Fractional Covers of Hypergraphs with Bounded Multi-Intersection

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Abstract

Fractional (hyper-)graph theory is concerned with the specific problems that arise when fractional analogues of otherwise integer-valued (hyper-)graph invariants are considered. The focus of this paper is on fractional edge covers of hypergraphs. Our main technical result generalizes and unifies previous conditions under which the size of the support of fractional edge covers is bounded independently of the size of the hypergraph itself. This allows us to extend previous tractability results for checking if the fractional hypertree width of a given hypergraph is $\leq k$ for some constant k. We also show how our results translate to fractional vertex covers.

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1 Introduction

Fractional (hyper-)graph theory [11] has evolved into a mature discipline in graph theory – building upon early research efforts that date back to the 1970s [2]. The crucial observation in this field is that many integer-valued (hyper-)graph invariants have a meaningful fractional analogue. Frequently, the integer-valued invariants are defined in terms of some integer linear program (ILP) and the fractional analogue is obtained by the fractional relaxation. Examples of problems which have been studied in fractional (hyper-)graph theory comprise matching problems, coloring problems, covering problems, and many more.

Covering problems come in two principal flavors, namely vertex covers and edge covers. We shall concentrate on edge covers in the first place, but we will later also mention how our results translate to vertex covers. *Fractional edge covers* have attracted a lot of attention

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in recent times. On the one hand, this is due to a deep connection between information theory and database theory. Indeed, the famous "AGM bound" – named after the authors of [1] – establishes a tight upper bound on the number of result tuples of relational joins in terms of fractional edge covers. On the other hand, fractional hypertree width (fhw) is to date the most general width-notion that allows one to define tractable fragments of solving Constraint Satisfaction Problems (CSPs), answering Conjunctive Queries (CQs), and solving the Homomorphism Problem [9]. The fractional hypertree width of a hypergraph is defined in terms of the size of fractional edge covers of the bags in a tree decomposition.

Fractional (hyper-)graph invariants give rise to new challenges that do not exist in the integral case. Intuitively, if a fractional (hyper-)graph invariant is obtained by the relaxation of a linear program (LP), one would expect things to become easier, since we move from the intractable problem of ILPs to the tractable problem of LPs. However, also the opposite may happen, namely that the fractional relaxation introduces complications not present in the integral case. To illustrate such an effect, we first recall some basic definitions.

▶ **Definition 1.** *A* hypergraph *H is a tuple $H = (V, E)$, consisting of a set of vertices V and a set of hyperedges (or simply "edges"), which are non-empty subsets of V.*

Let γ be a function of the form $\gamma\colon E \to [0, 1]$. Then the set of vertices "covered" by γ is defined as $B(\gamma) = \{v \in V \mid \sum_{e\in E, v\in e} \gamma(e) \geq 1\}$. Intuitively, γ assigns weights to the edges and a vertex v is covered if the total weight of the edges containing v is at least 1.

A fractional edge cover *of H is a function γ with $V \subseteq B(\gamma)$. An* integral edge cover *is obtained by restricting the function values of γ to $\{0, 1\}$. The support of γ is defined as* $\mathrm{support}(\gamma) = \{e \in E \mid \gamma(e) \neq 0\}$. *The weight of γ is defined as* $\mathrm{weight}(\gamma) = \sum_{e\in E} \gamma(e)$. *The minimum weight of a fractional (resp. integral) edge cover of a hypergraph H is referred to as the fractional (resp. integral) edge cover number of H.*

The following example adapted from [5] illustrates which complications may arise if we move from the integral to the fractional case.

▶ **Example 2.** Consider the family $(H_n)_{n\geq 2}$ of hypergraphs with $H_n = (V_n, E_n)$ defined as

$V_n = \{v_0, v_1, \ldots, v_n\}$

$E_n = \{e_0, e_1, \ldots, e_n\}$ with $e_0 = \{v_1, \ldots, v_n\}$ and $e_i = \{v_0, v_i\}$ for $i \in \{1, \ldots, n\}$.

The integral edge cover number of each H_n is 2 and an optimal integral edge cover can be obtained, e.g., by setting $\gamma_n(e_0) = \gamma_n(e_1) = 1$ and $\gamma_n(e) = 0$ for all other edges. In contrast, the fractional edge cover number is $2 - \frac{1}{n}$ and the unique optimal fractional edge cover is γ'_n with $\gamma'_n(e_0) = 1 - \frac{1}{n}$ and $\gamma'_n(e_i) = \frac{1}{n}$ for each $i \in \{1, \ldots, n\}$. For the support of these two covers, we have $|\mathrm{support}(\gamma_n)| = 2$ and $|\mathrm{support}(\gamma'_n)| = n + 1$. Hence, the support of the optimal edge covers is bounded in the integral case but unbounded in the fractional case. ◇

As mentioned above, fractional hypertree width (fhw) is to date the most general width-notion that allows one to define tractable fragments of classical NP-complete problems, such as CSP solving and CQ answering. However, recognizing if a given hypergraph H has $\mathrm{fhw}(H) \leq k$ for fixed $k \geq 1$ is itself an NP-complete problem [5]. It has recently been shown that the problem of checking $\mathrm{fhw}(H) \leq k$ becomes tractable if we can efficiently enumerate the fractional edge covers of size $\leq k$ [7]. This fact can be exploited to show that, for classes of hypergraphs with bounded rank (i.e., max. size of edges), bounded degree (i.e., max. number of edges containing a particular vertex), or bounded intersection (i.e., max. number of vertices in the intersection of two edges), checking $\mathrm{fhw}(H) \leq k$ becomes tractable. The size of the support has been recently [7] identified as a crucial parameter for the efficient enumeration of fractional edge covers of weight $\leq k$ for given $k \geq 1$.

The *overarching goal* of this work is to further extend and provide a uniform view of previously known structural properties of hypergraphs that guarantee a bound on the size of the support of fractional edge covers of a given weight. In particular, when looking at Example 2, we want to avoid the situation that the support of fractional edge covers increases with the size of the hypergraph. Our *main combinatorial result* (Theorem 5) will be that the size of the support of a fractional edge cover does not depend on the number of vertices or edges of a hypergraph but instead only on the weight of the cover as well as the structure of its edge intersections.

Formally, the structure of the edge intersections is captured by the so-called *Bounded-Multi-Intersection-Propery (BMIP)* [5]: a class $\mathcal{C}$ of hypergraphs has this property, if in every hypergraph $H \in \mathcal{C}$, the intersection of c edges of H has at most d elements, for constants $c \geq 2$ and $d \geq 0$. The BMIP thus generalizes all of the above mentioned hypergraph properties that ensure bounded support of fractional edge covers of given weight and, hence, also guarantee tractability of checking $\mathrm{fhw}(H) \leq k$, namely bounded rank, bounded degree, and bounded intersection. Moreover, when considering the incidence graph G of H, the BMIP corresponds to G not having large complete bipartite graphs. A notable result in the area of parameterized complexity [10] is the polynomial kernelizability of the Dominating Set Problem for graphs without $K_{c,d}$. A minor tweaking of the results yields a polynomial kernelization for the Set Cover Problem if the corresponding incidence graph does not contain $K_{c,d}$. Our result thus makes an interesting connection: it shows that a condition that enables efficient solving of the Set Cover problem also enables efficient checking of bounded fractional hypertree width.

In summary, the main results of this paper are as follows:

- First of all, we show that the size of the support of a fractional edge cover only depends on the weight of the cover and of the structure of its edge intersections (Theorem 5). More specifically, if the intersection of c edges of a hypergraph H has at most d elements, and H has a fractional edge cover of weight $\leq k$, then H also has a fractional edge cover of weight $\leq k$ with a support whose size only depends on c, d, and k.
- As an important consequence of this result, we show that the problem of checking if a given hypergraph H has $\mathrm{fhw}(H) \leq k$ is tractable for hypergraph classes satisfying the BMIP (Theorem 25). In particular, BMIP generalizes all previously known hypergraph classes with tractable fhw-checking, namely bounded rank, bounded degree, and bounded intersection.
- We transfer our results on fractional edge covers to fractional vertex covers, where we again vastly generalize previously known hypergraph classes (such as hypergraphs of bounded rank [6]) that guarantee a bound on the size of the support of fractional vertex covers (Theorem 28).

The paper is organized as follows: after recalling some basic notions and results in Section 2, we will present our main technical result on fractional edge covers in Section 3. The detailed proof of a crucial lemma is separated in Section 4. In Section 5, we apply our result on the bounded support of fractional edge covers to fractional hypertree width and fractional vertex covers. Finally, in Section 6, we summarize our results and discuss some directions for future research. Due to space limitations, some proofs are given only in the full version of this paper.

2 Preliminaries

Some general notation. It is convenient to use the following short-hand notation for various kinds of sets: we write $[n]$ for the set $\{1, \ldots, n\}$ of natural numbers. Let S be a set of sets. Then we write $\bigcap S$ and $\bigcup S$ for the intersection and union, respectively, of the sets in S, i.e., $\bigcap S = \{x \mid x \in s \text{ for all } s \in S\}$ and $\bigcup S = \{x \mid x \in s \text{ for some } s \in S\}$.

Hypergraphs. We recall some basic notions on hypergraphs next. We have already introduced in Section 1 hypergraphs as pairs (V, E) consisting of a set V of vertices and a set E of edges. Without loss of generality, we assume throughout this paper that a hypergraph neither contains isolated vertices (i.e., vertices that do not occur in any edge) nor empty edges. We call a hypergraph $H = (V, E)$ *reduced* if, in addition to these restrictions, it contains no two vertices of the same type, i.e., there do not exist $v_1 \neq v_2$ in V such that $\{e \in E \mid v_1 \in e\} = \{e \in E \mid v_2 \in e\}$. Note that, for computing (edge or vertex) covers, we may always assume that a hypergraph is reduced. It is sometimes convenient to identify a hypergraph with its set of edges E with the understanding that $V = \bigcup E$. A *subhypergraph* of a hypergraph H is obtained by taking a subset of the edges of H. By slight abuse of notation, we may thus write $H' \subseteq H$ for a subhypergraph H' of H.

Given a hypergraph $H = (V, E)$, the *dual hypergraph* $H^d = (W, F)$ is defined as $W = E$ and $F = \{\{e \in E \mid v \in e\} \mid v \in V\}$. If H is reduced, then we have $(H^d)^d = H$, i.e., the dual of the dual of H is H itself. The *incidence graph* of a hypergraph $H = (V, E)$ is a bipartite graph (W, F) with $W = V \cup E$, such that, for every $v \in V$ and $e \in E$, there is an edge $\{v, e\}$ in F iff $v \in e$. Note that a hypergraph H and its dual hypergraph H^d have the same incidence graph.

In this work, we are particularly interested in the structure of the edge intersections of a hypergraph. To this end, recall the notion of (c, d)-hypergraphs for integers $c \geq 1$ and $d \geq 0$ from [7]: $H = (V, E)$ is a (c, d)-hypergraph if the intersection of any c distinct edges in E has at most d elements, i.e., for every subset $E' \subseteq E$ with $|E'| = c$, we have $|\bigcap E'| \leq d$. A class $\mathcal{C}$ of hypergraphs is said to satisfy the *bounded multi-intersection property (BMIP)* [5], if there exist $c \geq 1$ and $d \geq 0$, such that every H in $\mathcal{C}$ is a (c, d)-hypergraph. As a special case studied in [4, 5], a class $\mathcal{C}$ of hypergraphs is said to satisfy the *bounded intersection property (BIP)*, if there exists $d \geq 0$, such that every H in $\mathcal{C}$ is a $(2, d)$-hypergraph.

We now recall tree decompositions, which form the basis of various notions of width. A tuple $(T, (B_u)_{u \in T})$ is a *tree decomposition (TD)* of a hypergraph $H = (V, E)$, if T is a tree, every B_u is a subset of V and the following two conditions are satisfied:

(a) For every edge $e \in E$ there is a node u in T, such that $e \subseteq B_u$, and

(b) for every vertex $v \in V$, $\{u \in T \mid v \in B_u\}$ is connected in T.

The vertex sets B_u are usually referred to as the *bags* of the TD. Note that, by slight abuse of notation, we write $u \in T$ to express that u is a node in T.

For a function $f\colon 2^V \to \mathbb{R}^+$, the *$f$-width* of a TD $(T, (B_u)_{u \in T})$ is defined as $\sup\{f(B_u) \mid u \in T\}$ and the f-width of a hypergraph is the minimal f-width over all its TDs.

An *edge weight function* is a function $\gamma : E \to [0, 1]$. We call γ a *fractional edge cover* of a set $X \subseteq V$ by edges in E, if for every $v \in X$, we have $\sum_{\{e \mid v \in e\}} \gamma(e) \geq 1$. The weight of a fractional edge cover is defined as $\text{weight}(\gamma) = \sum_{e \in E} \gamma(e)$. For a set $S \subseteq E$, we define $\gamma(S) = \sum_{e \in S} \gamma(e)$, i.e., the total weight of the edges in S. For $X \subseteq V$, we write $\rho^*_H(X)$ to denote the minimal weight over all fractional edge covers of X. The *fractional hypertree width (fhw)* of a hypergraph H, denoted $\text{fhw}(H)$, is then defined as the f-width for $f = \rho^*_H$. Likewise, the fhw of a TD of H is its ρ^*_H-width.

We state an important technical lemma for weight-functions of (c, d)-hypergraphs.

▶ **Lemma 3.** *There is a function $f(c, d, k)$ with the following property: let H be a (c, d)-hypergraph and let γ be an edge weight function with* $\text{weight}(\gamma) \leq k$. *Moreover, let $0 < \epsilon \leq 1$ and assume that, for each $e \in E$, $\gamma(e) \leq \frac{\epsilon}{2c}$. Let $B^\epsilon(\gamma)$ be the set of all vertices of weight at least ϵ. Then $|B^\epsilon(\gamma)| \leq f(c, d, k)$ holds.*

The above lemma is essentially an extract of Lemma 7.3 in [7]. A proof is available in the full version of the paper.

Linear Programs. We assume some familiarity with Linear Programs (LPs). Formally, we are dealing here with minimization problems of the form $c^T x = \min$ subject to $Ax \geq b$ and $x \geq 0$, where x is a vector of n variables, c is a vector of n constants, A is an $m \times n$ matrix, b is a vector of m constants, and 0 stands for the n-dimensional zero-vector. More specifically, for a hypergraph $H = (V, E)$ and vertices $X \subseteq V$, the fractional edge cover number $\rho^*_H(X)$ of X is obtained as the optimal value of the following LP: let $E = \{e_1, \ldots, e_n\}$ and $X = \{x_1, \ldots, x_m\}$, then c is the n-dimensional vector $(1, \ldots, 1)$, b is the m-dimensional vector $(1, \ldots, 1)$, and $A \in \{0, 1\}^{[m] \times [n]}$, such that $A_{ij} = 1$ if $x_i \in e_j$ and $A_{ij} = 0$ otherwise. In the sequel, we will refer to such LPs with $c \in \{1\}^n$, $b \in \{1\}^m$ and $A \in \{0, 1\}^{[m] \times [n]}$ as *unary* linear programs.

For given number n of edges, there are at most 2^n possible different inequalities of the form $A_i x \geq 1$. We thus get the following property of unary LPs, which intuitively states that if the optimum is bigger than some threshold k, then it exceeds k by some distance.

▶ **Lemma 4.** *For every positive integers n and k, there is an integer $D(n, k)$ such that for any unary LP Z of at most n variables if* $\text{OPT}(Z) > k$ *then* $\text{OPT}(Z) - k > \frac{1}{D(n,k)}$, *where* $\text{OPT}(Z)$ *denotes the minimum of the LP.*

3 Bounding the Support of Fractional Edge Covers

In this section we establish our main combinatorial result, Theorem 5. Every set of vertices in a (c, d)-hypergraph can be covered in a way such that the size of the support depends only on c, d, and the set's fractional edge cover number. Due to space constraints proofs of some statements have to be omitted and we refer to the full version of the paper for additional details.

▶ **Theorem 5.** *There is a function $h(c, d, k)$ such that the following is true. Let c, d, k be constants. Let $H = (V, E)$ be a (c, d)-hypergraph and let $\gamma\colon E \to [0, 1]$ Assume that* $\text{weight}(\gamma) \leq k$. *Then there exists an assignment $\nu\colon E \to [0, 1]$ such that weight$(\nu) \leq k$, $B(\gamma) \subseteq B(\nu)$ and* $|\text{support}(\nu)| \leq h(c, d, k)$.

The first step of our reasoning is to consider the situation where $|B(\gamma)|$ is bounded. In this case it is easy to transform γ into the desired ν. Partition all the hyperedges of H into equivalence classes corresponding to non-empty subsets of $B(\gamma)$ such that two edges e_1 and e_2 are equivalent if and only if $e_1 \cap B(\gamma) = e_2 \cap B(\gamma)$. Then let s_X be the total weight (under γ) of all the edges from the equivalence class where $e \cap B(\gamma) = X$. Identify one representative of each (non-empty) equivalence class and let e_X be the representative of the equivalence class corresponding to X. Then define ν as follows. For each X corresponding to a non-empty equivalence class, set $\nu(e_X) = s_X$. For each edge e whose weight has not been assigned in this way, set $\nu(e) = 0$. It is clear that $B(\gamma) \subseteq B(\nu)$ and that the support of ν is at most $2^{|B(\gamma)|}$, which is bounded by assumption.

Of course, in general we cannot assume that $|B(\gamma)|$ is bounded. Therefore, as the next step of our reasoning, we consider a more general situation where we have a bounded set $\mathbf{S} = \{S_1, \dots, S_r\}$ where each S_i is a set of at most c hyperedges such that the following conditions hold regarding $\mathbf{S}$: (i) for each $1 \leq i \leq r$, $\gamma(S_i) \geq 1$ and (ii) the set $U = B(\gamma) \setminus \bigcup_{i \in [r]} \bigcap S_i$ is of bounded size. Then the assignment ν as in Theorem 5 can be defined as follows. For each $e \in \bigcup \mathbf{S}$, set $\nu(e) = \gamma(e)$. Next, we observe that for the subhypergraph $H' = H - \bigcup \mathbf{S}$, $|B_{H'}(\gamma)|$ is bounded, where subscript H' means that we consider B for hypergraph H' and γ is restricted accordingly. Therefore, we define ν on the remaining edges as in the paragraph above. It is not hard to see that the support of the resulting ν is of size at most $c \cdot r + 2^{|U|}$. We are going to show that such a family of sets of edges can always be found for (c, d) hypergraphs (after a possible modification of γ).

▶ **Definition 6** (Well-formed pair). *Let $H = (V, E)$ be a hypergraph and let $\gamma \colon E \to [0, 1]$ be an edge weight function. We say $(\mathbf{S}, U)$ is a* well-formed pair *(with regard to γ) if it satisfies the following conditions:*

1. $U \subseteq B(\gamma)$
2. $\mathbf{S} = \{S_1, \dots, S_r\}$ *where each S_i is a set of at most c hyperedges of H.*
3. $B(\gamma) \setminus U \subseteq \bigcup_{i \in [r]} \bigcap S_i$.

We denote $\sum_{i \in [r]} |S_i| + 2^{|U|}$ by $n(\mathbf{S}, U)$ and refer to it as the size *of $(\mathbf{S}, U)$.*

▶ **Definition 7** (Perfect well-formed pairs). *A well-formed pair $(\mathbf{S}, U)$ is* perfect *if there is an assignment $\nu : E \to [0, 1]$ with* $\mathrm{weight}(\nu) \leq k$ *and* $|\mathrm{support}(\nu)| \leq n(\mathbf{S}, U)$ *such that* $\bigcup_{i \in [r]} \bigcap S_i \cup U \subseteq B(\nu)$.

Our aim now is to demonstrate the existence of a perfect pair $(\mathbf{S}, U)$ of size bounded by a function depending on c, d, and k. Clearly, this will imply Theorem 5.

In particular, we will define the *initial pair* which is a well-formed pair but not necessarily perfect. Then we will define two transformations from one well-formed pair into another and prove existence of a function transf so that if $(\mathbf{S_1}, U_1)$ is transformed into $(\mathbf{S_2}, U_2)$, then $n(\mathbf{S_2}, U_2) \leq \mathrm{transf}(n(\mathbf{S_1}, U_1))$. We will then prove that if we form a sequence of well-formed pairs starting from the initial pair and obtain every next element by a transformation of the last one then, after a bounded number of steps we obtain a perfect well-formed pair. We start by defining the initial pair.

▶ **Definition 8** (Initial pair). *The* initial pair *is $(\mathbf{S_0}, U_0)$ where $\mathbf{S_0} = \{\{e\} \mid \gamma(e) \geq 1/(2c)\}$ and $U_0 = B(\gamma) \setminus \bigcup_{\{e\} \in \mathbf{S_0}} e$.*

▶ **Lemma 9.** *There is a function* init *such that $n(\mathbf{S_0}, U_0) \leq \mathrm{init}(c, d, k)$.*

Proof. $|U_0| \leq f(c, d, k)$ where f is as in Lemma 3 (for $\epsilon = 1$) and $|\bigcup \mathbf{S_0}| \leq 2ck$ by construction. ◀

We now introduce two kinds of transformations, *folding* and *extension*. A folding removes a set S^* of c edges from $\mathbf{S}$ and adds to U the vertices in the intersection of the edges of S^*. In the resulting well-ordered pair $(\mathbf{S'}, U')$, $\mathbf{S'}$ has one less element than $\mathbf{S}$ and U', compared to U, has a bounded size increase of at most d vertices. Thus the action of folding gets the resulting well-formed pair closer to one with empty first component, which is a perfect pair according to the paragraph immediately after the statement of Theorem 5.

▶ **Definition 10** (Folding). *Let $(\mathbf{S}, U)$ be a well-formed pair such that $\mathbf{S}$ contains elements of size c. Let $S^* \in \mathbf{S}$ such that $|S^*| = c$. Let $\mathbf{S'} = \mathbf{S} \setminus \{S^*\}$ and $U' = U \cup (\bigcap S^* \cap B(\gamma))$. We call $(\mathbf{S'}, U')$ a* folding *of $(\mathbf{S}, U)$.*

The folding, however, is possible only if $\mathbf{S}$ has an element of size c. Otherwise, we need a more complicated transformation called an *extension*. The extension takes an element $S \in \mathbf{S}$ of size $r < c$ and expands it by replacing S with several subsets of E each containing all the edges of S plus one extra edge. This replacement may miss some of the elements v of $B(\gamma) \cap \bigcap S$ simply because v is not contained in any of these extra edges. This excess of missed elements is added to U and thus all the conditions of a well-formed pair are satisfied.

▶ **Definition 11** (Extension). *Let* $(\mathbf{S}, U)$ *be a well-formed pair with* $\mathbf{S} \neq \emptyset$ *such that every element of* $\mathbf{S}$ *is of size at most* $c - 1$*. For the extension, we identify* $S \in \mathbf{S}$ *be an element called the* extended element *and a set* S' *of hyperedges called the* extending set*. We refer to* $L = (\bigcap S \cap B(\gamma)) \setminus \bigcup S'$ *as the set of* light vertices*. An* extension *of* $(\mathbf{S}, U)$ *is* $(\mathbf{S'}, U')$ *where* $\mathbf{S'} = (\mathbf{S} \setminus \{S\}) \cup \{S \cup \{e\} \mid e \in S'\}$ *and* $U' = U \cup L$*.*

▶ **Proposition 12.** *With data as in Definition 11,* $(\mathbf{S'}, U')$ *is a well-formed pair.*

At the first glance the transformation performed by the extension is radically opposite to the one done by the folding: the first component grows rather than shrinks. Note, however, that the new sets replacing the removed one contain a larger number of edges and thus they are closer to being of size c at which stage the folding can be applied to them. Our claim is that after a sufficiently large number of foldings and extensions, a well-formed pair with empty first component is eventually obtained.

For our overall goal we then need to show that the size of the resulting perfect pair is indeed bounded by a function of c, d, and k. To that end, the following lemma first establishes that a single step in this process increases the size of the well-formed pair in a controlled manner. To streamline our path to the main result, the proof of the lemma is deferred to Section 4.

▶ **Lemma 13.** *There is a function* $\mathrm{ext}\colon \mathbb{N} \to \mathbb{N}$ *such that the following holds. Let* $(\mathbf{S}, U)$ *be a well-formed pair with* $\mathbf{S} \neq \emptyset$ *such that every element of* $\mathbf{S}$ *is of size at most* $c - 1$*. Then one of the following two statements is true.*

1. $(\mathbf{S}, U)$ *is a perfect pair.*
2. *There is an extension* $(\mathbf{S'}, U')$ *of* $(\mathbf{S}, U)$ *such that* $n(\mathbf{S'}, U') \leq \mathrm{ext}(n(\mathbf{S}, U))$*. We refer to* $(\mathbf{S'}, U')$ *as a* bounded extension *of* $(\mathbf{S}, U)$*.*

For the sake of syntactical convenience, we unify the notions of folding and bounded extension into a single notion of transformation and prove the related statement following from Lemma 13 and the definition of folding.

▶ **Definition 14** (Transformation). *Let* $(\mathbf{S}, U)$ *and* $(\mathbf{S'}, U')$ *be well-formed pairs. We say that* $(\mathbf{S'}, U')$ *is a* transformation *of* $(\mathbf{S}, U)$ *if it is either a folding or a bounded extension of* $(\mathbf{S}, U)$*.*

▶ **Lemma 15.** *There is a monotone function* $\mathrm{transf}\colon \mathbb{N} \to \mathbb{N}$ *with* $\mathrm{transf}(x) \geq x$ *for any natural number* x *such that the following holds. If* $(\mathbf{S}, U)$ *be a well-formed pair, then one of the following two statements is true.*

1. $(\mathbf{S}, U)$ *is a perfect pair.*
2. *There is a transformation* $(\mathbf{S'}, U')$ *of* $(\mathbf{S}, U)$ *such that* $n(\mathbf{S'}, U') \leq \mathrm{transf}(n(\mathbf{S}, U))$*.*

Proof. Assume that $(\mathbf{S}, U)$ is not a perfect pair. Then $|\mathbf{S}|$ is not empty (see the discussion at the beginning of this section). Suppose that an element of $\mathbf{S}$ is of size c. Then we set $(\mathbf{S'}, U')$ to be a folding of $(\mathbf{S}, U)$. By definition of the folding and of (c, d)-hypergraphs, $(\mathbf{S'}, U')$ is obtained from $(\mathbf{S}, U)$ by removal of an element from $\mathbf{S}$ and adding at most d vertices to U.

Hence the size of $(\mathbf{S'}, U')$ is clearly bounded in the size of $(\mathbf{S}, U)$. If all elements of $\mathbf{S}$ are of size at most $c-1$ then by Lemma 13, there is a bounded extension $(\mathbf{S'}, U')$ of $(\mathbf{S}, U)$.

Clearly, we can specify a function transf$'$ so that in both cases $n(\mathbf{S'}, U') \leq \text{transf}'(n(\mathbf{S}, U))$. To satisfy the requirement for transf, set $\text{transf}(x) = \max(x, \max_{i \in [x]} \text{transf}'(x))$ for each natural number x. ◀

Now that we know that each individual step on our path to a perfect pair increases the size only in a bounded fashion, we need to establish that the number of steps is also bounded by a function of c, d, and k. The following auxiliary theorem states that such a bound exists. A full proof of Theorem 17 is available in the full version of this paper.

▶ **Definition 16.** *A sequence of* $(\mathbf{S_1}, U_1), \ldots, (\mathbf{S_q}, U_q)$ *is a* sequence of transformations *if for each* $i \in [q-1]$ *the following two statements hold*

1. $(\mathbf{S_i}, U_i)$ *is not a perfect pair.*
2. $(\mathbf{S_{i+1}}, U_{i+1})$ *is a transformation of* $(\mathbf{S_i}, U_i)$*.*

▶ **Theorem 17.** *There is a monotone function* $\text{sl}\colon \mathbb{N} \rightarrow \mathbb{N}$ *such that the following is true. Let* $(\mathbf{S_1}, U_1), \ldots, (\mathbf{S_q}, U_q)$ *be a sequence of transformations. Then* $q \leq \text{sl}(n(\mathbf{S_1}, U_1))$*.*

In summary, we have shown that we can reach a perfect pair in a bounded number of transformations. Moreover, each transformation increases the size of a pair in a controlled manner. We are now ready to prove our main result.

Proof of Theorem 5. Consider the following algorithm.

1. Let $(\mathbf{S_0}, U_0)$ be the initial pair (see Definition 8).
2. $q \leftarrow 0$
3. While $(\mathbf{S_q}, U_q)$ is not a perfect pair
   a. $q \leftarrow q+1$
   b. Let $(\mathbf{S_q}, U_q)$ be a transformation of $(\mathbf{S_{q-1}}, U_{q-1})$ existing by Lemma 15

By Theorem 17, the above algorithm stops with the final q being at most $\text{sl}(n(\mathbf{S_0}, U_0))$. It follows from the description of the algorithm that $(\mathbf{S_q}, U_q)$ is a perfect pair. It remains to show that its size is bounded by a function of c, d, k.

$$q \leq \text{sl}(n(\mathbf{S_0}, U_0)) \leq \text{sl}(\text{init}(c, d, k)) \tag{1}$$

the second inequality follows from Lemma 9 and the monotonicity of sl. Next, by the properties of transf, an inductive application of Lemma 15 and Lemma 9 yields

$$n(\mathbf{S_q}, U_q) \leq \text{transf}^q(\text{init}(c, d, k)) \tag{2}$$

where superscript q means that the function transf is applied q times.

Let $h(c, d, k) = \text{transf}^{\text{sl}(\text{init}(c,d,k))}(\text{init}(c, d, k))$. It follows from combination of (1) and (2) that $n(\mathbf{S_q}, U_q) \leq h(c, d, k)$. ◀

4 Proof of Lemma 13

The first step of the proof is to define a unary linear program of bounded size associated with $(\mathbf{S}, U)$. Then we will demonstrate that if the optimal value of this linear program is at most k, then $(\mathbf{S}, U)$ is perfect. Otherwise, we show that a bounded extension can be constructed.

In order to define the linear program, we first formally define equivalence classes of edges covering U (see the informal discussion at the beginning of Section 3).

▶ **Definition 18** (Working subset, witnessing edge)**.** *A set of vertices $U' \subseteq U$ is called* working set *(for $(\mathbf{S}, U)$) if there is $e \in E \setminus \bigcup \mathbf{S}$ such that $e \cap U = U'$. This e is called a* witnessing edge *of U' and the set of all witnessing edges of U' is denoted by $W_{U'}$.*

Continuing on the previous definition, it is not hard to see that the sets $W_{U'}$ partition the set of edges of $E \setminus \bigcup \mathbf{S}$ having a non-empty intersection with U. Choose an arbitrary but fixed representative of each $W_{U'}$ and let A_U be the set of these representatives which we also refer to as the set of *witnessing representatives*. Now, we are ready to define the linear program.

▶ **Definition 19** ($LP(\mathbf{S}, U)$)**.** *The linear program $LP(\mathbf{S}, U)$ of $(\mathbf{S}, U)$ has the set of variables $X = \{x_e \mid e \in \bigcup \mathbf{S} \cup A_U\}$. The objective function is the minimization of $\sum_{x_e \in X} x_e$. The constraints are of the following two kinds.*

1. *$\{One_S \mid S \in \mathbf{S}\}$ where One_S is $\sum_{e \in S} x_e \geq 1$.*
2. *$\{One_u \mid u \in U\}$ where One_u is $\sum_{e \in E_u} x_e \geq 1$ where E_u is the subset of $\bigcup \mathbf{S} \cup A_U$ consisting of all the edges containing u.*

▶ **Lemma 20.** *Assume that the optimal solution of $LP(\mathbf{S}, U)$ is at most k. Then $(\mathbf{S}, U)$ is a perfect pair.*

Proof. Each variable x_e of $LP(\mathbf{S}, U)$ corresponds to an edge e and this correspondence is injective. For each x_e, let $\nu(e)$ be the value of x_e in the optimal solution. For each edge e not having a corresponding edge, set $\nu(e) = 0$. It follows from a direct inspection that $U \cup \bigcup_{i \in [r]} \bigcap S_i \subseteq B(\nu)$ and the size of support of ν is at most $n(\mathbf{S}, U)$. ◀

As stated above, in case the optimal value of $LP(\mathbf{S}, U)$ is greater than k we are going to demonstrate existence of a bounded extension of $(\mathbf{S}, U)$. The first step towards identifying such an extension is to identify the extending element of $\mathbf{S}$. Combining Lemma 4 from Section 2 with Lemma 21 below, we observe that $\mathbf{S}$ has an element S^* such that $\gamma(S^*)$ is *much* smaller than 1. This S^* will be the extended element.

▶ **Lemma 21.** *Let $(\mathbf{S}, U)$ be a well-formed pair. Let $\mathbf{S}^*$ be the subset of $\mathbf{S}$ consisting of all S such that $\gamma(S) < 1$. Let α be an optimal solution for $LP(\mathbf{S}, U)$. Then* $\mathrm{weight}(\alpha) \leq \mathrm{weight}(\gamma) + \sum_{S \in \mathbf{S}^*}(1 - \gamma(S))$.

Proof. Let β be an arbitrary assignment of weights to the hyperedges of H. We say that β satisfies a constraint One_S for $S \in \mathbf{S}$ if $\beta(S) \geq 1$ and that β satisfies the constraint One_u for $u \in U$ if $\beta(E_u) \geq 1$.

We are going to demonstrate an assignment of weights whose total weight exceeds that γ by at most $\sum_{S \in \mathbf{S}^*}(1 - \gamma(S))$ and that satisfies all the constraints One_S and One_v. Clearly, this will imply correctness of this theorem.

For each $S \in \mathbf{S}^*$ choose an arbitrary edge e_S and let $INCR$ be the set of all such edges. For each $e \in INCR$, let $incr_e = \max\{1 - \gamma(S) \mid e = e_S\}$.

Let γ' be obtained from γ as follows. If $e \in INCR$ then $\gamma'(e) = \gamma(e) + incr_e$. Otherwise, $\gamma'(e) = \gamma(e)$. It is not hard to see that γ' satisfies the constraints One_S for each $S \in \mathbf{S}$, that $\mathrm{weight}(\gamma') \leq \mathrm{weight}(\gamma) + \sum_{S \in \mathbf{S}^*}(1 - \gamma(S))$, and that, since γ' does not decrease the weight of any edge, $U \subseteq B(\gamma)$.

Let $\{U_1, \dots, U_a\}$ be all the working subsets of U and let $e_1, \dots, e_a$ be the respective witnessing representatives. Then the assignment γ'' of weights is defined as follows.

1. If there is $1 \leq i \leq a$ such that $e \in W_{U_i}$ then $\gamma''(e) = \gamma'(W_{u_i}) = \gamma(W_{U_i})$ if $e = e_i$ and $\gamma''(e) = 0$ otherwise.
2. Otherwise, $\gamma''(e) = \gamma'(e)$.

Let $W = \bigcup_{i\in[a]} W_{U_i}$. Note that, by construction, $\gamma'(W) = \gamma''(W)$ and the weights of edges outside W are the same under γ' and γ'' and thus, weight(γ') = weight(γ''). Moreover since $\bigcup \mathbf{S}$ does not intersect with W, γ'' satisfies the constraints One_S for all $S \in \mathbf{S}$.

It remains to show that γ'' satisfies the constraints One_u for each $u \in U$. Let $e_1, \dots, e_r$ be the edges of $\bigcup \mathbf{S}$ containing u, let $\{U_1, \dots, U_b\}$ be the working subsets of U containing u, and let $e'_1, \dots, e'_b$ be the respective witnessing representatives. As $u \in B(\gamma')$, it follows that $\sum_{i\in[r]} \gamma'(e_i) + \sum_{i\in[b]} \gamma'(W_{U_i}) \geq 1$. By construction, $\gamma''(e_i) = \gamma'(e_i)$ for each $1 \leq i \leq r$ and $\gamma''(e'_i) = \gamma'(W_{U_i})$ for each $1 \leq i \leq b$. Consequently, $\sum_{i\in[r]} \gamma''(e_i) + \sum_{i\in[b]} \gamma''(e'_i) \geq 1$. We conclude that γ'' satisfies One_u. ◀

Lemma 4 and Lemma 21 imply the following corollary.

▶ **Corollary 22.** *Let $(\mathbf{S}, U)$ be a well-formed pair. Assume that weight$(\gamma) \leq k$ while $OPT(LP(\mathbf{S}, U)) > k$. Let $n = n(\mathbf{S}, U)$. Then there is an $S^* \in \mathbf{S}$ with $1 - \gamma(S^*) > 1/(D(n,k) * |\mathbf{S}|)$. In particular this means that $\mathbf{S}^*$ is not empty where $\mathbf{S}^*$ is as in Lemma 21.*

Proof. Note that the number of variables of $LP(\mathbf{S}, U)$ is at most n. It follows from the combination of Lemma 4 and Lemma 21 that weight$(\gamma) + \sum_{S\in\mathbf{S}^*}(1 - \gamma(S)) > k + 1/D(n,k)$ and, since weight$(\gamma) \leq k$, $\sum_{S\in\mathbf{S}^*}(1 - \gamma(S)) > 1/D(n,k)$ and hence there is $S^* \in \mathbf{S}^*$ with $(1 - \gamma(S^*)) > \frac{1}{D(n,k)\cdot|\mathbf{S}^*|} \geq \frac{1}{D(n,k)\cdot|\mathbf{S}|}$. ◀

Proof of Lemma 13. If the value of the optimal solution of $LP(\mathbf{S}, U)$ is at most k, we are done by Proposition 20.

Otherwise, let $S^* \in \mathbf{S}$ be as in Corollary 22. Let $\epsilon = (D(n,k) \cdot |\mathbf{S}|)^{-1}$. It follows from Corollary 22 that vertices of $B(\gamma) \cap \bigcap S^*$ need weight contribution of at least ϵ from hyperedges of H other than S^*. We define the extending set S' to be the set of all hyperedges of H other than S^* whose weight is at least $\epsilon/2c$ and therefore $|S'| \leq 2ck/\epsilon$. Accordingly, we define the set L of light vertices to be the subset of $B(\gamma) \cap \bigcap S^*$ consisting of all vertices x that, besides S^* are contained only in hyperedges of weight smaller than $\epsilon/2c$. By Lemma 3, $|L| \leq f(c,d,k)$ and the size of S^* is clearly bounded by a function on n and c, d, k. It is not hard to see that the size of the resulting extension is bounded as well. ◀

5 Applications and Extensions

5.1 Checking Fractional Hypertree Width

Now that our main combinatorial result has been established we move our attention to an algorithmic application of the support bound. In particular, we are interested in the problem of deciding whether for an input hypergraph H and constant k we have fhw$(H) \leq k$. The problem is known to be NP-hard even for $k = 2$ [5]. However, as noted in the introduction, in recently published research we were able to show that for hypergraph classes which enjoy bounded intersection or bounded degree, it is indeed tractable to check fhw$(H) \leq k$ for constant k [7].

Due to limited space we will recall the main components of the framework for tractable width checking developed in [7] and use them in a black-box fashion.

▶ **Definition 23** (q-limited fractional hypertree width)**.** *Let $\rho^*_q(U)$ be the minimal weight of an assignment γ such that $U \subseteq B(\gamma)$ and $|\text{support}(\gamma)| \leq q$. We define the q-*limited *fractional hypertree width of a hypergraph H as its ρ^*_q-width.*

▶ **Lemma 24** (Theorem 4.5 & Lemma 6.2 in [7]). *Fix c, d, and q as constant integers. There is a polynomial time algorithm testing whether a given (c, d)-hypergraph has a q-limited fractional hypertree width at most k.*

The underlying intuition of q-limited fhw is that the bounded support allows for a polynomial time enumeration of all the (inclusion) maximal covers of sufficient weight. For (c, d)-hypergraphs it is then possible to compute a set of candidate bags such that a fitting tree decomposition, if one exists, uses bags only from this set. Deciding whether a tree decomposition can be created from a given set of candidate bags is tractable under some minor restrictions to the structure of the resulting decomposition (not of any concern to the case discussed here).

We now apply our main result and show that, under BMIP, there exists a constant q such that the q-limited fractional hypertree width always equals fractional hypertree width. From the previous lemma is then straightforward to arrive at the desired tractability result.

▶ **Theorem 25.** *There is a polynomial time algorithm for testing whether the* fhw *of the given (c, d)-hypergraph H is at most k (the degree of the polynomial is upper bounded by a fixed function depending on c, d, k).*

Proof. It follows from Theorem 5 that if $\text{fhw}(H) \leq k$ for a (c, d)-hypergraph H then the $h(c, d, k)$-limited fhw of H is also at most k.

Indeed, let $(T, (B_u)_{u \in T})$ be a tree decomposition with fhw at most k. Then, according to Theorem 5, for each node u in T there is an edge weight function γ with $|\text{support}(\gamma)| \leq h(c, d, k)$ such that $B_u \subseteq B(\gamma)$. In other words, it follows that $(T, (B_u)_{u \in T})$ has ρ^*_q-width at most k where q is $h(c, d, k)$. Thus, H also has $h(c, d, k)$-limited fractional hypertree width at most k.

Thus to test whether $\text{fhw}(H) \leq k$, it is enough to test whether the $h(c, d, k)$-limited fhw of H is at most k. This can be done in polynomial time according to Lemma 24. ◀

5.2 Extension to Fractional Vertex Cover

There are two natural dual concepts of fractional edge cover. One is the *fractional vertex cover* problem which is the dual in the sense that it is equivalent to the fractional edge cover on the dual hypergraph. The other, the *fractional independent set* problem, corresponds to the dual linear program of a linear programming formulation of finding an optimal fractional cover. Here we discuss how our results extend to vertex covers and discuss how the resulting statement generalizes, in a particular sense, a well-known statement of Füredi [6]. Some notes on connections to fractional independent sets are given in our discussion of future work in Section 6.

We start by giving a formal definition of the fractional vertex cover problem. Let $H = (V, E)$ be a hypergraph and $\beta : V \to [0, 1]$ be an assignment of weights to the vertices of H. Analogous to the definition of fractional edge covers we define

- $B_v(\beta) = \{e \in E \mid \sum_{v \in e} \beta(v) \geq 1\}$,
- $\text{vsupport}(\beta) = \{v \in V \mid \beta(v) > 0\}$,
- and $\text{weight}(\beta) = \sum_{v \in V} \beta(v)$.

A fractional vertex cover is also called a transversal in some contexts (cf. [11]). For a set of edges E' we denote the weight of the minimal fractional vertex cover β such that $E' \subseteq B_v(\beta)$ as $\tau^*(E')$. For hypergraph $H = (V, E)$, we say $\tau^*(H) = \tau^*(E)$. Recall, that we assume reduced hypergraphs and therefore there is a one-to-one correspondence of vertices in H and edges in H^d. We will make use of the following well-known fact about the connection of what we will call *dual weight assignments*.

▶ **Proposition 26.** *Let $H = (V, E)$ be a (reduced) hypergraph and $H^d = (W, F)$ its dual. We write f_v to identify the edge in F that corresponds to the vertex v in V. The following two statements hold:*

- *For every $\gamma : E \to [0, 1]$ and the function $\beta : W \to [0, 1]$ with $\beta(e) = \gamma(e)$ it holds that $B_v(\beta) = \{f_v \mid v \in B(\gamma)\}$.*
- *For every $\beta : V \to [0, 1]$ and the function $\gamma : F \to [0, 1]$ with $\gamma(f_v) = \beta(v)$ it holds that $B(\gamma) = \{v \mid f_v \in B_v(\beta)\}$.*

In the following we extend Theorem 5 to an analogous statement for fractional vertex covers thereby generalizing the previous proposition significantly. To derive the result we need a final observation about (c, d)-hypergraphs. In a sense, we show that bounded multi-intersection is its own dual property.

▶ **Lemma 27.** *Let H be a (c, d)-hypergraph. Then the dual hypergraph H^d is a $(d + 1, c)$-hypergraph.*[1]

Proof. Let $v_1, v_2, \ldots, v_{d+1}$ be $d + 1$ distinct arbitrary vertices of a (c, d)-hypergraph $H = (V, E)$. We write $I(v) = \{e \in E \mid v \in e\}$ for the set of edges incident to a vertex v. Since H is a (c, d)-hypergraph, it must hold that $X = \bigcap_{j\in[d+1]} I(v_j)$ has no more than c elements. Otherwise, there would be at least $c + 1$ edges in X that share $d + 1$ vertices, i.e., a contradiction to the assumption that H is a (c, d)-hypergraph.

Now, consider the edges $f_1, f_2 \ldots, f_{d+1}$ in $H^d = (W, F)$ that correspond to the vertices $v_1, v_2, \ldots, v_{d+1}$ in H. It follows from the definition of the dual hypergraph that $|\bigcap_{j\in[d+1]} f_j| = |X|$ since any two edges in H^d share exactly one vertex for each edge in H that they are both incident to. We know from above that $|X| \leq c$. As this applies to any choice of vertices in H, and thus also to any choice of $d + 1$ edges in H^d, we see that any intersection of $d + 1$ edges in H^d has cardinality less or equal c. ◀

▶ **Theorem 28.** *There is a function $h(c, d, k)$ such that the following is true. Let c, d, k be constants. Let H be a (c, d)-hypergraph and β be an assignment of weights to $V(H)$. Assume that weight$(\beta) \leq k$. Then there is an assignment ν of weights to $V(H)$ such that weight$(\nu) \leq k$, $B_v(\beta) \subseteq B_v(\nu)$ and $|\,\mathrm{vsupport}(\nu)| \leq h(c, d, k)$.*

Proof. Let γ be the dual weight assignment of β as in Proposition 26. That is, $\gamma : F \to [0, 1]$ is an edge weight assignment in the dual hypergraph $H^d = (W, F)$ with $|\,\mathrm{support}(\gamma)| = |\,\mathrm{vsupport}(\beta)|$ and $\mathrm{weight}(\gamma) = \mathrm{weight}(\beta)$.

From Lemma 27 we have that H^d is a $(d + 1, c)$-hypergraph and thus by Theorem 5 there is an edge weight function ν' with $B(\gamma) \subseteq B(\nu')$ and $|\,\mathrm{support}(\nu')| \leq h'(d + 1, c, k)$. Let ν now be the dual weight assignment of ν'. By Proposition 26 we then see that also $B_v(\beta) \subseteq B_v(\nu)$ and $|\,\mathrm{vsupport}(\nu)| = |\,\mathrm{support}(\nu')| \leq h'(d + 1, c, k)$. ◀

To conclude this section we wish to highlight the connection of Theorem 28 to a classical result on fractional edge covers. The following result is due to Füredi [6], who extended earlier results by Chung et al. [3].

▶ **Proposition 29** ([6], page 152, Proposition 5.11.(iii)). *For every hypergraph H of rank (i.e., maximal edge size) r, and every fractional vertex cover w for H satisfying* $\mathrm{weight}(w) = \tau^*(H)$, *the property $|\,\mathrm{vsupport}(w)| \leq r \cdot \tau^*(H)$ holds.*

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[1] Note that the superscript of H^d only signifies that it is the dual of H. It is not connected to the integer constant d used for the multi-intersection size of H.

Recall that a hypergraph H with rank r is also a $(1, r)$-hypergraph. Hence, the above proposition means that, for a $(1, r)$-hypergraph H, there is a fractional vertex cover of optimal weight whose support is bounded by a function of the weight and r. Theorem 28 generalizes Proposition 29 in two aspects. First, Theorem 28 considers (c, d)-hypergraphs with $c \geq 1$ and second, it applies to assignments of weights to vertices in general not just to those that establish an optimal fractional vertex cover. An important aspect of Proposition 29 not reflected in Theorem 28 is a concrete upper bound on the size of the support. Optimizing the upper bound following from Theorem 5 is left for future research.

6 Conclusion and Outlook

We have proved novel upper bounds on the size of the support of fractional edge covers and vertex covers. These bounds have then been fruitfully applied to the problem of checking $\mathrm{fhw}(H) \leq k$ for given hypergraph H. Recall that, without imposing any restrictions on the hypergraph H, this problem is NP-complete even for $k = 2$ [5], thus ruling out XP-membership. In contrast, for hypergraph classes that exhibit bounded multi-intersection, we have actually managed to establish XP-membership, that is, checking $\mathrm{fhw}(H) \leq k$ for hypergraphs in such a class is feasible in polynomial time for any constant k.

However, there is still room for improvement: first, our tractability result depends on a big constant $h(c, d, k)$. Hence, an important next step for future research will be a deeper investigation of algorithms for checking $\mathrm{fhw}(H) \leq k$ in case of the BMIP and either further improve the runtime or prove a matching lower bound. Moreover, XP-membership is only "second prize" in terms of a parameterized complexity result. It will be interesting to search for further restrictions on the hypergraphs to achieve fixed-parameter tractability (FPT).

Another major challenge for future research is the computation of $\mathrm{fhw}(H)$. Note that our tractability result refers to the *decision problem* of checking $\mathrm{fhw}(H) \leq k$. However, at its heart, dealing with fractional hypertree width is an *optimization problem*, namely computing the minimum possible width of all fractional hypertree decompositions of H. The difficulty here is that our bound $h(c, d, k)$ tends to infinity as k approaches the actual value of $\mathrm{fhw}(H)$. Substantial new ideas are required to overcome this problem.

Our bound on the support of fractional vertex covers generalizes a classical result by Füredi in two aspects. In future work, we plan to explore how this generalization can be applied to known consequences (cf. [6]) of Füredi's result. Finally, we have left open the extension to fractional independent sets. By use of the complementary slackness of linear programs our main result also implies structural restrictions for optimal fractional independent sets since there can only be a bounded number of constraints that have slack. We believe that an in-depth study of the connections to independent sets is merited.

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