Incidences Between Points and Curves with Almost Two Degrees of Freedom

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Abstract

We study incidences between points and (constant-degree algebraic) curves in three dimensions, taken from a family $\mathcal{C}$ of curves that have *almost two degrees of freedom*, meaning that (i) every pair of curves of $\mathcal{C}$ intersect in $O(1)$ points, (ii) for any pair of points p, q, there are only $O(1)$ curves of $\mathcal{C}$ that pass through both points, and (iii) a pair p, q of points admit a curve of $\mathcal{C}$ that passes through both of them if and only if $F(p,q)=0$ for some polynomial F of constant degree associated with the problem. (As an example, the family of unit circles in $\mathbb{R}^3$ that pass through some fixed point is such a family.)

We begin by studying two specific instances of this scenario. The first instance deals with the case of unit circles in $\mathbb{R}^3$ that pass through some fixed point (so called *anchored* unit circles). In the second case we consider tangencies between *directed points* and circles in the plane, where a directed point is a pair (p,u), where p is a point in the plane and u is a direction, and (p,u) is tangent to a circle γ if $p \in \gamma$ and u is the direction of the tangent to γ at p. A lifting transformation due to Ellenberg et al. maps these tangencies to incidences between points and curves ("lifted circles") in three dimensions. In both instances we have a family of curves in $\mathbb{R}^3$ with almost two degrees of freedom.

We show that the number of incidences between m points and n anchored unit circles in $\mathbb{R}^3$, as well as the number of tangencies between m directed points and n arbitrary circles in the plane, is $O(m^{3/5}n^{3/5}+m+n)$ in both cases.

We then derive a similar incidence bound, with a few additional terms, for more general families of curves in $\mathbb{R}^3$ with almost two degrees of freedom, under a few additional natural assumptions.

The proofs follow standard techniques, based on polynomial partitioning, but they face a critical novel issue involving the analysis of surfaces that are infinitely ruled by the respective family of curves, as well as of surfaces in a dual three-dimensional space that are infinitely ruled by the respective family of suitably defined dual curves. We either show that no such surfaces exist, or develop and adapt techniques for handling incidences on such surfaces.

The general bound that we obtain is $O(m^{3/5}n^{3/5}+m+n)$ plus additional terms that depend on how many curves or dual curves can lie on an infinitely-ruled surface.

2012 ACM Subject Classification Mathematics of computing → Combinatorics

Keywords and phrases Incidences, Polynomial partition, Degrees of freedom, Infinitely-ruled surfaces, Three dimensions

Digital Object Identifier 10.4230/LIPIcs.SoCG.2020.66

Related Version A full version of this paper is available at `https://arxiv.org/abs/2003.02190`.

Funding *Micha Sharir*: Partially supported by ISF Grant 260/18, by grant 1367/2016 from the German-Israeli Science Foundation (GIF), and by Blavatnik Research Fund in Computer Science at Tel Aviv University.

Acknowledgements The authors would like to thank Noam Solomon for many helpful discussions and for providing us with a copy of his ongoing work with Guth. Thanks are also due to the anonymous reviewers for their helpful comments.

36th International Symposium on Computational Geometry (SoCG 2020).
Editors: Sergio Cabello and Danny Z. Chen; Article No. 66; pp. 66:1–66:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

Our results: An overview. In this paper we study several incidence problems involving points and curves in three dimensions, where the curves are 3-parameterizable (each of them can be defined by three real parameters) and have *almost two degrees of freedom*, a notion that we discuss in detail below. We begin by deriving improved incidence bounds for two specific classes of such curves, one of which (studied in Section 2) is the class of *anchored unit circles* (unit circles that pass through some fixed point), and the other (studied in Section 3) is a class of "lifted circles" that arise in the context of tangencies between so-called *directed points* and circles in the plane. In both cases, the incidence bound, for m points and n curves, is $O(m^{3/5}n^{3/5} + m + n)$. We then study the problem for general curves that satisfy the above properties (and a few other natural assumptions), and derive the same bound as above, with additional terms that depend on various parameters associated with the problem. See Section 4 and the full version [12] for full details.

We begin with a review of the setup and of several basic features that arise in the analysis.

Incidence problems. Let P be a set of m points, and let C be a set of n algebraic curves of some bounded degree in $\mathbb{R}^3$. Let $I(P,C)$ denote the number of incidences between the points of P and the curves of C, i.e., $I(P,C) = |\{(p,c) \mid p \in P, c \in C, p \in c\}|$. The *incidence problem* for P and C is to bound $I(P,C)$. More precisely, we want to estimate $I(m,n) \coloneqq \max_{|P|=m,|C|=n} I(P,C)$, where the maximum is over all sets P of m points and C of n curves from some specific family of curves in $\mathbb{R}^3$ (such as lines, circles, etc.).

The simplest formulation of the incidence problem involves incidences between points and lines in the plane, where we have

▶ **Theorem 1** (Szemerédi and Trotter [13]). *For sets P of m points and L of n lines in the plane, we have $I(P,L) = O(m^{2/3}n^{2/3} + m + n)$, and the bound is tight in the worst case.*

The same asymptotic upper bound can be proven for unit circles as well, except that the matching lower bound is not known to hold, and is strongly suspected to be only close to linear. For general circles, of arbitrary radii, we have

▶ **Theorem 2** (Agarwal et al. [1] and Marcus and Tardos [8]). *For sets P of m points and C of n (arbitrary) circles in the plane we have $I(P,C) = O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\log^{2/11}(m^3/n) + m + n)$.*

Another variant of the incidence problem, which has recently been studied in Ellenberg et al. [3], and which is relevant to the study in this paper, is bounding the number of tangencies between lines and circles in the plane. In more detail, let a *directed point* in the plane be a pair (p,u), where $p \in \mathbb{R}^2$ and u is a direction (parameterized by its slope). A tangency occurs between a circle c and a directed point (p,u) when $p \in c$ and u is the direction of the tangent to c at p; see Figure 1. Unlike the standard case of point-circle incidences, there can be at most one circle that is tangent to a given pair of directed points (and in general there is no such circle). Ellenberg et al. [3] showed:

▶ **Theorem 3** (Ellenberg et al. [3]). *For a set P of m directed points and a set C of n (arbitrary) circles in the plane, there are $O(n^{3/2})$ tangencies between the circles in C and the directed points in P, assuming that each point of P is incident (i.e., tangent) to at least two circles.*

In fact, the bound in [3] also holds for more general sets of curves, and over fields other than $\mathbb{R}$. An immediate corollary of Theorem 3 is that the number of incidences between m directed points and n circles is $O(n^{3/2}+m)$. We will discuss this problem further in Section 3, where we obtain the improved bound $O(m^{3/5}n^{3/5}+m+n)$ mentioned above.

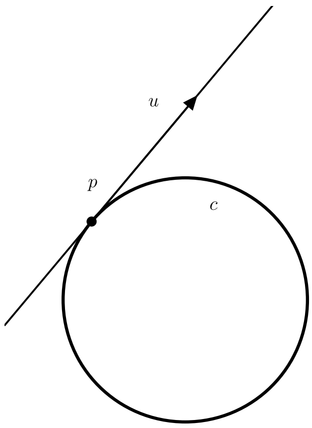

Figure 1 The tangent to circle c at point p has direction u. We then say that the directed point (p,u) is tangent (or incident) to c.

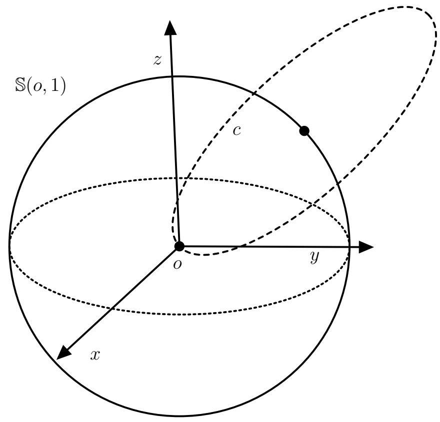

Figure 2 An anchored circle c. Its center is on the unit sphere $\mathbb{S}(o,1)$, and it passes through the origin o.

As has been observed, time and again, the result of Theorem 1, including both the upper and the lower bound, is applicable to point-line incidences $\mathbb{R}^3$ as well (and, in fact, in any higher-dimensional space $\mathbb{R}^d$), unless we impose some additional constraint on the number of coplanar input lines. The following celebrated theorem of Guth and Katz [5] gives such an improved bound[1].

▶ **Theorem 4** (Guth and Katz [5]). *Let P be a set of m points and L be a set of n lines in $\mathbb{R}^3$. Assume further that no plane in $\mathbb{R}^3$ contains more than q lines of L, for some parameter $q \le n$. Then $I(P,L)=O\left(m^{1/2}n^{3/4}+m^{2/3}n^{1/3}q^{1/3}+m+n\right)$. Moreover, the bound is tight in the worst case.*

A similar argument can be made for point-circle incidences in $\mathbb{R}^3$ (or again in any dimension ≥ 3) – here we need to constrain the number of input circles that can lie in any common plane or sphere. The best known upper bound, due to Sharir and Solomon [11], is (see also Sharir et al. [10] for an earlier, weaker bound).

▶ **Theorem 5** (Sharir and Solomon [11]). *Let P be a set of m points and let C be a set of n circles in $\mathbb{R}^3$, and let $q < n$ be an integer. If no sphere or plane contains more than q circles of C, then*

$$I(P,C)=O\left(m^{3/7}n^{6/7}+m^{2/3}n^{1/3}q^{1/3}+m^{6/11}n^{5/11}q^{4/11}\log^{2/11}(m^3/q)+m+n\right).$$

Polynomial partitioning. The polynomial partitioning technique is the most recently developed method for deriving incidence bounds (and many other results too), and it is due to Guth and Katz [5], with an extended version given later by Guth [4]. We use the following version (specialized to our needs), where $Z(f)$ denotes the zero set $\{z\in\mathbb{R}^3 \mid f(z)=0\}$ of a real (trivariate) polynomial f.

[1] The theorem is not stated explicitly in [5], but it is an immediate consequence of the analysis in [5].

▶ **Theorem 6** (Polynomial partitioning [4, 5]). *Let P be a set of m points and C be a set of n algebraic curves of some constant degree in $\mathbb{R}^3$. Then, for any $1 < D$ such that $D^3 < m$ and $D^2 < n$, there is a polynomial f of degree at most D such that each of the $O(D^3)$ (open) connected components of $\mathbb{R}^3 \setminus Z(f)$ contains at most $O(m/D^3)$ points of P, and is crossed by at most $O(n/D^2)$ curves of C.*

Note that the theorem has no guarantee regarding the number of points of P on $Z(f)$, or the number of curves of C that are contained in $Z(f)$.

One of the main techniques for proving incidence bounds via polynomial partitioning proceeds as follows. We first establish a simple (and weak) incidence bound (usually referred to as a *bootstrapping* bound) by some other method. Then we apply Theorem 6, and use the bootstrapping bound in every connected component (cell) of $\mathbb{R}^3 \setminus Z(f)$. Incidences between curves in C and points on $Z(f)$ must be treated separately, using a different set of tools and techniques, typically taken from algebraic geometry.

Degrees of freedom. We say that a family $\mathcal{C}$ of constant-degree irreducible algebraic curves in $\mathbb{R}^3$ has s *degrees of freedom* (of multiplicity μ) if:

1. each pair of curves of $\mathcal{C}$ intersect in at most μ points; and
2. for each s-tuple $p_1, \ldots, p_s$ of distinct points in $\mathbb{R}^3$ there are at most μ curves of $\mathcal{C}$ that pass through all these points.

The definition extends, verbatim, to curves in any other dimension or in the plane.

The notion of degrees of freedom can be defined for arbitrary families of curves (not necessarily algebraic). However, for various technical reasons, mainly to be able to apply Theorem 6, we confine ourselves to the case of constant-degree algebraic curves.

Many natural families of curves have a small number of degrees of freedom:

- Lines have two degrees of freedom with multiplicity one (in any space $\mathbb{R}^d$). Indeed, each pair of lines intersect in at most one point, and through any pair of points only a single line can be drawn.
- Similarly, unit circles in the plane have two degrees of freedom as well, with multiplicity two. (Note that unit circles in $\mathbb{R}^3$, or in any higher-dimensional space, do not have two degrees of freedom, but they have three degrees of freedom, as follows from the next example.)
- Circles of arbitrary radii, in any space $\mathbb{R}^d$, have three degrees of freedom.

The following theorem is a generalization of Theorem 1, and is due to Pach and Sharir [9]. The original bound applies to more general families of curves, but we stick to the algebraic setup.

▶ **Theorem 7** (Pach and Sharir [9]). *Let P be a set of m points in the plane, and let C be a set of n irreducible algebraic curves in the plane of degree at most k and with s degrees of freedom (with multiplicity μ); here k, s and μ are assumed to be constants. Then:*

$$I(P, C) = O\left(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n\right),$$

where the constant of proportionality depends on k, s and μ.

Note that this is the Szemerédi-Trotter bound for lines (for which $s = 2$), and also for unit circles in the plane.

▶ Remark. If we apply Theorem 7 to the family of circles of arbitrary radii, in any dimension (for which $s = 3$), we get the bound $I(P, C) = O(m^{3/5} n^{4/5} + m + n)$, which is weaker than the bound in Theorem 2.

Infinitely ruled surfaces. Extending the constraint that the parameter q imposes in Theorem 4, we use the following concept, studied by Sharir and Solomon in [11], adapting a similar reasoning from Guth and Zahl [6]. An algebraic surface V in $\mathbb{R}^3$ is *infinitely ruled* by a family $\mathcal{C}$ of curves, if each point $q \in V$ is incident to infinitely many curves of $\mathcal{C}$ that are fully contained in V. For example, the only surfaces that are infinitely ruled by lines are planes, and the only surfaces that are infinitely ruled by circles are planes and spheres; see Lubbes [7]. Sharir and Solomon have considered this notion in [11] to show:

▶ **Theorem 8** (Sharir and Solomon [11]). *Let P be a set of m points and C a set of n irreducible algebraic curves in $\mathbb{R}^3$, taken from a family $\mathcal{C}$, so that the curves of $\mathcal{C}$ are algebraic of constant degree, and with s degrees of freedom (of some multiplicity μ). If no surface that is infinitely ruled by curves of $\mathcal{C}$ contains more than q curves of C, for a parameter $q < n$, then $I(P,C) = O\left(m^{\frac{s}{3s-2}} n^{\frac{3s-3}{3s-2}} + m^{\frac{s}{2s-1}} n^{\frac{s-1}{2s-1}} q^{\frac{s-1}{2s-1}} + m + n\right)$, where the constant of proportionality depends on s, μ, and the degree of the curves in $\mathcal{C}$.*

Note that Theorem 4 is a special case of this result, with $s = 2$, where the infinitely ruled surfaces are planes.

An additional tool that we rely on is also due to Sharir and Solomon [11]. It is the following theorem, which is part of Theorem 1.13 in [11], and is a generalization of a result of Guth and Zahl [6] (that was stated there only for doubly ruled surfaces).

▶ **Theorem 9** (Sharir and Solomon [11]). *Let $\mathcal{C}$ be a family of algebraic curves in $\mathbb{R}^3$ of constant degree E. Let f be a complex irreducible polynomial of degree $D \gg E$. If $Z(f)$ is not infinitely ruled by curves from $\mathcal{C}$ then there exist absolute constants c, t, such that, except for at most cD^2 exceptional curves, every curve in $\mathcal{C}$ that is fully contained in $Z(f)$ is incident to at most cD t-rich points, namely points that are incident to at least t curves in $\mathcal{C}$ that are also fully contained in $Z(f)$.*

Almost two degrees of freedom. We introduce the following notion. A family $\mathcal{C}$ of algebraic irreducible curves in $\mathbb{R}^3$ has *almost s degrees of freedom* (of multiplicity μ) if:

1. each pair of curves of $\mathcal{C}$ intersect in at most μ points;
2. for each s-tuple $p_1, \ldots, p_s$ of distinct points in $\mathbb{R}^3$ there are at most μ curves of $\mathcal{C}$ that pass through all these points; and
3. there exists a curve of $\mathcal{C}$ that passes through $p_1, \ldots, p_s$, if and only if $F(p_1, \ldots, p_s) = 0$, where F is some $3s$-variate real polynomial of constant degree associated with $\mathcal{C}$.

With this definition we want to capture families $\mathcal{C}$ of curves that have some s degrees of freedom, but are such that for most s-tuples of points there is no curve of $\mathcal{C}$ that passes through all of them. As we demonstrate in this work, this additional restriction helps us improve the upper bound for incidences between points and curves from such a family.

As with the case of standard degrees of freedom, there are natural examples that fall under this definition. One such example is the family of unit circles in $\mathbb{R}^3$ (or in any $\mathbb{R}^d$, for $d \geq 3$), which, as is easily checked, has almost three degrees of freedom, with multiplicity two.

Our results. Although the above definition applies for general values of s and d, in this paper we focus on the special case $s = 2$ and $d = 3$.

In Section 2, we study the incidence problem between points and unit circles in three dimensions that pass through a fixed point (so-called *anchored* unit circles). With this additional constraint, this family has almost two degrees of freedom. We use this property to prove the bootstrapping bound $I(m,n) = O(m^{3/2} + n)$, which improves the naive bootstrapping bound $I(m,n) = O(m^2 + n)$ for general families of curves with two degrees of freedom. We then prove that no surface is infinitely ruled by this family of curves. Combining this with some additional arguments, most notably an argument that establishes the absence of infinitely ruled surfaces in a suitably defined dual context (needed to establish our improved bootstrapping bound), gives us the following incidence bound:

$$I(m,n) = O(m^{3/5}n^{3/5} + m + n).$$

We remark that Sharir et al. [10] have obtained the bound

$$I(m,n) = O^*(m^{5/11}n^{9/11} + m^{2/3}n^{1/2}q^{1/6} + m + n) \tag{1}$$

for m points and n *non-anchored* unit circles in $\mathbb{R}^3$ (where $O^*(\cdot)$ hides small sub-polynomial factors). While this bound applies to general families of unit circles, it does not imply our bound for anchored circles (and it depends on the threshold parameter q, of which our bound is independent).

In Section 3, we bound the number of tangencies between circles and directed points in the plane. We transform this problem to an incidence problem between points and curves with almost two degrees of freedom in $\mathbb{R}^3$, resulting from lifting the given circles to three dimensions, using a method of Ellenberg et al.In this case as well, we prove the bootstrapping bound[2] $I(m,n) = O(m^{3/2} + n)$, show that no surface is infinitely ruled by this family of curves, and combine these statements (with some other considerations) to get the same asymptotic bound $I(m,n) = O(m^{3/5}n^{3/5} + m + n)$.

In Section 4, we extend the proofs from Sections 2 and 3, for more general families of curves with almost two degrees of freedom in three dimensions. A large part of the analysis can be generalized directly, but in general, there may exist surfaces that are infinitely ruled by these families of curves. Additionally, as already noted, our analysis in Sections 2 and 3 also involves a stage where it studies the problem in a dual setting, and the existence of infinitely ruled surfaces is an issue that has to be dealt with in this setting too. As in Theorem 4, the bound depends on the maximum number of curves that can lie on a surface that is infinitely ruled by the given family of curves, and on a similar threshold parameter in the dual space. We also need to impose a few additional natural conditions on the family of curves to obtain our result.

The bound that we obtain is $O(m^{3/5}n^{3/5} + m + n)$ plus additional terms that depend on the threshold parameters for infinitely ruled surfaces, both in the primal and in the dual setups. These terms are subsumed in the bound just stated when the relevant parameters are sufficiently small. See Section 4 and the full version [12] for the precise bound.

We exemplify (in [12]) the general bound for families of lines in $\mathbb{R}^3$ that have almost two degrees of freedom, a problem that has also been looked at by Guth and Solomon (work in progress).

We conclude the paper in Section 5 by listing some open problems and suggesting directions for further research.

[2] Note the difference between this bound and the bound in Ellenberg et al. noted earlier. It is this stronger version that allows us to derive our bound, mentioned below.

2 Anchored unit circles in space

The setup. As stated in Section 1, unit circles in space have almost three degrees of freedom. We reduce the setup to one with almost two degrees of freedom, by considering only circles that pass through a fixed point, say the origin. We call such circles *anchored (unit) circles*. An anchored circle c has radius 1 and center on the unit sphere $\mathbb{S}(o,1)$ centered at o (see Figure 2). The main result of this section is

▶ **Theorem 10.** *The number of incidences between m points and n anchored circles in $\mathbb{R}^3$ is*

$$I(P,C) = O(m^{3/5}n^{3/5} + m + n).$$

2.1 Proof of Theorem 10

We obtain the desired bound by following the general approach in [11]. Using special properties of the underlying setup, we obtain the following improved bootstrapping bound (over the simple "naive" bound $O(m^2+n)$ used in [11]).

▶ **Lemma 11.** *The number of incidences between a set P of m points and a set C of n anchored unit circles in $\mathbb{R}^3$ is $I(P,C) = O(m^{3/2}+n)$.*

The proof of the lemma is given in Section 2.2 below. Assuming for now that the lemma holds, we apply the technique of [11], with suitable modifications, to derive the incidence bound in Theorem 10. We show, by induction on n, that $I(P,C) \le A\left(m^{3/5}n^{3/5}+m+n\right)$, for a suitable constant A. It is trivial to verify that this bound holds for n smaller than some constant threshold n_0, by choosing A sufficiently large, so we focus on the induction step.

We first construct, using Theorem 6, a partitioning polynomial f in $\mathbb{R}^3$, of some specified (maximum) degree D, so that each cell (connected component) of $\mathbb{R}^3 \setminus Z(f)$ contains at most $O(m/D^3)$ points of P, and is crossed by at most $O(n/D^2)$ circles of C.

For each (open) cell τ of the partition, let P_τ denote the set of points of P inside τ, and let C_τ denote the set of circles of C that cross τ; we have $m_\tau := |P_\tau| = O(m/D^3)$, and $n_\tau := |C_\tau| = O(n/D^2)$. We apply the bootstrapping bound of Lemma 11 within each cell τ, to obtain

$$I(P_\tau, C_\tau) = O\left(m_\tau^{3/2} + n_\tau\right) = O\left((m/D^3)^{3/2} + (n/D^2)\right) = O\left(m^{3/2}/D^{9/2} + n/D^2\right).$$

Multiplying by the number of cells, we get that the number of incidences within the cells is

$$\sum_\tau I(P_\tau, C_\tau) = O\left(D^3 \cdot \left(m^{3/2}/D^{9/2} + n/D^2\right)\right) = O\left(m^{3/2}/D^{3/2} + nD\right).$$

We choose $D = am^{3/5}/n^{2/5}$, for a sufficiently small constant a. For this to make sense, we require that $1 \le D \le a' \min\{m^{1/3}, n^{1/2}\}$, for another sufficiently small constant $a' > 0$, which holds when $b_1 n^{2/3} \le m \le b_2 n^{3/2}$, for suitable constants b_1, b_2 that depend on a and a'. For m in this range, the incidence bound is $O(m^{3/5}n^{3/5})$. As we detail in the full version [12], (i) when $m < b_1 n^{2/3}$, we apply Lemma 11 to the entire sets P and C, and get the bound $O(n)$, and (ii) when $m > b_2 n^{3/2}$, we choose $D = a'n^{1/2}$, for the a' used above, and get the bound $O(m)$. Combining all three cases, we obtain the overall within-cells bound

$$O\left(m^{3/5}n^{3/5} + m + n\right). \tag{2}$$

Consider next incidences involving points that lie on $Z(f)$. A circle γ that is not fully contained in $Z(f)$ crosses it in at most $O(D)$ points, which follows from Bézout's theorem (see, e.g., [2]). This yields a total of $O(nD) = O(m^{3/5}n^{3/5} + m)$ incidences, within the asymptotic bound in (2). It therefore remains to bound the number of incidences between the points of P on $Z(f)$ and the anchored circles that are fully contained in $Z(f)$.

We follow the proof of Theorem 1.4 in [11], which considers each irreducible component of $Z(f)$ separately, and distinguishes between components that are *infinitely ruled* by anchored circles, and components that are not. Let $\mathcal{C}$ denote the infinite family of all possible anchored (unit) circles. Fortunately for us, we have:

▶ **Lemma 12.** *No algebraic surface is infinitely ruled by anchored unit circles.*

Proof. Briefly, the only surfaces to consider are planes and spheres, and neither can be infinitely ruled by anchored unit circles. See the full version [12] for details. ◀

Write $m^* = |P \cap Z(f)|$ and $m_0 = |P \setminus Z(f)|$, so $m = m_0 + m^*$. The analysis in [11], which we follow here, handles each irreducible component of $Z(f)$ separately. Enumerate these components as $Z(f_1), \ldots, Z(f_k)$, for suitable irreducible polynomials $f_1, \ldots, f_k$, of respective degrees $D_1, \ldots, D_k$, where $\sum_{i=1}^{k} D_i \le D$. By Lemma 12, none of these components is infinitely ruled by anchored circles.

Let P_i (resp., C_i) denote the set of all points of P (resp., anchored circles of C) that are contained (resp., fully contained) in $Z(f_i)$, assigning each point and circle to the first such component (in the above order), when it is contained in more than one component. The "cross-incidences", between points and circles assigned to different components, occur at crossing points between circles and components that do not contain them, and their number is therefore $O(nD)$, which satisfies our asymptotic bound. It therefore suffices to bound the number of incidences between points and circles assigned to the same component.

By Theorem 9, there exist absolute constants c, t, such that there are at most cD_i^2 'exceptional' anchored circles in C_i, namely, anchored circles that contain more than cD_i *t-rich* points of $P \cap Z(f_i)$, namely points that are incident to at least t circles from C_i. Denote the number of t-rich points (resp., *t-poor* points, namely points that are not t-rich) as m_{rich} (resp., m_{poor}), so $m_{rich} + m_{poor} = m^*$. By choosing a and a' (in the definition of D) sufficiently small, we can ensure, as is easily checked, that $\sum_i D_i^2 \le (\sum_i D_i)^2 \le D^2 \le n/(2c)$.

The number of incidences on the non-exceptional circles, summed over all components $Z(f_i)$, is $O(m_{poor} + nD)$. Indeed, each non-exceptional circle contains at most cD_i t-rich points, for a total of $O(nD_i)$ incidences, and the sum of these bounds is $O(nD)$. Any t-poor point lies on at most t circles of C_i, for a total of $tm_{poor} = O(m_{poor})$ incidences (over all sets C_i).

For the exceptional circles, we apply the induction hypothesis, as their overall number is at most $c\sum_i D_i^2 \le cD^2 \le n/2$. Note that in this inductive step we only need to consider the t-rich points, as the t-poor points have already been taken care of. By the induction hypothesis, the corresponding incidence bound between the points and circles that were assigned to (the same) f_i is at most

$$A\left(m_i^{3/5}(cD_i^2)^{3/5} + m_i + cD_i^2\right),$$

where m_i is the number of t-rich points assigned to f_i. We now sum over i. Clearly, $\sum_i m_i = m_{rich}$. We also have $\sum_i cD_i^2 \le n/2$. As for the first term, we use Hölder's inequality:

$$\sum_i m_i^{3/5}(cD_i^2)^{3/5} = c^{3/5}\sum_i m_i^{3/5}D_i^{6/5} \le c^{3/5}\left(\sum_i m_i\right)^{3/5}\left(\sum_i D_i^3\right)^{2/5} \le c^{3/5}m^{3/5}\left(\sum_i D_i^3\right)^{2/5}.$$

Finally, using the fact that $\sum_i D_i^3 \le D^3$, we get the overall bound:

$$A\left(c^{3/5}m^{3/5}D^{6/5} + m_{rich} + n/2\right) \le A\left(\frac{m^{3/5}n^{3/5}}{2^{3/5}} + m_{rich} + n/2\right),$$

since $c^{3/5}D^{6/5} \le (n/2)^{3/5}$, by construction.

We now add to this quantity the bound for incidences within the cells, as well as the various other bounds involving points on $Z(f)$. Together, we can upper bound these bounds by $B\left(m^{3/5}n^{3/5} + n + m_0 + m_{poor}\right)$, for a suitable absolute constant B. By choosing A sufficiently large, the sum of all the bounds encountered in the analysis is at most $A\left(m^{3/5}n^{3/5} + m + n\right)$. This establishes the induction step, and thereby completes the proof of Theorem 10, modulo the still missing proof of Lemma 11, presented next.

2.2 Proof of Lemma 11

The lemma improves upon the naive (and standard) bootstrapping bound, used in [11], which is $O(m^2 + n)$, for m points and n anchored circles. We dualize the setup, exploiting the underlying geometry, mapping each circle $\gamma \in \mathcal{C}$ to a suitable algebraic representation of the point $q_\gamma = (\alpha_\gamma, \beta_\gamma, \phi_\gamma)$ in 3-space, where $(\alpha_\gamma, \beta_\gamma)$ represents the center of γ as a point on $\mathbb{S}(o,1)$, and ϕ_γ represents the angle by which the circle is rotated around the line connecting o to its center. We denote by $\mathcal{C}^*$ the infinite family of all these dual points q_γ (over all possible anchored unit circles γ).

We also map each point $p \in P$ to the locus h_p of all dual points q_γ that represent anchored circles γ that are incident to p, and argue (in [12]) that h_p is a one-dimensional curve.

Let $\mathcal{H}$ denote the family of all dual curves h_p for points $p \in \mathbb{R}^3$ (actually, only points in the ball of radius 2 around o, with o excluded, are relevant). We show (see [12]) that $\mathcal{H}$ has (almost) two degrees of freedom. Let $C^* \subset \mathcal{C}^*$ be the set of points q_γ dual to the anchored circles $\gamma \in C$, and let $H \subset \mathcal{H}$ be the set of curves h_p dual to the points $p \in P$. We have thus reduced our problem to that of bounding the number of incidences between C^* and H, to which we can apply Theorem 8, using the fact that the curves of $\mathcal{H}$ have two degrees of freedom, to get the bound

$$I(P,C) = I(C^*,H) = O\left(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}q^{1/3} + n + m\right), \tag{3}$$

where q is the maximum number of curves from H that lie on a common surface that is infinitely ruled by $\mathcal{H}$. Fortunately again for us, we have:

▶ **Lemma 13.** *No algebraic surface is infinitely ruled by $\mathcal{H}$.*

Proof. See the full version [12]. ◀

It thus follows that $I(P,C) = I(C^*,H) = O\left(n^{1/2}m^{3/4} + n + m\right)$, which is upper bounded by $O(n + m^{3/2})$. This completes the proof of Lemma 11, and, consequently, also of Theorem 10.

3 Point-circle tangencies in the plane

The setup. Let C be a set of n circles in the plane. A *directed point* in the plane is a pair (p,u), where $p \in \mathbb{R}^2$ and u is a direction, which we parameterize by its slope. A circle c is said to be *incident* (or *tangent*) to a directed point (p,u) if c passes through p, and c is tangent to the line emanating from p in direction u. See Figure 1.

As stated in Theorem 3, Ellenberg et al. [3] (using a somewhat different notation) have shown that the number of directed points that are incident to at least two circles of C is $O(n^{3/2})$. Using the main technical idea in [3], we represent directions by their slopes[3], and regard each directed point (p,u) as a point in $\mathbb{R}^3$, where the z-coordinate is the slope; from now on, we let the parameter u denote the slope. We map each circle c in $\mathbb{R}^2$ to the curve $c^* = \{(p,u) \mid c \text{ is incident to } (p,u)\}$, to which we refer as a *lifted circle*, or the *lifted image* of c. As is easily checked, c^* is an algebraic curve of degree 4.

Denote by $\mathcal{C}$ the infinite family of all possible lifted circles. It is easy to show that the curves of $\mathcal{C}$ have almost two degrees of freedom; see the full version [12] for details.

The setup then becomes similar to what we have seen in Section 2, and we have

▶ **Theorem 14.** *The number of incidences between m directed points and n circles in the plane is $O(m^{3/5}n^{3/5} + m + n)$.*

3.1 Proof of Theorem 14

We only give a brief sketch of the proof. Details can be found in [12]. The high-level approach in the proof of the theorem is very similar to the one presented in the previous section. We establish the improved bootstrapping bound.

▶ **Lemma 15.** *The number of incidences between m directed points and n circles in the plane is $O(m^{3/2} + n)$.*

Assuming that the lemma holds, we prove, by induction on n that, for $|P| = m$ and $|C| = n$, $I(P,C) \le A(m^{3/5}n^{3/5} + m + n)$, for a suitable absolute constant A. Again, the case of small n is trivial, with a suitable choice of A, and we only focus on the induction step.

As before, we first construct a partitioning polynomial f in $\mathbb{R}^3$, of the same maximum degree as in the previous section, and bound the number of incidences within the partition cells by $O(m^{3/5}n^{3/5} + m + n)$. Consider then incidences involving points that lie on $Z(f)$. A lifted circle c^* that is not fully contained in $Z(f)$ crosses it in at most $O(D)$ points, for an overall $O(nD)$ bound, which, as before, is asymptotically subsumed by the bound within the cells. It therefore remains to bound the number of incidences between the points of P on $Z(f)$ and the lifted circles that are fully contained in $Z(f)$.

Again, we handle each irreducible component of $Z(f)$ separately, and make use of the fortunate property in the following lemma. Its proof is given in the full version [12], and it strongly exploits the geometry of this setup.

▶ **Lemma 16.** *No algebraic surface is infinitely ruled by lifted circles.*

We can now continue exactly as in Section 2 and establish the asserted bound. See [12] for full details.

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[3] This excludes y-vertical directions from the analysis. We assume, without loss of generality, that no input directed point has vertical direction (i.e., slope $\pm\infty$).

3.2 Proof of Lemma 15

We dualize the setup, exploiting the underlying geometry in the plane, by mapping each circle $c \in C$, with center (ξ, η) and radius r, to the point $q_c = (\xi, \eta, \zeta)$, where $\zeta = r^2 - \xi^2 - \eta^2$, and by mapping each directed point (p, u) to the locus $h_{p,u}$ of all dual points that represent circles that are incident to (p, u). It is easily seen that $h_{p,u}$ is the intersection of two planes, and is therefore a line in $\mathbb{R}^3$. We have thus reduced the problem to that of incidences between n points (those dual to the circles of C) and m lines (the lines $h_{p,u}$, for $(p, u) \in P$) in three dimensions. We can apply the result of Guth and Katz [5] (see Theorem 4) for estimating the number of these incidences, and obtain

$$I(P, C) = I(C^*, H) = O\left(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}q^{1/3} + n + m\right), \tag{4}$$

where H is the set of the dual lines $h_{p,u}$, and q is the maximum number of lines of H that can lie in a common plane. This is a notable difference with the analysis in Section 2: There we showed that no surface is infinitely ruled by the dual curves, whereas here every plane is such a surface. Handling incidences on planes requires extra work, presented in the next subsection.

3.3 Coplanar lines

The gist of the analysis in this subsection is to control the value of q. For this, we distinguish between planes that contain at most q lines of H, for a suitable threshold value q that we will set later, and those that contain more than q lines. We handle the latter type of planes using a different technique that strongly exploits the geometry of the problem, and are then left with a subproblem in which (4) can be used.

Recall that if a circle c has center q and radius r then the *power* of a point w with respect to c is $|wq|^2 - r^2$. As is well known, and easy to see, the duality transform that we have used has the property that for each non-vertical plane π in $\mathbb{R}^3$ there exist a point w in $\mathbb{R}^2$ and a power ρ, such that the point dual to a circle c lies on π if and only if w has power ρ with respect to c.

Let π be any fixed non-vertical plane in $\mathbb{R}^3$, and let w and ρ be the corresponding point and power (in $\mathbb{R}^2$). We show in the full version [12] that a line $h_{p,u}$ is fully contained in π if and only if (a) p lies on the fixed circle $\gamma(w, \sqrt{\rho})$, with center w and radius $\sqrt{\rho}$, and (b) u is the direction of the line connecting p and w. See Figure 3.

Let $P^+ = \{(p, u)^+ = (p, -1/u) \mid (p, u) \in P\}$, and let $\mathcal{W}$ denote the set of all possible "power circles" $\gamma(w, \sqrt{\rho})$ as just defined. By construction, $h_{p,u}$ lies in a plane $\pi(w, \sqrt{\rho})$ if and only if $(p, u)^+$ is incident to the corresponding circle $\gamma(w, \sqrt{\rho})$.

We fix a threshold value q, to be determined shortly, and partition $\mathcal{W}$ into two subsets $\mathcal{W}^+$, $\mathcal{W}^-$, where $\mathcal{W}^+$ (resp., $\mathcal{W}^-$) consists of those circles in $\mathcal{W}$ that are incident to more than (resp., at most) q directed points of P^+. We refer to circles in $\mathcal{W}^+$ (resp., in $\mathcal{W}^-$) as being *q-rich* (resp., *q-poor*). The same notation carries over to the corresponding power planes in 3-space.

The analysis, whose full details are given in the full version [12], then shows that the number of incidences involving circles that are incident to at least one *q-rich point*, namely a point that is incident to at least one q-rich circle, is $O(m|\mathcal{W}^+| + n)$, and that $|\mathcal{W}^+| = O\left(\frac{m^2}{q^3} + \frac{m}{q}\right)$. Thus the number of these incidences is

$$O(m|\mathcal{W}^+| + n) = O\left(\frac{m^3}{q^3} + \frac{m^2}{q} + n\right).$$

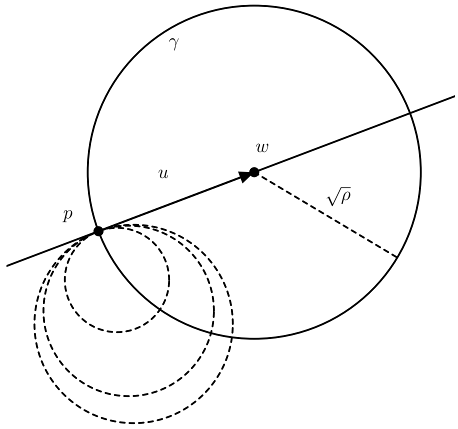

Figure 3 A circle $\gamma = \gamma(w, \sqrt{\rho})$ and a point p on γ with direction u to the center w of γ. Any circle incident to (p, u) (such as the dashed circles) has power ρ with respect to w.

The same (in fact, a smaller) bound applies for vertical planes π.

The remaining incidences only involve the surviving circles and the q-poor points. By construction, we now have that, in the dual 3-space, no plane contains more than q lines $h_{p,u}$, so by Guth and Katz's bound [5], the number of surviving incidences is $O\left(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}q^{1/3} + n + m\right)$, for a total number of incidences $O\left(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}q^{1/3} + n + \frac{m^3}{q^3} + \frac{m^2}{q}\right)$. By choosing the appropriate q (details in the full version), this bound becomes $O\left(n^{1/2}m^{3/4} + n\right) = O\left(m^{3/2} + n\right)$, thus completing the proof of the lemma.

4 Generalizations

Due to lack of space, we only state the main result, and leave all details to the full version [12].

▶ **Theorem 17.** *Let C be a set of n curves in $\mathbb{R}^3$ that are taken from a 3-parameterizable family $\mathcal{C}$ with almost two degrees of freedom, and let P be a set of m points in $\mathbb{R}^3$ whose duals (as defined in the previous sections) are all curves.*[4] *Assume that no surface that is infinitely ruled by the curves of $\mathcal{C}$ contains more than π curves from C. Let $\mathcal{P}^*$ be the family of curves in dual 3-space that are dual to the (suitable subset of) points of $\mathbb{R}^3$, with respect to the curves of $\mathcal{C}$, and assume that no surface that is infinitely ruled by curves of $\mathcal{P}^*$ contains more than δ curves dual to the points of P, and that not all pairs of curves of $\mathcal{C}$ intersect. Then*

$$I(P, C) = O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + n^{8/9})\delta^{1/3} + m^{2/3}n^{1/3}\pi^{1/3} + m + n\right).$$

If $\pi = O(n^{1/2})$ and $\delta = O(n^{1/3})$ then $I(P, C) = O(m^{3/5}n^{3/5} + m + n)$.

[4] In general, there might exist a lower-dimensional set of points whose duals are two-dimensional.

5 Conclusion

The elegant bound $O(m^{3/5}n^{3/5}+m+n)$ on the number of incidences, derived in Theorems 10 and 14 (and in the favorable subcase of Theorem 17), improves upon the best bounds for a family of curves with standard two degrees of freedom. Comparing this bound with the more cumbersome-looking general bound in Theorem 17, indicates that a major step in extending the technique of this paper to other instances of the problem, is analyzing the structure, or establishing the nonexistence, of surfaces that are infinitely ruled by the given curves or by the dual curves. This seems to be a rich area of further research, which calls for sophisticated tools from algebraic geometry.

Specific subproblems that are still not resolved, in their full generality, are: (a) Understand and characterize the existence of dual curves. (b) As just mentioned, understand and characterize the existence of surfaces that are infinitely ruled by the family of curves, as well as of dual surfaces that are infinitely ruled by the family of dual curves. (c) Obtain improved bounds, if at all possible, for the number of incidences between points and curves that lie on such a surface, both in the primal and in the dual setups.

In particular, it would be interesting to investigate whether ideas similar to those used in distinguishing between rich and poor points, given in Section 3.3, can be developed to reduce the threshold on the number of primal or dual curves that lie on a surface that is infinitely ruled by such curves.

A natural open problem, which we have yet to make progress on, is to generalize the bounds and techniques from this paper to families of curves in three dimension with almost s degrees of freedom, for larger constants $s \geq 3$. For instance, the problem of bounding the number of incidences between (non-anchored) unit circles and points in three dimensions falls under this general setup for $s = 3$, since unit circles (in any dimension) have almost three degrees of freedom. A specific goal here is to improve the bound (1) of [10] for non-anchored unit circles.

A simple, albeit unsatisfactory, way of handling the case $s \geq 3$ is to use anchoring. For example, for the case of unit circles in $\mathbb{R}^3$, we fix a point p_0 of P, consider the subfamily $\mathcal{C}_{p_0}$ of the unit circles that are incident to p_0, apply the bound obtained in Theorem 10 to P and the set C_{p_0} of the circles of C that are incident to p_0, and then combine these bounds, over all $p_0 \in P$, to obtain the desired bound. We believe that this coarse (and weak) approach can be considerably improved by replacing it by a direct approach that treats all the circles of C together, and leave this as yet another interesting open problem for further research.

A final open question is whether the bound $O(m^{3/5}n^{3/5}+m+n)$ is tight, for any instance of the setup considered in this paper. We strongly suspect that the bound is not tight.

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