Maximum Shallow Clique Minors in Preferential Attachment Graphs Have Polylogarithmic Size

Jan Dreier
Department of Computer Science, RWTH Aachen University, Germany
https://tcs.rwth-aachen.de/~dreier
dreier@cs.rwth-aachen.de

Philipp Kuinke
Department of Computer Science, RWTH Aachen University, Germany
https://tcs.rwth-aachen.de/~kuinke
kuinke@cs.rwth-aachen.de

Peter Rossmanith
Department of Computer Science, RWTH Aachen University, Germany
https://tcs.rwth-aachen.de
rossmani@cs.rwth-aachen.de

Abstract

Preferential attachment graphs are random graphs designed to mimic properties of real word networks. They are constructed by a random process that iteratively adds vertices and attaches them preferentially to vertices that already have high degree. We prove various structural asymptotic properties of this graph model. In particular, we show that the size of the largest r-shallow clique minor in G_m^n is at most $\log(n)^{O(r^2)}m^{O(r)}$. Furthermore, there exists a one-subdivided clique of size $\log(n)^{1/4}$. Therefore, preferential attachment graphs are asymptotically almost surely somewhere dense and algorithmic techniques developed for structurally sparse graph classes are not directly applicable. However, they are just barely somewhere dense. The removal of just slightly more than a polylogarithmic number of vertices asymptotically almost surely yields a graph with locally bounded treewidth.

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1 Introduction

Large scale complex networks occur for example in the context of social-, biological, and technical networks. Even though these are vastly different environments, all these networks follow the same common laws. Many models [18, 23, 19, 8, 20] have been proposed that aim to mimic complex networks. The most prominent model might be the *preferential attachment graph model* (also called Barabási–Albert model) [3]. These graphs are created by a random process that iteratively adds new vertices and randomly connects them to already existing ones. Every time a new vertex is added, it is connected to the remaining graph via m random edges. Thereby, the probability that an edge from the new vertex to another vertex v is drawn is proportional to the current degree of v (see Section 2.3 for a more rigorous definition). The vertices are denoted by $v_1, v_2, v_3, \ldots$ in order of insertion. We denote the preferential attachment graph with n vertices and nm edges by G_m^n.

This process creates a certain imbalance: The degree of low-degree vertices is unlikely to increase and the degree of high-degree vertices is likely to increase even further. This so called "the rich get richer"-effect has has been widely recognized as a reasonable explanation


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of the heavy tailed degree distribution of real networks. The preferential attachment model is particularly interesting from the point of mathematical analysis because of its simple formulation and interesting characteristics.

1.1 Previous Results

There exists a wide range of results on the asymptotic structure of preferential attachment graphs (for an overview see [31]). The radius of these graphs has been shown to be logarithmic [5, 12], which means they exhibit small world behavior [1, 21]. Various aspects of their degrees have been studied [13, 2, 24, 29, 28]. In particular, the degree distribution follows a power-law with exponent three [6]. However, unlike many real networks [32, 30], preferential attachment graphs have a vanishing clustering coefficient [7]. This result has been obtained by bounding the number of certain subgraphs in preferential attachment graphs. For example, the expected number of triangles or l-cycles in G_m^n has been shown to be $\log(n)^{\Theta(1)}$ for fixed l and m [7], which is lower than what is to be expected in real networks [22].

1.2 Our Results

In this work, we obtain new insights to the clustering behavior of preferential attachment graphs. We show that there is a cluster centered around a logarithmic number of tightly connected hubs that has previously not been discussed. On the other hand, removing these central hubs asymptotically yields a graph in which every r-neighborhood is a tree with at most one extra edge. In particular, at most two neighbors of a vertex are connected. This drastically bounds the local clustering coefficient. In summary, we show that the first logarithmically many vertices are the main source of structural complexity in preferential attachment graphs and removing them yields a locally extremely sparse graph.

We prove these results one by one. At first, we show that preferential attachment graphs contain one-subdivisions of cliques of logarithmic size. A one-subdivision is the graph obtained by replacing every edge with a path of length two.

▶ **Theorem 9.** *There exists a constant c such that for $m, n \in \mathbf{N}$, $m \geq 2$, $n \geq c$, the preferential attachment graph G_m^n contains a one-subdivided clique of size at least $\log(n)^{1/4}$ with a probability of at least $1 - e^{-\log(n)^{1/4}/c}$.*

Graph classes which contain subdivided cliques of logarithmic size cannot for example be planar or have bounded treewidth. In fact, they are *somewhere dense*, as we will discuss later on. The previous theorem is complemented by showing that removing the first polylogarithmically many vertices of a preferential attachment graph leaves a graph with a locally extremely sparse, tree-like structure.

▶ **Corollary 13.** *There exists a constant c such that for $l, n, m \in \mathbf{N}^+$ and $b = \lceil \log(n)^{cl^2} m^{cl} \rceil$ the graph $G_m^n[v_{b+1}, \ldots, v_n]$ contains a.a.s. no subgraph with up to l vertices and more edges than vertices.*

Forbidding subgraphs with more edges than vertices is quite restrictive. The previous statement states that for a fixed m, removing $\log(n)^{O(l^2)}$ vertices leads asymptotically to a graph in which all connected subgraphs with l vertices have either l or $l-1$ edges, i.e., they are either a tree or a tree with one extra edge. This means for example that the removal of $\log(n)^{O(1)}$ vertices yields asymptotically a graph with no cliques of size four. It also yields the following statement about tree-like neighborhoods in preferential attachment graphs.

▶ **Theorem 16.** *There exists a constant c such that for $r, n, m \in \mathbf{N}^+$ and $b = \lceil \log(n)^{cr^2} m^{rc} \rceil$, a.a.s. every r-neighborhood of $G_m^n[v_{b+1}, \dots, v_n]$ is either a tree or a tree with one additional edge.*

1.3 Sparsity

An often observed property of real networks is a relatively low average degree. For example in a social network, the vast majority of members only has relatively few neighbors, especially compared to the set of all potential neighbors. This leads us to a concept called *structural sparsity* [25]. Sparsity has been deeply studied and has lead to many interesting results. In particular, there exists a large body of work focusing on algorithmic applications of sparsity (for example [4, 9, 14, 16, 17]). Many graph problems that are hard for general graphs become easier on structurally sparse graph classes.

Sparsity is not a property of single graphs, but a property of graph classes. An important cornerstone of sparsity theory are *nowhere dense* and *somewhere dense* graph classes introduced by Nešetřil and Ossona de Mendez [25]. These graph classes are defined using *shallow topological clique minors*. A graph G contains an r-shallow topological clique minor of size k if a subgraph of G is isomorphic to a graph obtained by subdividing a k-clique up to $2r$ times. The size of the largest r-shallow topological clique minor of a graph G is denoted by $\omega(G \mathbin{\widetilde{\nabla}} r)$. A graph class is nowhere dense if for every radius r and every graph G in the graph class, $\omega(G \mathbin{\widetilde{\nabla}} r) \leq f(r)$ for some function $f(r)$ independent of the graph size. Conversely, if $\omega(G \mathbin{\widetilde{\nabla}} r)$ is unbounded for some r, the graph class is said to be somewhere dense. Every graph class is either nowhere or somewhere dense. For a rigorous definition of these two concepts see Section 2.4.

Nowhere dense graph classes generalize many different sparse graph properties such as bounded degree, planarity, bounded treewidth, bounded genus or bounded expansion [25]. An important algorithmic result is that every problem definable in first-order logic can be solved in almost linear time on nowhere dense graph classes [16]. On the other hand, somewhere density defines for many problems a natural barrier in tractability. The model checking problem for first-order logic on monotone somewhere dense graph classes is AW[$*$]-complete and therefore as hard as on general graphs [16].

In order to lift deep algorithmic results for sparse graph classes to random graphs, people started analyzing which random graph models are somewhere or nowhere dense *in the limit*. Roughly speaking, a random graph model is asymptotically almost surely (a.a.s.) nowhere dense if for large graphs $\omega(G \mathbin{\widetilde{\nabla}} r)$ is with probability one bounded by some function $f(r)$ independent of the graph size and a.a.s. somewhere dense if it is with probability one unbounded. If a random graph model is a.a.s. nowhere dense it is not a.a.s. somewhere dense, and vice versa. However, it is possible that a random graph model is neither a.a.s. somewhere nor a.a.s. nowhere dense. Again, for more details see Section 2.4.

Erdős–Rényi graphs, random intersection graphs, Chung–Lu graphs, and the configuration model have been classified with respect to a.a.s. somewhere and nowhere density [26, 10, 15]. This has lead directly to efficient algorithms on random graphs. Preferential attachment graphs are known not to be a.a.s. nowhere dense [10] but a complete classification remained open. In this work, we show that shallow topological clique minors have polylogarithmic size and therefore that the preferential attachment model is a.a.s. somewhere dense.

▶ **Theorem 1.** *Let $r \in \mathbf{N}^+$, $m \geq 2$. Then a.a.s. $\log(n)^{1/4} \leq \omega(G_m^n \mathbin{\widetilde{\nabla}} r) \leq \log(n)^{O(r^2)} m^{O(r)}$. In particular G_m^n is a.a.s. somewhere dense.*

This is a direct consequence of Theorem 9 and Theorem 15. Our result implies that algorithmic techniques developed for nowhere dense graph classes do not directly transfer to preferential attachment graphs. However, shallow topological clique minors have only polylogarithmic size. Hence, preferential attachment graphs are "just barely" somewhere dense. In particular, the removal of few vertices places the random graph model a.a.s. in a very sparse graph class, where all connected subgraphs of bounded diameter are "tree-like." Note that locally bounded treewidth implies nowhere denseness [25].

▶ **Theorem 17.** *Let $m \in \mathbf{N}^+$ and $g(n)\colon \mathbf{N} \to \mathbf{N}$ be a function with $g(n) = \omega(1)$. If we remove the first $\log(n)^{g(n)}$ vertices from G_m^n the remaining graph has a.a.s. locally bounded treewidth.*

2 Preliminaries

2.1 Graph Notation

We use common graph theory notation [11]. Furthermore, an *r-neighborhood* is an induced subgraph with radius at most r.

2.2 Probabilities and Random Graph Models

In this work, a random graph model (such as the preferential attachment model) is a sequence $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbf{N}}$, where $\mathcal{G}_n$ is a probability distribution over undirected simple graphs with n vertices (this means graphs have no self-loops or multi-edges). In slight abuse of notation, we also write $\mathcal{G}_n$ for the random variable which is distributed according to $\mathcal{G}_n$. This way, we can lift graph notation to notation for random variables of graphs: For example edge set and clique number of a random graph $\mathcal{G}_n$ are represented by random variables $E(\mathcal{G}_n)$ and $\omega(\mathcal{G}_n)$.

2.3 The Preferential Attachment Graph Model

In this work we focus on the *preferential attachment random graph model* which we describe in this subsection. It has been ambiguously defined in the original article by Barabási and Albert [3]. The model generates random graphs by iteratively inserting new vertices and edges. It depends on a parameter, usually denoted by m, which indicates the number of edges attached to each newly created vertex. We follow the rigorous definition of Bollobás et al. [6]: For a fixed parameter m the random process is defined by starting with a single vertex and iteratively adding vertices, thereby constructing a sequence of graphs $\bar{G}_m^1, \bar{G}_m^2, \ldots, \bar{G}_m^t$, where $\bar{G}_m^t$ has t vertices and mt edges (of which some may be self-loops). We define $d_m^t(v)$ to be the degree of vertex v in the graph $\bar{G}_m^t$. The random process for $m = 1$ works as follows. A random graph is started with one vertex v_1 that has exactly one self-loop. This graph is $\bar{G}_1^1$. We then define the graph process inductively: Given $\bar{G}_1^{t-1}$ with vertex set $\{v_1, \ldots, v_{t-1}\}$, we create $\bar{G}_1^t$ by adding a new vertex v_t together with a single edge from v_t to v_i, where i is chosen at random from $\{1, \ldots, t\}$ with

$$\Pr[i = s] = \begin{cases} \dfrac{d_1^{t-1}(v_s)}{2t-1} & 1 \le s \le t \\ \dfrac{1}{2t-1} & s = t. \end{cases}$$

This means we add an edge to a random vertex with a probability proportional to its degree at the time.

For $m > 1$, the process can be defined by merging sets of m consecutive vertices in $\bar{G}_1^{mt}$ to single vertices in $\bar{G}_m^t$ [6]. Let $v'_1, \dots, v'_{mt}$ be the vertices of $\bar{G}_1^{mt}$. The graph $\bar{G}_m^t$ with vertices $v_1, \dots, v_t$ is constructing by merging $v'_{(i-1)m+1}, \dots, v'_{im}$ into a single vertex v_i. The graph $\bar{G}_m^t$ is a multigraph. The number of edges between vertices v_i and v_j in $\bar{G}_m^t$ equals the number of edges between the corresponding sets of vertices in $\bar{G}_1^{mt}$. Self-loops are allowed. We obtain a simple random graph G_m^n from $\bar{G}_m^n$ by removing all self-loops and replacing multiple edges with a single edge.

2.4 Sparsity

At first, we define nowhere and somewhere density as a property of *graph classes* and then lift the notation to *random graph models*. There are various equivalent definitions and we use the most common definition based on shallow topological minors.

▶ **Definition 2** (Shallow topological minor [27]). *A graph H is an r-*shallow topological minor *of a graph G if a graph obtained from H by subdividing every edge up to $2r$ times is isomorphic to a subgraph of G. The set of all r-shallow topological minors of a graph G is denoted by $G \mathbin{\widetilde{\nabla}} r$. We define the maximum clique size over all shallow topological minors of G as*

$$\omega(G \mathbin{\widetilde{\nabla}} r) = \max_{H \in G \mathbin{\widetilde{\nabla}} r} \omega(H).$$

▶ **Definition 3** (Nowhere dense, somewhere dense [25]). *A graph class $\mathcal{G}$ is* nowhere dense *if there exists a function f, such that for all r and all $G \in \mathcal{G}$, $\omega(G \mathbin{\widetilde{\nabla}} r) \leq f(r)$. A graph class $\mathcal{G}$ is* somewhere dense *if for all functions f there exists an r and a $G \in \mathcal{G}$, such that $\omega(G \mathbin{\widetilde{\nabla}} r) > f(r)$.*

Observe that a graph class is somewhere dense if and only if it is not nowhere dense. We lift these notions to random graph models using the following two definitions.

▶ **Definition 4** (a.a.s. nowhere dense, a.a.s. somewhere dense). *A random graph model $\mathcal{G}$ is* a.a.s. nowhere dense *if there exists a function f such that for all r*

$$\lim_{n \to \infty} \Pr[\omega(\mathcal{G}_n \mathbin{\widetilde{\nabla}} r) \leq f(r)] = 1.$$

A random graph model $\mathcal{G}$ is a.a.s. somewhere dense *if for all functions f there is an r such that*

$$\lim_{n \to \infty} \Pr[\omega(\mathcal{G}_n \mathbin{\widetilde{\nabla}} r) > f(r)] = 1.$$

While for graph classes the concepts are complementary, a random graph model can be *neither* a.a.s. somewhere dense *nor* a.a.s. nowhere dense (e.g., if the random graph model is either the empty or the complete graph, both with a probability of $1/2$).

3 Lower Bounds

In this section we show that preferential attachment graphs are a.a.s. somewhere dense. We do so by analyzing the probability that a preferential attachment graph of size n contains a one-subdivided clique of size at least $k = \log(n)^{1/4}$ as a subgraph. Let this probability be p_n. We show that $\lim_{n \to \infty} p_n = 1$. The proof works as follows: We start with a small preferential attachment graph and pick a set of k vertices with high degree. These will be the principal vertices of the one-subdivided clique. We then add vertices to the graph according to the

preferential attachment process. A one-subdivided clique of size k arises if for every pair of principal vertices v and w, a new vertex u is added that is adjacent to both v and w. We show that after $n = k^4 2^{k^3}$ vertices have been inserted, with high probability there is at least one connecting vertex for every pair of principal vertices.

We now describe an urn experiment that illustrates for a pair of principal vertices v,w the probability that a new vertex u is connected to both u and v. This experiment has no connection to Pólya Urns and is solely used for illustrative purposes. The experiment consists of multiple rounds. In the ith round (we assume $i \geq 10$), we define an urn containing i balls, where $\sqrt{\lceil i \rceil}$ balls are red, $\sqrt{\lceil i \rceil}$ balls are blue, and the rest is black. In each round we draw two balls uniformly at random from the urn. The experiment succeeds if we draw a red and a blue ball in the same round. It is easy to see that the probability of success in the ith round equals $2(\sqrt{\lceil i \rceil}/i)^2$. We observe that eventually the experiment succeeds because

$$1 - \prod_{i=10}^{\infty} \left(1 - 2\Big(\frac{\sqrt{i}}{i}\Big)^2\right) = 1.$$

This experiment behaves similarly to the process of connecting two principal vertices. Two principal vertices are connected in round i if the vertex v_i is connected to both principal vertices. The expected degree of the first vertex in a preferential attachment graph of size i is proportional to $\sqrt{i}$. If we (naively) assume that the degrees of u and v at time i are at all times exactly $\sqrt{i}$ then a new vertex throws an edge to v or w with probability proportional to $\sqrt{i}/i$ and is connected to both with probability roughly $2(\sqrt{i}/i)^2$. Therefore, the probability that in the ith step a new vertex u connects both v and w is proportional to the probability that in the ith round of the urn experiment a red and a blue ball is drawn. Using similar arguments we show in this section that the success probability of building a one-subdivided clique also is high.

If we however alter the urn experiment and assume that in the ith round there are only about $\sqrt{i}/\log(i)$ red or blue balls we cannot guarantee success because

$$1 - \prod_{i=10}^{\infty} \left(1 - 2\Big(\frac{\sqrt{i}/\log(i)}{i}\Big)^2\right) \neq 1.$$

This means if the expected value of the degrees were just a logarithmic factor smaller then our proof would not work. This suggests that preferential attachment graphs are "just barely" a.a.s. somewhere dense. This also means we need lower bounds which guarantee that the degrees of our principal vertices are not much smaller than their expected value, e.g., at most a constant factor off. Bounds which guarantee a factor of $1/\log(i)$ would not be sufficient.

Unfortunately the probability distribution of the degree of a vertex is only well centered around its expected value if its initial degree is already large (Proposition 5). We therefore use Lemma 6 to find in a graph of size k^4 with high probability k vertices with a degree of roughly k. These vertices will be our principal vertices. For large k, their degree is centered closely around its expected value for our technique to work.

In our proof of Theorem 9 we argue that with high probability for every pair of principal vertices there will eventually be a vertex which is connected to both. One may try to prove this by showing that with high probability the degrees of the principal vertices are well centered at all times of the random process (roughly $\sqrt{i}$ in the ith round for every i) and then showing that the probability that the principal vertices will be connected under this condition is high. However, it is highly non-trivial to bound the probability that an edge is inserted during the preferential attachment process under the condition that something else will happen *afterwards*. Therefore this approach does not work easily. It is much easier to

condition under events that happen *before* in the random process. We therefore analyze the random process over time and *interleave* the concentration bounds of the principal vertices with bounds on the event that a new vertex connects a pair of principal vertices. This interleaved analysis works as follows: Let B_i be the event that the degree of the principal vertices is at least half their expected degree at time step i of the random process. We prove B_i to be likely. For j slightly larger than i, one can easily obtain a good bound on the probability that a new edge of the jth vertex is connected to a principal vertex under the assumption B_i. Our calculations work in a way where whenever we assume a new event B_i the small probability $\Pr[\bar{B}_i]$ is added to our failure probability (see Lemma 8). So if we were to assume all events $B_i, B_{i+1}, B_{i+2}, \dots$ our bound quickly becomes meaningless as the sum $\Pr[\bar{B}_i] + \Pr[\bar{B}_{i+1}] + \Pr[\bar{B}_{i+2}] + \cdots$ quickly becomes larger than one. But if we assume exponentially spaced events $B_i, B_{2i}, B_{4i}, B_{8i}, \dots$ our bound on the failure probability stays small enough and new vertices are still likely to be connected to our principal vertices, allowing us to show in Theorem 9 that G_2^n contains a large one-subdivided clique.

The results of this section build upon the following concentration bound for vertex degrees. We define $d_m^n(S) = \sum_{v \in S} d_m^n(v)$ as the summed degree of S in G_m^n.

▶ **Proposition 5** ([13], Theorem 19). *For $t, m, d \in \mathbf{N}^+$, $0 < \varepsilon \leq 1/2$, $S \subseteq \{v_1, \dots, v_t\}$ with $\Pr[d_m^t(S) = d] \neq 0$ and $d \geq \log(\log(3tm))\varepsilon^{-200}$*

$$\Pr\left[(1-\varepsilon)\sqrt{\frac{n}{t}}d < d_m^n(S) < (1+\varepsilon)\sqrt{\frac{n}{t}}d \text{ for all } n \geq t \;\middle|\; d_m^t(S) = d\right] \geq 1 - e^{-\varepsilon^{200}d}.$$

This bound is strong if the initial degree $d_m^t(S)$ is large. At first, we show that there are some vertices which have a reasonably high degree after a short number of steps.

▶ **Lemma 6.** *There exists a constant c such that for $k, m \in \mathbf{N}$, $k \geq c$ with probability at least $1 - ke^{-k/c}$ there exists a set of vertices $X \subseteq \{v_1, \dots, v_{k^2}\}$, $|X| = k$ such that $d_m^{k^4}(x) \geq mk/2$ for all $x \in X$.*

Proof. We partition the first k^2 vertices into k sets of k vertices. Let S be one of these sets. Since $|S| = k$ and every vertex has at least degree m, we know that $d_m^{k^2}(S) \geq mk$. Therefore

$$\Pr\left[d_m^{k^4}(S) \leq \frac{1}{2}mk^2\right] \leq \sum_{d=mk}^{\infty} \Pr\left[d_m^{k^2}(S) = d\right] \cdot \Pr\left[d_m^{k^4}(S) \leq \frac{1}{2}\sqrt{\frac{k^4}{k^2}}d_m^{k^2}(S) \;\middle|\; d_m^{k^2}(S) = d\right].$$

Since $k \geq c$, we can choose c sufficiently large to always guarantee $c \geq 2^{200}$ and $mk \geq 2^{200}\log\log(mk^2)$. Using $t = k^2$, $n = k^4$, and $\varepsilon = 1/2$, Proposition 5 bounds the term above by at most

$$\sum_{d=mk}^{\infty} \Pr\left[d_m^{k^2}(S) = d\right] \cdot e^{-\varepsilon^{200}d} \leq e^{-k/c}.$$

With a probability of at least $1 - ke^{-k/c}$ each of the k sets have at time k^4 a total degree of at least $mk^2/2$ by the union bound. We define x_i to be the vertex in the ith set that has the highest degree after k^4 steps. Since the set contains k vertices, x_i has a degree of at least $mk/2$. We set $X = \{x_1, \dots, x_k\}$. ◀

We now bound the probability that two principal vertices v_a, v_b become connected under the condition that they have high degree.

▶ **Lemma 7.** *We consider the preferential attachment process with $m \geq 2$. Let $k \in \mathbf{N}$ and $a, b \leq k^4$. Let B_i be the event that $d_m^i(v_a), d_m^i(v_b) \geq m\sqrt{i}/4k$. Let $A_{j,i}$ with $k^4 < j \leq i$ be the event that the first two edges of at least one of the vertices $v_j, \ldots, v_i$ are incident to v_a and v_b, respectively. Then $\Pr[\bar{A}_{i+1,2i} \mid \bar{A}_{j,i}, B_i] \leq e^{-\frac{1}{256k^2}}$ for $k^4 \leq i$ and $j \leq i$.*

Proof. Let $u > 0$. $\Pr[A_{i+u,i+u} \mid B_i]$ is the probability that vertex v_{i+u} is adjacent to both v_a and v_b under the condition that v_a and v_b have degree at least $m\sqrt{i}/4k$ at some earlier time i. When vertex v_{i+u} is inserted, the random process draws $m \geq 2$ edges from v_{i+u} to earlier vertices. The probability that some vertex is chosen equals its degree divided by the total number of edges in the graph at this time. The degree of v_a and v_b is at least $m\sqrt{i}/4k$ at this point in time. Also there is a total of at most $2(i+u)m$ edges in the graph. We can therefore bound

$$\Pr[A_{i+u,i+u} \mid B_i] \geq \Big(\frac{m\sqrt{i}/4k}{2(i+u)m}\Big)^2 = \Big(\frac{\sqrt{i}}{8(i+u)k}\Big)^2.$$

The same argument holds if we additionally assume some of the earlier vertices not to be adjacent to both v_a and v_b. Let $j < i$. Then

$$\Pr[A_{i+u,i+u} \mid \bar{A}_{j,i+u-1}, B_i] \geq \Big(\frac{\sqrt{i}}{8(i+u)k}\Big)^2.$$

We now consider the probability that no vertex in a sequence of vertices is adjacent to both v_a and v_b. The chain rule yields

$$\begin{aligned}\Pr[\bar{A}_{i+1,2i} \mid \bar{A}_{j,i}, B_i] &= \prod_{u=1}^{i} \Pr[\bar{A}_{i+u,i+u} \mid \bar{A}_{j,i+u-1}, B_i] \\ &\leq \prod_{u=1}^{i} \Big(1 - \Big(\frac{\sqrt{i}}{8(i+u)k}\Big)^2\Big) \\ &\leq \Big(1 - \Big(\frac{\sqrt{i}}{16ik}\Big)^2\Big)^i \leq \Big(1 - \frac{1}{256ik^2}\Big)^i \leq e^{-\frac{1}{256k^2}}.\end{aligned}$$

◀

Imagine a sequence of events $\mathcal{A}_0, \ldots, \mathcal{A}_{l-1}$ such that a preferential attachment graph contains a large subdivided clique if any one of these events occurs. This means it is sufficient to show that the probability $\Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{l-1}]$ is small. Assume we can only bound the probability of event $\bar{\mathcal{A}}_i$ under the condition $\mathcal{B}_i$. The following technical observation gives a good approximation of $\Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{l-1}]$ if the events $\mathcal{B}_i$ have a high probability.

▶ **Lemma 8.** *Let $\mathcal{A}_0, \ldots, \mathcal{A}_l, \mathcal{B}_0, \ldots, \mathcal{B}_l$ be events. Then*

$$\Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_l] \leq \sum_{0=1}^{l} \Pr[\bar{\mathcal{B}}_i] + \prod_{0=1}^{l} \Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1} \cap \mathcal{B}_i].$$

Proof. Let $i \leq l$. We apply the chain rule and the law of total probability. Then

$$\begin{aligned}\Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_i] &= \Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1}] \Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1}] \\ &= \Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1}] \Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1} \cap \mathcal{B}_i] \Pr[\mathcal{B}_i] \\ &\quad + \Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1}] \Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1} \cap \bar{\mathcal{B}}_i] \Pr[\bar{\mathcal{B}}_i] \\ &\leq \Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1}] \Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1} \cap \mathcal{B}_i] + \Pr[\bar{\mathcal{B}}_i].\end{aligned}$$

We can now recursively apply this inequality, and use an upper bound of 1 for all factors in front of $\Pr[\mathcal{B}_1], \ldots, \Pr[\mathcal{B}_l]$ when expanding the product, to get our claim. ◀

We now use Lemma 6 and Lemma 7 to prove the main result of this section.

▶ **Theorem 9.** *There exists a constant c such that for $m, n \in \mathbf{N}$, $m \geq 2$, $n \geq c$, the preferential attachment graph G_m^n contains a one-subdivided clique of size at least $\log(n)^{1/4}$ with a probability of at least $1 - e^{-\log(n)^{1/4}/c}$.*

Proof. Let $k \in \mathbf{N}$. We choose $n = k^4 2^{k^3}$ which implies $k \geq \log(n)^{1/4}$ for $k \geq 2$. We will prove this theorem by showing that k vertices in G_m^n are with high probability pairwise connected by a path of length two and thereby span a one-subdivided clique.

We know that if a vertex has high degree early on then their degree will be centered closely around its expected values in the future (Proposition 5). Let us therefore assume there is a set of vertices $X \subseteq \{v_1, \ldots, v_{k^2}\}$, $|X| = k$ such that $d_m^{k^4}(x) \geq mk/2$ for all $x \in X$. We call these vertices principal vertices. Since $n \geq c$, we have $k \geq \log(c)^{1/4}$. We can choose the constant c high enough such that according to Lemma 6 these principal vertices exist with a probability of at least $1 - ke^{-k/c}$.

Let us fix a pair of principal vertices v_a, v_b and show that with high probability there is a vertex that is adjacent to both v_a and v_b. The higher the degree of v_a and v_b the higher the probability that a new vertex is adjacent. As in Lemma 7, we define for $i \geq k^4$, B_i to be the event that $d_m^i(v_a), d_m^i(v_b) \geq m\sqrt{i}/4k$. Again using $k \geq \log(c)^{1/4}$, we can assume c large enough to guarantee $mk/2 \geq \log(\log(3k^4 m))2^{200}$. Since we have $d_m^{k^4}(v_a) \geq mk/2$, Proposition 5 states with $t = k^4$, $\varepsilon = 1/2$ and $d \geq mk/2$

$$\Pr\left[d_m^i(v_a) < \frac{m\sqrt{i}}{4k} = \frac{1}{2}\sqrt{\frac{i}{k^4}}mk/2\right] \leq e^{-\varepsilon^{200}mk/2} = e^{-|\Omega(k)|}.$$

This yields $\Pr[\bar{B}_i] = e^{-|\Omega(k)|}$. We further define $A_{j,i}$ with $k^4 < j \leq i$ to be the event that the first two edges of at least one of the vertices $v_j, \ldots, v_i$ is adjacent to both v_a and v_b. We will show that v_a and v_b have a connecting vertex in G_m^n by showing that $\Pr[\bar{A}_{k^4+1,n}]$ converges to zero. We divide our vertices from $k^4 + 1$ to $n = k^4 2^{k^3}$ into k^3 windows which double in size. For $0 \leq i < k^3$ we set $\bar{\mathcal{A}}_i = \bar{A}_{k^4 2^i + 1, k^4 2^{i+1}}$ and $\mathcal{B}_i = B_{k^4 2^i}$. Lemma 7 states $\Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1} \cap \mathcal{B}_i] \leq e^{-\frac{1}{256k^2}}$. By Lemma 8

$$\begin{aligned}
\Pr[\bar{A}_{k^4+1,n}] = \Pr[\bar{A}_{k^4+1,k^4 2^{k^3}}] &= \Pr[\bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{k^3-1}] \\
&\leq \sum_{i=0}^{k^3-1} \Pr[\bar{\mathcal{B}}_i] + \prod_{i=0}^{k^3-1} \Pr[\bar{\mathcal{A}}_i \mid \bar{\mathcal{A}}_0 \cap \cdots \cap \bar{\mathcal{A}}_{i-1} \cap \mathcal{B}_i] \\
&\leq \sum_{i=0}^{k^3-1} e^{-|\Omega(k)|} + \prod_{i=0}^{k^3-1} e^{-\frac{1}{256k^2}} \\
&\leq k^3 e^{-|\Omega(k)|} + e^{-\frac{k^3}{256k^2}} = k^3 e^{-|\Omega(k)|}.
\end{aligned}$$

This means that in G_m^n the probability that there exists a vertex which connects the principal vertices v_a and v_b is at least $1 - k^3 e^{-|\Omega(k)|}$. According to the union bound, the probability that for all $\binom{k}{2}$ pairs of principal vertices there exists a vertex which connects them is bounded by $1 - \binom{k}{2} k^3 e^{-|\Omega(k)|}$. In Lemma 7, only the first two edges of the connecting vertex are considered. Therefore each connecting vertex may only connect a single pair of principal vertices. This means every pair of principal vertices has a unique connecting vertex, i.e., the principal vertices span a one-subdivided clique with probability $1 - \binom{k}{2} k^3 e^{-|\Omega(k)|}$.

So far, all our calculations were based on the assumption that there are k principal vertices with reasonably high degree in the beginning. According to Lemma 6, the probability that k such vertices do not exist is at most $ke^{-k/c}$. So by law of total probability, we can add this error probability to the conditional bound to get an unconditional bound. This means that G_m^n contains no one-subdivided k clique with a probability of at most

$$\binom{k}{2}k^3e^{-|\Omega(k)|} + ke^{-k/c} = e^{-|\Omega(\log(n)^{1/4})|}. \qquad \blacktriangleleft$$

We can restate Theorem 9 as a lower bound on 1/2-shallow clique minors.

▶ **Corollary 10.** *Let $m \geq 2$. Then a.a.s. $\omega(G_m^n \mathbin{\widetilde{\nabla}} \frac{1}{2}) \geq \log(n)^{1/4}$. In particular, G_m^n is a.a.s. somewhere dense.*

4 Upper Bounds

In this section we prove polylogarithmic upper bounds on the density of shallow topological minors in preferential attachment graphs. In fact, we even show that it is sufficient to remove a polylogarithmic number of vertices to make these graphs look locally almost like trees. Our results are based on the following proposition that bounds the probability of edges occurring between fixed vertices.

▶ **Proposition 11** ([13], Lemma 10). *Let $n, m \in \mathbf{N}^+$, $n \geq 2$ and $E \subseteq \binom{v_1,\dots,v_n}{2}$. Then*

$$\Pr[E \subseteq E(G_m^n)] \leq \log(n)^{O(|E|)^2} m^{2|E|} \prod_{v_x v_y \in E} 1/\sqrt{xy}.$$

The main work is spend in Lemma 12, where we bound for $l, b \in \mathbf{N}$ the expected number of dense subgraphs of size l after removing the first b vertices of the graph. This is done by summing over all potential vertex sets $\binom{v_{b+1},\dots,v_n}{l}$ of a dense subgraph and using the previous proposition to bound the probability that these vertices induce a graph with many edges. This bound is quite strong and flexible, but its dependence on five parameters b, l, k, n, m makes it a bit unwieldy. We simplify it to obtain statements abound the size of r-shallow minors or the density of local regions.

▶ **Lemma 12.** *Let $b, l, k, n, m \in \mathbf{N}^+$ with $n \geq 2$. The expected number of subgraphs of $G_m^n[v_{b+1}, \dots, v_n]$ with at most l vertices and k more edges than vertices is at most $\log(n)^{O(l+k)^2} m^{O(l+k)} b^{-k}$.*

Proof. Let H be a graph with at most l vertices and k more edges than vertices. Let p be the expected number of subgraphs of $G_m^n[v_{b+1}, \dots, v_n]$ that are isomorphic to H. We want to find an upper bound for p. We iteratively remove every degree-one vertex from H. The resulting graph H' still has at most l vertices and k more edges than vertices. Let p' the expected number of subgraphs of $G_m^n[v_{b+1}, \dots, v_n]$ isomorphic to H'. Since H' is a subgraph of H, we have $p \leq p'$.

Let $V(H') = \{u_1, \dots, u_\gamma\}$ with $\gamma \leq l$ and let $\delta_1, \dots, \delta_\gamma$ be the degree sequence of $V(H')$. We state two inequalities that we will use later. Inequality (1) follows from $\delta_i \geq 2$ and (2) holds since the number of edges in H' is $\gamma + k = \sum_{i=1}^{\gamma} \delta_i/2$.

$$\sum_{y_i=b+1}^{n} \frac{1}{\sqrt{y_i}^{\delta_i}} \leq \int_b^n \frac{1}{\sqrt{x}^{\delta_i}} dx \leq 2b^{1-\delta_i/2} \log n \tag{1}$$

$$\sum_{i=1}^{\gamma} (-\delta_i/2 + 1) = -k \tag{2}$$

For integers $b < x_1, \dots, x_\gamma \leq n$, we consider an embedding of H' into the graph $G_m^n[v_{b+1}, \dots, v_n]$ that maps u_i to v_{x_i} (for $1 \leq i \leq \gamma$). According to Proposition 11, the probability that this embedding of H' is a subgraph of G_m^n is at most

$$\log(n)^{O(l+k)^2} m^{O(l+k)} \prod_{i=1}^{\gamma} \frac{1}{\sqrt{x_i}^{\delta_i}}.$$

We use the union bound over all valid embeddings, (1), and (2) to bound

$$\begin{aligned}
p \leq p' &\leq \sum_{x_1=b+1}^{n} \cdots \sum_{x_\gamma=b+1}^{n} \log(n)^{O(l+k)^2} m^{O(l+k)} \prod_{i=1}^{\gamma} \frac{1}{\sqrt{x_i}^{\delta_i}} \\
&= \log(n)^{O(l+k)^2} m^{O(l+k)} \sum_{x_1=b+1}^{n} \frac{1}{\sqrt{x_1}^{\delta_1}} \cdots \sum_{x_\gamma=b+1}^{n} \frac{1}{\sqrt{x_\gamma}^{\delta_\gamma}} \\
&\overset{(1)}{\leq} \log(n)^{O(l+k)^2} m^{O(l+k)} \prod_{i=1}^{\gamma} 2b^{1-\delta_i/2} \log(n) \\
&\overset{(2)}{\leq} \log(n)^{O(l+k)^2} m^{O(l+k)} 2^\gamma \log(n)^\gamma b^{-k} \leq \log(n)^{O(l+k)^2} m^{O(l+k)} b^{-k}.
\end{aligned}$$

For an arbitrary but fixed graph H with at most l vertices and k more edges than vertices we have bounded the expected number of valid embeddings p. There are at most 2^{l^2} graphs with l vertices. Therefore, the expected number of subgraphs of $G_m^n[v_{b+1}, \dots, v_n]$ with at most l vertices and k more edges than vertices is at most $2^{l^2} \log(n)^{O(l+k)^2} m^{O(l+k)} b^{-k} = \log(n)^{O(l+k)^2} m^{O(l+k)} b^{-k}$. ◀

The following corollary simplifies the last lemma by choosing the specified value for b.

▶ **Corollary 13.** *There exists a constant c such that for $l, n, m \in \mathbf{N}^+$ and $b = \lceil \log(n)^{cl^2} m^{cl} \rceil$ the graph $G_m^n[v_{b+1}, \dots, v_n]$ contains a.a.s. no subgraph with up to l vertices and more edges than vertices.*

▶ **Lemma 14.** *There is a constant c such that for all $r, n, m \in \mathbf{N}^+$ and $b = \lceil \log(n)^{cr^2 l^2} m^{crl} \rceil$, $G_m^n[v_{b+1}, \dots, v_n]$ contains a.a.s. no r-shallow topological minor of a graph with l vertices and more edges than vertices.*

Proof. If a graph contains an r-shallow topological minor of a graph with l vertices and more edges than vertices then it also contains a subgraph with at most $(2r+1)(l+1)$ vertices and more edges than vertices. Using this observation, this lemma follows directly from Corollary 13. ◀

▶ **Theorem 15.** *Let $r, n, m \in \mathbf{N}^+$. Every r-shallow topological clique minor in G_m^n has a.a.s. size at most $\log(n)^{O(r^2)} m^{O(r)}$.*

Proof. The previous lemma with $l = 4$ and $b = \lceil \log(n)^{16cr^2} m^{4cr} \rceil$ means that the graph $G_m^n[v_{b+1}, \dots, v_n]$ contains a.a.s. no r-shallow clique minor of size four. Since adding the vertices $v_1, \dots, v_b$ to the graph can increase the size of the maximum r-shallow topological clique minor by at most b, we immediately obtain this result. ◀

▶ **Lemma 16.** *There is a constant c such that for all $r, n, m \in \mathbf{N}^+$ and $b = \lceil \log(n)^{cr^2} m^{cr} \rceil$, a.a.s. every r-neighborhood of $G_m^n[v_{b+1}, \dots, v_n]$ is either a tree or a tree with one additional edge.*

Proof. Assume an r-neighborhood which is a tree with two or more additional edges. We build a breadth-first spanning tree T of the r-neighborhood with root v and radius r. We pick two edges a_1, a_2 and b_1, b_2 which are not in T. Now there are unique paths of length at most $2r+1$ in T between v and a_1, a_2, b_1, b_2. The two edges together with these four paths describe a graph with at most $4(2r+1)$ vertices and one more edge than vertices. The statement now follows from Corollary 13. ◀

▶ **Theorem 17.** *Let $m \in \mathbf{N}^+$ and $g(n)\colon \mathbf{N} \to \mathbf{N}$ be a function with $g(n) = \omega(1)$. If we remove the first $\log(n)^{g(n)}$ vertices from G_m^n the remaining graph has a.a.s. locally bounded treewidth.*

Proof. Let $G_m'^n$ be the graph obtained by removing the first $\log(n)^{g(n)}$ many vertices from G_m^n and let c be the constant from Lemma 16. For every $r \in \mathbf{N}$ there exists an $f(r)$ such that for $n \geq f(r)$, $G_m'^n$ was obtained by removing at least the first $\lceil \log(n)^{cr^2} m^{rc} \rceil$ vertices. Then by Lemma 16, a.a.s. every r-neighborhood of $G_m'^n$ is either a tree or a tree with an additional edge and therefore has treewidth at most 2. If $n \leq f(r)$ then trivially every r-neighborhood of $G_m'^n$ has treewidth at most $f(r)$. Thus, $G_m'^n$ a.a.s. has locally bounded treewidth. ◀

5 Conclusion

In our analysis of preferential attachment graphs we obtained a logarithmic lower bound on the size of shallow clique minors, that implies that preferential attachment graphs are a.a.s. somewhere dense. This implies that algorithmic techniques developed for nowhere dense graph classes are not directly applicable to preferential attachment graphs. This is complemented by matching polylogarithmic upper bounds. We further observed that the removal of a polylogarithmic number of vertices makes the graph locally extremely sparse. Real networks however, do not have this extremely sparse structure after removal of very few vertices. Our results therefore yield further evidence that the preferential attachment process alone is not sufficient to explain the structure of complex networks.

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