices satisfy the inequality $\pi_p + \pi_r + 1 < \pi_q + \pi_s$. Every vertex of G_k appears in exactly four inequalities of this form – twice on the left and twice on the right – so summing all k^2 such inequalities and canceling equal terms yields the obvious contradiction $1 < 0$. ◀

Every equilibrium stress on any graph G on any flat torus induces an equilibrium stress on the universal cover $\widetilde{G}$, which in turn induces a reciprocal diagram $(\widetilde{G})^*$, which is periodic. Typically, however, for almost all equilibrium stresses, $(\widetilde{G})^*$ is periodic with respect to a different lattice than $\widetilde{G}$. We describe a simple necessary and sufficient condition for an equilibrium stress to be reciprocal in Section 5.

4 Coherent iff Reciprocal

Unlike in the previous and following sections, the equivalence between coherent graphs and graphs with reciprocal diagrams generalizes fully from the plane to the torus.

4.1 Notation

In this section we fix a non-singular matrix $M = (u\ v)$ where $u, v \in \mathbb{R}^2$ are column vectors and $\det M > 0$. We primarily work with the universal cover $\widetilde{G}$ of G; if we are given a reciprocal embedding G^*, we also work with its universal cover $\widetilde{G}^*$ (which is reciprocal to $\widetilde{G}$). Vertices in $\widetilde{G}$ are denoted by the letters p and q and treated as column vectors in $\mathbb{R}^2$. A generic face in $\widetilde{G}$ is denoted by the letter f; the corresponding dual vertex in $\widetilde{G}^*$ is denoted f^* and interpreted as a row vector. To avoid nested subscripts when edges are indexed, we write $\Delta_i = \Delta_{e_i}$ and $\omega_i = \omega_{e_i}$, and therefore by Lemma 3.1, $\Delta_i^* = \omega_i \Delta_i^\perp$. For any integers a and b, the translation $p + au + bv$ of any vertex p of $\widetilde{G}$ is another vertex of $\widetilde{G}$, and the translation $f + au + bv$ of any face f of $\widetilde{G}$ is another face of $\widetilde{G}$.

4.2 Results

The following lemma follows directly from the definitions of weighted Delaunay graphs and their dual weighted Voronoi diagrams; see, for example, Aurenhammer [4, 6].

▶ **Lemma 4.1.** *Let G be a weighted Delaunay graph on some flat torus $\mathbb{T}$, and let G^* be the corresponding weighted Voronoi diagram on $\mathbb{T}$. Every edge e of G is orthogonal to its dual e^*. In short, every coherent torus graph is reciprocal.*

Maxwell's theorem implies a convex polyhedral lifting $z\colon \mathbb{R}^2 \to \mathbb{R}$ of the universal cover $\widetilde{G}$ of G, where the gradient vector $\nabla z|_f$ within any face f is equal to the coordinate vector of the dual vertex f^* in $\widetilde{G}^*$. To make this lifting unique, we fix a vertex o of $\widetilde{G}$ to lie at the origin $\binom{0}{0}$, and we require $z(o) = 0$.

Define the weight of each vertex $p \in \widetilde{G}$ as $\pi_p := \frac{1}{2}|p|^2 - z(p)$. The determinant conditions (2.1) and (2.2) for an edge to be locally Delaunay are both equivalent to interpreting $\frac{1}{2}|p|^2 - \pi_p$ as a z-coordinate and requiring that the induced lifting be locally convex at said edge. Because z is a convex polyhedral lifting, $\widetilde{G}$ is the intrinsic weighted Delaunay graph of its vertex set with respect to these weights.

To compute $z(q)$ for any point $q \in \mathbb{R}^2$, we choose an arbtirary face f containing q and identify the equation of the plane through the lift of f, that is, $z|_f(q) = \eta q + c$ where η is a row vector and $c \in \mathbb{R}$. Borcea and Streinu [11] give a calculation for η and c, which for our setting can be written as follows:

▶ **Lemma 4.2** **([11, Eq. 7]).** *For $q \in \mathbb{R}^2$, let f be a face containing q. The function $z|_f$ can be explicitly computed as follows:*

- *Pick an arbitrary* ***root*** *face f_0 incident to o.*
- *Pick an arbitrary path from f_0^* to f^* in $\widetilde{G}^*$, and let $e_1^*, \ldots, e_\ell^*$ be the dual edges along this path. By definition, $f^* = f_0^* + \sum_{i=1}^{\ell} \Delta_i^*$. Set $C(f) = z(o) + \sum_{i=1}^{\ell} \omega_i \,|p_i \; q_i|$, where $e_i = p_i {\to} q_i$ and $|p_i \; q_i| = \det\,(p_i \; q_i)$.*
- *Set $\eta = f^*$ and $c = C(f)$, implying that $z|_f(q) = f^* q + C(f)$. In particular, $C(f)$ is the intersection of this plane with the z-axis.*

Reciprocality of $\widetilde{G}^*$ implies that the actual choice of root face f_0^* and the path to f^* do not matter. We use this explicit computation to establish the existence of a translation of G^* such that $\pi_o = \pi_u = \pi_v = 0$. We then show that after this translation, every lift of the same vertex of G has the same Delaunay weight.

▶ **Lemma 4.3.** *There is a unique translation of $\widetilde{G}^*$ such that $\pi_u = \pi_v = 0$. Specifically, this translation places the dual vertex of the root face f_0 at the point*

$$f_0^* = \left(-\tfrac{1}{2}\left(|u|^2 \;\; |v|^2\right) - \left(C(f_0 + u) \;\; C(f_0 + v)\right)\right) M^{-1}.$$

Proof. Lemma 4.2 implies that

$$z(u) \;=\; (f_0 + u)^* u + C(f_0 + u) \;=\; f_0^* u + |u|^2 + C(f_0 + u),$$

and by definition, $\pi_u = 0$ if and only if $z(u) = \frac{1}{2}|u|^2$. Thus, $\pi_u = 0$ if and only if $f_0^* u = -\frac{1}{2}|u|^2 - C(f_0 + u)$. A symmetric argument implies $\pi_v = 0$ if and only if $f_0^* v = -\frac{1}{2}|v|^2 - C(f_0 + v)$. ◀

We defer the proof of the following lemma to the full version of the paper [33].

▶ **Lemma 4.4.** *If $\pi_o = \pi_u = \pi_v = 0$, then $\pi_p = \pi_{p+u} = \pi_{p+v}$ for all $p \in V(\widetilde{G})$. In other words, all lifts of any vertex of G have equal weight.*

The previous two lemmas establish the existence of a set of periodic weights with respect to which $\widetilde{G}$ is the weighted Delaunay complex of its point set, and a unique translation of $\widetilde{G}^*$ that is the corresponding intrinsic weighted Voronoi diagram. Projecting from the universal cover back to the torus, we conclude:

▶ **Theorem 4.5.** *Let G and G^* be reciprocal geodesic graphs on some flat torus $\mathbb{T}_M$. G is a weighted Delaunay complex, and a unique translation of G^* is the corresponding weighted Voronoi diagram. In short, every reciprocal torus graph is coherent.*

5 Equilibrium Implies Reciprocal, Sort Of

In this section, we will fix a positive equilibrium stress ω. It will be convenient to represent ω as the $E \times E$ diagonal stress *matrix* Ω whose diagonal entries are $\Omega_{e,e} = \omega_e$.

Let G be an essentially simple, essentially 3-connected geodesic graph on the *square* flat torus $\mathbb{T}_\square$, and let Δ be its $2 \times E$ displacement matrix. Our results are phrased in terms of the covariance matrix $\Delta\Omega\Delta^T = \left(\begin{smallmatrix}\alpha & \gamma\\ \gamma & \beta\end{smallmatrix}\right)$, where

$$\alpha = \sum_e \omega_e \Delta x_e^2, \qquad \beta = \sum_e \omega_e \Delta y_e^2, \qquad \gamma = \sum_e \omega_e \Delta x_e \Delta y_e. \tag{5.1}$$

Recall that $A^\perp = (JA)^T$.

5.1 The Square Flat Torus

Before considering arbitrary flat tori, as a warmup we first establish necessary and sufficient conditions for ω to be a reciprocal stress for G on the *square* flat torus $\mathbb{T}_\square$, in terms of the parameters α, β, and γ.

▶ **Lemma 5.1.** *If ω is a reciprocal stress for G on $\mathbb{T}_\square$, then $\Delta\Omega\Delta^T = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$.*

Proof. Suppose ω is a reciprocal stress for G on $\mathbb{T}_\square$. Then there is a geodesic embedding of the dual graph G^* on $\mathbb{T}_\square$ where $e \perp e^*$ and $|e^*| = \omega_e |e|$ for every edge e of G. Let $\Delta^* = (\Delta\Omega)^\perp$ denote the $E \times 2$ matrix whose rows are the displacement row vectors of G^*.

Recall from Lemma 2.5 that the first and second rows of Λ describe cocirculations of G with cohomology classes $(0\ 1)$ and $(-1\ 0)$, respectively. Applying Lemma 2.1 to G^* implies $\theta\Delta^* = [\theta]^*$ for any cocirculation θ in G. It follows immediately that $\Lambda\Delta^* = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right) = -J$.

Because the rows of Δ^* are displacement vectors of G^*, for every vertex p of G we have

$$\sum_{q\colon pq \in E} \Delta^*_{(p\to q)^*} \;=\; \sum_{d\colon\ tail(d)=p} \Delta^*_{d^*} \;=\; \sum_{d\colon\ left(d^*)=p^*} \Delta^*_{d^*} \;=\; (0\ 0)\,. \tag{5.2}$$

It follows that the *columns* of Δ^* describe circulations in G. Lemma 2.1 now implies that $\Delta\Delta^* = -J$. We conclude that $\Delta\Omega\Delta^T = \Delta\Delta^* J = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. ◀

▶ **Lemma 5.2.** *Fix an $E \times 2$ matrix Δ^*. If $\Lambda\Delta^* = -J$, then Δ^* is the displacement matrix of a geodesic drawing on $\mathbb{T}_\square$ that is dual to G. Moreover, if that drawing has an equilibrium stress, it is actually an embedding.*

Proof. Let λ_1 and λ_2 denote the rows of Λ. Rewriting the identity $\Lambda\Delta^* = -J$ in terms of these row vectors gives us $\sum_e \Delta_e^* \lambda_{1,e} = (0\ 1) = [\lambda_1]^*$ and $\sum_e \Delta_e^* \lambda_{2,e} = (-1\ 0) = [\lambda_2]^*$. Because $[\lambda_1]^*$ and $[\lambda_2]^*$ are linearly independent, we have $\sum_e \Delta_e^* \theta_e = [\theta]^*$ for any cocirculation θ in G^*. The result follows from Lemma 2.4. ◀

▶ **Lemma 5.3.** *If $\Delta\Omega\Delta^T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then ω is a reciprocal stress for G on $\mathbb{T}_\Box$.*

Proof. Set $\Delta^* = (\Delta\Omega)^\perp$. Because ω is an equilibrium stress in G, for every vertex p of G we have

$$\sum_{q:\, pq \in E} \Delta^*_{(p\to q)^*} = \sum_{q:\, pq\in E} \omega_{pq}\Delta_{p\to q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.3}$$

It follows that the columns of Δ^* describe circulations in G, and therefore Lemma 2.1 implies $\Lambda\Delta^* = \Delta\Delta^* = \Delta(\Delta\Omega)^\perp = \Delta\Omega\Delta^T J^T = -J$.

Lemma 5.2 now implies that Δ^* is the displacement matrix of an drawing G^* dual to G. Moreover, the stress vector ω^* defined by $\omega^*_{e^*} = 1/\omega_e$ is an equilibrium stress for G^*: under this stress vector, the darts leaving any dual vertex f^* are dual to the clockwise boundary cycle of face f in G. Thus G^* is in fact an embedding. By construction, each edge of G^* is orthogonal to the corresponding edge of G. ◀

5.2 Arbitrary Flat Tori

In the full version of the paper [33], we generalize our previous analysis to graphs on the flat torus $\mathbb{T}_M$ defined by an arbitrary non-singular matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. These results are stated in terms of the covariance parameters α, β, and γ, which are still defined in terms of $\mathbb{T}_\Box$.

▶ **Lemma 5.4.** *If ω is a reciprocal stress for the affine image of G on $\mathbb{T}_M$, then $\alpha\beta - \gamma^2 = 1$; in particular, if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then*

$$\alpha = \frac{b^2 + d^2}{ad - bc}, \qquad \beta = \frac{a^2 + c^2}{ad - bc}, \qquad \gamma = \frac{-(ab + cd)}{ad - bc}.$$

▶ **Corollary 5.5.** *If ω is a reciprocal stress for the image of G on $\mathbb{T}_M$, then $M = \sigma R \left(\begin{smallmatrix} \beta & -\gamma \\ 0 & 1 \end{smallmatrix}\right)$ for some 2×2 rotation matrix R and some real number $\sigma > 0$.*

▶ **Lemma 5.6.** *If $\alpha\beta - \gamma^2 = 1$ and $M = \sigma R \left(\begin{smallmatrix} \beta & -\gamma \\ 0 & 1 \end{smallmatrix}\right)$ for any 2×2 rotation matrix R and any real number $\sigma > 0$, then ω is a reciprocal stress for the image G on $\mathbb{T}_M$.*

▶ **Theorem 5.7.** *Let G be a geodesic graph on $\mathbb{T}_\Box$ with positive equilibrium stress ω. Let α, β, and γ be defined as in Equation (5.1). If $\alpha\beta - \gamma^2 = 1$, then ω is a reciprocal stress for the image of G on the flat torus $\mathbb{T}_M$ if and only if $M = \sigma R \left(\begin{smallmatrix} \beta & -\gamma \\ 0 & 1 \end{smallmatrix}\right)$ for some (in fact any) rotation matrix R and real number $\sigma > 0$. On the other hand, if $\alpha\beta - \gamma^2 \neq 1$, then ω is not a reciprocal stress for G on any flat torus $\mathbb{T}_M$.*

Theorem 5.7 immediately implies that every equilibrium graph on any flat torus has a coherent affine image on *some* flat torus. The requirement $\alpha\beta - \gamma^2 = 1$ is a necessary scaling condition: Given any equilibrium stress ω, the scaled equilibrium stress $\omega/\sqrt{\alpha\beta - \gamma^2}$ satisfies the requirement.

6 A Toroidal Steinitz Theorem

Finally, Theorem 2.3 and Theorem 5.7 immediately imply a natural generalization of Steinitz's theorem to graphs on the flat torus.

▶ **Theorem 6.1.** *Let G be any essentially simple, essentially 3-connected embedded graph on the square flat torus $\mathbb{T}_\square$, and let ω be* **any** *positive stress on the edges of G. Then G is homotopic to a geodesic embedding in $\mathbb{T}_\square$ whose image in some flat torus $\mathbb{T}_M$ is coherent.*

As we mentioned in the introduction, Mohar's generalization [55] of the Koebe-Andreev circle packing theorem already implies that every essentially simple, essentially 3-connected torus graph G is homotopic to *one* coherent homotopic embedding on *one* flat torus. In contrast, Lemma 3.1 and Theorem 6.1 characterize *all* coherent homotopic embeddings of G on *all* flat tori; *every* positive vector $\omega \in \mathbb{R}^E$ corresponds to such an embedding.

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