All Growth Rates of Abelian Exponents Are Attained by Infinite Binary Words

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Abstract

We consider repetitions in infinite words by making a novel inquiry to the maximum eventual growth rate of the exponents of abelian powers occurring in an infinite word. Given an increasing, unbounded function $f\colon \mathbb{N} \to \mathbb{R}$, we construct an infinite binary word whose abelian exponents have limit superior growth rate f. As a consequence, we obtain that every nonnegative real number is the critical abelian exponent of some infinite binary word.

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1 Introduction

Two finite words u and v are *abelian equivalent*, denoted $u \sim v$, if u is obtained from v by permuting its letters. For example, the words 01120 and 20011 are abelian equivalent. The study of abelian properties of words dates back to 1957 when P. Erdős asked for arbitrarily long 4-letter words that do not contain two consecutive abelian equivalent words [10], that is, he asked if abelian squares can be avoided on a 4-letter alphabet. The question was finally answered in the positive by V. Keränen in 1992 [14]. A word of the form $u_0 \cdots u_{e-1}$ is an *abelian power* of period m and exponent e if $u_0, \ldots, u_{e-1}$ have length m and $u_0 \sim \cdots \sim u_{e-1}$. For example, the word $010 \cdot 100 \cdot 010 \cdot 001$ is an abelian power of period 3 and exponent 4. During the last decade, various abelian properties of words have been studied; this includes not only research on avoidability of abelian powers or patterns (see, e.g., [3, Ch. 5]), but also on abelian complexity [26], abelian periods and period sets [29, 11, 19], abelian returns [24, 28], and abelian subshifts [12, 23]. Related algorithms have been developed as well; see, e.g., [15] and the references therein. Generalizations of abelian equivalence, such as k-abelian equivalence [13, 30] and k-binomial equivalence [27], have also been considered. See [23] for a recent survey on abelian properties of words.

In this paper, we focus on growth rates exponents of abelian powers occurring in infinite words. While a significant portion of the commonly studied words, such as Sturmian words, episturmian words, and automatic sequences, avoid ordinary powers with large enough exponent (in an ordinary power the adjacent words are identical), it is often the case that arbitrarily high abelian exponents occur. Indeed, the main result of [26] states that if the abelian complexity of an infinite word $\mathbf{w}$ is bounded (as in Sturmian and episturmian words mentioned above), then abelian powers of arbitrarily high exponent are found in $\mathbf{w}$.

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It is desirable to have more information than this; how fast or slowly do the exponents approach infinity? In this paper, we show that the growth rate of the exponents can be arbitrarily slow. Let $\mathscr{Ae}_{\mathbf{w}}(m)$ be the supremum of exponents of abelian powers of period m occurring in an infinite word $\mathbf{w}$.

▶ **Definition 1.** *Let* $\mathbf{w}$ *be an infinite word and* $f\colon \mathbb{N} \to \mathbb{R}$ *a function. We say that the abelian exponents of* $\mathbf{w}$ *have* growth rate f *if* $\limsup_{m\to\infty} \mathscr{Ae}_{\mathbf{w}}(m)/f(m) = 1$.

The question is now if we can find an infinite word whose abelian exponents have growth rate given by any (reasonable) function. We provide, given such a function, a constructive proof that such a word can be produced over an alphabet of optimal size. This is stated in the following theorem which is the main result of this paper.

▶ **Theorem 2.** *Let* $f\colon \mathbb{N} \to \mathbb{R}$ *be an unbounded increasing function. Then there exists an infinite binary word* $\mathbf{w}$ *such that the abelian exponents of* $\mathbf{w}$ *have growth rate* f.

Our construction also works if abelian powers are replaced in the definitions by ordinary powers or certain generalizations of abelian powers such as k-abelian powers or k-binomial powers. According to our knowledge, even the analogue of Theorem 2 for ordinary powers is new, so our inquiry to the growth rates of the exponents is novel. The closest result to ours seems to be the paper [16] of D. Krieger and J. Shallit where it is shown that for every real number $\alpha > 1$ there exists an infinite word $\mathbf{w}$ such that the supremum of exponents of fractional powers occurring $\mathbf{w}$ equals α. The number α is called the *critical exponent* of $\mathbf{w}$.

Let us then describe some earlier research and show its connection to Theorem 2. In order to study abelian powers in Sturmian words, it was proposed in [11] to define the *abelian critical exponent* $\mathscr{Ac}(\mathbf{w})$ of an infinite word $\mathbf{w}$ as

$$\mathscr{Ac}(\mathbf{w}) = \limsup_{m\to\infty} \frac{\mathscr{Ae}_{\mathbf{w}}(m)}{m}. \tag{1}$$

This quantity $\mathscr{Ac}(\mathbf{w})$ can be seen to measure linear growth of the exponents. The notion is particularly suitable for exponents of abelian powers in Sturmian words and leads to surprising results. A main result of [11] is that the set of finite abelian critical exponents of Sturmian words coincides with the so-called *Lagrange spectrum* $\mathcal{L}$, a mysterious set arising from Diophantine approximation theory [11, Thm. 5.10]. In the context of this paper, it suffices to say that $\mathcal{L}$ is a subset of $[\sqrt{5}, \infty)$ that has an initial discrete part in $[\sqrt{5}, 3]$ and contains the half-line $[c_F, \infty)$, where c_F is the Freiman constant

$$c_F = \frac{2221564096 + 283748\sqrt{462}}{491993569} = 4.5278295661\ldots.$$

There are many long-standing open problems concerning the set $\mathcal{L} \cap [3, c_F]$. For example, it is known that it contains gaps, but it is unknown if it contains an interval. Good sources on the Lagrange spectrum are the books [7, 1, 25]. For more recent results, see [18] and its references. The abelian critical exponent was studied in relation to k-abelian equivalence and Sturmian words in [22].

The result of [11] connecting the abelian critical exponents of Sturmian words to the Lagrange spectrum showed that every large enough real number is an abelian critical exponent of a Sturmian word. This raised the obvious question if for each nonnegative real number θ there exists an infinite word $\mathbf{w}$ such that $\mathscr{Ac}(\mathbf{w}) = \theta$. This was answered positively by the authors of this paper in [20] where it was proved that such a word $\mathbf{w}$ can be taken over an alphabet of at most 3 letters. The proof uses the deep fact that the Lagrange spectrum contains a half-line. It was stated as an open problem if the result can be improved by reducing the number of required letters from 3 to 2 (which is optimal). Our Theorem 2 is a much more general result concerning nonlinear growth rates as well, and it implies the following result that positively solves this open problem. This corollary is the main motivation for us to pursue proving Theorem 2.

▶ **Theorem 3.** *Let θ be a nonnegative real number. Then there exists an infinite binary word $\mathbf{w}$ such that $\mathcal{A}c(\mathbf{w}) = \theta$.*

Before the preliminary definitions and results, let us make a few remarks on our proof of Theorem 2. Our proof is constructive once the function f is given. The word $\mathbf{w}$ constructed for Theorem 2 is obtained by pasting together long repetitions of words with special properties. The special property here is *cyclic avoidance of abelian powers* with large enough exponent; see Section 2. This notion is developed here to suit our needs, but we believe that it can potentially be a useful tool for constructing words with prescribed properties outside the scope of this paper. Indeed, words avoiding abelian powers cyclically allow to control the propagation of abelian powers between two adjacent words. We provide some open problems related to these words in Section 5. Notice as well that we consider integer exponents for abelian powers. Fractional powers are often used in relation to ordinary powers [16], and some versions of fractional abelian exponents have been proposed in [5, 29]. Our results apply in the fractional setting as well since the statement of Theorem 2 is unchanged if a constant is added to $\mathcal{A}e_{\mathbf{w}}(m)$.

2 Preliminaries

We use here standard notation in combinatorics on words; a standard reference is [17, Ch. 1]. An *alphabet* A is a finite set of letters (symbols); here we focus on the binary alphabet $\{0, 1\}$. A (finite) *word* over A is a finite sequence of letters of A such as 00110. We denote by A^* the set of words over A. The length of w (the number of letters) is denoted by $|w|$. An *infinite word* is a mapping $\mathbb{N} \to A$ (we index words from 0). The concatenation of the words u and v is denoted by uv. A word u is a *factor* of a word w if $w = xuy$ for some words x and y. If x (resp. y) is empty, then u is a *prefix* (resp. *suffix*) of w. If u is a factor of w, we sometimes say that w contains u. If $w = u \cdots u$ where u is repeated N times, then w is an N-power, and we write $w = u^N$. For clarity, we sometimes refer to powers as ordinary powers. The infinite word $uu\cdots$ is denoted by u^ω. Let $w = a_0 \cdots a_{n-1}$ be a word of length n, and define $C(w)$ as the word $a_1 \cdots a_{n-1}a_0$, the left cyclic shift of w. The words w, $C(w)$, $C^2(w)$, $\ldots$ $C^{n-1}(w)$ are called the *conjugates* of w.

As was mentioned in the introduction, two words u and v are *abelian equivalent* if u is a permutation of v. An *abelian power* of period m and exponent e is a word $u_0 \cdots u_{e-1}$ such that $u_0, \ldots, u_{e-1}$ are abelian equivalent and $|u_0| = \ldots = |u_{e-1}| = m$; we also call this word an abelian e-power. If all factors of a word w that are abelian powers have exponent strictly less than N, then we say that w avoids abelian N-powers. The definitions of growth rate of abelian exponents of an infinite word and abelian critical exponent are given in the introduction.

Let A be an alphabet. A *substitution* σ is a map $A^* \to A^*$ such that $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in A^*$. We extend the action of σ on an infinite word $\mathbf{w} = a_0a_1 \cdots$ over A by setting $\sigma(\mathbf{w}) = \sigma(a_0)\sigma(a_1)\cdots$. If $\sigma(a)$ has prefix a and $\lim_{n\to\infty}|\sigma^n(a)| = \infty$, then we say that σ is *prolongable* on the letter a. If σ is prolongable on a, then repeated application of σ on the letter a produces an infinite word $\sigma^\omega(a)$ that is a fixed point of σ.

3 Avoiding Abelian Powers Cyclically

In order to construct the word $\mathbf{w}$ for Theorem 2, we need to insert into $\mathbf{w}$ abelian powers of the form $u_0 \cdots u_{e-1}$ with $|u_0| = \ldots = |u_{e-1}| = m$ and $e = f(m)$. The easiest way is to set $u_0 = \ldots = u_{e-1}$. This raises the difficulty of controlling the exponents of abelian powers with period different from m. Indeed, while a word x itself might avoid abelian N-powers with period less than $|x|$, its repetition might not. For example, if $x = 1000100$, then x and x^2 both avoid abelian 5-powers, but the word x^3 has an abelian 5-power of period 3 as a prefix. In order to work around this problem, we introduce the following notion of avoiding abelian powers in a cyclic sense. We believe that this notion is useful in other constructions outside the scope of this paper.

▶ **Definition 4.** *Let w be a word. Then w avoids abelian N-powers* cyclically *if every abelian power of period m with $m < |w|$ occurring in the infinite word w^ω has exponent less than N.*

For example, any letter avoids abelian N-powers cyclically. In the above discussion, we noted that the word x does not avoid abelian 5-powers cyclically. It does not avoid abelian 6-powers either, as x^4 contains an abelian 6-power of period 4 beginning from the second letter. By inspection, the word x does avoid abelian 7-powers cyclically. Notice that a word cannot avoid abelian N-powers cyclically for any N if it is conjugate to an abelian power.

In order to construct words avoiding abelian powers cyclically, we introduce the following notion.

▶ **Definition 5.** *A substitution $\sigma\colon A^* \to A^*$* preserves abelian N-powers *if the following is satisfied for all words $w \in A^*$: if $\sigma(w)$ contains an abelian N-power $u_0 \cdots u_{N-1}$, then w contains an abelian N-power $v_0 \cdots v_{N-1}$ such that $\sigma(v_0 \cdots v_{N-1})$ is a conjugate of $u_0 \cdots u_{N-1}$.*

In other words, a substitution σ preserves abelian N-powers if each abelian N-power in an image can be decoded by σ up to a cyclic shift. A similar but weaker notion is the notion of an abelian N-free substitution found in, e.g., [4].

Preserving abelian N-powers is by no means a trivial property to verify. We identify two examples from results of Dekking in [8].

▶ **Example 6.** Consider the substitutions

$$\tau_2\colon \begin{cases} 0 & \mapsto 011, \\ 1 & \mapsto 0001 \end{cases} \quad \text{and} \quad \tau_3\colon \begin{cases} 0 & \mapsto 0012, \\ 1 & \mapsto 112, \\ 2 & \mapsto 022. \end{cases} \tag{2}$$

Dekking proved that the fixed point $\tau_2^\omega(0)$ of τ_2 avoids abelian 4-powers, and the fixed point $\tau_3^\omega(0)$ of τ_3 avoids abelian 3-powers [8, Thms. 1, 2]. These parameters are optimal, as it is straightforward to verify that every binary word of length 10 contains an abelian 3-power and every ternary word of length 8 contains an abelian 2-power. A careful read of Dekking's article reveals, in fact, that τ_2 preserves abelian 4-powers and τ_3 preserves abelian 3-powers. Furthermore, any substitution satisfying the properties in [8, Lemma] for the parameter n preserves abelian n-powers.

Substitutions that preserve abelian N-powers have a property that is crucial for our arguments in the following section.

▶ **Lemma 7.** *Let $\sigma\colon A^* \to A^*$ be a substitution that preserves abelian N-powers and is prolongable on the letter 0. Then the sequence $(\sigma^n(0))_n$ is a sequence of words avoiding abelian N-powers cyclically.*

Proof. Let $z_n = \sigma^n(0)$, and set $\mathbf{z}_n = z_n^\omega$. Then $\mathbf{z}_n = \sigma(\mathbf{z}_{n-1})$ for all $n \geq 1$. We proceed by induction. Suppose, for the sake of a contradiction, that there exists a least n such that z_n does not cyclically avoid abelian N-powers. Since $z_0 = 0$, we have $n \geq 1$. This means that $\mathbf{z}_n$ contains an abelian N-power $u_0 \cdots u_{N-1}$ with period m, $m < |z_n|$. Since σ preserves abelian N-powers, the word $\mathbf{z}_{n-1}$ contains an abelian N-power $v_0 \cdots v_{N-1}$ such that $|\sigma(v_0)| = m < |z_n|$. By the minimality of n, it must be that $|v_0| \geq |z_{n-1}|$. Since $v_0 \cdots v_{N-1}$ is a factor of $\mathbf{z}_{n-1}$, it must be that v_0 has a conjugate z' of z_{n-1} as a factor. Therefore $m = |\sigma(v_0)| \geq |\sigma(z')| = |\sigma(z_{n-1})| = |z_n|$. This is a contradiction. ◀

Hence by applying Lemma 7 to the substitutions τ_2 and τ_3 defined in (2), we see that the next theorem is true. See the final section for a comment on the case of 4-letter alphabet and abelian 2-powers.

▶ **Theorem 8.** *The following holds:*
(i) *there exist arbitrarily long words over* $\{0, 1\}$ *that cyclically avoid abelian* 4*-powers and*
(ii) *there exist arbitrarily long words over* $\{0, 1, 2\}$ *that cyclically avoid abelian* 3*-powers.*

For the purposes of our main result, the following corollary of Theorem 8 is sufficient.

▶ **Corollary 9.** *There is an integer* N *such that there exist arbitrarily long binary words that cyclically avoid abelian* N*-powers.*

4 Proofs of Main Results

We are now ready to construct the word $\mathbf{w}$ required by Theorem 2. Let $f\colon \mathbb{N} \to \mathbb{R}$ be an increasing function such that $\lim_{n\to\infty} f(n) = \infty$, and let N be the number given by Corollary 9. We need an increasing sequence (n_i) of integers satisfying
(i) $f(n_i) \geq \max\{2N, N+2\}$ for all i;
(ii) $f(n_{i+1}) > f(n_i)$ for all i;
(iii) $n_{i+1} \geq \sum_{k=1}^{i} \lfloor f(n_k) \rfloor n_k$ for all $i \geq 1$; and
(iv) there exists a word x_i of length n_i avoiding abelian N-powers cyclically.
The existence of such a sequence (n_i) is guaranteed by Corollary 9 and the fact that f is unbounded. Having obtained the required sequences (n_i) and (x_i), we set

$$X_i = x_i^{\lfloor f(n_i) \rfloor}$$

and define an infinite binary word $\mathbf{w}$ as follows:

$$\mathbf{w} = \prod_{i=1}^{\infty} X_i.$$

▶ **Proposition 10.** *The abelian exponents of* $\mathbf{w}$ *have growth rate* f*.*

Proof. Let j be a fixed positive integer. Consider an abelian power z of period m and exponent e occurring in $\mathbf{w}$ such that $m \geq n_j$ (due to the definition of the growth rate of abelian exponents, we may ignore finitely many values of $\mathcal{A}e_{\mathbf{w}}(m)$), and set $z = u_0 \cdots u_{e-1}$ with $|u_0| = \ldots = |u_{e-1}| = m$. Let us suppose that $n_j \leq m < n_{j+1}$. Our aim is to show that $e \leq \lfloor f(n_j) \rfloor + N + 1$.

▷ Claim 11. The word z cannot have X_i with $i \geq j+1$ as a factor.

Proof. Suppose that z has X_i as a factor with $i \geq j+1$. This means that there exists a least ℓ such that the word X_i is a factor of a product of ℓ consecutive words u_i. Thus the word X_i contains at least $\ell - 2$ consecutive words u_i and, since x_i cyclically avoids abelian N-powers, we have $\ell - 2 < N$, that is, $\ell < N+2$. Now $m < n_{j+1} \leq n_i$ as the sequence (n_k) is increasing and, by assumption (i), we have $f(n_i) \geq N+2$. Therefore

$$(N+2)n_{j+1} \leq \lfloor f(n_i) \rfloor n_i = |X_i| < (N+2)m \leq (N+2)n_{j+1},$$

which is a contradiction. ◁

▷ Claim 12. If z is a factor of the suffix $\prod_{i=j+1}^{\infty} X_i$, then $e < 2N$.

Proof. Suppose that z is a factor of the suffix $\prod_{i=j+1}^{\infty} X_i$. If z is contained in a word X_i with $i \geq j+1$, then $e < N$ as x_i avoids abelian N-powers cyclically. By Claim 11, the factor z cannot contain X_i as a factor for $i \geq j+1$. Thus if z is not a factor of X_i with $i \geq j+1$, there exist $k \geq j+1$ and t such that $u_0 \cdots u_{t-1}$ is a factor of X_k and $u_{t+1} \cdots u_{e-1}$ is a factor of X_{k+1}. Since both x_k and x_{k+1} avoid abelian N-powers cyclically, it follows that $t < N$ and $e - 1 - (t+1) - 1 < N$, so $e < 2N$. ◁

Suppose that z is not a factor of the suffix $\prod_{i=j+1}^{\infty} X_i$, for otherwise $e < 2N$ by Claim 12. Then z is a factor of $(\prod_{i=1}^{j} X_i)X_{j+1}$ since z cannot have X_{j+1} as a factor by Claim 11. Our assumption (iii) means that $|\prod_{i=1}^{j-1} X_i| \leq n_j \leq m$. This indicates that the contribution of the prefix $\prod_{i=1}^{j-1} X_i$ of $\mathbf{w}$ to the exponent e of z is at most 1, so we may focus on the case that z occurs in the factor $X_j X_{j+1}$. Since x_{j+1} avoids abelian N-powers cyclically, we have $e < N$ if z is a factor of X_{j+1}, and thus we assume that z is not a factor of X_{j+1}. We now have two cases: either z is a factor of X_j or not. Let us first derive a helpful claim.

▷ Claim 13. Let t be an integer such that $0 \leq t < e$. If $u_0 \cdots u_{t-1}$ is a factor of X_j, then $t \leq \lfloor f(n_j) \rfloor$.

Proof. Suppose that $u_0 \cdots u_{t-1}$ is a factor of X_j with $t < e$. Then $\lfloor f(n_j) \rfloor n_j = |X_j| \geq |u_0 \cdots u_{t-1}| = tm \geq tn_j$, so $\lfloor f(n_j) \rfloor \geq t$. ◁

If z is a factor of X_j, then Claim 13 implies that $e \leq \lfloor f(n_j) \rfloor$. Taking into account the possible contribution of 1 to e, we have thus shown that $e \leq \lfloor f(n_j) \rfloor + 1$. Suppose then that z is not a factor of X_j. Since we assume that z is not a factor of X_{j+1}, there exists t such that $u_0 \cdots u_{t-1}$ is a factor of X_j and $u_{t+1} \cdots u_{e-1}$ is a factor of X_{j+1}. Claim 13 applied to the word $u_0 \cdots u_{t-1}$ yields $t \leq \lfloor f(n_j) \rfloor$. Since x_{j+1} avoids abelian N-powers cyclically, we see that $e - 1 - (t+1) + 1 < N$. Thus by taking into account the possible contribution of 1 to e, we see that $e \leq \lfloor f(n_j) \rfloor + N + 1$. Overall, we have thus shown that

$$e \leq \max\{N, 2N, \lfloor f(n_j) \rfloor + 1, \lfloor f(n_j) \rfloor + N + 1\} = \lfloor f(n_j) \rfloor + N + 1.$$

By repeating the preceding arguments for the values $j+1$, $j+2$, ... in place of j, we see that $\mathcal{A}e_{\mathbf{w}}(m) \leq \lfloor f(n_j) \rfloor + N + 1$ for all $m < n_{j+1}$ and j ignoring finitely many values of m (recall our assumption (ii)). This together with the fact that f is increasing, shows for $n_j \leq m < n_{j+1}$ that

$$\frac{\mathcal{A}e_{\mathbf{w}}(m)}{f(m)} \leq \frac{\lfloor f(n_j) \rfloor + N + 1}{f(n_j)} \xrightarrow{j \to \infty} 1.$$

Since X_j is a factor of $\mathbf{w}$, we have $\mathcal{A}e_{\mathbf{w}}(n_j) \geq \lfloor f(n_j) \rfloor$ for all j so, in conclusion, we have

$$\limsup_{m \to \infty} \frac{\mathcal{A}e_{\mathbf{w}}(m)}{f(m)} = 1,$$

that is, the abelian exponents of $\mathbf{w}$ have growth rate f. ◀

Proposition 10 now implies Theorem 2. Next we use Theorem 2 to prove Theorem 3. This strengthens [20, Thm. 1] and solves an open problem of [20] as stated in the introduction.

Proof of Theorem 3. As stated previously, the fixed point of Dekking's substitution τ_2 avoids abelian 4-powers, so the claim is clear if $\theta = 0$. Assume thus that $\theta > 0$. Define $f\colon \mathbb{N} \to \mathbb{R}$ by setting $f(n) = \theta n$. By Theorem 2, there exists an infinite binary word $\mathbf{w}$ having growth rate f. Thus

$$\limsup_{m\to\infty} \frac{\mathcal{Ae}_{\mathbf{w}}(m)}{\theta m} = 1$$

which implies that

$$\mathcal{Ac}(\mathbf{w}) = \limsup_{m\to\infty} \frac{\mathcal{Ae}_{\mathbf{w}}(m)}{m} = \theta. \qquad \blacktriangleleft$$

We note that Theorems 2 and 3 actually generalize for k-abelian equivalence and k-binomial equivalence. We discuss this in the next section.

5 Remarks and Open Problems

The proof of Theorem 2 applies analogously when abelian equivalence is replaced by other equivalence relations. Indeed, the required growth rate of $\mathbf{w}$ is attained by an abelian power that is also an ordinary power. Every ordinary power is an abelian power, so the arguments of the proof of Proposition 10 go through if we remove the word "abelian" from it and the appropriate definitions. The same applies for generalizations of abelian equivalence such as k-abelian equivalence [13, 30] and k-binomial equivalence [27]: k-abelian (resp. k-binomial) power is an abelian power and an ordinary power is a k-abelian (resp. k-binomial) power. In summary, we have the following result.

▶ **Theorem 14.** *Let $f\colon \mathbb{N} \to \mathbb{R}$ be an increasing function such that $\lim_{n\to\infty} f(n) = \infty$. Then there exists an infinite binary word $\mathbf{w}$ such that the exponents (resp. k-abelian exponents, k-binomial exponents) of $\mathbf{w}$ have growth rate f.*

Consequently an analogue of Theorem 3 is true for ordinary powers, k-abelian powers, and k-binomial powers. In the case of ordinary powers, it was shown in [22, Proposition 3.16] that for each nonnegative θ there exists a Sturmian word that has critical exponent θ (defined as in (1) by dropping the word "abelian").

While our proof works in these other settings, there is one drawback: the growth rates of abelian exponents and ordinary exponents of $\mathbf{w}$ coincide, but surely this is generally not necessary. For example, the Thue-Morse word $\mathbf{t}$, a fixed point of the substitution $0 \mapsto 01, 1 \mapsto 10$, is known not to contain ordinary 3-powers (see, e.g., [2, Sect. 4.2.3]) but, as the word $\mathbf{t}$ is a concatenation of the abelian equivalent words 01 and 10, we have $\mathcal{Ae}_{\mathbf{t}}(m) = \infty$ for all even m. It is conceivable that the growth rate of abelian exponents of an infinite word matches a prescribed function while the growth rate of ordinary exponents is bounded. We do not know how to approach this problem – our tool of cyclic avoidance of abelian powers is useless.

▶ **Question 15.** *Let $f\colon \mathbb{N} \to \mathbb{R}$ be an increasing function such that $\lim_{n\to\infty} f(n) = \infty$. Does there exist an infinite word $\mathbf{w}$ such that the ordinary exponents of $\mathbf{w}$ have bounded growth rate and the abelian exponents of $\mathbf{w}$ have growth rate f?*

The result [20, Thm. 1] is weaker than Theorem 3: three letters are required instead of two, but the constructed words have better properties than the word $\mathbf{w}$ constructed here. Indeed, the words constructed for [20, Thm. 1] are images of Sturmian words by a uniform substitution meaning that they are, for example, uniformly recurrent. An infinite word is uniformly recurrent if each of its factors occurs infinitely many times and for each factor u there exists a constant B such that the distance between two occurrences of u is at most B. In fact, it can be shown that the constructed words are even linearly recurrent (the slopes of the Sturmian words used in the construction have bounded partial quotients and such Sturmian words are always linearly recurrent [9, Proposition 5.1]). The word $\mathbf{w}$ constructed here is not uniformly recurrent. This raises the following question we are unable to answer.

▶ **Question 16.** *Let $f\colon \mathbb{N} \to \mathbb{R}$ be an increasing function such that $\lim_{n\to\infty} f(n) = \infty$. Does there exist a uniformly recurrent (or a linearly recurrent) infinite binary word $\mathbf{w}$ such that the abelian exponents of $\mathbf{w}$ have growth rate f?*

In order to prove our main results, we exhibited arbitrarily long binary words avoiding abelian 4-powers cyclically. This was sufficient for our purposes, but raises the obvious question if such words exist for all lengths. The answer is "no". It is straightforward to check that there exists no binary word of length 8 avoiding abelian 4-powers cyclically. Bizarrely for lengths $n = 9, \ldots, 150$, computations show that such a word exists. This makes us believe that the answer to the above question is "yes" with the exception of $n = 8$. We formulate this as the following question we are unable to answer.

▶ **Question 17.** *Does there exist a binary word of length n avoiding abelian 4-powers cyclically for all $n \neq 8$?*

Recall that there exist arbitarily long factors of the fixed point $\tau_2^\omega(0)$ of Dekking's substitution τ_2 that avoid abelian 4-powers cyclically. However, computer experiments suggest that it is often the case that no such factor of length n exists. The factors of the fixed point of the substitution

$$0 \mapsto 0010001011101000101100010$$
$$1 \mapsto 1101110100010111010011101$$

that is known to avoid abelian 4-powers [6] provide more examples, but neither substitution yields such a factor of length 22, for instance. However, the word

$$0001000100011101110111$$

of length 22 avoids abelian 4-powers cyclically.

Questions analogous to Question 17 can be asked for alphabets of size 3 and 4. In the case of a 4-letter alphabet, we do not know if arbitrarily long words avoiding abelian 2-powers cyclically exist. Keränen provided in [14] a 85-uniform substitution σ defined on a 4-letter alphabet whose fixed point avoids abelian 2-powers. However, for all letters a, none of the words $\sigma^n(a)$, $n \geq 1$, avoid abelian 2-powers cyclically because the image $\sigma(a)$ begins and ends with the same letter.

Note added in proof: The research on cyclic avoidance of abelian powers is continued in the recent preprint [21]. [21, Thm. 1.2] shows that there exist arbitrarily long words over a 4-letter alphabet that avoid abelian 2-powers cyclically. Question 17 is still open.

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