Non Axiomatisability of Positive Relation Algebras with Constants, via Graph Homomorphisms

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Abstract

We study the equational theories of composition and intersection on binary relations, with or without their associated neutral elements (identity and full relation). Without these constants, the equational theory coincides with that of semilattice-ordered semigroups. We show that the equational theory is no longer finitely based when adding one or the other constant, refuting a conjecture from the literature. Our proofs exploit a characterisation in terms of graphs and homomorphisms, which we show how to adapt in order to capture standard equational theories over the considered signatures.

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics

Keywords and phrases Relation algebra, graph homomorphisms, (in)equational theories

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2020.29

Related Version Appendix available at `https://hal.archives-ouvertes.fr/hal-02870687`.

Funding This work has been funded by the European Research Council (ERC-StG CoVeCe 678157) and the French National Research Agency (ANR-10-LABX-0070 ANR-11-IDEX-0007).

1 Introduction

Several operations and constants are used frequently on binary relations: set-theoretic operations (intersection, union, complement, empty and full relations), relational composition and identity relation, and converse (transpose). Amongst others, Tarski studied those operations and analysed their expressiveness and the equational laws they satisfy [25, 26, 24, 19]. It turns out that these basic operations already make it possible to encode Peano arithmetic in a purely algebraic setting, without variables. As a consequence, the corresponding equational theory is undecidable and not finitely based [21].

The situation changes when considering *positive* fragments [13, 4, 1, 20, 2], where the complement operation is removed. Indeed, the equational theory of those fragments is decidable, even in the presence of additional operations like reflexive transitive closure [6, 22]. However, results concerning finite axiomatisations are more on the negative side. Hodkinson and Mikulás proved that one cannot obtain a finite firstorder axiomatisation whenever the operations of composition, intersection and converse are present [18]. When only two of those operations are considered, we get positive results: the problem is straightforward for converse and intersection; the case of composition and converse is more subtle and covered in [4, 11, 1]; and the equational theory of composition and intersection coincides with that of *semilattice-ordered semigroups* [3].

However, understanding the laws satisfied by the identity and the full-relation constants is difficult. They are neutral elements for composition and intersection, respectively, but they also satisfy rather unexpected laws, so that the equational theories depart from those of semilattice-ordered monoids and bounded-semilattice-ordered semigroups. The case of

31st International Conference on Concurrency Theory (CONCUR 2020).
Editors: Igor Konnov and Laura Kovács; Article No. 29; pp. 29:1–29:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

composition, intersection, and identity was thought to be finitely based, with an explicit candidate [2], but there was an error in the completeness proof so that it remained as a conjecture. Our first contribution consists in refuting this conjecture: the equational theory of this fragment is not finitely based. Our second contribution is that adding the full relation to composition and intersection also yields an equational theory which is not finitely based.

The reasons for non finite axiomatisability of those two fragments are quite different. Still, our two proofs rely on a graph-theoretical characterisation the equational theory of binary relations [13, 1]. First, terms u, v built over a set of variables and the signature consisting of composition, intersection, and their neutral elements, can be used to denote graphs. The class of *expressible graphs*, those that may be denoted via a term, strongly depends on the considered fragment: they are always of treewidth at most two [8], they are also acyclic unless we have the identity constant, they are also connected unless we have the full-relation constant. The key result shared for all fragments is that a law $u \leq v$ is valid for relations if and only if there exists a homomorphism from the graph of v to the graph of u.

We prove the first negative result as follows: we first show that if we had a finite and equational axiomatisation, then we would be able to decompose every homomorphism between two expressible graphs while remaining within the class of expressible graphs – a similar idea is used in [13] for representable allegories; we formalise it in Section 3. Then we provide a counter-example: an infinite sequence of homomorphisms that cannot be decomposed accordingly, by exploiting a necessary condition for a graph to be expressible (Section 4).

We do not think a similar argument can used for the second negative result. Instead, we give directly an infinite sequence of homomorphisms and we show that for each of them, the corresponding law essentially has to be included into any sound and complete axiomatisation (Section 5).

As mentioned above, when neutral elements are taken into account, the equational theory of binary relations differs from that of natural algebraic structures extending semilattice-ordered semigroups (semilattice-ordered monoids, bounded-semilattice-ordered semigroups, and bounded-semilattice-ordered monoids). Our last contribution consists in providing graph-theoretical characterisations for those structures, yielding decidability in polynomial time of their equational theories (Section 6).

From the concurrency theory point of view, the structures considered here should not be confused with the ones studied in the litterature on pomsets [15, 14] or concurrent Kleene algebra [17]. Indeed, two forms of composition are also put forward in those lines of work: sequential and parallel composition, and they resemble the operations of relational composition and intersection we consider in the present paper (e.g., they form monoids related by the "weak exchange" law). However, the operation of intersection we use in the present paper is idempotent, and thus induces a partial order, which is not the case for parallel composition in concurrency theory. Accordingly, one should consider intersection as an operator for combining specifications rather than a program construction for concurrency, like with allegories and some of its extensions [13, 6, 23, 9].

2 Preliminaries

2.1 Terms

We fix in the rest of the paper an infinite alphabet A, and we let $a, b \dots$ range over its letters. SP1⊤ *terms* (series-parallel with one and ⊤) are generated by the following syntax

$$e, f ::= e \cdot f \;\mid\; e \cap f \;\mid\; 1 \;\mid\; \top \mid a \qquad (a \in A)$$

We denote their set by $\mathsf{SP1}\top$ and we often write ef for $e \cdot f$. We moreover assign priorities so that $ab \cap c$ reads as $(a \cdot b) \cap c$. We define $\mathsf{SP1}$, $\mathsf{SP}\top$ and SP to be respectively the set of $\mathsf{SP1}\top$ terms not containing $\top$, 1 and neither of them. We also use those symbols to denote the associated signatures.

2.2 Relational interpretation

We are primarily interested in the relational interpretation of terms, where $\cdot$ is relational composition, $\cap$ is set-theoretic intersection, 1 is the identity relation, and $\top$ is the full relation. Given an interpretation $\sigma : A \to \mathcal{P}(S \times S)$ of letters into some space of binary relations, we write $\widehat{\sigma} : \mathsf{SP1}\top \to \mathcal{P}(S \times S)$ for the corresponding extension to terms.

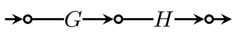
An inequation between two terms u and v is *valid*, written $\mathcal{R}el \models u \leq v$, if for every such interpretation σ we have $\widehat{\sigma}(u) \subseteq \widehat{\sigma}(v)$. We call *(in)equational theory of binary relations* the set $\mathcal{R}el$ of valid inequations. (We focus on inequations in the present work; note however that those are equivalent to equations: we have $\mathcal{R}el \models u \leq v$ iff $\mathcal{R}el \models u \cap v = u$, where the latter symbol is defined as expected.)

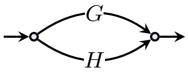
2.3 Graphs

As explained in the introduction, terms also make it possible to denote graphs – more precisely, directed multigraphs with edges labelled in A and two designated vertices. Formally, those are tuples $\langle V, E, s, t, l, \iota, o\rangle$ with V (resp. E) a finite set of vertices (resp. edges), $s, t : E \to V$ the *source* and *target* functions, $l : E \to A$ the *labelling* function, and $\iota, o \in V$ two distinguished vertices, respectively called *input* and *output*. We simply call them *graphs* in the sequel; we depict them as expected, with unlabelled ingoing and outgoing arrows to denote the input and the output, respectively.

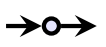
Vertices distinct from input and output are called *inner* vertices. A vertex without incident edges is *isolated*.

Graphs can be composed in series or in parallel, as depicted below:

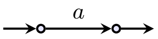
$G \cdot H \triangleq$ 

$G \cap H \triangleq$ 

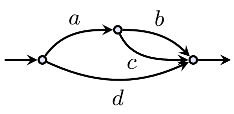
Those operations do have neutral elements, which are edge-less graphs:

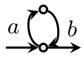
$1 \triangleq$  $\top \triangleq$ 

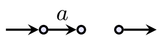
We can thus recursively associate to every term u a graph $\mathcal{G}(u)$ called the *graph of* u, where the graph of a letter $a \in A$ is

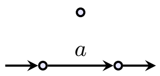
$\mathcal{G}(a) \triangleq$ 

Here are, from left to right, the graphs of $a\,(b \cap c) \cap d$, $ab \cap 1$, $a\top$ and $a \cap \top\top$:









We say that a graph is SP (resp. $\mathsf{SP1}$, $\mathsf{SP}\top$, $\mathsf{SP1}\top$) if it is the graph of some SP (resp. $\mathsf{SP1}$, $\mathsf{SP}\top$, $\mathsf{SP1}\top$) term. The SP graphs are the acyclic and series-parallel graphs with all edges directed from the input towards the output. In a $\mathsf{SP1}$ graph, every vertex belongs to a directed path from the input towards the output; unlike SP graphs, they may contain cycles. In contrast, $\mathsf{SP}\top$ graphs remain acyclic but they are not necessarily connected. $\mathsf{SP1}\top$ have treewidth at most two, i.e., they are K_4-free [8].

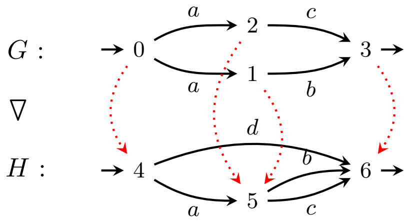
2.4 Homomorphisms

Graph homomorphisms play a central role in the paper; they are defined as follows:

▶ **Definition 1** (Graph homomorphism). *Given two graphs* $G = \langle V, E, s, t, l, \iota, o\rangle$ *and* $G' = \langle V', E', s', t', l', \iota', o'\rangle$*, a* (graph) homomorphism $h : G \to H$ *is a pair* $\langle h_v, h_e\rangle$ *of functions* $h_v : V \to V'$ *and* $h_e : E \to E'$ *that respect the various components:* $s' \circ h_e = h_v \circ s$, $t' \circ h_e = h_v \circ t$, $l = l' \circ h_e$, $\iota' = h_v(\iota)$*, and* $o' = h_v(o)$*.*

We write $H \lhd G$ *if there exists a graph homomorphism from* G *to* H*. Like other relations on graphs, we sometimes use this relation directly on terms, writing* $u \lhd v$ *for* $\mathcal{G}(u) \lhd \mathcal{G}(v)$*.*

The vertex component of such a homomorphism is depicted below



A pleasant way to think about graph homomorphisms is the following: we have $H \lhd G$ if H is obtained from G by merging (or identifying) some vertices and some edges, and by adding some extra vertices and edges. For instance, the graph H in the example above is obtained from G by merging vertices 1 and 2 and the two a-labelled edges, and by adding a d-labelled edge from the input to the output.

We write $h \circ g$ for the pointwise composition of the two components of two homomorphisms, which yields a homomorphism as expected.

A homomorphism is *injective* (resp. *surjective*, *bijective*) when its two components are so. We write $G \hookrightarrow H$ if there exists an injective homomorphism from G to H; in such a case, we say that G is a *subgraph* of H. We write $G \simeq H$ when there exists a bijective homomorphism from G to H (an *isomorphism*). The aforementioned intuition about homomorphisms is reflected by the epi-mono factorisation property: every homomorphism $h : G \to H$ factors uniquely into a surjective homomorphism followed by an injective homomorphism:

$$G \twoheadrightarrow h(G) \hookrightarrow H$$

The intermediate graph $h(G)$ is the *image of* h; it is a subgraph of H by definition.

The key result we exploit in the present paper is the following characterisation:

▶ **Theorem 2** ([1, Thm. 1], [13, p. 208]). *For all terms* u, v, $\mathcal{R}el \models u \leq v$ *iff* $u \lhd v$*.*

Thus, analysing the inequational theory of binary relations amounts to analysing homomorphism between graphs denoted by terms.

2.5 Closure under taking subgraphs

We show below that, SP⊤ (resp. SP1⊤), seen as a class of graphs, is the closure of SP (resp. SP1) under taking subgraphs. This property is convenient in the sequel.

▶ **Proposition 3.** *G is* SP⊤ *(resp.* SP1⊤*) iff G is a subgraph of a* SP *(resp.* SP1*) graph.*

Proof. We can reason mostly on terms. The forward implication is easy: given a $\mathsf{SP}\top$ (resp. $\mathsf{SP1}\top$) term u, replacing all occurrences of $\top$ with an arbitrary letter yields a SP (resp. $\mathsf{SP1}$) term u' such that $\mathcal{G}(u) \hookrightarrow \mathcal{G}(u')$.

For the converse implication, we proceed in two steps. Assume $h : G \hookrightarrow \mathcal{G}(u')$ for some SP (resp. $\mathsf{SP1}$) term u'. First observe that the edges in $\mathcal{G}(u')$ are in one-to-one correspondence with the occurrences of letters in u'. By replacing with $\top$ all occurrences of letters in u' that are not in the image of h through this correspondence, we obtain a $\mathsf{SP}\top$ (resp. $\mathsf{SP1}\top$) term u_0 such that h corestricts to $h_0 : G \hookrightarrow \mathcal{G}(u_0)$, which is actually bijective on edges. It remains to get rid of the vertices $\mathcal{G}(u_0)$ which are not in the image of h_0. Those are necessarily isolated inner vertices in $\mathcal{G}(u_0)$ since h_0 is bijective on edges. Roughly speaking, those arise via subterms of the shape $\top\top$ in u_0, which we can replace with $\top$ to obtain a $\mathsf{SP}\top$ (resp. $\mathsf{SP1}\top$) term u whose graph is G. The formal argument is slightly more involved; we give it in [10, Appendix A] ◀

▶ **Corollary 4.** *The classes of graphs* $\mathsf{SP}\top$ *and* $\mathsf{SP1}\top$ *are closed under taking subgraphs.*

2.6 Inequational reasoning

Let us define what we mean by axiomatisation in the present context, where we focus on inequations rather than equations.

Assume a signature Σ, and consider in this subsection terms u, v built over this signature and variables in the alphabet A. We let σ, θ range over *substitutions* assigning a terms to letters in A, and we write $u\sigma$ for the result of applying such a substitution σ to a term u. A *renaming* is a possibly non-injective substitution whose range consists only of letters. We let C range over *contexts*, i.e., terms with exactly one occurrence of a special letter $\bullet$ called the *hole*. We write $C[u]$ for the term obtained by replacing the hole of a context C by a term u.

An *inequation* is a pair of terms, which we denote by $u \leq v$. An *inequational theory* is a set of inequations which forms a pre-order and which is stable under contexts and substitutions. For instance, the set of inequations such that $\mathcal{R}el \models u \leq v$ is an inequational theory.

Given a set $\mathcal{H}$ of inequations, the *axioms*, the *inequational theory of* $\mathcal{H}$ is the least inequational theory containing $\mathcal{H}$. We write $\mathcal{H} \vdash u \leq v$ when the inequation $u \leq v$ belongs to the inequational theory of $\mathcal{H}$, or, equivalently, if it can be derived using the following rules:

$$\frac{}{u \leq v}\;\mathcal{H} \qquad \frac{}{u \leq u} \qquad \frac{u \leq v \quad v \leq w}{u \leq w} \qquad \frac{u \leq v}{u\sigma \leq v\sigma} \qquad \frac{u \leq v}{C[u] \leq C[v]}$$

An inequational theory is *finitely based* if it can be generated by a finite set of axioms.

Standard algebraic structures

Inequational theories as defined above can be presented as equational theories as soon as the signature Σ contains a binary symbol $\cap$ and $\mathcal{H}$ contains the following finite set of inequations:

$$\mathcal{P} \triangleq \{a \leq a \cap a,\ \ a \cap b \leq a,\ \ a \cap b \leq b\}$$

Indeed, in such a case, $\cap$ turns the partial order $\leq$ into an *inf-semilattice*, and inf-semilattices can be defined algebraically as commutative idempotent semigroups: the partial order can be defined as $u \leq v \triangleq (u \cap v = u)$. The other operations in the signature must all be monotone, which can be expressed algebraically by adding equations of the form $f(a \cap b) \cap f(a) = f(a)$, say, for a unary symbol f. Conversely, any equational theory with a commutative idempotent semigroup symbol and where all operations are monotone w.r.t. the associated partial order can be represented as an inequational theory in the previous sense. (Those conversions preserve the finiteness of the considered set of axioms, so that an inequational theory is finitely based iff its associated equational theory is finitely based, and vice versa.)

In particular, we capture other standard algebraic structures as follows. Define the following (finite) sets of inequations, where an equation is a shorthand for the corresponding two inequations:

$$\mathcal{SP} \triangleq \mathcal{P} \cup \{a \cdot (b \cdot c) = (a \cdot b) \cdot c\} \qquad \mathcal{SP}^{\top} \triangleq \mathcal{SP} \cup \{a \leq \top\}$$

$$\mathcal{SP}^{1} \triangleq \mathcal{SP} \cup \{a \cdot 1 = a, 1 \cdot a = a\} \qquad \mathcal{SP}^{1\top} \triangleq \mathcal{SP}^{1} \cup \mathcal{SP}^{\top}$$

(Note that these sets are implicitly associated to the four signatures we consider in the present paper: for instance, when writing $\mathcal{SP}^1 \vdash u \leq v$, we mean that u and v are SP1 terms and that the derivation mentions only SP1 terms, contexts, and substitutions.)

We have that

- $\mathcal{SP}$ axiomatises *semilattice-ordered semigroups (sl-semigroups)*;
- $\mathcal{SP}^1$ axiomatises *semilattice-ordered monoids (sl-monoids)*;
- $\mathcal{SP}^{\top}$ axiomatises *bounded-semilattice-ordered semigroups (bsl-semigroups)*;
- $\mathcal{SP}^{1\top}$ axiomatises *bounded-semilattice-ordered monoids (bsl-monoids)*.

Axiomatisability of relations

Given a subsignature X of SP1⊤, we say that a set $\mathcal{H}$ of inequations on X *axiomatises relations on* X if

$$\text{for all terms } u, v \text{ on } X, \qquad \mathcal{H} \vdash u \leq v \quad \text{iff} \quad \mathcal{R}el \models u \leq v \ .$$

Bredihin and Schein proved that the equational theory of relations on SP coincides with that of sl-semigroups [3], which means in the above terminology that $\mathcal{SP}$ axiomatises relations on SP.

In contrast, $\mathcal{SP}^1$ and $\mathcal{SP}^{\top}$ do not suffice to axiomatise relations on SP1 or SP⊤. For instance, relations satisfy the laws below (by Theorem 2, this can be proved by providing appropriate homomorphisms – most of them actually are isomorphisms here) but there are bsl-monoids violating those laws[1].

$$\mathcal{R}el \vdash a\top \cap bc = (a\top \cap b)c \qquad \mathcal{R}el \vdash \top a \top b \top = \top b \top a \top$$

$$\mathcal{R}el \vdash a \cap b \cap 1 = (a \cap 1)(b \cap 1) \qquad \mathcal{R}el \vdash (a \cap 1)b \cap c = (a \cap 1)(b \cap c) \qquad \mathcal{R}el \vdash a \cap 1 \leq aa$$

In fact, as shown in the sequel, $\mathcal{R}el$ is not finitely based on SP1, SP⊤, and SP1⊤ (so that the corresponding equational theories are not finitely based either).

3 Decomposability

We fix a signature $X \in \{\mathsf{SP}, \mathsf{SP1}, \mathsf{SP}\top, \mathsf{SP1}\top\}$ in this section, and we provide a necessary condition for finite axiomatisability of relations on X (and thus existence of graph homomorphisms between X graphs). This is essentially the same condition as the one used by Freyd and Scedrov for representable allegories [13, pp 208–210]. We generalise it here so that it fits our needs, providing a different proof and more explicit treatment.

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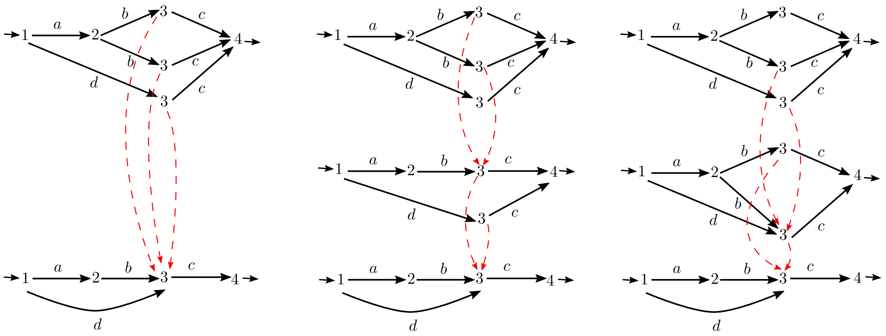
[1] even finite ones, that can easily be found with tools such as Mace4.

Recall that a homomorphism can be seen as the action of merging several vertices and edges of the source graph and adding some vertices and edges. The *degree* of a homomorphism is the number of vertices it merges:

▶ **Definition 5** (Degree). *Let $h = (h_e, h_v)$ be a graph homomorphism. The* degree *of h, denoted $\deg(h)$ is the number $\#\{u \mid \exists w, w \neq u \text{ and } h_v(u) = h_v(w)\}$. We write $H \lhd_n G$ when there exists $h : G \to H$ with $\deg(h) \leq n$.*

Since the vertices of a graph can always be merged two at a time, every homomorphism can be decomposed into a sequence of homomorphisms of degree at most two. However, intermediate graphs in this decomposition are not necessarily in X, even if the endpoints are.

For instance, we depict on the left below a homomorphism of degree three between two SP graphs (top-down). This homomorphism can be decomposed into two sequences of homomorphisms of degree two, given in the middle and on the right. The intermediate graph in the middle is SP, while the intermediate graph on the right is not SP.



We write $u \lhd_n^X v$ when u, v are terms of X and $\mathcal{G}(u) \lhd_n \mathcal{G}(v)$; we write $\lhd_n^{X*}$ for the reflexive transitive closure of this relation.

In the above example, the valid law corresponding to $(ab \cap d)c \lhd a(bc \cap bc) \cap dc$ can be decomposed into two valid laws $(ab \cap d)c \lhd_2^{\mathsf{SP}} abc \cap dc$ and $abc \cap dc \lhd_2^{\mathsf{SP}} a(bc \cap bc) \cap dc$. These latter laws are intuitively simpler: they can be justified by homomorphisms with a smaller degree.

We need the following assumption about X to obtain Proposition 8 below.

▶ **Assumption 6.** *There is an integer k such that for every term u of X, we have $u \lhd_k^{X*} u \cap u$.*

This is a rather mild assumption, which is satisfied in the context of the present paper:

▶ **Fact 7.** *Assumption 6 is satisfied with $k = 2$ for the four values of X considered here.*

Proof. By a straightforward induction on u in each case. ◀

▶ **Proposition 8.** *Under Assumption 6, if $\mathcal{R}el$ is finitely based on X, then there exists an integer n such that for all terms u, v in X, $u \lhd v$ entails $u \lhd_n^{X*} v$.*

Proof. Suppose that we have a finite axiomatisation $\mathcal{H}$. By soundness (and Theorem 2), each axiom of $\mathcal{H}$ gives rise to a graph homomorphism. Let n' be the maximal degree of these homomorphisms, and let $n = \max\{k, n'\}$. Now suppose $u \lhd v$ for some terms u, v of X. By completeness (and Theorem 2), we get a derivation $\mathcal{H} \vdash u \leq v$. We prove $u \lhd_n^{X*} v$ by induction on this derivation. The only interesting case is that of the substitution rule. In this case, we have $u \lhd_n^{X*} v$ by induction, and we must prove $u\sigma \lhd_n^{X*} v\sigma$ for a given substitution σ. W.l.o.g., we can assume $u \lhd_n^X v$, and we consider the underlying homomorphism $h : v \to u$, of degree at most n. Observe that h can be extended into a homomorphism $h\sigma : v\sigma \to u\sigma$. If h is injective on edges, then $\deg(h\sigma) = \deg(h) \leq n$ and we are done. If instead h merges two a-labelled edges, then the degree of $h\sigma$ is at least the number of inner vertices of $\sigma(a)$, which is not bounded by n, *a priori*. In this latter case, we use the homomorphism $h' : v \to u'$ obtained from h by duplicating all edges in the image graph as many times as necessary to get injectivity on edges. The corresponding term u' is obtained from u be replacing some occurrences of letters, say a, with intersections of the same letter (e.g., $a \cap a \cap a$). The homomorphism $h'\sigma : v\sigma \to u'\sigma$ satisfies $\deg(h'\sigma) = \deg(h') = \deg(h) \leq n$, so that $u'\sigma \lhd_n^{X^*} v\sigma$ We finally obtain $u\sigma \lhd_k^{X^*} u'\sigma$ by repeatedly using Assumption 6. ◀

By contraposition, to prove non-finite-axiomatisability, it suffices to find a sequence of homomorphisms $(e_n \lhd f_n)_{n\in\omega}$ between terms of X such that for all n, $e_n \lhd_n^{X^*} f_n$ does not hold. We define such a sequence in the following section, for SP1 and SP1⊤.

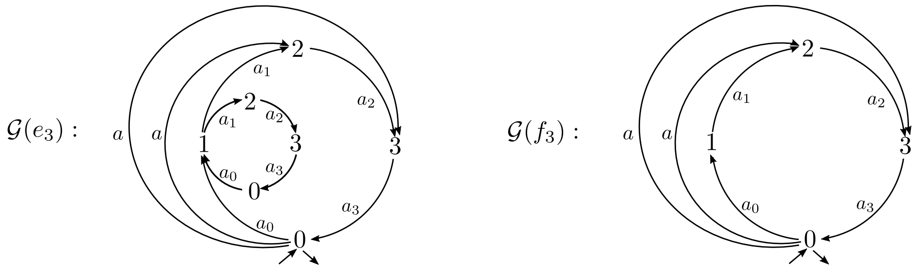
4 The fragments SP1 and SP1T

We show in this section that $\mathcal{R}el$ is not finitely based on SP1 and SP1⊤.

▶ **Definition 9.** *Let $a, a_0, a_1, \ldots$ be a fixed sequence of pairwise disjoint letters, and let n be a strictly positive integer. Let c_n be the term $(a_1a_2\ldots a_na_0) \cap 1$. We define the terms e_n and f_n with the help of the families $(g_i^n)_{i\in[1,n-1]}$ and $(h_i^n)_{i\in[1,n-1]}$ respectively as follows:*

$$g_0^n = a_0c_n, \qquad g_{i+1}^n = (g_i^na_{i+1}) \cap a, \qquad e_n = (g_{n-1}^na_n) \cap 1.$$
$$h_0^n = a_0, \qquad h_{i+1}^n = (h_i^na_{i+1}) \cap a, \qquad f_n = (h_{n-1}^na_n) \cap 1.$$

For instance, the graphs of e_3 and f_3 are:



There is a homomorphism from $\mathcal{G}(e_3)$ to $\mathcal{G}(f_3)$, which maps the nodes of $\mathcal{G}(e_3)$ tagged by $i \in [0,3]$ to the node of $\mathcal{G}(f_3)$ tagged i. More generally, it is not hard to see that, for every $n \in \omega$, there is a (unique) homomorphism from $\mathcal{G}(e_n)$ to $\mathcal{G}(f_n)$, that is $f_n \lhd e_n$. Let us state the main proposition of this section:

▶ **Proposition 10.** *For every $n \in \omega$, $f_n \not\lhd_n^{\mathsf{SP1}\top *} e_n$.*

Together with Proposition 8, this entails that $\mathcal{R}el$ is not finitely based on SP1⊤. Since e_n and f_n actually are SP1 terms, this also entails that it is not finitely based on SP1.

We prove Proposition 10 below, by contradiction. In order to ease this proof, we first establish a property verified by SP1⊤ graphs; this will allow us to reach a contradiction in the two main cases of the proof.

▶ **Definition 11** (Back pattern). *A* back pattern *in a graph is a pair of distinct nodes m, n together with three directed paths: π from the input to m, κ from n to m, and ρ from n to the output, such that π and κ intersect exactly on m and κ and ρ intersect exactly on n.*

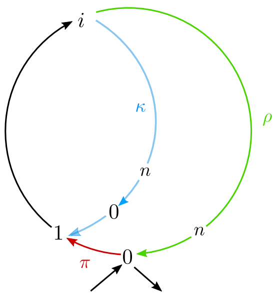


Figure 1 First case in proof of Prop. 10.

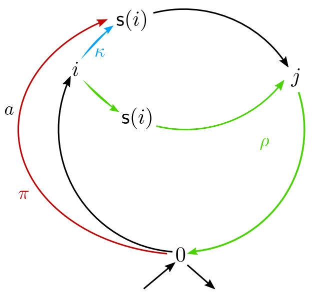


Figure 2 Second case in proof of Prop. 10.

Such a back pattern can be depicted as follows: $\iota \xrightarrow{\pi} m \xleftarrow{\kappa} n \xrightarrow{\rho} o$. They cannot arise in SP1⊤ graphs. Intuitively, although SP1 graphs are not acyclic (unlike SP graphs), they are nevertheless oriented from the input towards the output.

▶ **Proposition 12.** SP1⊤ *graphs do not contain back patterns.*

A proof is given found in [10, Appendix B]; it is an easy induction on the structure of SP1⊤ terms, after adding some other forbidden patterns for the induction to go through. Proposition 12 can be extended to characterise SP1 graphs (and thus SP1⊤ graphs via Corollary 4) using a slight relaxation of the definition of back patterns. Such a characterisation is not needed here, however, and we can now prove Proposition 10.

Proof of Proposition 10. Let h be the (unique) homomorphism from $\mathcal{G}(e_n)$ to $\mathcal{G}(f_n)$. To simplify the presentation, we label the nodes of e_n and f_n by integers from $[0, n]$, in the same way as in the above example for $n = 3$: the input is labelled 0, and for every $i \in [0, n-1]$, if a node is labelled i, then its a_i successors are labelled $i + 1$. Note that, for every $i \in [1, n]$, there is exactly one node labelled i in f_n; its pre-image by h consists of those nodes labelled by i in $\mathcal{G}(e_n)$.

Suppose by contradiction that $(f_n, e_n) \in \lhd_n^*$. Thus, we can find SP1⊤ terms $(g_i)_{i \in [0,m+1]}$ and homomorphisms $(h_i : g_i \to g_{i+1})_{i \in [0,m]}$ of degree at most n, and such that $g_0 = e_n$ and $g_{m+1} = f_n$. The composition of the homomorphisms $(h_i)_{i \in [0,m]}$ yields h.

Let k be an index such that $e_n \hookrightarrow g_k$ and $e_n \not\hookrightarrow g_{k+1}$. This index exists since $e_n \hookrightarrow g_0$ and $e_n \not\hookrightarrow g_{m+1}$. As $\mathcal{G}(e_n)$ is a sub-graph of $\mathcal{G}(g_k)$, we label the nodes of $\mathcal{G}(g_k)$ accordingly: keep the same labels for nodes in $\mathcal{G}(e_n)$, and do not label the other nodes.

The fact that $e_n \not\hookrightarrow g_{k+1}$ means that the homomorphism h_k merged some labelled nodes. Note that h_k cannot merge nodes tagged by different integers. Otherwise, the composition of the homomorphisms $(h_i)_{i \in [0,m]}$ would also merge nodes tagged by different labels, which is not possible. We label the nodes of $\mathcal{G}(g_{k+1})$ according to h_k: a node is labelled i if it is the image of a node labelled i by h_k.

Let us show that g_{k+1} cannot be an SP1⊤ term. Let i be the largest integer in $[1, n]$ such that for every $j \in [1, i]$ the nodes labelled by j in $\mathcal{G}(g_k)$ have been merged by h_k.

We define the function $\mathsf{s} : [1, n] \to [0, n]$ as follows: $\mathsf{s}(x) = x + 1$ if $x \in [1, n-1]$ and $\mathsf{s}(n) = 0$. Note that the nodes labelled by $\mathsf{s}(i)$ in $\mathcal{G}(g_k)$ are not merged by h_k. If this was the case, then either $i \in [1, n-1]$, which would contradict maximality of i, or $i = n$, meaning that the degree of the homomorphism is at least $n + 1$, contradicting our hypothesis.

We distinguish two cases:

- For every $j \in [i + 1, n] \cup \{0\}$, the nodes labelled j in $\mathcal{G}(g_k)$ are not merged by h_k. In this case, $i \neq 1$, otherwise no labelled nodes of $\mathcal{G}(g_k)$ would be merged by h_k. There are two distinct directed paths from i to 1 in $\mathcal{G}(g_{k+1})$: one that visits the input and one that does not. We name the first one κ. The second can be decomposed into a path from i to the input, we name it ρ, and a path from the input to 1, we name it π.

Then the nodes labelled i and 1, together with the paths π, κ, ρ form a back-pattern in $\mathcal{G}(g_{k+1})$, as illustrated in Figure 1.

- There is $j \in [i+1, n] \cup \{0\}$ such that the nodes labelled j are merged. Recall that the nodes labelled $\mathsf{s}(i)$ are not merged by h_k. We call m the node of $\mathcal{G}(g_{k+1})$, which is labelled $\mathsf{s}(i)$ and which is the a successor of the input. Note that since the nodes labelled j are merged, there is a path in $\mathcal{G}(g_{k+1})$ connecting the node labelled i to the input (which in this case coincide with the output), which does not go through m. We call this path ρ. We call π the path labelled a from the input to m and κ the path labelled a_i from i to $\mathsf{s}(i)$. The nodes $\mathsf{s}(i)$ and i, together with the paths π, κ and ρ form a back-pattern in $\mathcal{G}(g_{k+1})$, as illustrated in Figure 2. ◀

5 The fragment SPT

We show in this section that $\mathcal{R}el$ is not finitely based on $\mathsf{SP}\top$. We do not adopt the same strategy as for $\mathsf{SP1}$[2]. Instead, our proof in this case is in two steps. First, we show that every axiomatisation can be turned into one with a very constrained shape, called *simple axiomatisation*. In a second step, we exhibit an infinite collection of inequations $(f_n \leq e_n)_{n\in\omega}$ which any simple axiomatisation should contain in a certain sense.

5.1 Dealing with idempotency

Simple axiomatisations, which we introduce in the following, are axiomatisations where idempotency of intersection is used in a very controlled way, following Freyd and Scedrov' idea of *separatedness* [13, page 208].

We call *idempotency axiom* an inequation of the form $a \leq a \cap a$ for some letter a.

▶ **Definition 13.** *A term v is* simple *if every letter appears at most once in v. An inequation $u \leq v$ is* simple *if v is simple. An axiomatisation $\mathcal{H}$ is* simple *if contains only simple axioms and idempotency axioms.*

The key intuition about simple axioms is that the corresponding homomorphisms cannot merge edges; in a sense, they are idempotency-free.

For every term v, there is a simple term v' and a renaming θ_v such that $v = v'\theta_v$ (e.g., for $v = aa$, take $v' = a_1a_2$ and $\theta_v = \{a_1, a_2 \mapsto a\}$). In such a case, write θ_v^{-1} for the following substitution: $\theta_v^{-1}(a) = \bigcap_{\theta_v(a')=a} a'$. For every term u such that $u \lhd v$, we have $u\theta_v^{-1} \lhd v'$.

We can thus turn any sound axiomatisation $\mathcal{H}$ into a simple axiomatisation $\mathcal{H}'$ as follows: adjoin an idempotency axiom, and replace each non-simple axiom $u \leq v$ of $\mathcal{H}$ with $u\theta_v^{-1} \leq v'$. Every axiom of $\mathcal{H}$ is derivable in $\mathcal{H}'$ using its simple counterpart and the idempotency axiom; therefore if $\mathcal{H}$ axiomatises $\mathcal{R}el$ then so does $\mathcal{H}'$. Note that $\mathcal{H}'$ is finite whenever $\mathcal{H}$ is finite.

Idempotency can thus be isolated from the other axioms. We go further and show that its use may actually be pushed towards the leaves. Let $\mathcal{I}$ be the following set of inequations

$$a \leq a \cap a \qquad \top \leq \top \cap \top \qquad (a \cap c)(b \cap d) \leq ab \cap cd \qquad (a \cap c) \cap (b \cap d) \leq (a \cap b) \cap (c \cap d)$$

These axioms are sound w.r.t. $\mathcal{R}el$; we need these axioms to obtain Proposition 15 below: every derivation can be seen as a sequence of rewriting steps where idempotency axioms are used only on letters (i.e., to merge parallel edges).

---

[2] We actually conjecture that for all terms u, v in $\mathsf{SP}\top$, $u \lhd v$ entails $u \lhd_2^{\mathsf{SP}\top *} v$.

▶ **Definition 14.** *Given two terms* e, f*, if* $e = C[u\sigma]$ *and* $f = C[v\sigma]$ *for some context* C*, substitution* σ*, and terms* u, v*, we say that* (C, σ) *is a* unifying context-substitution for e and f with inner terms u and v*.*

▶ **Proposition 15.** *If* $\mathcal{I} \subseteq \mathcal{H}$ *and* $\mathcal{H} \vdash f \leq e$*, then there is a sequence* $g_0, \dots, g_m$ *of terms such that* $e = g_0$*,* $g_m = f$*, and for all* $i < m$*, there is a unifying context-substitution for* g_i *and* g_{i+1} *with inner terms* u *and* v *such that* $(u \leq v) \in \mathcal{H}$*, and the substitution is a renaming if the latter axiom is an idempotency axiom.*

Proof. A simple induction on the derivation yields a sequence as in the statement, but without the constraint idempotency axioms. We refine this sequence by using the following property:

> For every SP⊤ term e there is a sequence $g_0, \dots, g_m$ of terms such that $e \cap e = g_0$, $g_n = e$, and for all $i < m$, there is a unifying context-substitution for g_i and g_{i+1} with inner terms u and v such that $(u \leq v) \in \mathcal{I}$, and the substitution is a renaming if the latter axiom is the idempotency axiom of $\mathcal{I}$.

This property is proved by an easy induction on e, using in each case the corresponding axiom of $\mathcal{I}$, e.g., in the product case,

$$ef \leq (e \cap e)f \leq (e \cap e)(f \cap f) \leq ef \cap ef$$

where the first two (sequences of) steps are obtained by induction hypothesis and the third step is an instance of the third axiom of $\mathcal{I}$. ◀

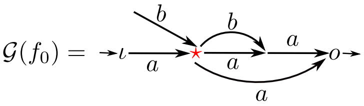
5.2 The counter-example

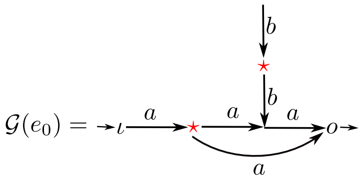
We can finally give the counter-example.

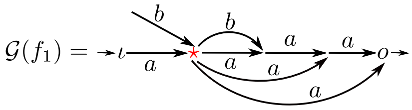
▶ **Definition 16.** *Let* a, b *be fixed letters. Given* $n \in \omega$*, the* SP⊤ *terms* e_n *and* f_n *are defined as follows, with the help of two sequences* $(u_i^n)_{i \leq n}$ $(v_i^n)_{i \leq n}$ *parameterised by* n*:*

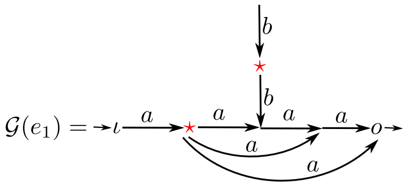
$$u_0^n = (\top bb \cap a)a \cap a \qquad u_{i+1}^n = (u_i^n a) \cap a \qquad e_n = a u_n^n$$
$$v_0^n = (a \cap b)a \cap a \qquad v_{i+1}^n = (v_i^n a) \cap a \qquad f_n = (\top b \cap a) v_n^n$$

The graphs of f_0, e_0 and f_1, e_1 are depicted below.









There is a unique homomorphism $!_n$ from e_n to f_n. This homomorphism is surjective, it just merges two vertices of the graph of e_n, which are tagged by red stars in the picture above.

The key property of the sequences $(e_n)_{n \in \omega}$ and $(f_n)_{n \in \omega}$ is stated in Proposition 18 below: every inequation $e_n \leq f_n$ can be injected up-to renaming into an inequation of $\mathcal{H}$.

▶ **Definition 17.** *We say that v* injects up-to renaming into u, *written $v \hookrightarrow_\alpha u$, if $v \hookrightarrow u\theta$ for some renaming θ.*

▶ **Proposition 18.** *If $\mathcal{H}$ is a simple axiomatisation of relations on* $\mathsf{SP}\top$*, then for every $n \in \omega$, there are* $\mathsf{SP}\top$ *terms e'_n and f'_n such that $f_n \hookrightarrow_\alpha f'_n$, $e_n \hookrightarrow_\alpha e'_n$ and $(f'_n \leq e'_n) \in \mathcal{H}$.*

As the size of the graphs of e_n and f_n grows infinitely, the set of inequations $(f'_n \leq e'_n)_{n\in\omega}$ is infinite. Thus $\mathcal{R}el$ is not finitely based on $\mathsf{SP}\top$. (Note that the above proposition is stronger than necessary: we could focus on one side of the equations.)

The proof of Proposition 18 relies on the three following lemmas, whose proofs can be found in [10, Appendix C]. The first one says that if $!_n : e_n \twoheadrightarrow f_n$ decomposes into a sequence of homomorphisms, then of one them must essentially act like $!_n$ under some irrelevant context.

▶ **Lemma 19.** *Fix $n \in \omega$ and assume $!_n : e_n \twoheadrightarrow f_n$ decomposes into a sequence $h_m \circ \cdots \circ h_0$. For $i < m$, write $h_{<i}$ for the partial composition $h_{i-1} \circ \cdots \circ h_0$. There exists an index $i < m$ such that $h_{<i}$ is injective and the image of $h_{<i+1}$ is the graph of f_n.*

In other words, in the above statement, $h_{<i+1}$ factors through f_n as in the diagram below.

$$\begin{tikzcd}
f_n \arrow[dr, hook] & & & e_n \arrow[lll, two heads, "!_n"'] \arrow[dl, hook', "h_{<i}"] \\
& \cdot & \cdot \arrow[l, "h_i"'] &
\end{tikzcd}$$

The second lemma essentially says that if there is a unifying context-substitution (Definition 14) for e_n and f_n whose inner terms are related by a homomorphism, then the context is necessarily trivial and e_n and f_n can be injected up-to renaming into the inner terms. We need to be slightly more flexible and we use the following notation: we write $u \mathrel{\dot{\hookrightarrow}} v$ if there is a homomorphism from u to v which is bijective on edges and injective on vertices. In other words, $u \mathrel{\dot{\hookrightarrow}} v$ when v is u with some additional isolated vertices.

▶ **Lemma 20.** *Fix $n \in \omega$ and suppose that there is a context C, a substitution σ and two terms u and v such that $f_n \mathrel{\dot{\hookrightarrow}} C[v\sigma]$, $e_n \mathrel{\dot{\hookrightarrow}} C[u\sigma]$, and $v \lhd u$. Then $f_n \hookrightarrow_\alpha v$ and $e_n \hookrightarrow_\alpha u$.*

The last ingredient says that when a term e can be injected in a term of the form $C[u\sigma]$, for a simple term u, then we can restrict C, u and σ to match e up to some isolated vertices.

▶ **Lemma 21.** *If $\iota : e \hookrightarrow C[u\sigma]$ with u simple, then there is a context $C' \hookrightarrow C$, a substitution $\sigma' \hookrightarrow \sigma$ and a term $u' \hookrightarrow u$ such that ι decomposes into $e \mathrel{\dot{\hookrightarrow}} C'[u'\sigma'] \hookrightarrow C[u\sigma]$. (Where we extend $\hookrightarrow$ to substitutions componentwise: $\sigma \hookrightarrow \gamma$ if $dom(\sigma) \subseteq dom(\gamma)$ and $\forall a \in dom(\sigma), \sigma(a) \hookrightarrow \gamma(a)$.)*

Note that the requirement that u must be simple cannot be dropped (because, e.g., $bc \hookrightarrow (b \cap c)(b \cap c) = (aa)\{a \mapsto b \cap c\}$).

Proof of Proposition 18. Let $\mathcal{H}$ be a simple axiomatisation of relations on $\mathsf{SP}\top$. Since $\mathcal{I}$ is simple and none of the $f_n \leq e_n$ injects up-to renaming into $\mathcal{I}$, we can assume w.l.o.g that $\mathcal{H}$ contains $\mathcal{I}$.

Let $n \in \omega$. In the rest of this proof we write e, f, and $!$ for e_n, f_n, and $!_n$, respectively.

Since $f \lhd e$, we have $\mathcal{H} \vdash f \leq e$ by completeness (and Theorem 2). By Proposition 15, we obtain a sequence $g_0, \ldots, g_m$ of terms such that $e = g_0$, $g_m = f$, and for all $i < m$, there is a unifying-context substitution (C_i, σ_i) for g_i and g_{i+1} with inner terms u_i and v_i (i.e., $g_i = C_i[u_i\sigma_i]$ and $g_{i+1} = C_i[v_i\sigma_i]$) such that either $(v_i \leq u_i) \in \mathcal{H}$ is simple or this is an idempotency axiom and σ_i is a renaming.

By soundness (and Theorem 2), there are homomorphisms $h_i : u_i \to v_i$ for all $i < m$. Each of these homomorphisms can be extended, through the context C_i and the substitution σ_i, into a homomorphism $h_i^\uparrow : g_i \to g_{i+1}$. The composition of these homomorphisms must be the only homomorphism $! : e \twoheadrightarrow f$. Hence, by Lemma 19, there is an index i such that $h_{<i}^\uparrow$ is injective and the image of $h_{<i+1}^\uparrow$ is f. Since h_i merges two vertices, it cannot be the case that this step is an idempotency step: since those are restricted to letters, they only merge edges. Therefore u_i must be simple.

Since $h_{<i}^\uparrow : e \hookrightarrow g_i = C_i[u_i\sigma_i]$, Lemma 21 gives us a context $C \hookrightarrow C_i$, a substitution $\sigma \hookrightarrow \sigma_i$ and a term $u \hookrightarrow u_i$ such that $\iota(e) \dot{\hookrightarrow} C[u\sigma]$. Call ι the injective homomorphism $u \hookrightarrow u_i$.

The image of $h_i \circ \iota : u \to v_i$, as a subgraph of v_i, must be a SP⊤ term v by Proposition 3. This decomposition corresponds to the commuting diagram on the left below, and assembling the various ingredients collected so far, we obtain the commuting diagram on the right.

$$\begin{tikzcd}
v_i & u_i \arrow[l, "h_i"'] & u \arrow[l, hook', "\iota"'] \arrow[dl, two heads] \\
& v \arrow[ul, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
C_i[v_i\sigma_i] & C_i[u_i\sigma_i] \arrow[l, "h_i^\uparrow"'] & & C[u\sigma] \arrow[ll, hook'] \arrow[dll, two heads] & e \arrow[l, hook', "\cdot"] \arrow[lll, hook', bend right, "h_{<i}^\uparrow"'] \\
& C[v\sigma] \arrow[ul, hook]
\end{tikzcd}$$

Since f is the image of $h_{<i+1}^\uparrow$, we deduce $f \dot{\hookrightarrow} C[v\sigma]$.

We can finally use Lemma 20 to deduce $e \hookrightarrow_\alpha u$ and $f \hookrightarrow_\alpha v$. As $u \hookrightarrow u_i$ and $v \hookrightarrow v_i$, we also have that $e \hookrightarrow_\alpha u_i$ and $f \hookrightarrow_\alpha v_i$. Since $(v_i \leq u_i) \in \mathcal{H}$ this concludes the proof. ◀

6 Graph theoretical characterisation for natural structures

The characterisation of $\mathcal{R}el$ in terms of graph homomorphisms (Theorem 2) works for all SP1⊤ terms. This inequational theory coincides with that of sl-semigroups ($\mathcal{SP}$) for SP terms, but we have seen that it departs from related algebraic structures (sl-monoids, bsl-semigroups and bsl-monoids) for SP1, SP⊤, and SP1⊤ terms. We show in this section that we can nevertheless obtain simple graph theoretical characterisations of the (in)equational theory of those three algebraic structures.

Let us focus on the SP case first, for which the notions we have used so far just work:

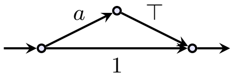
▶ **Proposition 22.** *For all $u, v \in$ SP, $\mathcal{SP} \vdash u \leq v$ iff $\mathcal{G}(u) \lhd \mathcal{G}(v)$.*

This result is a consequence of Theorem 2 and [3], but we give below a direct proof due to Brunet [5]. We say that a graph *goes forward* if every vertex belongs to a simple directed path from the input to the output, and so does every edge. All SP graphs go forward, which make it possible to obtain the following property:

▶ **Lemma 23.** *For all SP terms u, v_1, v_2 such that $\mathcal{G}(u) \lhd \mathcal{G}(v_1v_2)$, there are SP terms u_1, u_2 such that $\mathcal{G}(u_1) \lhd \mathcal{G}(v_1)$, $\mathcal{G}(u_2) \lhd \mathcal{G}(v_2)$, and $\mathcal{SP} \vdash u \leq u_1u_2$.*

Proposition 22 follows easily (see [10, Appendix D]).

The above lemma cannot be adapted directly in the presence of 1 and ⊤, because the graphs no longer go forward, Instead, we start from a simpler interpretation of terms into graphs, ensuring that we keep forward graphs: we simply interpret 1 and ⊤ as letters, so that graphs are now labelled in $A \uplus \{1\} \uplus \{\top\}$. We write $\mathcal{G}'(\cdot)$ for the corresponding interpretation function. For instance, $\mathcal{G}'(a\top \cap 1)$ is the following graph:



The previous notion of homomorphism is of course too strict, as it leaves the constants 1 and $\top$ uninterpreted. We thus adjust the notion of homomorphism below. We first define some notations: given two vertices x, y in a graph as above, we write

- $x \Rightarrow y$ if there is a directed path from x to y along edges labelled with 1;
- $x \overset{a}{\Rightarrow} y$ if $x \Rightarrow x' \overset{a}{\to} y' \Rightarrow y$ for some vertices x' and y';
- $x \to^* y$ if there is a directed path from x to y along arbitrary edges;
- $x \to^+ y$ if there is a non-empty directed path from x to y along arbitrary edges.

▶ **Definition 24.** *Fix X in $\{1, \top, 1\top\}$ and two graphs G, H as above. An X-homomorphism from G to H is a function h from the vertices of G to those of H preserving input and output, and such that:*

(a) *if $x \overset{a}{\to} y$ in G then $h(x) \overset{a}{\Rightarrow} h(y)$ in H;*
(1) *for $X \in \{1, 1\top\}$, if $x \overset{1}{\to} y$ in G then $h(x) \Rightarrow h(y)$ in H;*
($\top$) *for $X = \top$, if $x \overset{\top}{\to} y$ in G then $h(x) \to^+ h(y)$ in H;*
($\top'$) *for $X = 1\top$, if $x \overset{\top}{\to} y$ in G then $h(x) \to^* h(y)$ in H;*

We write $H \lhd^X G$ when there exists such a X-homomorphism.

We finally state our three characterisations, for sl-monoids, bsl-semigroups, and bsl-monoids.

▶ **Theorem 25.** *For all X in $\{1, \top, 1\top\}$, for all $u, v \in \mathsf{SP}X$, $\mathcal{SP}^X \vdash u \leq v$ iff $\mathcal{G}'(u) \lhd^X \mathcal{G}'(v)$.*

The distinction in the clause for $\top$-labelled edges in the definitions of $\top$- and $1\top$-homomorphisms is required because bsl-monoids validate $\top \leq \top\top$ (for instance, because $\top = 1\top \leq \top\top$) while bsl-semigroups do not: allowing to use empty-paths with $1\top$-homomorphisms makes it possible to absorb one of the two edges of $\mathcal{G}'(\top\top)$, while this not possible with $\top$-homomorphisms.

Lemma 23 extends as follows:

▶ **Lemma 26.** *For all X in $\{1, \top, 1\top\}$, for all $\mathsf{SP}X$ terms u, v_1, v_2 s.t. $\mathcal{G}'(u) \lhd^X \mathcal{G}'(v_1 v_2)$, there are $\mathsf{SP}X$ terms u_1, u_2 such that $\mathcal{G}'(u_1) \lhd^X \mathcal{G}'(v_1)$, $\mathcal{G}'(u_2) \lhd \mathcal{G}'(v_2)$, and $\mathcal{SP} \vdash u \leq u_1 u_2$.*

Like in the SP case, this lemma is proved by induction on the size of u, producing terms u_1, u_2 such that $\mathcal{G}'(u_1 u_2)$ is a subgraph of $\mathcal{G}'(u)$ (in the strict sense), this is why we can get a derivation in $\mathcal{SP}$ rather than in $\mathcal{SP}^X$.

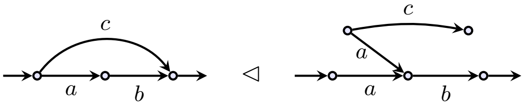
Theorem 25 follows like in the SP case, by two inductions. The base case for letters is slightly more involved in the presence of 1; we give all details in [10, Appendix D].

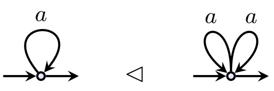
▶ **Corollary 27.** *The (in)equational theories of sl-monoids, bsl-semigroups, and bsl-monoids are decidable in polynomial time.*

Proof. The existence of an X-homomorphism from G to H is equivalent to the existence of a homomorphism from G to an appropriate closure of H. The graph of a term can obviously be computed in polynomial time, as well as its closure. The graph-homomorphism problem can be solved in polynomial time when the source graph has bounded treewidth [12, 7, 16], which is the case here (series-parallel graphs have treewidth at most two). ◀

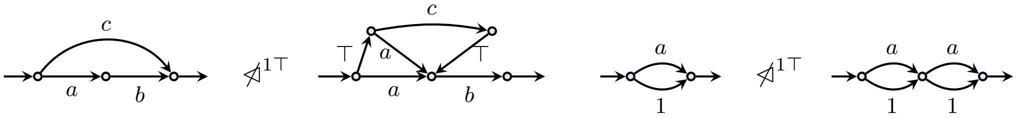
As an example, consider the following homomorphisms, establishing two valid laws of $\mathcal{R}el$:

$$\mathcal{R}el \models ab \cap c \leq (a \cap \top(a \cap c\top))b \qquad \mathcal{R}el \models 1 \cap a \leq (1 \cap a)(1 \cap a)$$





Those are not laws of bsl-monoids: under the simpler interpretation $\mathcal{G}'$, we obtain the following pairs of graphs, which are not related by $\lhd^{1\top}$.



7 Conclusion

We have shown that on the signatures SP1, SP⊤ and SP1⊤, $\mathcal{R}el$ is not finitely axiomatisable with a set of *(in)equations*. Does this change if we are more flexible on the shape of axioms? For example if Horn sentences or first-order formulas are allowed as axioms?

We have given a necessary condition on signatures for $\mathcal{R}el$ to be finitely based. This is the decomposability condition given in Proposition 8. Can we obtain a full characterisation?

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