On the Axiomatisability of Parallel Composition: A Journey in the Spectrum

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Abstract

This paper studies the existence of finite equational axiomatisations of the *interleaving parallel composition operator* modulo the behavioural equivalences in van Glabbeek's *linear time-branching time spectrum*. In the setting of the process algebra BCCSP over a finite set of actions, we provide *finite*, ground-complete axiomatisations for various simulation and (decorated) trace semantics. On the other hand, we show that no congruence over that language that includes bisimilarity and is included in possible futures equivalence has a finite, ground-complete axiomatisation. This *negative result* applies to all the nested trace and nested simulation semantics.

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1 Introduction

Process algebras [4, 6] are prototype specification languages allowing for the description and analysis of concurrent and distributed systems, or simply *processes*. Briefly, the *operational semantics* [26] of a process is modelled via a *labelled transition system* (LTS) [20] in which the computational steps are abstracted into state-to-state transitions having actions as labels. Notably, in order to model the concurrent interaction between processes, the majority of process algebras include some form of *parallel composition operator*, also known as *merge*.

Behavioural equivalences have then been introduced as simple and elegant tools for comparing the behaviour of processes. These are equivalence relations defined on the states of LTSs allowing one to establish whether two processes have the same *observable behaviour*. Different notions of observability correspond to different levels of abstraction from the information carried by the LTS, which can either be considered irrelevant in a given application context, or be unavailable to an external observer.

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In [16], van Glabbeek presented the *linear time-branching time spectrum*, namely a taxonomy of behavioural equivalences based on their distinguishing power. He carried out his study in the setting of the process algebra BCCSP, which consists of the basic operators from CCS [21] and CSP [19], and he proposed *ground-complete axiomatisations* for most of the congruences in the spectrum over this language. (An axiomatisation is ground-complete if it can prove all the valid equations relating terms that do not contain process variables.) The presented ground-complete axiomatisations are *finite* if so is the set of actions. For ready simulation, ready trace and failure trace equivalences, the axiomatisation in [16] made use of conditional equations. Blom, Fokkink and Nain gave purely equational, finite axiomatisations of those equivalences in [7]. Then, the works in [1], on nested semantics, and in [8], on impossible futures semantics, completed the studies of the axiomatisability of behavioural congruences over BCCSP by providing *negative* results: neither impossible futures nor any of the nested semantics have a finite, ground-complete axiomatisation over BCCSP.

Obtaining a complete axiomatisation of a behavioural congruence is a classic, key problem in concurrency theory, as it allows for characterising the semantics of a process algebra in a purely syntactic fashion. Hence, this characterisation becomes independent of the details of the definition of the process semantics of interest.

All the results mentioned so far were obtained over the algebra BCCSP that does not include any operator for the parallel composition of processes. Considering the crucial role of such an operator, it is natural to ask which of those results would still hold over a process algebra including it.

In the literature, we can find a wealth of studies on the axiomatisability of parallel composition modulo *bisimulation semantics* [25]. Briefly, in the seminal work [18], Hennessy and Milner proposed a ground-complete axiomatisation of (a part of) CCS modulo bisimilarity. That axiomatisation, however, included infinitely many axioms, which corresponded to instances of the *expansion law* used to express equationally the semantics of the merge operator. Then, Bergstra and Klop showed in [5] that a finite ground-complete axiomatisation modulo bisimilarity can be obtained by enriching CCS with two auxiliary operators, i.e., the *left merge* $\mathbin{\|\!\_}$ and the *communication merge* $|$. Later, Moller proved that the use of auxiliary operators is indeed necessary to obtain a finite equational axiomatisation of bisimilarity in [22–24].

To the best of our knowledge, no systematic study of the axiomatisability of the parallel composition operator modulo the other semantics in the spectrum has been presented so far.

Our contribution. We consider the process algebra $\mathrm{BCCSP}_{\|}$, namely BCCSP enriched with the interleaving parallel composition operator, and we study the existence of finite equational axiomatisations of the behavioural congruences in the linear time-branching time spectrum over it. Our results delineate the boundary between finite and non-finite axiomatisability of the congruences in the spectrum over the language $\mathrm{BCCSP}_{\|}$. (See Figure 1.)

We start by providing a *finite*, *ground-complete* axiomatisations for *ready simulation* semantics. The axiomatisation is obtained by extending the one for BCCSP with a few axioms expressing equationally the behaviour of interleaving modulo the considered congruence. The added axioms allow us to eliminate all occurrences of the interleaving operator from $\mathrm{BCCSP}_{\|}$ processes, thus reducing ground-completeness over $\mathrm{BCCSP}_{\|}$ to ground-completeness over BCCSP [7, 16]. Since the axioms for the elimination of parallel composition modulo ready simulation equivalence are of course sound with respect to the coarser equivalences, the reduction works for all behavioural equivalences below ready simulation equivalence. Nevertheless, we shall find more elegant ways to do the reduction for the coarser equivalences in the spectrum. We shall then observe a sort of parallelism between the axiomatisations for the notions of simulation and the corresponding decorated trace semantics: the axioms used to express equationally the interleaving operator in a decorated trace semantics can be seen as the *linear counterpart* of those used in the corresponding notion of simulation semantics. For instance, while the axioms for ready simulation impose constraints on the form of both arguments of the interleaving operator to trigger the reductions, those for ready trace equivalence impose similar constraints but only on one argument.

Then, we complete our journey in the spectrum by showing that *nested simulation* and *nested trace* semantics do not have a finite axiomatisation over BCCSP$_\|$. To this end, firstly we adapt Moller's arguments to the effect that bisimilarity is not finitely based over CCS to obtain the *negative result* for *possible futures equivalence*, also known as 2-*nested trace equivalence*. Informally, the negative result is obtained by providing an infinite family of equations that are all sound modulo possible futures equivalence but that cannot all be derived from any finite sound axiom system. Then, we exploit the soundness of the equations in the family modulo bisimilarity to extend the negative result to all the congruences that are finer than possible futures and coarser than bisimilarity, thus including all nested trace and nested simulation semantics.

Organisation of contents. After reviewing some basic notions on behavioural equivalences and equational logic in Section 2, we start our journey in the spectrum by providing a finite, ground-complete axiomatisation for ready simulation equivalence over BCCSP$_\|$ in Section 3. In Section 4 we discuss how it is possible to refine the axioms for ready simulation to obtain finite, ground-complete axiomatisations for completed simulation and simulation equivalences. Then, in Section 5 similar refinements are provided for the (decorate) trace equivalences, thus completing the presentation of our positive results. We end our journey in Section 6 with the presentation of the negative results, namely that the nested simulation and nested trace equivalences do not have a finite axiomatiosation over BCCSP$_\|$. Finally, in Section 7 we draw some conclusions and discuss avenues for future work.

2 Background

The language BCCSP$_\|$. The language BCCSP$_\|$ extends BCCSP with parallel composition. Formally, BCCSP$_\|$ consists of basic operators from CCS [21] and CSP [19], with the purely *interleaving* parallel composition operator $\|$, and is given by the following grammar:

$$t ::= \mathbf{0} \mid x \mid a.t \mid t+t \mid t \| t$$

where a ranges over a set of actions $\mathcal{A}$ and x ranges over a countably infinite set of variables $\mathcal{V}$. In what follows, we assume that the set of actions $\mathcal{A}$ is *finite*.

We shall use the meta-variables $t, u, \dots$ to range over BCCSP$_\|$ terms, and write $\mathrm{var}(t)$ for the collection of variables occurring in the term t. We also adopt the standard convention that prefixing binds strongest and $+$ binds weakest. Moreover, trailing $\mathbf{0}$'s will often be omitted from terms. We use a *summation* $\sum_{i\in\{1,\dots,k\}} t_i$ to denote the term $t = t_1 + \cdots + t_k$, where the empty sum represents $\mathbf{0}$. We can also assume that the terms t_i, for $i \in \{1,\dots,k\}$, do not have $+$ as head operator, and refer to them as the *summands* of t. The *size* of a term t, denoted by $\mathrm{size}(t)$, is the number of operator symbols in it.

A BCCSP$_\|$ term is *closed* if it does not contain any variables. We shall, sometimes, refer to closed terms simply as *processes*. We let $\mathcal{P}$ denote the set of BCCSP$_\|$ processes and let $p, q, \dots$ range over it. We use the *Structural Operational Semantics* (SOS) framework [26]

Table 1 Operational semantics of BCCSP$_\|$.

$$\frac{}{a.x \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'} \qquad \frac{x \xrightarrow{a} x'}{x \| y \xrightarrow{a} x' \| y} \qquad \frac{y \xrightarrow{a} y'}{x \| y \xrightarrow{a} x \| y'}$$

to equip processes with an operational semantics. A *literal* is an expression of the form $t \xrightarrow{a} t'$ for some process terms t, t' and action $a \in \mathcal{A}$. It is *closed* if both t, t' are closed terms. The inference rules for *prefixing* $a.\_$, *nondeterministic choice* $+$ and *interleaving parallel composition* $\|$ are reported in Table 1. A *substitution* σ is a mapping from variables to terms. It extends to terms, literals and rules in the usual way and it is *closed* if it maps every variable to a process.

The inference rules in Table 1 induce the $\mathcal{A}$-*labelled transition system* [20] $(\mathcal{P}, \mathcal{A}, \rightarrow)$ whose transition relation $\rightarrow \subseteq \mathcal{P} \times \mathcal{A} \times \mathcal{P}$ contains exactly the closed literals that can be derived using the rules in Table 1. As usual, we write $p \xrightarrow{a} p'$ in lieu of $(p, a, p') \in \rightarrow$. For each $p \in \mathcal{P}$ and $a \in \mathcal{A}$, we write $p \xrightarrow{a}$ if $p \xrightarrow{a} p'$ holds for some p', and $p \not\xrightarrow{a}$ otherwise. The *initials* of p are the actions that label the outgoing transitions of p, that is, $\mathtt{I}(p) = \{a \mid p \xrightarrow{a}\}$. For a sequence of actions $\alpha = a_1 \cdots a_k$ $(k \geq 0)$, and processes p, p', we write $p \xrightarrow{\alpha} p'$ if and only if there exists a sequence of transitions $p = p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} p_k = p'$. If $p \xrightarrow{\alpha} p'$ holds for some process p', then α is a *trace* of p, and p' is a *derivative* of p. Moreover, we say that α is a *completed trace* of p if $\mathtt{I}(p') = \emptyset$. We let $\mathtt{T}(p)$ denote the set of traces of p, and we let $\mathtt{CT}(p) \subseteq \mathtt{T}(p)$ denote the set of completed traces of p. We let ε denote the empty trace, and $|\alpha|$ denote the length of trace α. It is well known, and easy to show, that $\mathtt{T}(p)$ is finite for each BCCSP$_\|$ process p. It follows that we can define the *depth* of a process p, denoted by $\mathrm{depth}(p)$, as the length of a *longest* completed trace of p. Formally, $\mathrm{depth}(p) = \max\{|\alpha| \mid \alpha \in \mathtt{CT}(p)\}$. Similarly, the *norm* of a process p, denoted by $\mathrm{norm}(p)$, is the length of a *shortest* completed trace of p, i.e. $\mathrm{norm}(p) = \min\{|\alpha| \mid \alpha \in \mathtt{CT}(p)\}$.

Behavioural equivalences. *Behavioural equivalences* have been introduced to establish whether the behaviours of two processes are *indistinguishable for their observers*. Roughly, they allow us to check whether the *observable* semantics of two processes is *the same*. In the literature we can find several notions of behavioural equivalence based on the observations that an external observer can make on the process. In his seminal article [16], van Glabbeek gave a taxonomy of the behavioural equivalences discussed in the literature on concurrency theory, which is now called the *linear time-branching time spectrum* (see Figure 1).

One of the main concerns in the development of a meta-theory of process languages is to guarantee their *compositionality*, i.e., that the *replacement* of a component of a system with an $\mathcal{R}$-equivalent one, for a chosen behavioural equivalence $\mathcal{R}$, does not affect the behaviour of that system. In algebraic terms, this is known as the *congruence property* of $\mathcal{R}$ with respect to all language operators, which consists in verifying whether

$$f(t_1, \ldots, t_n) \,\mathcal{R}\, f(t_1', \ldots, t_n') \text{ for any } n\text{-ary operator } f \text{ whenever } t_i \,\mathcal{R}\, t_i' \text{ for all } i = 1, \ldots, n \ .$$

Since BCCSP$_\|$ operators are defined by inference rules in the de Simone format [12], by [14, Theorem 4] we have that all the equivalences in the spectrum in Figure 1 are congruences with respect to them. Our aim in this paper is to investigate the existence of a finite equational axiomatisation of BCCSP$_\|$ modulo all those congruences.

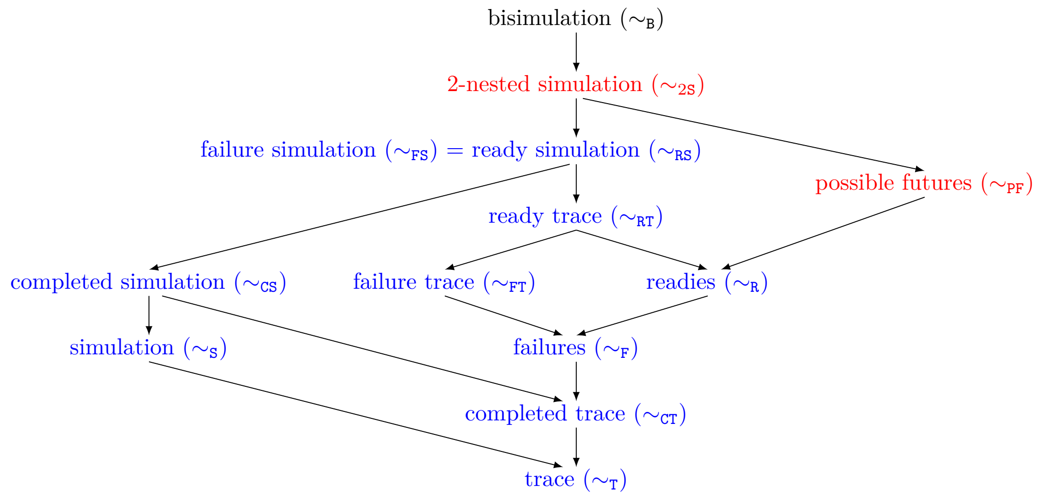

Figure 1 The linear time-branching time spectrum [16]. For the equivalence relations in blue we provide a finite, ground-complete axiomatization. For the ones in red, we provide a negative result. The case of bisimulation is known from the literature.

Table 2 The rules of equational logic.

$$(e_1)\ t \approx t \qquad (e_2)\ \frac{t \approx u}{u \approx t} \qquad (e_3)\ \frac{t \approx u \quad u \approx v}{t \approx v} \qquad (e_4)\ \frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$(e_5)\ \frac{t \approx u}{a.t \approx a.u} \qquad (e_6)\ \frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'} \qquad (e_8)\ \frac{t \approx u \quad t' \approx u'}{t \parallel t' \approx u \parallel u'}\ .$$

Equational Logic. An *axiom system* $\mathcal{E}$ is a collection of *equations* $t \approx u$ over $\text{BCCSP}_{\parallel}$. An equation $t \approx u$ is *derivable* from an axiom system $\mathcal{E}$, notation $\mathcal{E} \vdash t \approx u$, if there is an *equational proof* for it from $\mathcal{E}$, namely if $t \approx u$ can be inferred from the axioms in $\mathcal{E}$ using the *rules* of *equational logic*, which express reflexivity, symmetry, transitivity, substitution and closure under $\text{BCCSP}_{\parallel}$ contexts and are reported in Table 2.

We are interested in equations that are valid modulo some congruence relation $\mathcal{R}$ over closed terms. The equation $t \approx u$ is said to be *sound* modulo $\mathcal{R}$ if $\sigma(t)\ \mathcal{R}\ \sigma(u)$ for all closed substitutions σ. For simplicity, if $t \approx u$ is sound modulo $\mathcal{R}$, then we write $t\ \mathcal{R}\ u$. An axiom system is *sound* modulo $\mathcal{R}$ if, and only if, all of its equations are sound modulo $\mathcal{R}$. Conversely, we say that $\mathcal{E}$ is *ground-complete* modulo $\mathcal{R}$ if $p\ \mathcal{R}\ q$ implies $\mathcal{E} \vdash p \approx q$ for all closed terms p, q. We say that $\mathcal{R}$ has a *finite* ground-complete axiomatisation, if there is a *finite* axiom system $\mathcal{E}$ that is sound and ground-complete for $\mathcal{R}$.

In Table 3 we present some basic axioms for $\text{BCCSP}_{\parallel}$ that are sound with respect to all the behavioural equivalences in Figure 1. Henceforth, we will let $\mathcal{E}_0 = \{\text{A0}, \text{A1}, \text{A2}, \text{A3}\}$, and we will denote by $\mathcal{E}_1$ the axiom system consisting of all the axioms in Table 3, namely $\mathcal{E}_1 = \mathcal{E}_0 \cup \{\text{P0}, \text{P1}\}$.

To be able to eliminate the interleaving parallel composition operator from closed terms we will make use of two refinements EL1 and EL2 of EL3, which is the classic expansion law [18] (see Table 4). We remark that the actions occurring in the three axioms in Table 4 are not action variables. Hence, when we write that an axiom system $\mathcal{E}$ includes one of these axioms, we mean that it includes all possible instances of that axiom with respect to the

Table 3 Basic axioms for BCCSP$_\parallel$. We define $\mathcal{E}_0 = \{\mathrm{A0}, \mathrm{A1}, \mathrm{A2}, \mathrm{A3}\}$ and $\mathcal{E}_1 = \mathcal{E}_0 \cup \{\mathrm{P0}, \mathrm{P1}\}$.

(A0)	$x + \mathbf{0} \approx x$	(P0)	$x \parallel \mathbf{0} \approx x$
(A1)	$x + y \approx y + x$	(P1)	$x \parallel y \approx y \parallel x$
(A2)	$(x + y) + z \approx x + (y + z)$		
(A3)	$x + x \approx x$		

Table 4 The different instantiations of the expansion law.

(EL1)	$ax \parallel by \approx a(x \parallel by) + b(ax \parallel y)$
(EL2)	$\sum_{i\in I} a_i x_i \parallel \sum_{j\in J} b_j y_j \approx \sum_{i\in I} a_i(x_i \parallel \sum_{j\in J} b_j y_j) + \sum_{j\in J} b_j(\sum_{i\in I} a_i x_i \parallel y_j)$ with $a_i \neq a_k$ whenever $i \neq k$ and $b_j \neq b_h$ whenever $j \neq h$, $\forall\, i, k \in I, \forall\, j, h \in J$
(EL3)	$\sum_{i\in I} a_i x_i \parallel \sum_{j\in J} b_j y_j \approx \sum_{i\in I} a_i(x_i \parallel \sum_{j\in J} b_j y_j) + \sum_{j\in J} b_j(\sum_{i\in I} a_i x_i \parallel y_j)$

actions in $\mathcal{A}$. In particular, EL3 is a schema that generates infinitely many axioms, regardless of the cardinality of the set of actions. This is due to the fact that we can have arbitrary summations in the two arguments of the parallel composition in the left hand side of EL3. Conversely, when the set of actions is assumed to be finite, we are guaranteed that there are only finitely many instances of EL1 and EL2. Indeed, EL1 is a particular instance of EL2, i.e., the one in which both summations are over singletons. The reason for considering both is that, as we will see, EL1 is enough to obtain the elimination result when combined with axioms allowing us to reduce any process of the form $(\sum_{i\in I} a_i p_i) \parallel (\sum_{j\in J} b_j q_j)$ to $\sum_{i\in I, j\in J}(a_i p_i \parallel b_j q_j)$. Conversely, EL2 is needed when this reduction is not sound modulo the considered semantics.

3 The first stage: ready simulation

In this section we study the equational theory of *ready simulation*, whose formal definition is recalled below together with those of *completed simulation* and *simulation* equivalence.

▶ **Definition 1** (Simulation equivalences).

- *A* simulation *is a binary relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ such that, whenever $p \,\mathcal{R}\, q$ and $p \xrightarrow{a} p'$, then there is some q' such that $q \xrightarrow{a} q'$ and $p' \,\mathcal{R}\, q'$. We write $p \sqsubseteq_{\mathtt{S}} q$ if there is a simulation $\mathcal{R}$ such that $p \,\mathcal{R}\, q$. We say that p is* simulation equivalent *to q, notation $p \sim_{\mathtt{S}} q$, if $p \sqsubseteq_{\mathtt{S}} q$ and $q \sqsubseteq_{\mathtt{S}} p$.*
- *A* completed simulation *is a simulation $\mathcal{R}$ such that, whenever $p \,\mathcal{R}\, q$ and $\mathtt{I}(p) = \emptyset$, then $\mathtt{I}(q) = \emptyset$. We write $p \sqsubseteq_{\mathtt{CS}} q$ if there is a completed simulation $\mathcal{R}$ such that $p \,\mathcal{R}\, q$. We say that p is* completed simulation equivalent *to q, notation $p \sim_{\mathtt{CS}} q$, if $p \sqsubseteq_{\mathtt{CS}} q$ and $q \sqsubseteq_{\mathtt{CS}} p$.*
- *A* ready simulation *is a simulation $\mathcal{R}$ such that, whenever $p \,\mathcal{R}\, q$ then $\mathtt{I}(p) = \mathtt{I}(q)$. We write $p \sqsubseteq_{\mathtt{RS}} q$ if there is a ready simulation $\mathcal{R}$ such that $p \,\mathcal{R}\, q$. We say that p is* ready simulation equivalent *to q, notation $p \sim_{\mathtt{RS}} q$, if $p \sqsubseteq_{\mathtt{RS}} q$ and $q \sqsubseteq_{\mathtt{RS}} p$.*

In [15] the notion of *failure simulation* was also introduced as a simulation $\mathcal{R}$ such that, whenever $p \,\mathcal{R}\, q$ and $\mathtt{I}(p) \cap X = \emptyset$, for some $X \subseteq \mathcal{A}$, then $\mathtt{I}(q) \cap X = \emptyset$. Then, in [14] it was proved that the notion of failure simulation coincides with that of ready simulation.

Our aim is to provide a *finite*, *ground-complete* axiomatisation of BCCSP$_\parallel$ modulo ready simulation equivalence. To this end, we recall that in [16] it was proved that the axiom system consisting of $\mathcal{E}_0$ together with axiom RS in Table 5 is a ground-complete axiomatisation of BCCSP, namely BCCSP$_\parallel$ without any occurrence of $\parallel$, modulo ready simulation equivalence.

Table 5 Additional axioms for (ready, completed) simulation equivalence.

(RS)	$a(bx+by+z) \approx a(bx+by+z)+a(bx+z)$
(RSP1)	$(ax+ay+u) \parallel (bz+bw+v) \approx (ax+u) \parallel (bz+bw+v)+(ay+u) \parallel (bz+bw+v)+$ $+(ax+ay+u) \parallel (bz+v)+(ax+ay+u) \parallel (bw+v)$
(RSP2)	$\left(\sum_{i\in I} a_i x_i\right) \parallel (by+bz+w) \approx \sum_{i\in I} a_i\,(x_i \parallel (by+bz+w))+$ $+\left(\sum_{i\in I} a_i x_i\right) \parallel (by+w)+\left(\sum_{i\in I} a_i x_i\right) \parallel (bz+w)$ where $a_j \neq a_k$ whenever $j \neq k$ for $j,k \in I$
$\mathcal{E}_{\mathtt{RS}} = \mathcal{E}_1 \cup \{\text{RS,RSP1,RSP2,EL2}\}$	
(CS)	$a(bx+y+z) \approx a(bx+y+z)+a(bx+z)$
(CSP1)	$(ax+by+u) \parallel (cz+dw+v) \approx (ax+u) \parallel (cz+dw+v)+(by+u) \parallel (cz+dw+v)+$ $+(ax+by+u) \parallel (cz+v)+(ax+by+u) \parallel (dw+v)$
(CSP2)	$ax \parallel (by+cz+w) \approx a(x \parallel (by+cz+w))+ax \parallel (by+w)+ax \parallel (cz+w)$
$\mathcal{E}_{\mathtt{CS}} = \mathcal{E}_1 \cup \{\text{CS,CSP1,CSP2,EL1}\}$	
(S)	$a(x+y) \approx a(x+y)+ax$
(SP1)	$(x+y) \parallel (z+w) \approx x \parallel (z+w)+y \parallel (z+w)+(x+y) \parallel z+(x+y) \parallel w$
(SP2)	$ax \parallel (y+z) \approx a(x \parallel (y+z))+ax \parallel y+ax \parallel z$
$\mathcal{E}_{\mathtt{S}} = \mathcal{E}_1 \cup \{\text{S,SP1,SP2,EL1}\}$	

Hence, to obtain a finite, ground-complete axiomatisation of BCCSP$_\parallel$ modulo $\sim_{\mathtt{RS}}$ it suffices to enrich the axiom system $\mathcal{E}_1 \cup \{\mathrm{RS}\}$ with finitely many axioms allowing one to eliminate all occurrences of $\parallel$ from closed BCCSP$_\parallel$ terms. In fact, by letting $\mathcal{E}_{\mathtt{RS}}$ denote the axiom system $\mathcal{E}_1 \cup \{\mathrm{RS}\}$ enriched with the necessary axioms, the elimination result consists in proving that for every closed BCCSP$_\parallel$ term p there is a closed BCCSP term q (i.e., without any occurrence of $\parallel$ in it) such that $\mathcal{E}_{\mathtt{RS}} \vdash p \approx q$. Therefore, the completeness of the proposed axiom system over BCCSP$_\parallel$ is a direct consequence of that over BCCSP proved in [16].

Clearly, EL3 would allow us to obtain the desired elimination, but, as previously outlined, it is a schema that finitely presents as infinite collection of equations, and thus an axiom system including it is not finite. In order to obtain the elimination result using only finitely many axioms we will characterise the distributivity properties of $\parallel$ over $+$ modulo ready simulation equivalence. This is done by axioms RSP1 and RSP2 in Table 5.

First of all, we notice that the axiom system $\mathcal{E}_{\mathtt{RS}} = \mathcal{E}_1 \cup \{\mathrm{RS}, \mathrm{RSP1}, \mathrm{RSP2}, \mathrm{EL2}\}$ is sound modulo ready simulation equivalence.

▶ **Theorem 2** ($\mathcal{E}_{\mathtt{RS}}$ soundness). *The axiom system $\mathcal{E}_{\mathtt{RS}}$ is sound for* BCCSP$_\parallel$ *modulo ready simulation equivalence, namely whenever $\mathcal{E}_{\mathtt{RS}} \vdash p \approx q$ then $p \sim_{\mathtt{RS}} q$.*

Let us focus now on ground-completeness. Intuitively, RSP1 and RSP2 have been constructed in such a way that the set of initial actions of the two arguments of $\parallel$ is preserved, while the initial term is reduced to a sum of terms of smaller size. Briefly, according to the main features of ready simulation semantics, axiom RSP1 allows us to distribute $\parallel$ over $+$ when both arguments of $\parallel$ have nondeterministic choices among summands having the same initial action. Conversely, axiom RSP2 deals with the case in which only one argument of $\parallel$ has summands with the same initial action. In order to preserve the branching structure of the process, which is fundamental to guarantee the soundness of the axioms modulo $\sim_{\mathtt{RS}}$, both RSP1 and RSP2 take into account the behaviour of both arguments of $\|$: the terms in the right-hand side of both axioms are such that whenever the initial nondeterministic choice of one argument of $\|$ is resolved, the entire behaviour of the other argument is preserved. In fact, we stress that a simplified version of, e.g., RSP1 in which only one argument of $\|$ distributes over $+$ would not be sound modulo $\sim_{\mathtt{RS}}$. Consider, for instance, the process $p = (ap_1 + ap_2 + b) \| c$, with $p_1 \not\sim_{\mathtt{RS}} p_2$. It is immediate to verify that $p \not\sim_{\mathtt{RS}} (ap_1 + b) \| c + (ap_2 + b) \| c$.

The idea is that by (repeatedly) applying axioms RSP1 and RSP2, from left to right, we are able to reduce a process of the form $(\sum_{i\in I} p_i) \| (\sum_{j\in J} p_j)$ to a process of the form $\sum_{k\in K} p_k$ such that whenever p_k has $\|$ as head operator then $p_k = \sum_{h\in H} a_h p_h \| \sum_{l\in L} b_l p_l$, with $a_h \neq a_{h'}$ for $h \neq h'$, and $b_l \neq b_{l'}$ for $l \neq l'$, for some closed $\mathrm{BCCSP}_{\|}$ terms p_h, p_l. The elimination of $\|$ from these terms can then proceed by means of the finitary refinement EL2 of the expansion law presented in Table 4. In particular, we notice that RSP2 is needed because RSP1 alone does not allow us to reduce all processes of the form $(\sum_{i\in I} p_i) \| (\sum_{j\in J} p_j)$ into a sum of processes to which EL2 can be applied. This is mainly due to the fact that, in order to be sound modulo $\sim_{\mathtt{RS}}$, RSP1 imposes constraints on the form of both arguments of a process $(\sum_{i\in I} p_i) \| (\sum_{j\in J} p_j)$.

We can then proceed to prove the elimination result.

▶ **Proposition 3** ($\mathcal{E}_{\mathtt{RS}}$ elimination). *For every closed* $\mathrm{BCCSP}_{\|}$ *term p there exists a* BCCSP *term q such that $\mathcal{E}_{\mathtt{RS}} \vdash p \approx q$.*

The ground-completeness of $\mathcal{E}_{\mathtt{RS}}$ then follows from the ground-completeness of $\mathcal{E}_0 \cup \{\mathrm{RS}\}$ over BCCSP [16].

▶ **Theorem 4** ($\mathcal{E}_{\mathtt{RS}}$ completeness). *The axiom system $\mathcal{E}_{\mathtt{RS}}$ is a ground-complete axiomatisation of* $\mathrm{BCCSP}_{\|}$ *modulo ready simulation equivalence, i.e., whenever $p \sim_{\mathtt{RS}} q$ then $\mathcal{E}_{\mathtt{RS}} \vdash p \approx q$.*

We remark that since axioms RSP1, RSP2, and EL2 are sound modulo ready simulation equivalence, they are automatically sound modulo all the equivalences in the spectrum that are coarser than $\sim_{\mathtt{RS}}$, namely the completed simulation, simulation, and (decorated) trace equivalences. Hence, we can easily obtain finite, ground-complete axiomatisations of $\mathrm{BCCSP}_{\|}$ modulo each of those equivalences by adding RSP1, RSP2 and EL2 to the respective ground-complete axiomatisations of BCCSP that have been proposed in the literature [7, 16]. However, for each of those equivalences we can provide stronger axioms that give a more elegant characterisation of the distributivity properties of $\|$ over $+$. In particular, the axiom schemata RSP2 and EL2 both generate $2^{|\mathcal{A}|}$ equational axioms. By exploiting the various forms of distributivity of parallel composition over choice, we can obtain more concise ground-complete axiomatisations of $\mathrm{BCCSP}_{\|}$ modulo the coarser equivalences. We dedicate the next two sections to the presentation of these results.

4 Completed simulation and simulation

In this section we refine the axiom system $\mathcal{E}_{\mathtt{RS}}$ to obtain finite, ground-complete axiomatisations of $\mathrm{BCCSP}_{\|}$ modulo completed simulation and simulation equivalences. To this end, we replace RSP1 and RSP2 with new axioms, tailored for the considered semantics, that allow us to obtain the elimination of $\|$ from closed $\mathrm{BCCSP}_{\|}$ terms, while imposing less restrictive constraints on the distributivity of $\|$ over $+$.

Let us focus first on completed simulation equivalence. We can use axioms CSP1 and CSP2 in Table 5 to characterise the distributivity of $\|$ over $+$ modulo $\sim_{\mathtt{CS}}$. Intuitively, CSP1 is the *completed simulation counterpart* of RSP1, and CSP2 is that of RSP2. Notice that both

CSP1 and CSP2 are such that when distributing $\parallel$ over $+$ we never get $\mathbf{0}$ as an argument of $\parallel$, thus guaranteeing the soundness of the reduction modulo $\sim_{\mathtt{CS}}$. Moreover, we stress that CSP1 and CSP2 are not sound modulo ready simulation equivalence. This is due to the fact that both axioms allow for distributing $\parallel$ over $+$ regardless of the initial actions of the summands. It is then immediate to check that, for instance, $a \parallel (b + c) \not\sim_{\mathtt{RS}} a \parallel b + a \parallel c + a \parallel (b + c)$, whereas $a \parallel (b + c) \sim_{\mathtt{CS}} a \parallel b + a \parallel c + a \parallel (b + c)$. Interestingly, due to the relaxed constraints on distributivity, by (repeatedly) applying CSP1 and CSP2, from left to right, we are able to reduce a $\mathrm{BCCSP}_{\parallel}$ process of the form $(\sum_{i\in I} p_i) \parallel (\sum_{j\in J} p_j)$ to a $\mathrm{BCCSP}_{\parallel}$ process of the form $\sum_{k\in K} p_k$ such that whenever p_k has $\parallel$ as head operator then $p_k = a_k q_k \parallel b_k q'_k$ for some q_k, q'_k. We can then use the refinement EL1 of the expansion law to proceed with the elimination of $\parallel$ from these terms.

Consider the axiom system $\mathcal{E}_{\mathtt{CS}} = \mathcal{E}_1 \cup \{\mathrm{CS},\mathrm{CSP1},\mathrm{CSP2},\mathrm{EL1}\}$. We can formalise the elimination result for $\sim_{\mathtt{CS}}$ in the following proposition.

▶ **Proposition 5** ($\mathcal{E}_{\mathtt{CS}}$ elimination)**.** *For every closed* $\mathrm{BCCSP}_{\parallel}$ *term p there exists a* BCCSP *term q such that $\mathcal{E}_{\mathtt{CS}} \vdash p \approx q$.*

A similar reasoning could be applied to obtain the elimination result for simulation equivalence. Although this result could be directly derived by the soundness of CSP1 and CSP2 modulo simulation equivalence, we can provide stronger axioms for the distributivity of $\parallel$ over summation modulo $\sim_{\mathtt{S}}$. Hence, we replace CSP1 and CSP2 by axioms SP1 and SP2 in Table 5 and we combine them with EL1 to eliminate all occurrences of $\parallel$ from the closed $\mathrm{BCCSP}_{\parallel}$ terms. However, it is also possible to obtain the elimination result for simulation equivalence as a corollary of that for completed simulation. Consider the axiom system $\mathcal{E}_{\mathtt{S}} = \mathcal{E}_1 \cup \{\mathrm{S},\mathrm{SP1},\mathrm{SP2},\mathrm{EL1}\}$. We can show that the axioms in $\mathcal{E}_{\mathtt{CS}}$ are all provable from the axiom system $\mathcal{E}_{\mathtt{S}}$.

▶ **Lemma 6.** *The axioms of the system $\mathcal{E}_{\mathtt{CS}}$ are derivable from the axiom system $\mathcal{E}_{\mathtt{S}}$, namely:*

1. $\mathcal{E}_{\mathtt{S}} \vdash \mathrm{CS}$*,*
2. $\mathcal{E}_{\mathtt{S}} \vdash \mathrm{CSP1}$*, and*
3. $\mathcal{E}_{\mathtt{S}} \vdash \mathrm{CSP2}$*.*

▶ **Proposition 7** ($\mathcal{E}_{\mathtt{S}}$ elimination)**.** *For every closed* $\mathrm{BCCSP}_{\parallel}$ *term p there exists a closed* BCCSP *term q such that $\mathcal{E}_{\mathtt{S}} \vdash p \approx q$.*

▶ Remark 8. A natural question that may arise is whether a similar derivation is possible for $\mathcal{E}_{\mathtt{RS}}$ from $\mathcal{E}_{\mathtt{CS}}$. We conjecture that the answer is negative. In particular, axiom RSP2 cannot be derived from the axioms in $\mathcal{E}_{\mathtt{CS}}$.

In light of the results above, and those in [16] showing that $\mathcal{E}_0 \cup \{\mathrm{CS}\}$ and $\mathcal{E}_0 \cup \{\mathrm{S}\}$ are sound and ground-complete axiomatisations of BCCSP modulo $\sim_{\mathtt{CS}}$ and $\sim_{\mathtt{S}}$, respectively, we can infer that $\mathcal{E}_{\mathtt{CS}}$ and $\mathcal{E}_{\mathtt{S}}$ are ground-complete axiomatisations of $\mathrm{BCCSP}_{\parallel}$ modulo completed simulation equivalence and simulation equivalence, respectively.

▶ **Theorem 9** (Soundness and completeness of $\mathcal{E}_{\mathtt{CS}}$ and $\mathcal{E}_{\mathtt{S}}$)**.** *Let $\mathtt{X} \in \{\mathtt{CS},\mathtt{S}\}$. The axiom system $\mathcal{E}_{\mathtt{X}}$ is a sound, ground-complete axiomatisation of* $\mathrm{BCCSP}_{\parallel}$ *modulo $\sim_{\mathtt{X}}$, i.e., $p \sim_{\mathtt{X}} q$ if and only if $\mathcal{E}_{\mathtt{X}} \vdash p \approx q$.*

Table 6 Additional axioms for trace and decorated trace equivalences.

(RT) $a\left(\sum_{i=1}^{\lvert\mathcal{A}\rvert}(b_i x_i + b_i y_i) + z\right) \approx a\left(\sum_{i=1}^{\lvert\mathcal{A}\rvert} b_i x_i + z\right) + a\left(\sum_{i=1}^{\lvert\mathcal{A}\rvert} b_i y_i + z\right)$
(FP) $(ax + ay + w) \parallel z \approx (ax + w) \parallel z + (ay + w) \parallel z$
$\mathcal{E}_{\mathtt{RT}} = \mathcal{E}_1 \cup \{\text{RT,FP,EL2}\}$
(FT) $ax + ay \approx ax + ay + a(x + y)$
$\mathcal{E}_{\mathtt{FT}} = \mathcal{E}_1 \cup \{\text{FT,RS,FP,EL2}\}$
(R) $a(bx + z) + a(by + w) \approx a(bx + by + z) + a(by + w)$
$\mathcal{E}_{\mathtt{R}} = \mathcal{E}_1 \cup \{\text{R,FP,EL2}\}$
(F) $ax + a(y + z) \approx ax + a(x + y) + a(y + z)$
$\mathcal{E}_{\mathtt{F}} = \mathcal{E}_1 \cup \{\text{F,R,FP,EL2}\}$
(CT) $a(bx + z) + a(cy + w) \approx a(bx + cy + z + w)$
(CTP) $(ax + by + w) \parallel z \approx (ax + w) \parallel z + (by + w) \parallel z$
$\mathcal{E}_{\mathtt{CT}} = \mathcal{E}_1 \cup \{\text{CT,CTP,EL1}\}$
(T) $ax + ay \approx a(x + y)$
(TP) $(x + y) \parallel z \approx x \parallel z + y \parallel z$
$\mathcal{E}_{\mathtt{T}} = \mathcal{E}_1 \cup \{\text{T,TP,EL1}\}$

5 Linear semantics: from ready traces to traces

We continue our journey in the spectrum by moving to the linear-time semantics. In this section we consider trace semantics and all of its decorated versions, and we provide a finite, ground-complete axiomatisation for each of them (see Table 6).

From a technical point of view, we can split the results of this section into two parts:

1. those for ready trace, failure trace, ready, and failures equivalence, and
2. those for completed trace, and trace equivalence.

In both parts we prove the elimination result only for the finest semantics, namely ready trace (Proposition 11) and completed trace (Proposition 17) respectively. We then obtain the remaining elimination results by showing that all the axioms in $\mathcal{E}_{\mathtt{X}}$ are provable from $\mathcal{E}_{\mathtt{Y}}$, where $\mathtt{X}$ is finer than $\mathtt{Y}$ in the considered part.

5.1 From ready traces to failures

First we deal with the decorated trace semantics based on the comparison of the failure and ready sets of processes.

▶ **Definition 10** (Readiness and failures equivalences).

- *A* failure pair *of a process p is a pair (α, X), with $\alpha \in \mathcal{A}^*$ and $X \subseteq \mathcal{A}$, such that $p \xrightarrow{\alpha} q$ for some process q with $\mathtt{I}(q) \cap X = \emptyset$. We denote by $\mathtt{F}(p)$ the set of failure pairs of p. Two processes p and q are* failures equivalent*, denoted $p \sim_{\mathtt{F}} q$, if $\mathtt{F}(p) = \mathtt{F}(q)$.*
- *A* ready pair *of a process p is a pair (α, X), with $\alpha \in \mathcal{A}^*$ and $X \subseteq \mathcal{A}$, such that $p \xrightarrow{\alpha} q$ for some process q with $\mathtt{I}(q) = X$. We let $\mathtt{R}(p)$ denote the set of ready pairs of p. Two processes p and q are* ready equivalent*, written $p \sim_{\mathtt{R}} q$, if $\mathtt{R}(p) = \mathtt{R}(q)$.*

- *A* failure trace *of a process* p *is a sequence* $X_0a_1X_1 \dots a_nX_n$*, with* $X_i \subseteq \mathcal{A}$ *and* $a_i \in \mathcal{A}$*, such that there are* $p_1, \dots, p_n \in \mathcal{P}$ *with* $p = p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} p_n$ *and* $\mathtt{I}(p_i) \cap X_i = \emptyset$ *for all* $0 \leq i \leq n$*. We write* $\mathtt{FT}(p)$ *for the set of failure traces of* p*. Two processes* p *and* q *are* failure trace equivalent*, denoted* $p \sim_{\mathtt{FT}} q$*, if* $\mathtt{FT}(p) = \mathtt{FT}(q)$*.*
- *A* ready trace *of a process* p *is a sequence* $X_0a_1X_1 \dots a_nX_n$*, for* $X_i \subseteq \mathcal{A}$ *and* $a_i \in \mathcal{A}$*, such that there are* $p_1, \dots p_n \in \mathcal{P}$ *with* $p = p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} p_n$ *and* $\mathtt{I}(p_i) = X_i$ *for all* $0 \leq i \leq n$*. We write* $\mathtt{RT}(p)$ *for the set of ready traces of* p*. Two processes* p *and* q *are* ready trace equivalent*, denoted* $p \sim_{\mathtt{RT}} q$*, if* $\mathtt{RT}(p) = \mathtt{RT}(q)$*.*

We consider first the finest equivalence among those in Definition 10, namely ready trace equivalence. This can be considered as the linear counterpart of ready simulation: we focus on the current execution of the process and we require that each step is mimicked by reaching processes having the same sets of initial actions. Interestingly, we can find a similar correlation between the axioms characterising the distributivity of $\|$ over $+$ modulo the two semantics. Consider axiom FP in Table 6. We can see this axiom as the *linear* counterpart of RSP1: since in the linear semantics we are interested only in the current execution of a process, we can characterise the distributivity of $\|$ over $+$ by treating the two arguments of $\|$ independently from one another. To obtain the elimination result for $\sim_{\mathtt{RT}}$ we do not need to introduce the linear counterpart of axiom RSP2. In fact, FP imposes constraints on the form of only one argument of $\|$. Hence, it is possible to use it to reduce any process of the form $(\sum_{i \in I} p_i) \| (\sum_{j \in J} p_j)$ into a sum of processes to which EL2 can be applied. We can in fact prove that the axioms in the system $\mathcal{E}_{\mathtt{RT}} = \mathcal{E}_1 \cup \{\mathrm{RT}, \mathrm{FP}, \mathrm{EL2}\}$ are sufficient to eliminate all occurrences of $\|$ from closed BCCSP$_{\|}$ terms.

▶ **Proposition 11** ($\mathcal{E}_{\mathtt{RT}}$ elimination)**.** *For every closed* BCCSP$_{\|}$ *term* p *there is a closed* BCCSP *term* q *such that* $\mathcal{E}_{\mathtt{RT}} \vdash p \approx q$*.*

▶ Remark 12. Similarly to the case of completed simulation (cf. Remark 8), the reason why we propose to prove directly the elimination result for ready trace equivalence is that we did not manage to derive the axioms in $\mathcal{E}_{\mathtt{RS}}$ from those in $\mathcal{E}_{\mathtt{RT}}$. Once again, the main issue is that axiom RSP2 cannot be derived from those in $\mathcal{E}_{\mathtt{RT}}$, even though all its closed instantiations can. We leave a formal analysis of this issue as future work.

Interestingly, axiom FP also characterises the distributivity of $\|$ over $+$ modulo $\sim_{\mathtt{FT}}, \sim_{\mathtt{R}}$ and $\sim_{\mathtt{F}}$, in the sense that the constraints that it imposes on the form of the arguments of $\|$ to trigger the reduction cannot be relaxed when considering the above-mentioned coarser semantics. Consider the axiom systems $\mathcal{E}_{\mathtt{FT}} = \mathcal{E}_1 \cup \{\mathrm{FT,RS,FP,EL2}\}$, $\mathcal{E}_{\mathtt{R}} = \mathcal{E}_1 \cup \{\mathrm{R,FP,EL2}\}$ and $\mathcal{E}_{\mathtt{F}} = \mathcal{E}_1 \cup \{\mathrm{F,R,FP,EL2}\}$. The following derivability relations among them and $\mathcal{E}_{\mathtt{RT}}$ are then easy to check.

▶ **Lemma 13.**

1. *The axioms in the system* $\mathcal{E}_{\mathtt{RT}}$ *are derivable from* $\mathcal{E}_{\mathtt{FT}}$*, namely* $\mathcal{E}_{\mathtt{FT}} \vdash \mathrm{RT}$*.*
2. *The axioms in the system* $\mathcal{E}_{\mathtt{RT}}$ *are derivable from* $\mathcal{E}_{\mathtt{R}}$*, namely* $\mathcal{E}_{\mathtt{R}} \vdash \mathrm{RT}$*.*
3. *The axioms in the system* $\mathcal{E}_{\mathtt{FT}}$ *are derivable from* $\mathcal{E}_{\mathtt{F}}$*, namely,*
   a. $\mathcal{E}_{\mathtt{F}} \vdash \mathrm{FT}$*, and*
   b. $\mathcal{E}_{\mathtt{F}} \vdash \mathrm{RS}$*.*

   Moreover, also the axioms in the system $E_{\mathtt{R}}$ *are derivable from* $\mathcal{E}_{\mathtt{F}}$*.*

The next proposition is then a corollary of Proposition 11 and Lemma 13.

▶ **Proposition 14** ($\mathcal{E}_{\mathtt{FT}}$, $\mathcal{E}_{\mathtt{R}}$, $\mathcal{E}_{\mathtt{F}}$ elimination)**.** *Let* $\mathtt{X} \in \{\mathtt{FT}, \mathtt{R}, \mathtt{F}\}$*. For every* BCCSP$_{\|}$ *term* p *there is a closed* BCCSP *term* q *such that* $\mathcal{E}_{\mathtt{X}} \vdash p \approx q$*.*

In [7] it was proved that, under the assumption that $\mathcal{A}$ is finite, the axiom system $\mathcal{E}_0 \cup \{\text{RT}\}$ is a ground-complete axiomatisation of BCCSP modulo $\sim_{\mathtt{RT}}$. Moreover, it was also proved that $\mathcal{E}_0 \cup \{\text{FT,RS}\}$ is a ground-complete axiomatisation of BCCSP modulo $\sim_{\mathtt{FT}}$. The ground-completeness of $\mathcal{E}_0 \cup \{\text{R}\}$, modulo $\sim_{\mathtt{R}}$, and that of $\mathcal{E}_0 \cup \{\text{F}, \text{R}\}$, modulo $\sim_{\mathtt{F}}$, over BCCSP were proved in [16]. Consequently, the soundness and ground-completeness of the proposed axioms systems can then be derived from the elimination results above and the completeness results given in [7, 16].

▶ **Theorem 15** (Soundness and completeness of $\mathcal{E}_{\mathtt{RT}}$, $\mathcal{E}_{\mathtt{FT}}$, $\mathcal{E}_{\mathtt{R}}$ and $\mathcal{E}_{\mathtt{F}}$)**.** *Let* $\mathtt{X} \in \{\mathtt{RT}, \mathtt{FT}, \mathtt{R}, \mathtt{F}\}$*. The axiom system* $\mathcal{E}_{\mathtt{X}}$ *is a sound, ground-complete axiomatisation of* $\text{BCCSP}_{\|}$ *modulo* $\sim_{\mathtt{X}}$*, i.e.,* $p \sim_{\mathtt{X}} q$ *if and only if* $\mathcal{E}_{\mathtt{X}} \vdash p \approx q$*.*

5.2 Completed traces and traces

It remains to consider completed trace equivalence and trace equivalence.

▶ **Definition 16** (Trace and completed trace equivalences)**.** *Two processes p and q are* trace equivalent*, denoted* $p \sim_{\mathtt{T}} q$*, if* $\mathtt{T}(p) = \mathtt{T}(q)$*. If, in addition, it holds that* $\mathtt{CT}(p) = \mathtt{CT}(q)$*, then p and q are* completed trace equivalent*, denoted* $p \sim_{\mathtt{CT}} q$*.*

Consider the axiom systems $\mathcal{E}_{\mathtt{CT}} = \mathcal{E}_1 \cup \{\text{CT,CTP,EL1}\}$ and $\mathcal{E}_{\mathtt{T}} = \mathcal{E}_1 \cup \{\text{T,TP,EL1}\}$, presented in Table 6. In the same way that axiom FP is the linear counterpart of RSP1 and RSP2, we have that CTP is the linear counterpart of CSP1 and CSP2, and TP is that of SP1 and SP2. It is then easy to check that we can use the axioms in $\mathcal{E}_{\mathtt{CT}}$ to obtain the elimination result for $\sim_{\mathtt{CT}}$.

▶ **Proposition 17** ($\mathcal{E}_{\mathtt{CT}}$ elimination)**.** *For every closed* $\text{BCCSP}_{\|}$ *term p there is a closed BCCSP term q such that* $\mathcal{E}_{\mathtt{CT}} \vdash p \approx q$*.*

Moreover, the elimination for $\sim_{\mathtt{T}}$ follows from the fact that the axioms in $\mathcal{E}_{\mathtt{CT}}$ are derivable from those in $\mathcal{E}_{\mathtt{T}}$.

▶ **Lemma 18.** *The axioms in the system* $\mathcal{E}_{\mathtt{CT}}$ *are derivable from* $\mathcal{E}_{\mathtt{T}}$*, namely,*

1. $\mathcal{E}_{\mathtt{T}} \vdash \text{CT}$*, and*
2. $\mathcal{E}_{\mathtt{T}} \vdash \text{CTP}$*.*

▶ **Proposition 19** ($\mathcal{E}_{\mathtt{T}}$ elimination)**.** *For every closed* $\text{BCCSP}_{\|}$ *term p there exists a closed* BCCSP *term q such that* $\mathcal{E}_{\mathtt{T}} \vdash p \approx q$*.*

▶ Remark 20. The precise relationship between $\mathcal{E}_{\mathtt{CT}}$ on the one hand, and $\mathcal{E}_{\mathtt{RT}}$ and $\mathcal{E}_{\mathtt{CS}}$ on the other hand still needs to be investigated further. We conjecture that the axioms of $\mathcal{E}_{\mathtt{RT}}$ are derivable from $\mathcal{E}_{\mathtt{CT}}$ and that those of $\mathcal{E}_{\mathtt{CS}}$ are not.

In light of Proposition 17, the ground-completeness of $\mathcal{E}_{\mathtt{CT}}$ over $\text{BCCSP}_{\|}$ modulo $\sim_{\mathtt{CT}}$ follows from that of $\mathcal{E}_0 \cup \{\text{CT}\}$ over BCCSP provided in [16]. Similarly, the ground-completeness of $\mathcal{E}_0 \cup \{\text{T}\}$ over BCCSP proved in [16] and Proposition 19 give us the ground-completeness of $\mathcal{E}_{\mathtt{T}}$ over $\text{BCCSP}_{\|}$.

▶ **Theorem 21** (Soundness and completeness of $\mathcal{E}_{\mathtt{CT}}$ and $\mathcal{E}_{\mathtt{T}}$)**.** *Let* $\mathtt{X} \in \{\mathtt{CT}, \mathtt{T}\}$*. The axiom system* $\mathcal{E}_{\mathtt{X}}$ *is a ground-complete axiomatisation of* $\text{BCCSP}_{\|}$ *modulo* $\sim_{\mathtt{X}}$*, i.e.,* $p \sim_{\mathtt{X}} q$ *if and only if* $\mathcal{E}_{\mathtt{X}} \vdash p \approx q$*.*

6 The negative results

We dedicate this section to the negative results: we prove that all the congruences between possible futures equivalence ($\sim_{\mathtt{PF}}$) and bisimilarity ($\sim_{\mathtt{B}}$) do not admit a finite, ground-complete axiomatisation over BCCSP$_{\|}$. This includes all the nested trace and nested simulation equivalences. In [1] it was shown that, even if the set of actions is a singleton, the nested semantics admit no finite axiomatisation over BCCSP. Indeed, the presence of the additional operator $\|$ might, at least in principle, allow us to finitely axiomatise the equations over closed BCCSP terms that are valid modulo the considered equivalences. Hence, we prove these results directly.

In detail, firstly we focus on the negative result for possible futures semantics, corresponding to the 2-nested trace semantics [18]. To obtain it, we apply the general technique used by Moller to prove that interleaving is not finitely axiomatisable modulo bisimilarity [22–24]. Briefly, the main idea is to identify a *witness property*. This is a specific property of BCCSP$_{\|}$ terms, say W_N for $N \geq 0$, that, when N is *large enough*, is an invariant that is preserved by provability from finite, sound axiom systems. Roughly, this means that if $\mathcal{E}$ is a finite set of axioms that are sound modulo possible futures equivalence, the equation $p \approx q$ can be derived from $\mathcal{E}$, and N is larger than the size of all the terms in the equations in $\mathcal{E}$, then either both p and q satisfy W_N, or none of them does. Then, we exhibit an infinite family of valid equations, called the *witness family of equations*, in which W_N is not preserved, namely it is satisfied only by one side of each equation.

Afterwards, we exploit the soundness modulo bisimilarity of the equations in the witness family to lift the negative result for $\sim_{\mathtt{PF}}$ to all congruences between $\sim_{\mathtt{B}}$ and $\sim_{\mathtt{PF}}$.

Differently from the aforementioned negative results over BCCSP, ours are obtained assuming that the set of actions contains at least two distinct elements. In fact, when the action set is a singleton, and *only* in that case, the axiom

$$ax \| (ay + az) \approx ax \| (ay + a(y + z)) + ax \| (az + a(y + z))$$

is sound modulo $\sim_{\mathtt{PF}}$. Due to this axiom we were not able to prove the negative result for $\sim_{\mathtt{PF}}$ in the case that $|\mathcal{A}| = 1$, which we leave as an open problem for future work.

6.1 Possible futures equivalence

According to possible futures equivalence [27] two processes are deemed equivalent if, by performing the same traces, they reach processes that are trace equivalent. For this reason, possible futures equivalence is also known as the 2-*nested trace equivalence* [18].

▶ **Definition 22** (Possible futures equivalence). *A* possible future *of a process p is a pair (α, X) where $\alpha \in \mathcal{A}^*$ and $X \subseteq \mathcal{A}^*$ such that $p \xrightarrow{\alpha} p'$ for some p' with $X = \mathtt{T}(p')$. We write $\mathtt{PF}(p)$ for the set of possible futures of p. Two processes p and q are said to be* possible futures equivalent*, denoted $p \sim_{\mathtt{PF}} q$, if $\mathtt{PF}(p) = \mathtt{PF}(q)$.*

Our order of business is to prove the following result.

▶ **Theorem 23.** *Assume that $|\mathcal{A}| \geq 2$. Possible futures equivalence has no finite, ground-complete, equational axiomatisation over the language* BCCSP$_{\|}$.

In what follows, for actions $a, b \in \mathcal{A}$ and $i \geq 0$, we let $b^0 a$ denote $a.0$ and $b^{i+1}a$ stand for $b(b^i a)$. Consider now the infinite family of equations $\{e_N \mid N \geq 1\}$ given, for $a \neq b$, by:

$$p_N = \sum_{i=1}^{N} b^i a \qquad (N \geq 1)$$

$$e_N \ : \ a \parallel p_N \approx a p_N + \sum_{i=1}^{N} b(a \parallel b^{i-1}a) \qquad\qquad (N \geq 1) \ .$$

Notice that the equations e_N are sound modulo $\sim_{\mathtt{PF}}$ for all $N \geq 1$.

We also notice that none of the summands in the right-hand side of equation e_N is, alone, possible futures equivalent to $a \parallel p_N$. However, we now proceed to show that, when N is large enough, having a summand possible futures equivalent to $a \parallel p_N$ is an invariant under provability from finite sound axiom systems, and it will thus play the role of witness property for our negative result.

To this end, we introduce first some basic notions and results on $\sim_{\mathtt{PF}}$.

▶ **Definition 24.** *We say that a* $\mathrm{BCCSP}_{\parallel}$ *term t has a* $\mathbf{0}$ *factor if it contains a subterm of the form $t_1 \parallel t_2$, where either t_1 or t_2 is possible futures equivalent to* $\mathbf{0}$.

Next, we characterise closed $\mathrm{BCCSP}_{\parallel}$ terms that are possible futures equivalent to p_N.

▶ **Lemma 25.** *Let q be a* $\mathrm{BCCSP}_{\parallel}$ *term that does not have* $\mathbf{0}$ *summands or factors and such that* $\mathtt{CT}(q) = \mathtt{CT}(p_N)$ *for some* $N \geq 1$. *Then q does not contain any occurrence of* $\parallel$. *Moreover* $q \sim_{\mathtt{PF}} p_N$ *if and only if* $q = \sum_{j \in J} q_j$ *for some terms q_j such that none of them has* $+$ *as head operator and:*

- *for each* $i \in \{1, \ldots, N\}$ *there is some* $j \in J$ *such that* $b^i a \sim_{\mathtt{PF}} q_j$*;*
- *for each* $j \in J$ *there is some* $i \in \{1, \ldots, N\}$ *such that* $q_j \sim_{\mathtt{PF}} b^i a$.

In light of Lemma 25, we can also provide a decomposition-like characterisation of closed $\mathrm{BCCSP}_{\parallel}$ terms that are possible futures equivalent to $a \parallel p_N$.

▶ **Proposition 26.** *Assume that p, q are two* $\mathrm{BCCSP}_{\parallel}$ *processes such that* $p, q \not\sim_{\mathtt{PF}} \mathbf{0}$, *$p, q$ do not have* $\mathbf{0}$ *summands or factors, and* $p \parallel q \sim_{\mathtt{PF}} a \parallel p_N$, *for some* $N > 1$. *Then either* $p \sim_{\mathtt{PF}} a$ *and* $q \sim_{\mathtt{PF}} p_N$, *or* $p \sim_{\mathtt{PF}} p_N$ *and* $q \sim_{\mathtt{PF}} a$.

The following lemma characterises the open $\mathrm{BCCSP}_{\parallel}$ terms whose substitution instances can be equivalent in possible futures semantics to terms having at least two summands of p_N $(N > 1)$ as their summands.

▶ **Lemma 27.** *Let t be a* $\mathrm{BCCSP}_{\parallel}$ *term that does not have* $+$ *as head operator. Let* $m > 1$ *and* σ *be a closed substitution such that* $\sigma(t)$ *has no* $\mathbf{0}$ *summands or factors. If* $\sigma(t) \sim_{\mathtt{PF}} \sum_{k=1}^{m} b^{i_k} a$, *for some* $1 \leq i_1 < \cdots < i_m$, *then* $t = x$ *for some variable x.*

We now have all the ingredients necessary to prove Theorem 23. To streamline our presentation, we split the proof of into two main parts: Proposition 28 deals with the preservation of the witness property under provability from the substitution rule of equational logic. Theorem 29 builds on Proposition 28 and proves the witness property to be an invariant under provability from finite sound axiom systems. The full proofs of these two results are provided in the Appendix.

▶ **Proposition 28.** *Let* $t \approx u$ *be an equation over* $\mathrm{BCCSP}_{\parallel}$ *that is sound modulo* $\sim_{\mathtt{PF}}$. *Let* σ *be a closed substitution with* $p = \sigma(t)$ *and* $q = \sigma(u)$. *Suppose that p and q have neither* $\mathbf{0}$ *summands nor* $\mathbf{0}$ *factors, and that* $p, q \sim_{\mathtt{PF}} a \parallel p_N$ *for some N larger than the sizes of t and u. If p has a summand possible futures equivalent to* $a \parallel p_N$, *then so does q.*

▶ **Theorem 29.** *Let* $\mathcal{E}$ *be a finite axiom system over* $\mathrm{BCCSP}_{\parallel}$ *that is sound modulo* $\sim_{\mathtt{PF}}$. *Let N be larger than the size of each term in the equations in* $\mathcal{E}$. *Assume that p and q are closed terms that contain no occurrences of* $\mathbf{0}$ *as a summand or factor, and that* $p, q \sim_{\mathtt{PF}} a \parallel p_N$. *If* $\mathcal{E} \vdash p \approx q$ *and p has a summand possible futures equivalent to* $a \parallel p_N$, *then so does q.*

As the left-hand side of equation e_N, i.e., the term $a \parallel p_N$, has a summand possible futures equivalent to $a \parallel p_N$, whilst the right-hand side, i.e., the term $ap_N + \sum_{i=1}^{N} b(a \parallel b^{i-1}a)$, does not, we can conclude that the collection of infinitely many equations e_N $(N \geq 1)$ is the desired witness family. This concludes the proof of Theorem 23.

6.2 Extending the negative result

It is easy to check that the equations e_N $(N \geq 1)$ in the witness family of the negative result for $\sim_{\mathtt{PF}}$ are all sound modulo bisimilarity, i.e., the largest symmetric simulation. Consequently, they are also sound modulo any congruence $\mathcal{R}$ such that $\sim_{\mathtt{B}} \subseteq \mathcal{R} \subseteq \sim_{\mathtt{PF}}$. Hence, the negative result for all these equivalences can be derived from that for $\sim_{\mathtt{PF}}$, by exploiting this fact and that any finite axiom system that is sound modulo $\mathcal{R}$ is also sound modulo $\sim_{\mathtt{PF}}$.

▶ **Theorem 30.** *Assume that $|\mathcal{A}| \geq 2$. Let $\mathcal{R}$ be a congruence such that $\sim_{\mathtt{B}} \subseteq \mathcal{R} \subseteq \sim_{\mathtt{PF}}$. Then $\mathcal{R}$ has no finite, ground-complete, equational axiomatisation over the language* $\mathrm{BCCSP}_{\parallel}$.

Theorem 30 can be applied to establish for $n \geq 2$ that the n-nested trace and simulation semantics have no finite, ground-complete equational axiomatisation over $\mathrm{BCCSP}_{\parallel}$. The n-nested trace equivalences were introduced in [18] as an alternative tool to define bisimilarity. The hierarchy of n-nested simulations, namely simulation relations contained in a (nested) simulation equivalence, was introduced in [17].

▶ **Definition 31** (n-nested semantics)**.** *For $n \geq 0$, the relation $\sim_{\mathtt{T}}^{n}$ over $\mathcal{P}$, called the* n-nested trace equivalence*, is defined inductively as follows:*

- *$p \sim_{\mathtt{T}}^{0} q$ for all $p, q \in \mathcal{P}$,*
- *$p \sim_{\mathtt{T}}^{n+1} q$ if and only if for all traces $\alpha \in \mathcal{A}^*$:*
  - *if $p \xrightarrow{\alpha} p'$ then there is a q' such that $q \xrightarrow{\alpha} q'$ and $p' \sim_{\mathtt{T}}^{n} q'$, and*
  - *if $q \xrightarrow{\alpha} q'$ then there is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \sim_{\mathtt{T}}^{n} q'$.*

For $n \geq 0$, the relation $\sqsubseteq_{\mathtt{S}}^{n}$ over $\mathcal{P}$ is defined inductively as follows:

- *$p \sqsubseteq_{\mathtt{S}}^{0} q$ for all $p, q \in \mathcal{P}$,*
- *$p \sqsubseteq_{\mathtt{S}}^{n+1} q$ if and only if $p \mathcal{R} q$ for some simulation $\mathcal{R}$, with $\mathcal{R}^{-1}$ included in $\sqsubseteq_{\mathtt{S}}^{n}$.*

n-nested simulation equivalence *is the kernel of $\sqsubseteq_{\mathtt{S}}^{n}$, i.e., the equivalence $\sim_{\mathtt{S}}^{n} = \sqsubseteq_{\mathtt{S}}^{n} \cap (\sqsubseteq_{\mathtt{S}}^{n})^{-1}$.*

Notably, $\sim_{\mathtt{T}}^{1}$ corresponds to trace equivalence, $\sim_{\mathtt{T}}^{2}$ is possible futures equivalence, and $\sim_{\mathtt{S}}^{1}$ is simulation equivalence. The following theorem is a corollary of Theorems 23 and 30.

▶ **Theorem 32.** *Assume that $|\mathcal{A}| \geq 2$. Let $n \geq 2$. Then, n-nested trace equivalence and n-nested simulation equivalence admit no finite, ground-complete, equational axiomatisation over the language* $\mathrm{BCCSP}_{\parallel}$.

7 Concluding remarks

We have studied the finite axiomatisability of the language $\mathrm{BCCSP}_{\parallel}$ modulo the behavioural equivalences in the linear time-branching time spectrum. On the one hand we have obtained finite, ground-complete axiomatisations modulo the (decorated) trace and simulation semantics in the spectrum. On the other hand we have proved that for all equivalences that are finer than possible futures equivalence and coarser than bisimilarity a finite ground-complete axiomatisation does not exist.

Since our ground-completeness proof for ready simulation equivalence proceeds via elimination of $\parallel$ from closed terms (Proposition 3), and all behavioural equivalences in the linear time-branching time spectrum that include ready simulation have a finite ground-complete axiomatisation over BCCSP, it immediately follows from the elimination result that all these behavioural equivalences have a finite ground-complete axiomatisation over $\mathrm{BCCSP}_{\|}$. Exploiting various forms of distributivity of parallel composition over choice, we were able to present more concise and elegant axiomatisations for the coarser behavioural equivalences. We did not succeed to equationally derive the axioms of ready simulation equivalence from the axiomatisations of the coarser equivalences. In fact, we conjecture that this is not possible, and leave it for future research to find a proof.

The parallel composition operator we have considered in this paper implements interleaving without synchronisation between parallel components. It is natural to consider extensions of our result to parallel composition operators with some form synchronisation. We expect that extension with CCS-style synchronisation is straightforward, both for the positive and the negative results. Whether this is also the case for extension with ACP-style or CSP-style synchronisation we leave as a topic for future investigations.

As previously outlined, in [1] it was proved that the nested semantics admit no finite axiomatisation over BCCSP. However, our negative results cannot be reduced to a mere lifting of those in [1], as the presence of the additional operator $\|$ might, at least in principle, allow us to finitely axiomatise the equations over BCCSP processes that are valid modulo the considered nested semantics. Indeed, auxiliary operators can be added to some language in order to obtain a finite axiomatisation of some congruence relation (see, e.g. the classic example given in [5]). Understanding whether it is possible to lift non-finite axiomatisability results among different algebras, and under which constraints this can be done, is an interesting research avenue and we aim to investigate it in future work. A methodology for transferring non-finite-axiomatisability results across languages was presented in [3], where a reduction-based approach was proposed. However, that method has some limitations and thus further studies are needed.

A behavioural equivalence is *finitely based* if it has a finite equational axiomatisation from which all valid equations between open terms are derivable. In [13] and [2] finite bases for bisimilarity with respect to PA and $\mathrm{BCCSP}_{\|}$ extended with the auxiliary operators left merge and communication merge were presented. Furthermore, in [9] an overview was given of which behavioural equivalences in the linear time-branching time spectrum are finitely based with respect to BCCSP. The negative results in Section 6 imply that none of the behavioural equivalences between possible futures equivalence and bisimilarity is finitely based with respect to $\mathrm{BCCSP}_{\|}$. An interesting question is which of the behavioural equivalences including ready simulation semantics is finitely based with respect to $\mathrm{BCCSP}_{\|}$.

In [11] an alternative classification of the equivalences in the spectrum with respect to [16] was proposed. In order to obtain a general, unified, view of process semantics, the spectrum was divided into layers, each corresponding to a different notion of constrained simulation [10]. There are pleasing connections between the different layers and the partition they induce over on the congruences in the spectrum, as given in [11], and the relationships between the axioms for the interleaving operator we have presented in this study.

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A Proof of Theorem 23

Before proceeding to the proof we introduce some auxiliary results.

For $k \geq 0$, we denote by $\mathrm{var}_k(t)$ the set of variables occurring in the k-derivatives of t, namely $\mathrm{var}_k(t) = \{x \in \mathrm{var}(t') \mid t \xrightarrow{\alpha} t', |\alpha| = k\}$.

▶ **Lemma 33.** *Let t, u be two $\mathrm{BCCSP}_{\|}$ terms. If $t \sim_{\mathtt{PF}} u$ then:*

1. *For each $k \geq 0$ it holds that $\mathrm{var}_k(t) = \mathrm{var}_k(u)$.*
2. *t has a summand x, for some variable x, if and only if u does.*
3. $\mathrm{norm}(t) = \mathrm{norm}(u)$ *and* $\mathrm{depth}(t) = \mathrm{depth}(u)$.

The following result is immediate.

▶ **Lemma 34.** *Let t be a $\mathrm{BCCSP}_{\|}$ term, and let σ be a closed substitution. If $x \in \mathrm{var}(t)$ then $\mathrm{depth}(\sigma(t)) \geq \mathrm{depth}(\sigma(x))$.*

▶ **Proposition 28.** *Let $t \approx u$ be an equation over $\mathrm{BCCSP}_{\|}$ that is sound modulo $\sim_{\mathtt{PF}}$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor $\mathbf{0}$ factors, and that $p, q \sim_{\mathtt{PF}} a \| p_N$ for some N larger than the sizes of t and u. If p has a summand possible futures equivalent to $a \| p_N$, then so does q.*

Proof. Observe, first of all, that since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, then neither do t and u. We can therefore assume that, for some finite index sets $I, J \neq \emptyset$,

$$t = \sum_{i \in I} t_i \quad \text{and} \quad u = \sum_{j \in J} u_j \ , \tag{1}$$

where none of the t_i $(i \in I)$ and u_j $(j \in J)$ is $\mathbf{0}$ or has $+$ as its head operator. Note that, as t and u have no $\mathbf{0}$ summands or factors, then none of the t_i $(i \in I)$ and u_j $(j \in J)$ does either.

Since $p = \sigma(t)$ has a summand that is possible futures equivalent to $a \| p_N$, there is an index $i \in I$ such that $\sigma(t_i) \sim_{\mathtt{PF}} a \| p_N$. Our aim is now to show that there is an index $j \in J$ such that $\sigma(u_j) \sim_{\mathtt{PF}} a \| p_N$, proving that $q = \sigma(u)$ has the required summand. This we proceed to do by a case analysis on the form t_i may have.

1. Case $t_i = x$ for some variable x. In this case, we have that $\sigma(x) \sim_{\mathtt{PF}} a \| p_N$ and t has x as a summand. As $t \approx u$ is sound with respect to possible futures equivalence, from $t \sim_{\mathtt{PF}} u$ we get $t \sim_{\mathtt{CT}} u$. Hence, by Lemma 33.2, we obtain that u has a summand x as well, namely there is an index $j \in J$ such that $u_j = x$. It is then immediate to conclude that $q = \sigma(u)$ has a summand which is possible futures equivalent to $a \| p_N$.
2. Case $t_i = ct'$ for some action $c \in \{a, b\}$ and term t'. This case is vacuous because, since $\sigma(t_i) = c\sigma(t') \xrightarrow{c} \sigma(t')$ is the only transition afforded by $\sigma(t_i)$, this term cannot be possible futures equivalent to $a \| p_N$.

3. Case $t_i = t' \| t''$ for some terms t', t''. We have that $\sigma(t_i) = \sigma(t') \| \sigma(t'') \sim_{\mathtt{PF}} a \| p_N$. As $\sigma(t_i)$ has no $\mathbf{0}$ factors, it follows that $\sigma(t') \not\sim_{\mathtt{PF}} \mathbf{0}$ and $\sigma(t'') \not\sim_{\mathtt{PF}} \mathbf{0}$. Thus, by Proposition 26, we can infer that, without loss of generality, $\sigma(t') \sim_{\mathtt{PF}} a$ and $\sigma(t'') \sim_{\mathtt{PF}} p_N$. Notice that $\sigma(t'') \sim_{\mathtt{PF}} p_N$ implies $\mathtt{CT}(\sigma(t'')) = \mathtt{CT}(p_N)$. Now, t'' can be written in the general form $t'' = v_1 + \cdots + v_l$ for some $l > 0$, where none of the summands v_h is $\mathbf{0}$ or a sum. By Lemma 25, $\sigma(t'') \sim_{\mathtt{PF}} p_N$ implies that for each $i \in \{1, \ldots, N\}$ there is a summand r_i of $\sigma(t'')$ such that $b^i a \sim_{\mathtt{PF}} r_i$, and for each summand r of $\sigma(t'')$ there is an $i_r \in \{1, \ldots, N\}$ such that $r \sim_{\mathtt{PF}} b^i a$. Observe that, since N is larger than the size of t, we have that $l < N$. Hence, there must be some $h \in \{1, \ldots, l\}$ such that $\sigma(v_h) \sim_{\mathtt{S}} \sum_{k=1}^{m} b^{i_k} a$ for some $m > 1$ and $1 \leq i_1 < \ldots < i_m \leq N$. The term $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, or else, so would $\sigma(t'')$ and $\sigma(t)$. By Lemma 27, it follows that v_h can only be a variable x and thus that

$$\sigma(x) \sim_{\mathtt{PF}} \sum_{k=1}^{m} b^{i_k} a \quad . \tag{2}$$

Observe, for later use, that the above equation yields that $x \notin \mathrm{var}(t')$, or else $\sigma(t') \not\sim_{\mathtt{PF}} a$ due to Lemma 34. So, modulo possible futures equivalence, t_i has the form $t' \parallel (x + t''')$, for some term t''', with $x \notin \mathrm{var}(t')$, $\sigma(t') \sim_{\mathtt{PF}} a$ and $\sigma(x + t'') \sim_{\mathtt{PF}} p_N$.

Our order of business will now be to show that u has a summand u_j such that $\sigma(u_j)$ is possible futures equivalent to $a \parallel p_N$. We recall that $t \sim_{\mathtt{PF}} u$ implies $t \sim_{\mathtt{CT}} u$. Thus, by Lemma 33.1 we obtain that $\mathrm{var}_k(t) = \mathrm{var}_k(u)$ for all $k \geq 0$. Hence, from $x \in \mathrm{var}_0(t_i) = \mathrm{var}(t_i)$ we get that there is at least one $j \in J$ such that $x \in \mathrm{var}_0(u_j) = \mathrm{var}(u_j)$.

So, firstly, we show that x cannot occur in the scope of prefixing in u_j, namely u_j cannot be of the form $c.u'$ or $(c.u' + u'') \parallel u'''$ for some $c \in \{a, b\}$ and u' with $x \in \mathrm{var}(u')$. We proceed by a case analysis:

a. $c = b$ and $u_j = (b.u' + u'') \parallel u'''$ for some $u', u'', u''' \in \mathrm{BCCSP}_{\|}$ with $x \in \mathrm{var}(u')$. As $\sigma(u)$ does not have $\mathbf{0}$ summands or factors we have that $\sigma(u''') \not\sim_{\mathtt{PF}} \mathbf{0}$. Let $D = \max\{d \mid x \in \mathrm{var}_d(u')\}$. From $\sigma(x) \sim_{\mathtt{PF}} \sum_{k=1}^{m} b^{i_k} a$ and $\mathtt{CT}(\sigma(u)) = \mathtt{CT}(a \parallel p_N)$ we can infer that the completed traces of $\sigma(u''')$ are of the form $b^i a$, for some $i \in \{0, \ldots, N - i_m - D - 1\}$. Let $\alpha \in \mathtt{T}(\sigma(u'))$ be such that $|\alpha| = D$ and $u' \xrightarrow{\alpha} w$ with $x \in \mathrm{var}(w)$. By the choice of D, we can infer that x does not occur in the scope of prefixing in w, and thus $\mathtt{T}(\sigma(x)) \subseteq \mathtt{T}(\sigma(w))$. Then we get that $(b^i ab\alpha, \mathtt{T}(\sigma(w))) \in \mathtt{PF}(\sigma(u))$, where $b^i a \in \mathtt{CT}(\sigma(u'''))$. However, as $m \geq 2$, there is no p' such that $a \parallel p_N \xrightarrow{b^i ab\alpha} p'$ and $\mathtt{T}(\sigma(x)) \subseteq \mathtt{T}(p')$, thus giving $(b^i ab\alpha, \mathtt{T}(\sigma(w))) \notin \mathtt{PF}(a \parallel p_N)$. This contradicts $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$.

b. $c = b$ and $u_j = b.u'$ for some $\mathrm{BCCSP}_{\|}$ term u' with $x \in \mathrm{var}(u')$. The proof is similar to the one of the previous case and it is therefore omitted.

c. $c = a$ and $u_j = (a.u' + u'') \parallel u'''$ for some $u', u'', u''' \in \mathrm{BCCSP}_{\|}$ with $x \in \mathrm{var}(u')$. As $\sigma(u)$ does not have $\mathbf{0}$ summands or factors we have that $\sigma(u''') \not\sim_{\mathtt{PF}} \mathbf{0}$. From $\sigma(x) \sim_{\mathtt{PF}} \sum_{k=1}^{m} b^{i_k} a$ we infer that $\mathtt{T}(a.\sigma(u'))$ includes traces having two occurrences of action a. Since $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$, this implies that there is no $\alpha \in \mathtt{T}(\sigma(u'''))$ such that α contains an occurrence of action a, for otherwise $\sigma(u)$ could perform a trace having 3 occurrences of that action. In particular, this implies that the last symbol in each trace of $\sigma(u''')$ must be action b. This gives that there is at least one completed trace of $\sigma(u_j)$, and thus of $\sigma(u)$, whose last symbol is action b. Hence we get $\mathtt{CT}(\sigma(u)) \neq \mathtt{CT}(a \parallel p_N)$, which contradicts $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$.

d. $c = a$ and $u_j = a.u'$ for some $\mathrm{BCCSP}_{\|}$ term u' with $x \in \mathrm{var}(u')$. In this case we are going to prove a slightly weaker property, namely that not all summands u_j with $x \in \mathrm{var}(u_j)$ can be of this form. Consider the closed substitution σ' defined by

$$\sigma'(y) = \begin{cases} ap_N & \text{if } y = x \\ \sigma(y) & \text{otherwise.} \end{cases}$$

Then we have that $\sigma'(t_i) = \sigma'(t') \parallel \sigma'(x) + \sigma'(t''') \xrightarrow{a} \sigma(t') \parallel p_N \sim_{\mathtt{PF}} a \parallel p_N$. Since $\sigma'(t) \sim_{\mathtt{PF}} \sigma'(u)$ then there is a process r such that $\sigma'(u) \xrightarrow{a} r$ and $\mathtt{T}(r) = \mathtt{T}(a \parallel p_N)$. In particular, this means that $\mathrm{depth}(r) = N + 2$. Hence, from the choices of N, σ and σ', we can infer that such an a-move by $\sigma'(u)$ can only stem from a summand u_j such that $x \in \mathrm{var}(u_j)$. Assume, towards a contradiction, that all such summands u_j are of the form $a.u'_j$ for some BCCSP$_\parallel$ term u'_j with $x \in \mathrm{var}(u'_j)$ and $r = \sigma'(u'_j)$. As $\mathrm{depth}(\sigma'(u'_j)) = N + 2 = \mathrm{depth}(\sigma'(x))$, by Lemma 34 we get that u'_j can only be of the form $u'_j = x + w_j$ for some BCCSP$_\parallel$ term w_j with $\mathrm{depth}(\sigma'(w_j)) \leq N + 2$. Notice that $\mathtt{T}(\sigma'(x)) \subset \mathtt{T}(a \parallel p_N)$. Hence $\sigma'(w_j) \neq \mathbf{0}$. More precisely, $\sigma'(x) = ap_N$ implies that $\{b\alpha \mid b\alpha \in \mathtt{T}(a \parallel p_N)\} \subseteq \mathtt{T}(\sigma'(w_j)) \subseteq \mathtt{T}(a \parallel p_N)$. Clearly, no trace starting with action b can stem from $\sigma'(x)$ and we can then infer, in light of Lemma 34, that $x \notin \mathrm{var}(w_j)$, as $\mathrm{depth}(\sigma'(w_j)) \leq N + 2$. This implies that $\sigma'(w_j) = \sigma(w_j)$ and thus $\{b\alpha \mid b\alpha \in \mathtt{T}(a \parallel p_N)\} \subseteq \mathtt{T}(\sigma(w_j)) \subseteq \mathtt{T}(a \parallel p_N)$. In particular, $\sigma(w_j)$ can perform at least one (completed) trace of the form $b\alpha$ where α contains two occurrences of action a. From $\sigma(u_j) = a.(\sigma(x) + \sigma(w_j))$, then get that $(ab\alpha, \emptyset) \in \mathtt{PF}(\sigma(u))$, namely $\sigma(u)$ can perform at least one (completed) trace containing 3 occurrences of action a. This gives a contradiction with $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$.

We have therefore obtained that x does not occur in the scope of prefixing in (at least one) u_j. We proceed now by a case analysis on the possible forms of this summand.

a. $u_j = x$. Then, modulo possible futures equivalence, $\sigma(u)$ has the form $r' + \sum_{k=1}^{m} b^{i_k} a$ for some r'. We show that this contradicts $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$. This follows directly by noticing that, due to the summand $b^{i_1}a$, we have that $(b^{i_1}a, \emptyset) \in \mathtt{PF}(\sigma(u))$. However, $(b^{i_1}a, \emptyset) \notin \mathtt{PF}(a \parallel p_N)$, since $a \parallel p_N$ by performing the trace $b^{i_1}a$ can reach either a process that can perform an a (in case the first b-move is performed by the summand $b^{i_1}a$ of p_N) or a b (in case the first b-move is performed by a summand $b^{i}a$ of p_N such that $i > i_1$).

b. $u_j = (x + w) \parallel w'$, for some terms w, w' with $w' \not\sim_{\mathtt{PF}} \mathbf{0}$. From $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$, we infer that $\mathtt{CT}(\sigma(u_j)) \subseteq \mathtt{CT}(a \parallel p_N)$. We recall that no completed trace of $a \parallel p_N$ has b as last symbol and, moreover, in all the completed traces of $a \parallel p_N$ there are exactly two occurrences of a. Hence, all (nonempty) completed traces of $\sigma(x), \sigma(w)$ and $\sigma(w')$ must have exactly one occurrence of a and this occurrence must be as the last symbol in the completed trace.

We now proceed to show that $\sigma(w')$ has a summand a and $a \notin \mathtt{I}(\sigma(x) + \sigma(w))$. We start by noticing that it cannot be the case that $a \in \mathtt{I}(\sigma(x) + \sigma(w)) \cap \mathtt{I}(\sigma(w'))$, for otherwise we would have $a^2 \in \mathtt{T}(\sigma(u_j)) \subseteq \mathtt{T}(\sigma(u))$, thus contradicting $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$. Assume now, towards a contradiction, that $\mathtt{I}(\sigma(w')) = \{b\}$. Then, due to summand $b^{i_m}a$ of $\sigma(x)$, we have that $\sigma(u_j) \xrightarrow{b^{i_m - 1}} ba \parallel \sigma(w')$ and $a\alpha \notin \mathtt{T}(ba \parallel \sigma(w'))$ for any trace $\alpha \in \mathcal{A}^*$. Clearly, $(b^{i_m - 1}, \mathtt{T}(ba \parallel \sigma(w'))) \in \mathtt{PF}(\sigma(u_j))$, and thus it is also a possible future of $\sigma(u)$. However, $(b^{i_m - 1}, \mathtt{T}(ba \parallel \sigma(w'))) \notin \mathtt{PF}(a \parallel p_N)$, as the interleaving of p_N with a guarantees that after an initial trace of an arbitrary number of b-transitions it is always possible to perform a trace starting with a. This gives a contradiction with $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$. We have obtained that $a \in \mathtt{I}(\sigma(w'))$. More precisely, from the constraints on the completed traces of $\sigma(w')$, we infer that $\sigma(w')$ has a summand a.

Our order of business will now be to show that $\sigma(w') \sim_{\mathtt{PF}} a$. Since $\sigma(w') \xrightarrow{a} \mathbf{0}$, we have that $\sigma(u_j) \xrightarrow{a} (\sigma(x)+\sigma(w)) \parallel \mathbf{0} \sim_{\mathtt{PF}} \sigma(x)+\sigma(w)$. Thus, $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$ implies that $a \parallel p_N \xrightarrow{a} r$ for some r with $\mathtt{T}(r) = \mathtt{T}(\sigma(x)+\sigma(w))$. Since $a \parallel p_N$ has only one possible initial a-transition, namely $a \parallel p_N \xrightarrow{a} \mathbf{0} \parallel p_N$, we get that $r \sim_{\mathtt{PF}} p_N$ and thus $\mathtt{T}(\sigma(x)+\sigma(w)) = \mathtt{T}(p_N)$. In particular, this implies that $\text{depth}(\sigma(x)+\sigma(w)) = N+1$. Therefore, we have

$$\begin{aligned}
1 \le \text{depth}(\sigma(w')) &= \text{depth}(\sigma(u_j)) - \text{depth}(\sigma(x)+\sigma(w)) \\
&= \text{depth}(\sigma(u_j)) - (N+1) \\
&\le \text{depth}(\sigma(u)) - (N+1) \\
&= \text{depth}(a \parallel p_N) - (N+1) && \text{(by Lem. 33.3)} \\
&= N+2-(N+1) \\
&= 1
\end{aligned}$$

and we can therefore conclude that $\sigma(w') \sim_{\mathtt{PF}} a$. Furthermore, it is not difficult to prove that $\mathtt{CT}(\sigma(x)+\sigma(w)) = \mathtt{CT}(p_N)$, for otherwise we get a contradiction with $\sigma(u) \sim_{\mathtt{PF}} a \parallel p_N$.
So far we have obtained that, modulo possible futures equivalence,

$$\sigma(u_j) \sim_{\mathtt{PF}} \left(\sum_{k=1}^{m} b^{i_k}a + \sigma(w)\right) \parallel a \ \text{ and } \ \mathtt{CT}(\sum_{k=1}^{m} b^{i_k}a + \sigma(w)) = \{b^i a \mid i \in \{1,\dots,N\}\} \ .$$

To conclude the proof, we need to show that $\sum_{k=1}^{m} b^{i_k}a + \sigma(w) \sim_{\mathtt{PF}} p_N$. Let $I_m = \{i_1,\dots,i_m\}$ and $I_N = \{1,\dots,N\}$. Assume, towards a contradiction, that $\sum_{k=1}^{m} b^{i_k}a + \sigma(w) \not\sim_{\mathtt{PF}} p_N$. Notice that $\sigma(w)$ can be written in the general form $\sigma(w) = \sum_{l\in L} q_l$ for some terms q_l that do not have $+$ as head operator nor contain any occurrence of $\parallel$. By Lemma 25, this means that either there is an $i \in I_N \setminus I_m$ such that $b^i a \not\sim_{\mathtt{PF}} q_l$ for any $l \in L$, or that there is a summand q_l of $\sigma(w)$ such that $q_l \not\sim_{\mathtt{PF}} b^i a$ for any $i \in I_N$. In both cases, we obtain that there is (at least) a summand q_l of $\sigma(w)$ such that $b^k a, b^h a \in \mathtt{CT}(q_l)$ for some $k \neq h, h, k \in I_N$. We can then proceed as in the proof of Lemma 25 to prove that this gives the desired contradiction. We have therefore obtained that $\sum_{k=1}^{m} b^{i_k}a + \sigma(w) \sim_{\mathtt{PF}} p_N$. Hence, by congruence closure, we get that $\sigma(u_j) \sim_{\mathtt{PF}} a \parallel p_N$ and we can therefore conclude that $\sigma(u)$ has the desired summand.
This concludes the proof. ◀

Finally, we can formally prove Theorem 29.

▶ **Theorem 29.** *Let $\mathcal{E}$ be a finite axiom system over $\text{BCCSP}_{\parallel}$ that is sound modulo $\sim_{\mathtt{PF}}$. Let N be larger than the size of each term in the equations in $\mathcal{E}$. Assume that p and q are closed terms that contain no occurrences of $\mathbf{0}$ as a summand or factor, and that $p, q \sim_{\mathtt{PF}} a \parallel p_N$. If $\mathcal{E} \vdash p \approx q$ and p has a summand possible futures equivalent to $a \parallel p_N$, then so does q.*

Proof. Assume that $\mathcal{E}$ is a finite axiom system over the language $\text{BCCSP}_{\parallel}$ that is sound modulo possible futures equivalence, and that the following hold, for some closed terms p and q and positive integer N larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \sim_{\mathtt{PF}} q \sim_{\mathtt{PF}} a \parallel p_N$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand possible futures equivalent to $a \parallel p_N$.

We prove that q also has a summand possible futures equivalent to $a \parallel p_N$ by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$. Without loss of generality, we may assume that the closed terms involved in the proof of the equation $p \approx q$ have no $\mathbf{0}$ summands or factors, and that applications of symmetry happen first in equational proofs (that is, $\mathcal{E}$ is closed with respect to symmetry).

We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows immediately by using the inductive hypothesis twice. Below we only consider the other possibilities.

- Case $E \vdash p \approx q$, because $\sigma(t) = p$ and $\sigma(u) = q$ for some equation $(t \approx u) \in E$ and closed substitution σ. Since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, and N is larger than the size of each term mentioned in equations in $\mathcal{E}$, the claim follows by Proposition 28.
- Case $E \vdash p \approx q$, because $p = cp'$ and $q = cq'$ for some p', q' such that $E \vdash p' \approx q'$, and for some action c. This case is vacuous because $p = cp' \not\sim_{\mathtt{PF}} a \parallel p_N$, and thus p does not have a summand possible futures equivalent to $a \parallel p_N$.
- Case $E \vdash p \approx q$, because $p = p' + p''$ and $q = q' + q''$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since p has a summand possible futures equivalent to $a \parallel p_N$, we have that so does either p' or p''. Assume, without loss of generality, that p' has a summand possible futures equivalent to $a \parallel p_N$. Since p is possible futures equivalent to $a \parallel p_N$, so is p'. Using the soundness of $\mathcal{E}$ modulo possible futures equivalence, it follows that $q' \sim_{\mathtt{PF}} a \parallel p_N$. The inductive hypothesis now yields that q' has a summand possible futures equivalent to $a \parallel p_N$. Hence, q has a summand possible futures equivalent to $a \parallel p_N$, which was to be shown.
- Case $E \vdash p \approx q$, because $p = p' \parallel p''$ and $q = q' \parallel q''$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since the proof involves no uses of $\mathbf{0}$ as a summand or a factor, we have that $p', p'' \not\sim_{\mathtt{PF}} \mathbf{0}$ and $q', q'' \not\sim_{\mathtt{PF}} \mathbf{0}$. It follows that q is a summand of itself. By our assumptions, $q' \parallel q'' \sim_{\mathtt{PF}} a \parallel p_N$ which, by Proposition 26 gives that either $q' \sim_{\mathtt{S}} a$ and $q'' \sim_{\mathtt{S}} p_N$, or $q' \sim_{\mathtt{S}} p_N$ and $q'' \sim_{\mathtt{S}} a$. In both cases, we can conclude that q has itself as summand of the required form.

This completes the proof of Theorem 29 and thus of Theorem 23. ◀