Layered Fan-Planar Graph Drawings

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Abstract

In a *fan-planar drawing* of a graph an edge can cross only edges with a common end-vertex. In this paper, we study fan-planar drawings that use h (horizontal) layers and are proper, i.e., edges connect adjacent layers. We show that if the embedding of the graph is fixed, then testing the existence of such drawings is fixed-parameter tractable in h, via a reduction to a similar result for planar graphs by Dujmović et al. If the embedding is not fixed, then we give partial results for $h = 2$: It was already known how to test the existence of fan-planar proper 2-layer drawings for 2-connected graphs, and we show here how to test this for trees. Along the way, we exhibit other interesting results for graphs with a fan-planar proper h-layer drawing; in particular we bound their pathwidth and show that they have a bar-1-visibility representation.

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1 Introduction

A *proper h-layer drawing* of a graph is a drawing where each vertex is on one of h distinct horizontal lines (*layers*) and edges connect pairs of vertices on consecutive layers. (Detailed definitions are in the next section.) In a seminal paper, Dujmović et al. [4] showed that testing whether a planar graph has a planar proper h-layer drawings is fixed-parameter tractable in h. This is of interest since finding a proper layered drawing of minimum height is NP-hard [6]. Dujmović et al. also study some variations, such as having an overall constant number of crossings or permitting flat edges (i.e., with endpoints on the same layer) and long edges (i.e., with endpoints connecting non-consecutive layers).

In this paper, we aim to generalize their results to so-called *beyond-planar graphs*, i.e., non-planar graphs that admit drawings with some restrictions on how crossings may occur (rather than how many). Such graphs have been the object of great interest in graph drawing and graph theory in recent years (refer to [3, 8] for surveys). In particular, we study two central families of beyond-planar graphs, namely *1-planar graphs*, which can be drawn such that every edge has at most one crossing (but the overall drawing may have linearly many crossings), and *fan-planar graphs*, which can be drawn such that an edge e may have many crossings but all the edges crossed by e have a common endpoint. Our main result is that for a fan-planar graph G with a fixed embedding, we can test in time fixed-parameter tractable in h whether G has an embedding-preserving proper h-layer drawing. Our approach is to modify G to obtain a planar graph G' that has a planar $f(h)$-layer drawing if and only if G has a fan-plane h-layer drawing. We then appeal to the result by Dujmović et al. [4]. Nearly the same approach also works for short drawings where flat edges are allowed, and for 1-planar graphs it also works for long edges when drawn as y-monotone polylines.

The above algorithms crucially rely on the given embedding. For fan-planar graphs where the embedding can be chosen, the problem appears much harder; the only result we know of is to test the existence of fan-planar proper 2-layer drawings for 2-connected fan-planar graphs [2]. Based on their insights, we give here an algorithm to solve the problem for trees.

One crucial ingredient for the algorithm by Dujmović et al. [4] is that a graph with a planar proper h-layer drawing has pathwidth at most $h-1$, and this bound is tight. We similarly can bound the pathwidth for graphs that have a fan-planar proper h-layer drawing, and again the bound is tight. The proof uses a detour: we show that graphs with a fan-planar proper layered drawing have a *bar-1-visibility representation*, a result of independent interest.

Paper organization. After reviewing definitions (Section 2), we start with the results about bar-1-visibility representations and pathwidth (Section 3), since these are convenient warm-ups for dealing with fan-planar proper layered drawings. We then give the reduction from fan-plane proper h-layer drawing to planar proper $f(h)$-layer drawing and hence prove fixed-parameter tractability of the existence of fan-plane proper h-layer drawing (Section 4). Finally we turn towards fan-planar proper 2-layer drawings, and show how to test the existence of such drawings for trees in linear time (Section 5). All our algorithms are constructive, i.e., give such drawings in case of a positive answer. We conclude with open problems (Section 6).

2 Preliminaries

We assume familiarity with basic graph theoretic notions. Let $G=(V,E)$ be a graph. We assume throughout that G is connected and simple.

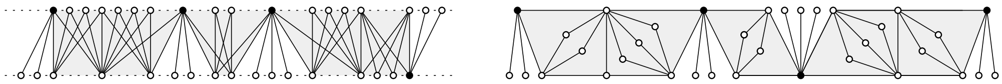


Figure 1 A fan-planar proper 2-layer drawing and its graph (a stegosaurus [2]).

A *path decomposition* P of G is a sequence $P_1, \ldots, P_p$ of vertex sets ("*bags*") that satisfies: (1) every vertex is in at least one bag, (2) for every edge (v, w) at least one bag contains both v and w, and (3) for every vertex v the bags containing v are contiguous in the sequence. The *width* of a path decomposition is $\max\{|P_t| - 1 : 1 \leq t \leq p\}$. The *pathwidth* $pw(G)$ of G is the minimum width of any path decomposition of G.

Embeddings and drawings that respect them. We mostly follow the notations in [9]. Let Γ be a *drawing* of G, i.e., an assignment of distinct points to vertices and non-self-intersecting curves connecting the endpoints to each edge. In what follows, we usually identify the graph-theoretic object (vertex, edge) with the geometric object (point, curve) that represents it. All drawings are assumed to be *good*: (i) No edge contains a vertex (except at its endpoints), (ii) two edges share at most one point, which is either a common endpoint or an interior point (called *crossing*) where the two edges cross transversely, and (iii) no three edges cross at the same point. An *edge-segment* is a maximal (open) subset of an edge that contains no crossing or vertex. The *rotation* at a vertex v in the drawing is the cyclic order in which the incident edges end at v. (Often we list the neighbours rather than the edges.) The *rotation system* of a drawing consists of the set of rotations at all vertices. A *region* of a drawing Γ is a maximal connected part of $\mathbb{R}^2 \setminus \Gamma$; it can be identified by listing the edge-segments, crossings and vertices on it in clockwise order. The *planarization* of a drawing is obtained by replacing every crossing by a new vertex of degree 4 (called a *(crossing)-dummy-vertex*).

A graph is called *k-planar* (or simply *planar* for k=0) if it has a *k-planar drawing* where every edge has at most k crossings. In a planar drawing the regions are called *faces* and the infinite region is called the *outer-face*. A drawing of G is called *fan-planar* if it has a *fan-planar drawing* where for any edge e, all edges $e_1, \ldots, e_d$ that are crossed by e have a common endpoint v.[1] The set $\{e_1, \ldots, e_d\}$ is also called a *fan* with *center-vertex* v.

A *planar embedding* of a graph G consists of the rotation system obtained from some planar drawing of G as well as a specification of outer-face. An *embedding* of a graph G consists of a graph G_P with a planar embedding that is the planarization of some drawing of G. Put differently, an embedding of G specifies the rotation system, the pairs of edges that cross, the order in which the crossings occur along each edge, and the infinite region. A drawing of a graph G with a specified embedding is called *embedding-preserving* if its planarization is G_P. We use *plane/1-plane/fan-plane* for a graph G together with an embedding corresponding to a planar/1-planar/fan-planar drawing, and also for an embedding-preserving drawing of G.

Layered drawings. Let $h \geq 1$ be an integer. An *h-layer drawing* of a graph G is a drawing where the vertices are on one of h distinct horizontal lines $L_1, \ldots, L_h$, called *layers*, and edges are drawn as y-monotone polylines for which all bends lie on layers. We enumerate the layers top-to-bottom. Layered drawings are further distinguished by what types of edges

[1] There are further restrictions, see e.g. [7]. These are automatically satisfied if the graph has a proper layered drawing and so will not be reviewed here.

Figure 2 A crossing-patch in a graph that is not fan-planar, and how to contract it.

are allowed; the following notation is from [10]. An edge is called *flat* if its endpoints lie on the same layer, *proper* if its endpoints lie on adjacent layers, and *long* otherwise. A *proper* h-layer drawing contains only proper edges, a *short* h-layer drawing contains no long edges, an *upright* h-layer drawing contains no flat edges, and an *unconstrained* h-layer drawing permits any type of edge.[2] Any graph with a planar unconstrained h-layer drawing has pathwidth at most h [5], and at most $h-1$ if a planar upright h-layer drawing exists [4]. Any graph with a fan-planar proper 2-layer drawing is a subgraph of a so-called stegosaurus (see Fig. 1 and Section 5) [2]; those have pathwidth 2.

A key concept for us is where crossings can be in proper layered drawings and how to group them. Let G_P be the planarization of some graph G with a fixed embedding. As in Fig. 2, a *crossing-patch* $\mathcal{C}$ is a maximal connected subgraph of G_P for which all vertices are crossing-dummy-vertices. Let $E_\mathcal{C}$ be the edges of G that have crossings in $\mathcal{C}$, let $V_\mathcal{C}$ be the endpoints of $E_\mathcal{C}$, and let $G_\mathcal{C}$ be the graph $(V_\mathcal{C}, E_\mathcal{C})$. Since any edge connects two adjacent layers, and a crossing-patch is connected, we can observe:

▶ **Observation 1.** *If G has a proper embedding-preserving layered drawing Γ then all crossings of a crossing-patch $\mathcal{C}$ lie strictly between two consecutive layers, and the vertices in $V_\mathcal{C}$ lie on those layers.*

3 Bar-Visibility Representations and Pathwidth

We show that for a graph G with a fan-planar short h-layer drawing, we have $pw(G) \leq 2h-1$ (and $pw(G) \leq 2h-2$ if the drawing is proper). The proof uses a *bar-c-visibility representation*, which assigns to every vertex a horizontal line segment (*bar*) and to every edge a vertical line segment connecting the bars of its endpoints in such a way that bars are disjoint and every edge-segment contains at most c points (excluding the endpoints) that belong to bars.

▶ **Theorem 1.** *If G has fan-planar proper h-layer drawing Γ, then G has a bar-1-visibility representation. Moreover, any vertical line intersects at most $2h-1$ bars of the representation.*

Proof. In the first step, make Γ maximal, i.e., insert all edges that can be added while keeping a fan-planar proper h-layer drawing. In the resulting drawing every crossing-patch is enclosed by two planar edges (shown thick blue in Fig. 3). The subgraph between two such planar edges consists (if it has crossings at all) of two crossing fans; we call this a *fan-subgraph*. Studying all possible positions of these two fans, we see that the two center-vertices include exactly one of the following two vertices: the top vertex of the left planar edge, or the bottom vertex of the right planar edge. We remove the crossed edges incident to this center-vertex in the fan-subgraph; see Fig. 3 where removed edges are red (dashed). The remaining graph G'

[2] The terminology is slightly different in [4]; for them *any* h-layer drawing was required to be short.

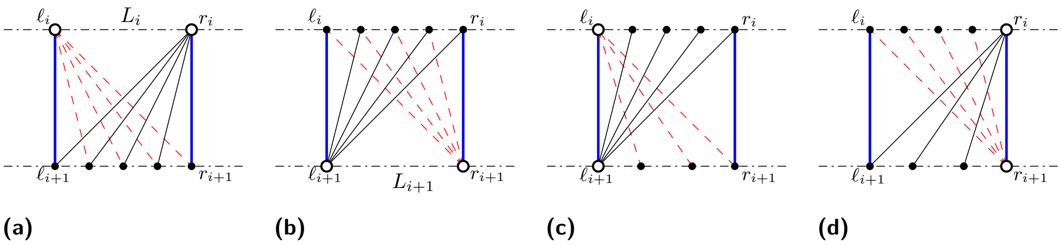


Figure 3 A fan-planar proper 2-layer drawing; planar edges that separate fan-subgraphs are blue (thick). [For labelling-purposes we show the planar edges as vertex-disjoint, but consecutive ones could have vertices in common.] We show the four possible locations of center-vertices (white).

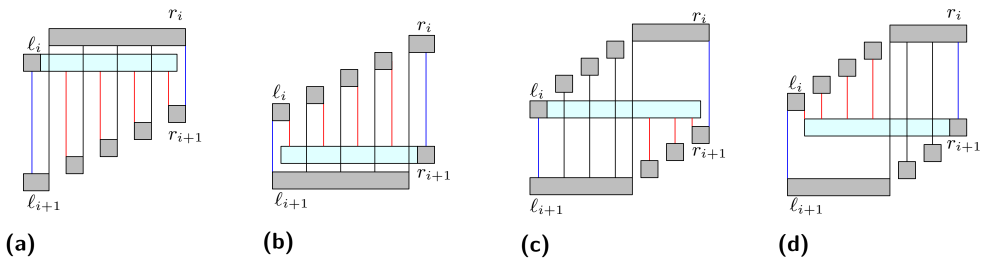


Figure 4 Bar-1-visibility representation for different types of fan-subgraphs.

is planar and has a planar proper h-layer drawing. We can convert this into a bar-0-visibility representation Γ' where the layer-assignment and the order within layers is unchanged [1]; in particular any vertical line intersects at most h bars.

Next, shift bars upward until bars of each layer lie "diagonally", see the dark gray bars in Fig. 4. More precisely, we process layers from bottom to top. For each layer we assign increasing y-coordinates to the bars from left to right such that every bar has its own y-coordinate.

Let the planar edges to the left and right of a fan-subgraph be (ℓ_i, ℓ_{i+1}) and (r_i, r_{i+1}), with vertices indexed by layer. The process of removing edges ensures that all of the missing edges are incident to r_{i+1} or ℓ_i. If they were incident to ℓ_i, then we extend ℓ_i to the right until it vertically sees its diagonally opposite corner r_{i+1}. Otherwise, we extend r_{i+1} to the left until it vertically sees its diagonally opposite corner ℓ_i. This extension (shown light blue in Fig. 4) realizes all removed edges of the fan-subgraph, since the extended bar can see vertically all other bars of vertices of the fan-subgraph. By our construction, the extended bars do not cross the planar edges between ℓ_i and ℓ_{i+1}, or between r_i and r_{i+1}. Since for each fan-subgraph there is only one extended bar, the edges of G that belong to G' go through at most one extended bar. Therefore the computed representation is a bar-1-visibility representation of G. In each fan-subgraph only one bar is extended, hence every vertical line intersects at most h bars from the h layers and at most $h-1$ bars from the $h-1$ fan-subgraphs that it traverses. ◀

With a minor change, we can prove a similar result for short layered drawings.

▶ **Theorem 2.** *If G has a fan-planar short h-layer drawing Γ, then G has a bar-1-visibility representation where any vertical line intersects at most $2h$ bars of the visibility representation.*

Proof. Let G^- be the graph obtained by removing all flat edges; this has a fan-planar proper h-layer drawing and therefore a bar-1-visibility representation using Theorem 1. Let Γ' be the visibility representation (of some subgraph of G^-) used as intermediate step in this proof.

Lengthen the bars of Γ' maximally so that within any layer, the bar of one vertex v ends exactly where the bar of the next vertex w begins. (Note that no vertical edge-segment lies between the bars of v and w since there are no long edges.) We have some choice in how much to extend v vs. how much to extend w into the gap between them, and do this such that no two points where bars begin/end have the same x-coordinate.

Now convert this visibility representation into a bar-1-visibility representation Γ^- of G^- exactly as before. We claim that this is the desired bar-1-visibility representation of G. Consider a flat edge $e = (v, w)$, with (say) v left of w on their common layer. Let X_e be the x-coordinate where the bar of v ends and the bar of w begins in the modified Γ'. To obtain Γ^-, these bars are first shifted to different y-coordinates (without changing x-coordinates of their endpoints). Since v and w are consecutive within one layer of Γ', they end on consecutive layers of Γ^-. Next the bars are (possibly) lengthened, but never shortened. Therefore edge (v, w) can be inserted with x-coordinate X_e to connect the bars of v and w.

It was argued in Theorem 1 that any vertical line intersects at most $2h - 1$ bars in that construction. The only change in our construction is that sometimes endpoints of bars may have the *same* x-coordinate X_e (for some flat edge e), which means that the vertical line with x-coordinate X_e now may intersect more bars. However, we ensured that $X_e \neq X_{e'}$ for any two flat edges e, e', which means that even at x-coordinate X_e the vertical line intersects at most $2h$ bars. ◀

▶ **Corollary 3.** *If G has a fan-planar proper h-layer drawing, then $pw(G) \leq 2h - 2$. If G has a fan-planar short h-layer drawing, then $pw(G) \leq 2h - 1$.*

Proof. Take the bar-1-visibility representation of G from Theorem 1 [respectively Theorem 2] and read a path decomposition $\mathcal{P}$ from it. To do so, sweep a vertical line ℓ from left to right. Whenever ℓ reaches the x-coordinate of an edge-segment, attach a new bag P at the right end of $\mathcal{P}$ and insert all vertices that are intersected by ℓ. The properties of a path decomposition are easily verified since bars span a contiguous set of x-coordinates, and for every edge (v, w) the line through the edge-segment intersects both bars of v and w. Since any vertical line intersects at most $2h - 1$ [$2h$, respectively] bars, each bag has size at most $2h - 1$ [$2h$] and the width of the decomposition is at most $2h - 2$ [$2h - 1$]. ◀

We now show that the bounds of Corollary 3 are tight, even for trees.

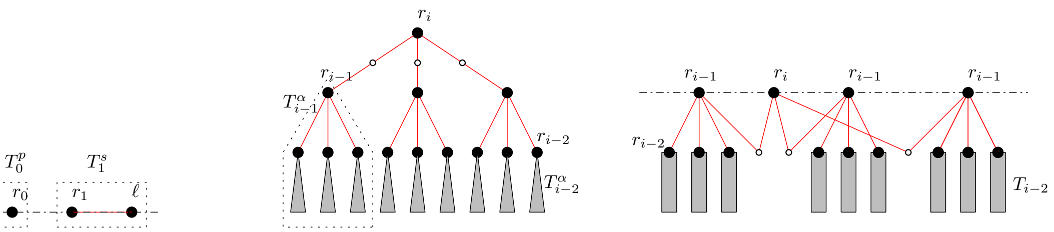
▶ **Theorem 4.** *For any $h \geq 1$, there are trees T^p_{2h-2} and T^s_{2h-1} such that*

- *T^p_{2h-2} has a fan-planar proper h-layer drawing and $pw(T^p_{2h-2}) \geq 2h-2$,*
- *T^s_{2h-1} has a fan-planar short h-layer drawing and $pw(T^s_{2h-1}) \geq 2h-1$.*

Proof. Roughly speaking, for $\alpha \in \{s, p\}$, T^α_i is the complete ternary tree with some (but not all) edges subdivided. To be more precise, for $h = 1$, define T^p_0 to be a single node r_0, which can drawn on one layer and has pathwidth $0 = 2h - 2$. Define T^s_1 to be an edge (r_1, ℓ), which can be drawn as a flat edge on one layer and has pathwidth $1 = 2h - 1$.

For $\alpha \in \{s, p\}$ and any i where T^α_i is not yet defined, set $\overline{T}^\alpha_i$ to be a new vertex r_i with three children, and make each child a root of T^α_{i-1}. Clearly $pw(\overline{T}^\alpha_i) \geq pw(T^\alpha_{i-1}) + 1$, since removing r_i from $\overline{T}^\alpha_i$ gives three components that each contain T^α_{i-1}. To obtain T^α_i from $\overline{T}^\alpha_i$ we subdivide the edges incident to r_i. This cannot decrease the pathwidth, so using induction one shows that $pw(T^\alpha_i) \geq i$.

Figure 5 shows that for all i where $\overline{T}^\alpha_{i-2}$ is defined, $\overline{T}^\alpha_i$ has a fan-planar drawing with one more layer than used by $\overline{T}^\alpha_{i-2}$. Furthermore, r_i is in the top row, and every edge is drawn properly. Using induction therefore T^p_{2h-2} and T^s_{2h-1} have fan-planar h-layer drawings. ◀



(a) **(b)** **(c)**

Figure 5 (a) The trees for $h = 1$. (b) Constructing T_i^α from T_{i-2}^α and (c) drawing it using one added layer.

(a) **(b)**

Figure 6 (a) Constructing F_i^α from F_{i-3}^α and (b) drawing it using two added layers.

Note that the drawing in Fig. 5(c) is fan-planar, but not 1-planar. This naturally raises the question: What is the pathwidth of a graph that has a 1-planar h-layer drawing? We suspect that it cannot be more than $\approx \frac{3}{2}h$ (this remains open), and can show that for the above trees (subdivided differently) this bound would be tight. Refer to Fig. 6.

▶ **Theorem 5** (*). *For any odd $h \geq 1$ (say $h = 2k + 1$ with $k \geq 0$), there are trees F_{3k}^p and F_{3k+1}^s such that*

- *F_{3k}^p has a 1-planar proper h-layer drawing and $pw(F_{3k}^p) \geq 3k = \frac{3}{2}h - \frac{3}{2}$, and*
- *F_{3k+1}^s has a 1-planar short h-layer drawing and $pw(F_{3k+1}^s) \geq 3k+1 = \frac{3}{2}h - \frac{1}{2}$.*

4 Testing Algorithm for Embedded Graphs

This section presents FPT-algorithms to determine whether an embedded graph G has an embedding-preserving h-layer drawing. The first algorithm tests the existence of a *proper* drawing, and can be applied to fan-planar graphs. (In fact, the algorithm works for any embedded graph if we allow the order of crossings along an edge to change.) A minor change allows to test the existence of *short* drawings instead. For the smaller class of 1-planar graphs, yet another change allows to test the existence of an *unconstrained* drawing. All algorithms require crucially that the embedding is fixed.

Recall that Dujmović et al. [4] gave an algorithm for this problem for planar graphs where the embedding is not fixed; in the following we refer to their algorithm as PlanarDP. The idea for our algorithm is to convert G into a *planar* graph G' such that G has an embedding-preserving h-layer drawing if and only if G' has a plane h'-layer drawing (where $h' = 2h-1$). One might be tempted to then appeal to PlanarDP. However, it is not at all clear whether PlanarDP could be modified to guarantee that the planar embedding is respected. We therefore further modify G' (in two steps) into a planar graph G''' that has a planar h'''-layer drawing (where $h''' = 12h'+1$) if and only if G' has a plane h'-layer drawing. Then call PlanarDP on G'''.

This latter step is of interest in its own right: For plane graphs, we can test the existence of a *plane* h-layer drawing in time FPT in h. This improves on PlanarDP, which permitted changes of the embedding.

To simplify the reductions, we observe that PlanarDP allows further restrictions. We will not review all details of PlanarDP (it is quite complicated), but need a few properties. It is a dynamic program with table-entries indexed (among other things) by the bags of a path decomposition $\mathcal{P}$ and specifying (among other properties) the layer for each vertex in the bag. Hence it is possible to impose restrictions on the layers that a vertex may be on, or even on the layers for a group of vertices, as long as they all appear within one bag.

4.1 Proper drawings: contracting crossing patches

This section applies when we want to test the existence of a *proper* h-layer drawing (i.e., no long or flat edges are allowed). Observation 1 implies:

▶ **Lemma 6 (*).** *Let G be an embedded graph with a crossing-patch $\mathcal{C}$, and assume G has an embedding-preserving proper h-layer drawing Γ. Then in the embedding of $G_{\mathcal{C}}$ induced by the one of G, all vertices of $V_{\mathcal{C}}$ are on the infinite region.*

Note that the conclusion of Lemma 6 depends only on the embedding of G, not on Γ, and as such can be tested given the embedding of G. In the rest of this subsection we assume that it holds for all crossing-patches, as otherwise G has no embedding-preserving proper layered drawing and we can stop.

As depicted in Fig. 2, the operation of *contracting a crossing-patch* $\mathcal{C}$ consists of contracting all the edge-segments within $\mathcal{C}$ to obtain one vertex c that is adjacent to all of $V_{\mathcal{C}}$. Hence, the rotation at c lists the vertices of $V_{\mathcal{C}}$ in the order in which they appeared on the infinite region of $G_{\mathcal{C}}$. As Fig. 2 suggests, we can convert a proper layered drawing Γ of G into a layered drawing Γ' of G' with roughly twice as many layers. To be able to undo such a conversion, observe that Γ' has special properties. First, it is *2-proper*, by which we mean that for any edge (v,w) of G the vertices v and w are exactly two layers apart, and the edges incident to a contracted vertex c are proper. It also *preserves monotonicity*: for any edge (v,w) of G that had a crossing, the edges (v,c) and (c,w) are drawn such that their union is a y-monotone curve.[3] Since G' is obtained from G by contracting all crossing-patches, and each contracted vertex c can be placed at a dummy-layer between the two layers surrounding the crossing-patches, we have:

▶ **Lemma 7.** *Let G be an embedded graph with an embedding-preserving proper h-layer drawing Γ. Then the graph G' obtained from G by contracting crossing-patches has a plane monotonicity-preserving 2-proper $(2h-1)$-layer drawing.*

The other direction is not true. It is easy to convert a plane monotonicity-preserving 2-proper $(2h-1)$-layer drawing of G' to an h-layer drawing of G with the correct rotation system and pairs of crossing edges (the drawing is *weakly isomorphic* [9]). But the *order* of crossings may change when connecting vertices by straight-line segments. For example, in Fig. 2(a), moving the top left vertex much farther left would change the order of crossings while keeping the rotation scheme unchanged. So we give the other direction only for fan-planar graphs, where this is impossible.[4]

[3] As discussed later these properties can be tested within PlanarDP.

[4] Another resolution would be to use polylines between two layers, without requiring their bends to be on layers. One can argue that if G had a straight-line embedding-preserving drawing, then such curves could be made y-monotone.

▶ **Lemma 8.** *Let G be a fan-plane graph, and let G' be the graph obtained by contracting all crossing-patches. If G' has a plane monotonicity-preserving 2-proper $(2h{-}1)$-layer drawing Γ' then G has a fan-plane proper h-layer drawing.*

Proof. Consider any crossing patch $\mathcal{C}$ of G that was contracted into vertex c, say c is on layer L_i in Γ'. Since the drawing is 2-proper, all neighbours of c are on L_{i-1} or L_{i+1}. Since for any edge (v, w) in $E_\mathcal{C}$ the endpoints are two layers apart, we have $v \in L_{i-1}$ and $w \in L_{i+1}$ or vice versa. Remove the edges incident to c and re-insert the edges in $E_\mathcal{C}$ as straight-line segments. Since the rotation at c is respected, the order of $V_\mathcal{C}$ on $L_{i-1} \cup L_{i+1}$ reflects the order along the infinite region of $G_\mathcal{C}$. Two edges e, e' in $E_\mathcal{C}$ crossed in G if and only if their endpoints alternated in the order along the infinite region of $G_\mathcal{C}$, and so they cross in the resulting drawing as needed.

Assume an edge $e = (u, w)$ in $E_\mathcal{C}$ crosses edges $e_1, \dots, e_d$ in G, in this order while walking from u to w. It suffices to argue that the same order of crossings happens in the created drawing. Let v be the common endpoint of $e_1, \dots, e_d$, say $e_i = (v, w_i)$ for $i = 1, \dots, d$. We know that endpoints of $e, e_1, \dots, e_d$ are on the infinite region of $G_\mathcal{C}$ since they belong to $V_\mathcal{C}$. Furthermore, their (clockwise or counter-clockwise) order along the infinite region must be exactly $v, u, w_1, \dots, w_d, w$ since we have a good drawing. Namely, for any $i \in 1, \dots, d$ vertex v must be separated from w_i in the order by $\{u, w\}$, otherwise e and e_i would have to cross twice since they cross at least once. Also, for any $i < j$, if the order along the infinite region is u, w_j, w_i, v while the order along e is u, e_i, e_j, v, then e_j and e_i would have to cross each other between where they cross e and their endpoints w_i and w_j. In a good drawing no two edges cross twice and edges with a common endpoint do not cross, so both are impossible.

Assume up to symmetry that $v \in L_{i-1}$, which means that $w_1, \dots, w_d$ are on L_{i+1}. Since the rotation at c contains $v, u, w_1, \dots, w_d, w$ in this order, $w_1, \dots, w_d$ are on layer L_{i+1} in this order, and edge e crosses $e_1, \dots, e_d$ in this order as desired.

Repeating this operation at all crossing patches hence gives a drawing of G that respects the embedding. After deleting even-indexed layers (which contained no vertices of G), we obtain a fan-plane proper h-layer drawing of G. ◀

4.2 Flat and long edges

We will discuss in a moment how to test whether a graph has a plane $(2h{-}1)$-layer drawing that is monotonicity-preserving and 2-proper, but first study modifications that allow us to test for short drawings (i.e., to allow flat edges) and unconstrained drawings.

Only minimal changes are needed when flat edges are allowed. Observation 1, and therefore Lemma 6 continue to hold. When there are no long edges, flat edges never have crossings. So it suffices to allow edges without crossings to be flat in G'. We say that a layered drawing Γ' of G' is *2-short* if for any edge (v, w) of G the vertices v, w are either zero or two layers apart, and the edges incident to a contracted vertex c are proper. As in Lemma 7 and 8 one shows:

▶ **Lemma 9 (*).** *Let G be a fan-plane graph, and let G' be the graph obtained by contracting crossing-patches. Then, G has a fan-plane short h-layer drawing if and only if G' has a plane monotonicity-preserving 2-short $(2h{-}1)$-layer drawing.*

Long edges pose difficulties because Observation 1 no longer holds. However, in a 1-plane graph G crossing-patches are single crossings, i.e., contracting crossing-patches is simply planarizing G. A crossing either lies between two layers or (if a long edge crosses a flat edge) exactly on a layer. A drawing of G' is called *2-unconstrained* if every vertex of G lies on an odd-indexed layer. The next lemma is shown almost exactly as Lemmas 7–9; we omit the details.

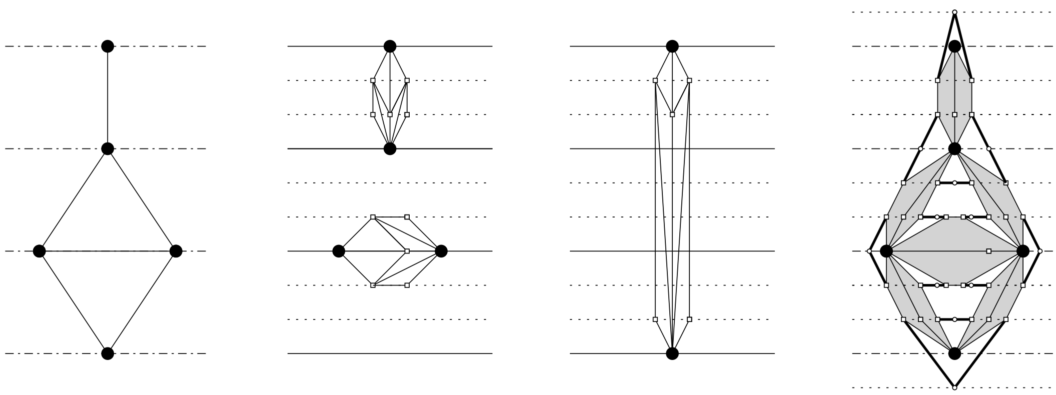


Figure 7 (a) A very small plane graph G'. (b)-(c) Replacing a proper, flat or long edge with a tripler-graph. (d) Graph G'' obtained by filling all angles. Some edges from the tripler-graphs are not shown. Filler paths are given in bold.

▶ **Lemma 10.** *Let G be a 1-plane graph and let G' be its planarization. Then G has a 1-plane unconstrained h-layer drawing if and only if G' has a plane monotonicity-preserving 2-unconstrained $(2h-1)$-layer drawing.*

4.3 Enforcing a planar embedding

Recall that we want a *plane* drawing of G' while PlanarDP tests the existence of *planar* drawings. As the next step we hence turn G' into a graph G'' that is a subdivision of a 3-connected planar graph (hence has a unique planar rotation scheme). There are many ways of making a planar graph 3-connected (e.g. we could triangulate the graph or stellate every face), but we need to use a technique here that allows to relate the height of layered drawings of G' and G'', and this seems hard when using triangulation or stellation.

Instead we use a different idea, which is easier to describe from the point of view of *angles* of G', i.e., incidences between a vertex v and a region f. The operation of *filling the angles of* G' consists of two steps. First, replace every edge e of G' by a *tripler-graph* H (see Fig. 7(b)). Then connect the tripler-graphs incident to each face via *filler paths* of length 2. One can argue that G'' is a subdivision of a 3-connected planar graph, and as Fig. 7 illustrates, it can be drawn using three times as many layers.

Recall that we had some restrictions on drawings of G', such as being 2-proper and monotonicity-preserving. All of them can be expressed as a *subgraph-restriction*, where we are given a (connected, constant-sized) subgraph H of G' and restrict the indices of layers used by $V(H)$. Such restrictions can naturally be translated to G'', since layer-indices relate via $i \leftrightarrow 3i-2$ in drawings of G' and G''. We add as further restrictions to G'' that vertices of G' can only be on every third layer and the length-2 paths that replace edges of G' must be drawn y-monotonically. One can then easily argue:

▶ **Lemma 11 (*).** *Let G' be a plane graph. Let G'' be a graph obtained by filling the angles of G'. Then G' has a plane subgraph-restricted h-layer drawing if and only if G'' has a plane subgraph-restricted $(3h)$-layer drawing.*

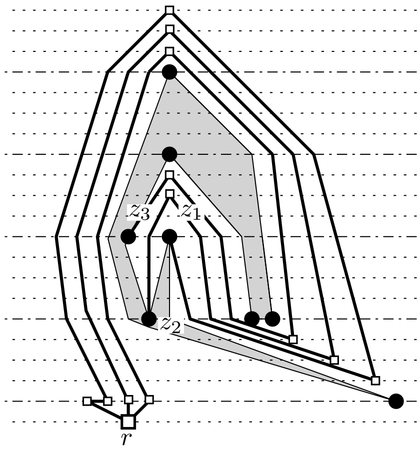


Figure 8 Routing the escape-paths (thick solid) along the outer-face. For illustration purposes we chose paths that are longer than needed.

We can enforce (see also Fig. 8) that the drawing respects a given outer-face f by inserting a new vertex r and adding three *escape-paths* from r to three vertices on the face f. The resulting graph G''' then can be drawn using roughly 4 times as many layers as G'', and the relationship goes both ways if we restrict vertices of G'' to use only every fourth layer and r to be on the bottommost layer.

4.4 Putting it all together

▶ **Theorem 12.** *There are $O(f(h)\,\mathrm{poly}(n))$ time algorithms to test the following:*

- *Given a fan-plane graph G, does it have a fan-plane proper h-layer drawing?*
- *Given a fan-plane graph G, does it have a fan-plane short h-layer drawing?*
- *Given a 1-plane graph G, does it have a 1-plane unconstrained h-layer drawing?*

Proof. First test whether the conclusion of Lemma 6 is satisfied for all crossing-patches (this is trivially true for 1-plane graphs). If not, abort. Otherwise contract the crossing-patches of G to obtain G', and add the subgraph-restrictions that G' must be drawn monotonicity-preserving and 2-proper/2-short/2-unconstrained. Fill the angles of G' to obtain G'', and add escape paths to obtain G'''. Inherit the above subgraph-restrictions into G'' and G'''. Also add the restrictions discussed when building G'' and G'''. We have argued that G''' contains a planar subgraph-restricted $(24h-11)$-layer drawing if and only if G has the desired embedding-preserving h-layer drawing.

We can test for the existence of a planar $(24h-11)$-layer drawing of G''' using PlanarDP, the algorithm from [4]. To ensure subgraph-restrictions $H_1, \ldots, H_d$, we proceed as follows. Observe that every edge of G''' belongs to a constant number of subgraph-restrictions, and that each H_j has constant size. Let $\mathcal{P}$ be a path decomposition of G''' of width at most $24h$ (this must exist, otherwise G''' has no $(24h-11)$-layer drawing). The path decomposition $\mathcal{P}$ is found as part of PlanarDP. Modify $\mathcal{P}$ as follows: For each H_j that is not a single vertex, and every bag P that contains at least one edge of H_j, add *all* vertices of H_j to P. The result $\mathcal{P}'$ is a path decomposition since H_j is connected. Since bag P represents $O(h)$ edges (it induces a planar graph), and edges belong to constant number of restriction subgraphs of constant size, the bags of $\mathcal{P}'$ have size $O(h)$. Call the dynamic programming algorithm PlanarDP on G''' using this path decomposition $\mathcal{P}'$. Recall that each table-entry of PlanarDP specifies the layer-assignment, and since each H_j appears in at least one bag P of $\mathcal{P}'$, we can enforce the subgraph-restriction by permitting (among the table-entries indexed by bag P) only those that satisfy the restriction on H_j. ◀

Sadly, our results are mostly of theoretical interest. Algorithm PlanarDP is FPT in h, but the dependency on h is a very large function. Our algorithm, where h gets replaced by $24h$ and then increased by another constant factor to accommodate the subgraph-restrictions, makes this even larger.

5 Testing Algorithm for 2-Layer Fan-planarity

Finally we turn to fan-planar drawings when the embedding is not fixed.

▶ **Theorem 13 (*).** *Let T be a tree with n vertices. We can test in $O(n)$ time whether T admits a fan-planar proper 2-layer drawing.*

Proof sketch. We know that T admits a fan-planar proper 2-layer drawing if and only if it is a subgraph of a so-called stegosaurus [2]. This imposes severe restrictions on the possible degrees in T. In particular, if T' is the subtree obtained by deleting all leaves of T, and Π is the longest path in T', then all vertices of T' have degree at most 4, and vertices of $T' \setminus \Pi$ of degree 3 or more can only occur in specific places in a stegosaurus. We now parse the vertices of Π in order and reconstruct at each vertex how this could possibly have fit into the structure of a stegosaurus (there may be multiple ways of doing this; we find the one that is "best" in the sense that it leaves the most space for future insertions). There are numerous cases here, making the analysis lengthy. In all cases, we either conclude that T was not a subgraph of a stegosaurus or we find the best-possible way in which T' can fit into a stegosaurus. Then we re-insert the leaves of T while maintaining a stegosaurus (or conclude that T was not a subgraph of a stegosaurus, since we found the best-possible one). Finally, we obtain a fan-planar proper 2-layer drawing using the result in [2]. ◀

6 Summary and Future Directions

We studied layered drawings of fan-planar graphs. Motivated by the algorithm by Dujmović et al. [4], and using it as a subroutine, we gave an algorithm that tests the existence of a fan-plane proper h-layer drawing and is fixed-parameter tractable in h. (Variations can handle fan-plane short or 1-plane unconstrained drawings.) For the situation where the embedding of the graph is not fixed, we studied the existence of fan-planar proper 2-layer drawings for trees. Along the way, we also bounded the pathwidth of graphs that have a fan-planar (short or proper) h-layer drawing, and argued that such graphs have a bar-1-visibility representation. Many open problems remain.

- Are there FPT algorithms to test whether a graph has a fan-planar h-layer drawing for $h > 2$, presuming we can change the embedding? This problem was non-trivial even for trees, $h = 2$, and proper drawings.
- Our FPT algorithm for fan-plane drawings only worked for proper or short drawings. Is there an FPT algorithm if long edges are allowed?
- Does every graph with a fan-planar unrestricted h-layer drawing have pathwidth $O(h)$? Does it have a bar-1-visibility representation? Note that fan-planar graphs are not closed under edge subdivision, so we cannot subdivide long edges.
- Last but not least, what do we know about layered drawings of k-planar graphs for $k > 1$? Note that these are not necessarily fan-planar.

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