Minimum Neighboring Degree Realization in Graphs and Trees

Amotz Bar-Noy
City University of New York (CUNY), NY, USA
amotz@sci.brooklyn.cuny.edu

Keerti Choudhary
Tel Aviv University, Israel
keerti.choudhary@cs.tau.ac.il

Avi Cohen
Weizmann Institute of Science, Rehovot, Israel
avi.cohen@weizmann.ac.il

David Peleg
Weizmann Institute of Science, Rehovot, Israel
david.peleg@weizmann.ac.il

Dror Rawitz
Bar-Ilan University, Ramat-Gan, Israel
dror.rawitz@biu.ac.il

Abstract

We study a graph realization problem that pertains to degrees in vertex neighborhoods. The classical problem of *degree sequence realizability* asks whether or not a given sequence of n positive integers is equal to the degree sequence of some n-vertex undirected simple graph. While the realizability problem of degree sequences has been well studied for different classes of graphs, there has been relatively little work concerning the realizability of other types of information *profiles*, such as the vertex neighborhood profiles.

In this paper we introduce and explore the *minimum degrees in vertex neighborhood profile* as it is one of the most natural extensions of the classical degree profile to *vertex neighboring degree profiles*. Given a graph $G = (V, E)$, the min-degree of a vertex $v \in V$, namely $\text{MinND}(v)$, is given by $\min\{deg(w) \mid w \in N[v]\}$. Our input is a sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$, where $d_{i+1} > d_i$ and each n_i is a positive integer. We provide some necessary and sufficient conditions for σ to be realizable. Furthermore, under the restriction that the realization is acyclic, i.e., a tree or a forest, we provide a full characterization of realizable sequences, along with a corresponding constructive algorithm.

We believe our results are a crucial step towards understanding extremal neighborhood degree relations in graphs.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms

Keywords and phrases Graph realization, neighborhood profile, graph algorithms, degree sequences

Digital Object Identifier 10.4230/LIPIcs.ESA.2020.10

Funding US-Israel BSF grant 2018043; ARL Network Science CTA W911NF-09-2-0053.

1 Introduction

Background and Motivation. Vertex degrees occur as a central and natural parameter in many network applications, and provide information on the significance, centrality, connectedness and influence of each vertex in the network, contributing to our understanding of the network structure and properties. The m *degree sequence* of an n-vertex graph G consists of its vertex degrees, $\text{Deg}(G) = (d_1, \ldots, d_n)$. It is a straightforwad task to extract the degree sequence of a given graph G from its adjacency matrix or adjacency lists. A more

28th Annual European Symposium on Algorithms (ESA 2020).
Editors: Fabrizio Grandoni, Grzegorz Herman, and Peter Sanders; Article No. 10; pp. 10:1–10:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

interesting and challenging task, known as the *realization* problem, concerns the opposite situation where, given a sequence of non-negative integers D, it is necessary to decide whether there exists a graph whose degree sequence conforms to D. A sequence that admits such a realization is called *graphic*. A necessary and sufficient condition for a given sequence of integers to be graphic (also implying an $O(n)$ decision algorithm) was presented by Erdös and Gallai in [10]. Havel and Hakimi [12, 14] described an algorithm that given a sequence of integers computes in $O(m)$ time an m-edge graph realizing it, or proves that the given sequence is not graphic. Over the years, a number of extensions of the degree realization problem were studied as well, e.g., [1, 3, 23].

The current work is motivated by the fact that similar realization questions arise naturally in a variety of *other* contexts. Typically, some type of information profile, specifying some desired vertex property (related to degrees, distances, centrality, connectedness, etc), is given to us, and we are asked to find a graph conforming to the specified profile. Questions of this type span a wide research area, which was so far studied only sparsely. The current paper makes a step towards studying one specific information profile, from the family of *neighborhood degree* profiles. Such profiles arise in the context of social networks, where it is common to look at vertex degrees as representing influence or centrality, and neighboring degrees as representing proximity to power. Neighborhood degrees were considered before in [6], but there each vertex i is associated with the *list* of degrees of all vertices in its neighborhood. Our profiles are leaner, and provide a single parameter per vertex. In [5], we studied maximum-neighborhood-degree (MaxND) profiles, in which each vertex i is associated with the *maximum* degree of the vertices in its (closed) neighborhood.

A natural problem in this direction concerns the *minimum* degrees in the vertex neighborhoods. For each vertex i, let d_i denote the minimum vertex degree in i's closed neighborhood (i.e., including the vertex i itself). Then $\text{MinND}(G) = (d_1, \ldots, d_n)$ is the minimum-neighborhood-degree profile of G.

The same realizability questions asked above for degree sequences can be posed for neighborhood degree profiles as well. This brings us to the following central question of our work:

Minimum Neighborhood Degree Realization

Input: A sequence $D = (d_1, \ldots, d_n)$ of non-negative integers.

Question: Is there a graph G of size n such that the *minimum* degree in the closed neighborhood of the *i-th* vertex in G is exactly equal to d_i ?

Our Contributions. We now discuss our contributions in detail. For simplicity, we represent the input vector D alternatively in a more compact format as $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$, where n_i's are positive integers with $n(\sigma) = \sum_{i=1}^{\ell} n_i = n$; here the specification requires that G contains exactly n_i vertices whose minimum degree in neighborhood is d_i. We may assume that $d_\ell > d_{\ell-1} > \cdots > d_1 \geq 1$ (noting that vertices with minimum-neighborhood-degree zero are necessarily singletons and can be handled separately).

Conditions. We show the following necessary and sufficient conditions for $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ to be MinND realizable. The necessary condition is that

$$d_i \leq n_1 + n_2 + \ldots + n_i - 1 \text{ , for } i \in [1, \ell], \quad \text{and} \tag{NC1}$$

$$d_\ell \leq \left\lfloor \frac{n_1 d_1}{d_1 + 1} \right\rfloor + \left\lfloor \frac{n_2 d_2}{d_2 + 1} \right\rfloor + \ldots + \left\lfloor \frac{n_\ell d_\ell}{d_\ell + 1} \right\rfloor . \tag{NC2}$$

The sufficient condition is that

$$d_i \leq \left\lfloor \frac{n_1 d_1}{d_1+1} \right\rfloor + \left\lfloor \frac{n_2 d_2}{d_2+1} \right\rfloor + \ldots + \left\lfloor \frac{n_i d_i}{d_i+1} \right\rfloor \ , \ \text{for } i \in [1, \ell] \ . \tag{SC}$$

We remark that these conditions can be computed in polynomial time, and the realizing graphs, when any exist, can be constructed in polynomial time.

Approximation bound. For any sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying the first necessary condition (NC1), the sequence $\sigma^\gamma = (d_\ell^{\lceil \gamma n_\ell \rceil}, \ldots, d_1^{\lceil \gamma n_1 \rceil})$, where $\gamma = (d_1+1)/d_1$ satisfies[1] the sufficient condition (SC), thus our necessary and sufficient conditions differ by a factor of at most 2 in the n_i's.

We leave it as an open question to resolve the problem exactly over general graphs.

▶ **Open Question.** *Does there exist a closed-form characterization for realizing* MinND *profiles for general graphs?*

For the special case of ℓ bounded by 3, we show that $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ is MinND-realizable if and only if along with (NC1) and (NC2) the following condition is satisfied:

$$d_2 \leq \left\lfloor \frac{n_1 d_1}{d_1+1} \right\rfloor + \left\lfloor \frac{n_2 d_2}{d_2+1} \right\rfloor, \ \textit{or} \ d_3 + 1 \leq n_1 + n_2 + n_3 - \left(1 + \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil \right) \tag{NC3}$$

Acyclic Realization. When the required graph G is acyclic (that is, G is a tree or a forest), we give tight bounds for realizability (in the form of a constructive algorithm as well as a matching lower bound). For a sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$, let

$$\phi(\sigma) = d_\ell^2 + 1 + \sum_{i=1}^{\ell} (n_i - 1)(d_i - 1)^2 + \sum_{i=1}^{\ell-1} d_i(d_i - 1) \ .$$

We show that σ is MinND-realizable by a tree if and only if the following conditions are met:

$$d_1 \ = \ 1 \qquad \text{and} \qquad \phi(\sigma) \ \leq \ n(\sigma) \ . \tag{NC-Tree}$$

Recall that $n(\sigma) = \sum_{i=1}^{\ell} n_i$. Observe that when the profile is (1^n), condition (NC-Tree) is equivalent to claiming that (1^n) is realizable for any $n \geq 2$. Indeed, the star graph provides such a realization. Next, note that d_1 and n_1 do not appear in $\phi(\sigma)$ when $\ell > 1$, because of the terms $d_i - 1$. However, n_1 is part of $n(\sigma)$, and it must be large enough to satisfy the condition. Therefore, condition (NC-Tree) can be rewritten as $\phi(\sigma) - \sum_{i=2}^{\ell} n_i \leq n_1$, where the left hand side is effectively independent of d_1 and n_1. That is, any sub-profile $\sigma' = (d_\ell^{n_\ell}, \cdots, d_2^{n_2})$ of σ can be realized if it is expanded into a full profile $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ for which n_1 is large enough. Hence in a sense, these n_1 vertices, which are leaves or neighbors of leaves, "control" the realizability of the profile.

[1] For further explanation, see the text just before Corollary 10.

The MaxND profile. We remark that in the companion paper [5] we studied the dual MaxND realization problem, which turns out to exhibit radically different behavior from the MinND realization problem, and requires different techniques. In the MaxND profile, d_i specifies the *maximum* degree in the neighborhood of the ith vertex in G. [5] gives tight bounds for realizations by an arbitrary graph and by a connected graph. However, the question of realizations with trees is left open.

It is interesting to contrast the behavior of the MinND and MaxND profiles. For general graphs, MinND appears to be more difficult, since it is nonmonotone when edges are added or deleted, while the MaxND profile is monotone. For trees, on the other hand, the realizability of the MinND profile depends only on the leaves and their parents, which simplifies the analysis; no analogous simplifying property was found for the MaxND profile.

Applicability. Realization questions may potentially be applicable in two general settings. The first involves scientific contexts, where the information profile may consist of measurement results obtained by observing some natural network of unknown structure and our goal is to build a model (possibly explaining the measurements). The second involves engineering contexts, where the profile is derived from a given specification and the goal is to implement a network abiding by the specification.

One of the concrete uses for degree realization techniques is within the framework of generating random graphs with specific given properties. In particular, given the ability to efficiently generate a graph with a given degree sequence, one can design methods for generating a random graph with a specific degree distribution based on first generating a random degree sequence from the given distribution. As happened with degree realization, one may expect that efficient solutions for the problem of realizing certain neighborhood degree profiles may lead to improved techniques for generating and simulating social networks with prescribed neighborhood degree profiles.

Finally, a popular sampling technique that takes advantage of the Friendship Paradox [11] is based on sampling a *random neighbor* of a *random vertex*. While the average of the degrees in the traditional degree profile is the expected degree of a random vertex, the lower and upper bounds on the expected degree of the random neighbor are the averages of the degrees in the MinND and MaxND profiles respectively. Providing realizations and characterizing realizable profiles may be useful in exploring and analyzing the performance of this sampling technique.

Related Work. Many works have addressed related questions such as finding all the (non-isomorphic) graphs that realize a given degree sequence, counting all the (non-isomorphic) realizing graphs of a given degree sequence, sampling a random realization for a given degree sequence as uniformly as possible, or determining the conditions under which a given degree sequence defines a unique realizing graph , cf. [8, 10, 12, 13, 14, 15, 18, 19, 21, 20, 22, 24]. Other works such as [7, 9, 16] studied interesting applications in the context of social networks.

To the best of our knowledge, the MinND realization problems have not been explored so far. There are two other related problems that we are aware of. The first is the *shotgun assembly* problem [17], where the characteristic associated with the vertex i is some description of its neighborhood up to radius r. The second is the *neighborhood degree lists* problem [6], where the characteristic associated with the vertex i is the list of degrees of all vertices in i's neighborhood. We point out that in contrast to these studies, our MinND problem applies to a more restricted profile (with a single number characterizing each vertex), and the techniques involved are totally different from those of [6, 17]. Several other realization problems are surveyed in [2, 4].

2 Preliminaries

Let H be an undirected graph. We use $V(H)$ and $E(H)$ to respectively denote the vertex set and the edge set of the graph H. For a vertex $x \in V(H)$, let $deg_H(x)$ denote the degree of x in H. Let $N_H[x] = \{x\} \cup \{y \mid (x,y) \in E(H)\}$ be the (closed) neighborhood of x in H. For a set $W \subseteq V(H)$, we denote by $N_H(W)$, the set of all the vertices lying outside the set W that are adjacent to some vertex in W, that is, $N_H(W) = (\bigcup_{w \in W} N[w]) \setminus W$. Given a vertex v in H, the minimum degree in the neighborhood of v, namely $\text{MinND}_H(v)$, is defined to be the minimum over the degrees of all the vertices in the neighborhood of v. Given a set of vertices A in a graph H, we denote by $H[A]$ the subgraph of H induced by the vertices of A. For a set A and a vertex $x \in V(H)$, we denote by $A \cup x$ and $A \setminus x$, respectively, the sets $A \cup \{x\}$ and $A \setminus \{x\}$. When the graph is clear from context, for simplicity, we omit the subscripts H in all our notations. Finally, given two integers $i \leq j$, we define $[i,j] = \{i, i+1, \ldots, j\}$.

Consider a profile $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying $d_\ell > d_{\ell-1} > \cdots > d_1 > 0$. Denote its size by $n(\sigma) = \sum_{i=1}^{\ell} n_i$. The profile σ is said to be *MinND realizable* if there exists an $n(\sigma)$-vertex graph G such that $|\{v \in V(G) : \text{MinND}(v) = d_i\}| = n_i$, namely, G contains exactly n_i vertices whose MinND is d_i, for every $i \in [1,\ell]$. Figure 1 depicts a MinND realization of $(2^3, 1^2)$. (The numbers represent vertex degrees.)

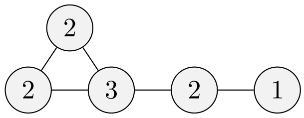

Figure 1 A MinND realization of $(2^3, 1^2)$.

3 Realizations on Acyclic graphs

In this section, we provide a complete characterisation for realizability on acyclic graphs.

3.1 Constructive Algorithm

▶ **Proposition 1.** *Any sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying $d_1 = 1$ and $\phi(\sigma) \leq n(\sigma)$ is MinND-realizable over trees.*

Proof. Initialize T to be a star with a root r and d_ℓ leaves. Let the initial set of $d_\ell + 1$ vertices be X_0. Notice $|X_0 \setminus \{r\}| = d_\ell > \ell - 1$, since $d_1 = 1$.

Partition $X_0 \setminus \{r\}$ into two sets Z_1 and Z_2, respectively of size $\ell - 1$ and $d_\ell - \ell + 1$. We label the $(i-1)^{th}$ vertex in Z_1 as $v_{i,1}$, for $i \in [2,\ell]$. Observe $|Z_2| \geq 1$.

Our algorithm (to iteratively build T) proceeds in ℓ rounds: $i = \ell, \ldots, 1$. (See Algorithm 1 for a pseudocode).

We will maintain the following invariant in our algorithm.

Invariant. *Before the beginning of round i, the vertex $v_{i,1}$ is a leaf node in the partially constructed tree T, and its neighbor r (always) has degree at least $d_\ell \geq d_i$.*

Description of round i ($i > 1$). Take the leaf node $v_{i,1} \in X_0$. Add $n_i - 1$ new vertices, namely $v_{i,2}, \ldots, v_{i,n_i}$ and connect each $v_{i,j}$ to $v_{i,j-1}$, for $2 \leq j \leq n_i$. Let V_i represent the set $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$. Notice that V_i forms a simple path. Recall by our invariant that the neighbor of $v_{i,1}$ (other than $v_{i,2}$ in T) had degree already at least d_i. We will ensure next the following:

C1: All vertices in V_i have degree d_i.

C2: All neighbors of vertices in V_i have degree at least d_i.

To ensure condition C1, we proceed as follows: (i) Since the vertices $v_{i,1}, \ldots, v_{i,n_i-1}$ already have degree 2 in the current T_i, they are connected to $d_i - 2$ *new* vertices, and (ii) the vertex v_{i,n_i} is connected to $d_i - 1$ *new* vertices. In the process we add in total $n_i(d_i - 2) + 1$ new vertices. Let these be represented by the set A_i.

To ensure condition C2, we connect each $a \in A_i$ to an additional $d_i - 1$ *new* vertices. Let B_i be the set of new vertices added. Then, $|B_i| = |A_i| \cdot (d_i - 1)$.

We now compute the size of $V_i \cup A_i \cup B_i$.

$$
\begin{aligned}
|V_i \cup A_i \cup B_i| &= n_i + \big(n_i(d_i - 2) + 1\big) + \big(n_i(d_i - 2) + 1\big) \cdot (d_i - 1) \\
&= n_i + d_i\big(n_i(d_i - 2) + 1\big) \\
&= n_i\big(d_i - 1\big)^2 + d_i
\end{aligned}
$$

Description of round 1. Finally, in round $i = 1$, add a set Y_0 of $n(\sigma) - \phi(\sigma)$ *new* vertices to T. Observe that $n(\sigma) - \phi(\sigma) \geq 0$ due to the assumption. Connect the root node $r \in X_0$ in T to each of the vertices in Y_0.

We next show that our construction satisfies $|V(T)| = n(\sigma)$.

$$
\begin{aligned}
|V(T)| = |X_0| + \sum_{i=2}^{\ell} \big(|V_i \cup A_i \cup B_i| - 1\big) + |Y_0| \\
&= d_\ell + 1 + \sum_{i=2}^{\ell} \big[n_i(d_i - 1)^2 + d_i - 1\big] + n(\sigma) - \phi(\sigma) \\
&= d_\ell^2 + 1 + \sum_{i=2}^{\ell} (n_i - 1)(d_i - 1)^2 + \sum_{i=2}^{\ell-1} d_i(d_i - 1) + n(\sigma) - \phi(\sigma) \\
&= n(\sigma)
\end{aligned}
$$

Algorithm 1 Computing a tree MinND-realization for a given realizable σ.

Input: A sequence $\sigma = (d_\ell^{n_\ell} \cdots d_1^{n_1})$ satisfying $d_1 = 1$ and $n(\sigma) \geq \phi(\sigma)$.

Initialize T to be a star with a root r and d_ℓ leaves.
Label the i^{th} leaf in T as $v_{i,1}$, for $i \in [2, \ell]$.
for $i = \ell$ to 2 **do**
    Add $n_i - 1$ new vertices to T, namely $v_{i,2}, \ldots, v_{i,n_i}$.
    Connect each $v_{i,j}$ to $v_{i,j-1}$, for $2 \leq j \leq n_i$.
    Add to T a set A_i of $n_i(d_i - 2) + 1$ new vertices.
    Connect each $v_{i,j}$, for $1 \leq j \leq n_i - 1$, to $d_i - 2$ isolated vertices in A_i.
    Connect v_{i,n_i} to $d_i - 1$ isolated vertices in A_i.
    Add to T a set B_i of $|A_i| \cdot (d_i - 1)$ new vertices.
    Connect each $a \in A_i$ to $d_i - 1$ isolated vertices in B_i.
Add $n(\sigma) - \phi(\sigma)$ new vertices to T as children of the root r.
Output T.

Correctness Analysis

Let V_1 denote the set $V(T) \setminus \bigcup_{i=2}^{\ell} V_i$. Clearly, $|V_i| = n_i$ for $i \in [2,\ell]$, and since $|V(T)| = n(\sigma)$ it follows that $|V_1| = n(\sigma) - \sum_{i=2}^{\ell} n_i = n_1$. Therefore, if we show that for every $u \in V_i$, $\textsc{MinND}(u) = d_i$, for $i \in [1,\ell]$, then we are done.

Observe that the degrees of vertices in $V_i \cup A_i$ do not alter after round i, so C1 and C2 continue to hold for each V_i, $i \in [2,\ell]$. This shows that for every $u \in V_i$, $\textsc{MinND}(u) = d_i$, for $i \in [2,\ell]$. We are left to analyse set V_1. We have:

$$\begin{aligned} V_1 &= \Big(X_0 \setminus \cup_{i=2}^{\ell}\{v_{i,1}\}\Big) \cup Y_0 \cup \Big(\cup_{i=2}^{\ell} (A_i \cup B_i)\Big) \\ &= \{r\} \cup Z_2 \cup Y_0 \cup \Big(\cup_{i=2}^{\ell} (A_i \cup B_i)\Big) \end{aligned}$$

For $2 \le i \le \ell$, the set B_i contains only leaves, and each node in A_i must have a neighbor in B_i. Thus, vertices in $\cup_{i=2}^{\ell}(A_i \cup B_i)$ have MinND exactly 1.

So it is left to consider the vertices of $\{r\} \cup Z_2 \cup Y_0$, of which the vertices in $Z_2 \cup Y_0$ have already degree 1. Now recall $Z_2 \neq \emptyset$, and r is adjacent to degree-1 vertices in Z_2, thus MinND of r is 1 as well.

This completes the correctness analysis. ◀

3.2 Tightness Criterion

We next show that our construction is tight, i.e., a sequence is MinND-realizable over trees if and only if it is realizable by the procedure of Proposition 1.

▶ **Proposition 2.** *For a sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying $d_1 = 1$, a necessary condition of MinND-realizability over trees is $\phi(\sigma) \le n(\sigma)$.*

Proof. Consider a profile $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$, and let T be a MinND tree-realization of σ on V. Let $r \in V(T)$ be a vertex that satisfies $\textsc{MinND}(u) = d_\ell$. Root T at node r. For $i = 1, \ldots, \ell$, let $V_i = \{v \in V(T) \mid \textsc{MinND}(v) = d_i\}$. Observe that for each $i < \ell$, there exists (at least) one edge, denoted $(y_i, x_i) \in E(T)$, where y_i is the parent of x_i, satisfying the condition that (i) $x_i \in V_i$, and (ii) none of the vertices in the tree-path $(r \rightsquigarrow_T y_i)$ lie in V_i. These edges play a crucial role in our tight bound on $\phi(\sigma)$.

Let

$$\begin{aligned} A &= \{x_i \mid \textsc{MinND}(y_i) < d_i, \text{ for } i < \ell\}, \\ B &= \{x_i \mid \textsc{MinND}(y_i) > d_i, \text{ for } i < \ell\}. \end{aligned}$$

For each $w \in V(T)$, let $\mathcal{C}_w$ and $\mathcal{GC}_w$, respectively, be the set consisting of the children and grand-children of w in T. Also let $\mathcal{C}_A = \cup_{w \in A} \mathcal{C}_w$.

Now we define a function $\Gamma : V \mapsto 2^V$ as follows (see example in Figure 2):

$$\Gamma(w) = \begin{cases} \{r\} \cup \mathcal{C}_r \cup \mathcal{GC}_r, & \text{if } w = r, \\ \mathcal{C}_w \cup (\mathcal{GC}_w \setminus \mathcal{C}_A), & \text{if } w \in A, \\ \mathcal{GC}_w \setminus \mathcal{C}_A, & \text{otherwise.} \end{cases}$$

Figure 3 illustrates the subtree induced over $\{w\} \cup \mathcal{C}_w \cup \mathcal{GC}_w$, for some node w.

▷ Claim. $\Gamma(w) \cap \Gamma(v) = \emptyset$ for every $v, w \in V(T)$ such that $v \neq w$.

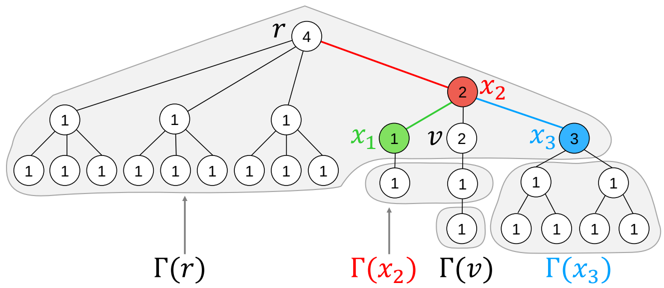

Figure 2 Tree MinND-realization of $\sigma = (4^1 3^1 2^2 1^{22})$. The number in each vertex denotes its MinND. The edges (y_1, x_1), (y_2, x_2) and (y_3, x_3) are colored green, red and blue, respectively. Further, $A = \{x_3\}$ and $B = \{x_1, x_2\}$.

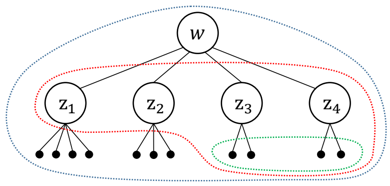

Figure 3 Illustration of a subtree induced over $\{w\} \cup \mathcal{C}_w \cup \mathcal{GC}_w$, for some node w. $\Gamma(w)$ is the set of vertices in the blue dotted line if $w = r$, in the red if $w \in A$, and in the green otherwise (assuming $z_1, z_2 \in A$).

Proof. Let us first consider the case $v = r$. The result is obviously true if $w \notin \mathcal{C}_r$, or $w \notin A$. Now if $w \in \mathcal{C}_r$, then $\text{MinND}(r) \geq \text{MinND}(w)$, thereby implying $w \notin A$.

Next consider any two vertices $v \neq w \in V(T) \setminus \{r\}$. Assume towards contradiction $\Gamma(v) \cap \Gamma(w)$ contains a node z. Then z must be a child of exactly one of the nodes v or w, and the corresponding node must lie in A. Assume $z \in \mathcal{C}_w$, and $w \in A$. Since $z \in \mathcal{C}_w \subseteq \mathcal{C}_A$, we have $z \notin \mathcal{GC}_v \setminus \mathcal{C}_A$, and also z cannot be a child of v, thereby implying $z \notin \Gamma(v)$. Hence, $\Gamma(v) \cap \Gamma(w)$ must be empty. ◁

▷ Claim. For every $v \in V_i$, $1 \leq i \leq \ell$, we have

$$|\Gamma(v)| \geq \begin{cases} d_i(d_i - 1), & \text{if } v \in A \cup B, \\ (d_i - 1)^2, & \text{if } v \notin \{r\} \cup A \cup B. \end{cases}$$

Proof. Consider a node $v \in V_i$, for some $i \leq \ell$. Observe that each $u \in \mathcal{C}_v$ must have degree at least d_i, and thus satisfy $|\mathcal{C}_u| \geq d_i - 1$. Let z_0 be v's parent and $z_1, \ldots, z_t$ be the nodes in $\mathcal{C}_v \cap A$. Since z_0 is an ancestor of $z_1, \ldots, z_t$, by definition of $A \cup B$, the MinND of all the vertices $z_0, z_1, \ldots, z_t$ must be distinct. Without loss of generality, we can assume $\text{MinND}(z_t) > \cdots > \text{MinND}(z_1)$. By definition of A, $\text{MinND}(z_1) > d_i$. Let $\Delta = \max_{j=0}^{t} \text{MinND}(z_j)$. Then $deg(v) \geq \Delta$. Consequently we have $\Delta \geq d_i + t$, since $\Delta \geq \text{MinND}(z_t) > \cdots > \text{MinND}(z_1) > d_i$. So

$$|\mathcal{C}_v \setminus A| = deg(v) - t - 1 \geq \Delta - t - 1 \geq (d_i - 1)\ . \tag{1}$$

We now consider three cases, according to whether v lies in A, B, or $V \setminus (\{r\} \cup A \cup B)$.

1. $v \in A$: By Eq. (1), $|\mathcal{GC}_v \setminus \mathcal{C}_A| \geq (d_i - 1)^2$, and also $|\mathcal{C}_v| \geq d_i - 1$. Combined, we get that $|\Gamma(v)| = |\mathcal{GC}_v \setminus \mathcal{C}_A| + |\mathcal{C}_v| \geq d_i(d_i - 1)$.

2. $v \in B$: By definition of B, MinND$(z_0) > d_i$. Thus, MinND$(z_j) > d_i$ for $j \in [0, t]$. Also, MinND of $z_0, \ldots, z_t$ are distinct. Hence, $\Delta \geq d_i + (t+1)$. So $|\mathcal{C}_v \setminus A| = deg(v) - t \geq \Delta - t \geq d_i$. This implies $|\Gamma(v)| = |\mathcal{GC}_v \setminus \mathcal{C}_A| \geq d_i(d_i - 1)$.
3. $v \notin \{r\} \cup A \cup B$: By Eq. (1), $|\mathcal{GC}_v \setminus \mathcal{C}_A| \geq (d_i - 1)^2$, implying $|\Gamma(v)| \geq (d_i - 1)^2$.

The claim follows. ◁

Note that $\Gamma(r)$ contains at least $d_\ell^2 + 1$ nodes, since the degrees of r and of its children are at least d_ℓ. Now, we are ready to prove the bound over $\phi(\sigma)$. In our calculations we use x_ℓ to denote the node r.

$$\begin{aligned} n(\sigma) = |V(T)| &\geq |\Gamma(r)| + \sum_{i=1}^{\ell} \sum_{v \in V_i \setminus \{x_i\}} |\Gamma(v)| + \sum_{i=1}^{\ell-1} |\Gamma(x_i)| \\ &\geq d_\ell^2 + 1 + \sum_{i=1}^{\ell} (n_i - 1)(d_i - 1)^2 + \sum_{i=1}^{\ell-1} d_i(d_i - 1) \\ &= \phi(\sigma) \ . \end{aligned}$$

This completes our proof of $n(\sigma) \geq \phi(\sigma)$. ◀

▶ **Corollary 3.** *For a sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying $d_1 = 1$, a necessary condition of MinND-realizability over forests is $\phi(\sigma) \leq n(\sigma)$.*

Proof. Given a sequence σ that is MinND-realizable as a forest, it can be partitioned into subsequences $\sigma_1, \ldots, \sigma_k$ corresponding to each of its connected components. By Proposition 2, $n(\sigma_i) \geq \phi(\sigma_i)$ for $i \in [1, k]$. Therefore, $n(\sigma) = \sum_{i=1}^{k} n(\sigma_i) \geq \sum_{i=1}^{k} \phi(\sigma_i) \geq \phi(\sigma)$, where the last inequality follows immediately from the definition of ϕ. ◀

By Proposition 1 and Corollary 3, and the fact that a tree always contains vertices of degree one (and hence also MinND one), the following is immediate.

▶ **Theorem 4.** *The sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ is MinND-realizable over acyclic graphs if and only if $d_1 = 1$, and $\phi(\sigma) \leq n(\sigma)$.*

4 Realizations in General graphs

We first define the notion of *leader* and *follower* crucial to our construction. Let $G = (V, E)$ be any graph. For any vertex $v \in V$, we define *leader*(v) to be a vertex in $N[v]$ of minimum degree, if there is more than one choice we pick the leader arbitrarily (these arbitrarily chosen leaders do not have to be consistent between neighbors, e.g., it is possible that two vertices u and v are the leaders of each other). In other words, $leader(v) \in \arg\min\{deg(w) \mid w \in N[v]\}$. Next let $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ be the min-degree sequence of G. We define V_i to be the set of those vertices in G whose minimum-degree in the closed neighborhood is exactly d_i, so $|V_i| = n_i$. Also, let L_i be the set of those vertices in G who are leaders of at least one vertex in V_i, equivalently, $L_i = \{leader(v) \mid v \in V_i\}$, and denote by $L = \cup_{i=1}^{\ell} L_i$ the set of all the leaders in G. Observe that the sets $V_1, \ldots, V_\ell$ form a partition of the vertex-set of G.

A vertex v in G is said to be a *follower*, if $leader(v) \neq v$. Let $F_i = \{v \in V_i \mid v \neq leader(v)\}$ be the set of all the followers in V_i. Finally we define $R = V \setminus L$ to be the set of all the non-leaders, and $F = \cup_{i=1}^{\ell} F_i$ to be the set of all the followers.

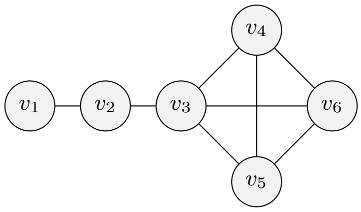

Figure 4 MinND-realization of sequence $\sigma = (3^3 2^1 1^2)$. Here MinND(v_1) = MinND(v_2) = $deg(v_1) = 1$, MinND$(v_3) = deg(v_2) = 2$, and MinND$(v_i) = 3$, for $i \in \{4,5,6\}$. Since $leader(v_2) = v_1$ and $leader(v_3) = v_2$, thus, v_2 is a leader as well as a follower.

We point here that there exist realizable sequences σ for which any graph G realizing σ and any leader function over G, the sets L and F have non-empty intersection. For example, consider the sequence $\sigma = (3^3 2^1 1^2)$ in Figure 4. It can be easily checked that σ has only one realizing graph, and in it, the leader-set and follower-set are non-disjoint.

We classify the sequences that admit disjoint leader and follower sets as follows.

▶ **Definition 5.** *A sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ is said to admit a Disjoint Leader-Follower (DLF) MinND-realization if there exists a graph G realizing σ and a leader function under which the sets L and F are mutually disjoint, that is, $L \cap F = \emptyset$.*[2]

▶ **Theorem 6.** *For any $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ that is MinND-realizable by a graph, say G, the following conditions must be satisfied.*

(NC1) $d_i \le \left(\sum_{j=1}^{i} n_j\right) - 1$, *for* $i \in [1, \ell]$;

(NC2) $d_\ell \le \sum_{j=1}^{\ell} \left\lfloor \frac{n_j d_j}{d_j+1} \right\rfloor$.

Further, for any leader function defined over G, and $i < \ell$, if $L_i \cap V_i \neq \emptyset$ then $d_i \le \sum_{j=1}^{i} \left\lfloor \frac{n_j d_j}{d_j+1} \right\rfloor$.

Proof. We provide first a lower bound on the size of the leader set L_i. We show for each $i \in [1,\ell]$, we have $|L_i| \ge \left\lceil \dfrac{n_i}{d_i+1} \right\rceil$. Consider a vertex $a \in L_i$. Since $|N[a]| = d_i + 1$, vertex a can serve as leader for at most $d_i + 1$ vertices. This shows that $|L_i| \ge \frac{n_i}{d_i+1}$. The claim follows from the fact that $|L_i|$ is integer.

Proof of (NC1). Let w be any vertex in G such that $deg(w) = d_i$. Then w as well as all the neighbors of w must be contained in $\cup_{j=1}^{i} V_j$, therefore, we have: $d_i + 1 = |N[w]| \le |\cup_{j=1}^{i} V_j| = \sum_{j=1}^{i} n_j$, thereby proving condition (NC1).

Proof of (NC2). Now suppose w is a vertex in G such that MinND$(w) = d_\ell$. Then $N[w]$ cannot contain vertices of degree less than d_ℓ, so $N[w] \cap L_i = \emptyset$, for each $i < \ell$. Therefore, $|N[w]| \le n - \sum_{i=1}^{\ell-1} |L_i|$. Also $deg(w)$ must be at least d_ℓ. We thus get,

$$d_\ell + 1 \le |N[w]| \le n - \sum_{i=1}^{\ell-1} |L_i| = n_\ell + \sum_{i=1}^{\ell-1} (n_i - |L_i|) \le n_\ell + \sum_{i=1}^{\ell} \left\lfloor \frac{n_i d_i}{d_i + 1} \right\rfloor .$$

Now if $n_\ell \le d_\ell$, then $n_\ell - 1 = \left\lfloor \frac{n_\ell d_\ell}{d_\ell+1} \right\rfloor$, and so $d_\ell \le \sum_{i=1}^{\ell} \left\lfloor \frac{n_i d_i}{d_i+1} \right\rfloor$. If $n_\ell \ge d_\ell + 1$, then $\frac{n_\ell d_\ell}{d_\ell+1} \ge d_\ell$ which implies $d_\ell \le \left\lfloor \frac{n_\ell d_\ell}{d_\ell+1} \right\rfloor$ since d_ℓ is integral.

[2] In Section 4.1, we show that our construction realizes a DLF MinND-realization whenever one exists. In other words, the sufficient condition (SC) is also necessary for sequences that admit a DLF MinND-realization. Nevertheless, there exist MinND-realizable sequences that do not admit a DLF MinND-realization. These sequences may violate the sufficient condition (SC) despite being MinND-realizable.

Proof of last claim. Let w be any vertex lying in $L_i \cap V_i$, so $\textsc{MinND}(w) = deg(w) = d_i$. Recall for each $j < i$, vertices in the set L_j have degree strictly less than d_i. Since $N[w]$ cannot contain vertices of degree less than d_i, thus for each $j < i$, $N[w] \cap L_j = \emptyset$. Also vertices in $V_{i+1} \cup \ldots \cup V_\ell$ cannot be adjacent to any vertex in $\{w\} \cup \left(\cup_{j=1}^{i-1} L_j\right)$, therefore, $N[w]$ as well as $\cup_{j=1}^{i-1} L_j$ are contained in union $\cup_{j=1}^{i} V_j$. We thus get,

$$d_i + 1 = |N[w]| \leq \left|\bigcup_{j=1}^{i} V_j\right| - \left|\bigcup_{j=1}^{i-1} L_j\right| = n_i + \sum_{j=1}^{i-1}(n_i - |L_j|) \leq n_i + \sum_{j=1}^{i-1} \left\lfloor \frac{n_j d_j}{d_j+1} \right\rfloor .$$

If $n_i \leq d_i$, then $n_i - 1 = n_i - \left\lceil \frac{n_i}{d_i+1} \right\rceil = \left\lfloor \frac{n_i d_i}{d_i+1} \right\rfloor$, and so $d_i \leq \sum_{j=1}^{i} \left\lfloor \frac{n_j d_j}{d_j+1} \right\rfloor$. If $n_i \geq d_i + 1$, then the bound trivially holds since $\frac{n_i d_i}{d_i+1} \geq d_i$ which from the fact that d_i is integral implies $d_i \leq \left\lfloor \frac{n_i d_i}{d_i+1} \right\rfloor$. ◀

We next prove the following theorem.

▶ **Theorem 7** (Sufficient condition SC). *Any sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying*

$$d_i \leq \sum_{j=1}^{i} \left\lfloor \frac{n_j d_j}{d_j+1} \right\rfloor, \text{ for } i \in [1, \ell] ,$$

is MinND*-realizable. Further, we can always compute a realizing graph, say G, and a leader function defined over G that satisfies $L \cap F = \emptyset$.*

Proof. We begin with the simple case of realizing uniform sequences, and then consider the scenario of general sequences.

Uniform Sequences. Consider for simplicity first the sequence $\sigma = (d^n)$. We provide a realization for σ if $n \geq d+1$. Let $q \geq 1$ and $r \in [0, d]$ be integers satisfying $n = (q)(d+1) - r$. Take a set A of q vertices, namely a_i ($i \in [1, q]$), and another set B of dq vertices, namely b_{ij} ($i \in [1, q], j \in [1, d]$). Connect each a_i to the vertices $b_{i1}, \ldots, b_{id}$. So vertices in A have degree exactly d and vertices in B have in their neighborhood a vertex of degree d. Next if $r > 0$, then we merge b_{1j} with b_{2j}, for $j \in [1, r]$, thereby reducing r vertices in B. (Notice that b_{1j} and b_{2j} exists because $r > 0$ only when $q \geq 2$.) Thus $|A| + |B| = n$ and each vertex in A still has degree exactly d. So $|A| = \frac{n+r}{d+1} = \left\lceil \frac{n}{d+1} \right\rceil$ and $|B| = n - |A| = \left\lfloor \frac{nd}{d+1} \right\rfloor \geq d$. Finally, we add edges between each pair of vertices in B to make it a clique of size at least d; this will imply that the vertices in set B have degree at least d. It is easy to check that $\textsc{MinND}(v)$ for each $v \in A \cup B$ in our constructed graph is d. In our construction A forms the leader set, and B forms the follower set.

In the rest of proof, we use $\textsc{graph}(n, d, A, B)$ to denote a function that returns the edges of the graph as constructed above (over A and B) whenever provided with four parameters n, d, A, B satisfying $n \geq d+1$, $|A| = \left\lceil \frac{n}{d+1} \right\rceil$, and $|B| = \left\lfloor \frac{nd}{d+1} \right\rfloor$.

General Sequences. We now consider the case $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$. Initialize G to be an empty graph. Our algorithm proceeds in ℓ rounds. (See Algorithm 2 for a pseudocode.) In each round, we first add to G a set V_i of n_i new vertices and partition V_i into two sets L_i and R_i of sizes respectively $\left\lceil \frac{n_i}{d_i+1} \right\rceil$ and $\left\lfloor \frac{n_i d_i}{d_i+1} \right\rfloor$. Now if $n_i > d_i + 1$, then we solve this round independently by adding to G all the edges returned by $\textsc{graph}(n_i, d_i, L_i, R_i)$. Notice that if $n_i \leq d_i + 1$, then L_i will contain only one vertex, say a_i. In such a case, we add edges between a_i and all the vertices in the set R_i. Also, we add edges between a_i and any arbitrarily chosen $d_i + 1 - n_i$ vertices in $\cup_{j<i} R_j$. This is possible since $d_i + 1 - n_i = d_i - \left\lfloor \frac{n_i d_i}{d_i+1} \right\rfloor \leq \sum_{j=1}^{i-1} \left\lfloor \frac{n_j d_j}{d_j+1} \right\rfloor = \sum_{j=1}^{i-1} |R_j|$. Finally, after the ℓ rounds are completed, we add edges between each pair of vertices in set $R = \cup_{i=1}^{\ell} R_i$ to make it a clique.

Let us now show bounds on the degree of vertices in sets L_i and R_i.

1. Each vertex in L_i has degree exactly d_i : Recall we add edges to vertices in L_i only in the i^{th} iteration of the for loop. If $n_i > d_i + 1$, then the degree of each vertex in L_i is exactly d_i. If $|L_i| = 1$ or, equivalently, $n_i \leq d_i + 1$, then $|R_i| = n_i - |L_i| = n_i - 1$, and so the degree of the vertex $a_i \in L_i$ is $(n_i - 1) + (d_i + 1 - n_i) = d_i$.
2. Vertices in R have degree at least d_ℓ : For any $i \in [1, \ell]$, if $n_i > d_i + 1$, then $|R_i| = \lceil \frac{n_i d_i}{d_i+1} \rceil$, and even in the case $n_i \leq d_i + 1$, we have $|R_i| = n_i - |L_i| = n_i - \lceil \frac{n_i}{d_i+1} \rceil = \lceil \frac{n_i d_i}{d_i+1} \rceil$. Thus $|R| = \sum_{i=1}^{\ell} |R_i| = \sum_{i=1}^{\ell} \lceil \frac{n_i d_i}{d_i+1} \rceil$ which is bounded below by d_i. Since $|R| \geq d_\ell$, and each vertex in R is adjacent to at least one vertex in $\cup_i L_i$, the degree of vertices in R is at least d_ℓ.

Algorithm 2 Computing a MinND-realization for a given special σ.

Input: A sequence $\sigma = (d_\ell^{n_\ell} \cdots d_1^{n_1})$ satisfying $d_i \leq \sum_{j=1}^{i} \lfloor \frac{n_j d_j}{d_j+1} \rfloor$, for $1 \leq i \leq \ell$.

Initialize G to be an empty graph.
for $i = 1$ *to* ℓ **do**
    Add to G a set V_i of n_i new vertices.
    Partition V_i in two sets L_i, R_i such that $|L_i| = \lceil \frac{n_i}{d_i+1} \rceil$ and $|R_i| = \lfloor \frac{n_i d_i}{d_i+1} \rfloor$.
    if ($n_i > d_i + 1$, or equivalently, $|L_i| > 1$) **then**
        Add to G all the edges returned by GRAPH(n_i, d_i, L_i, R_i).
    else if ($|L_i| = 1$) **then**
        Let a_i be the only vertex in L_i.
        Connect a_i to all vertices in R_i, and any arbitrary $d_i + 1 - n_i$ vertices in $\cup_{j<i} R_j$.
Add edges between each pair of vertices in $R = \cup_{i=1}^{\ell} R_i$ to make it a clique.
Output G.

We next show that for any vertex $v \in V_i$, MinND$(v) = d_i$, where $i \in [1, \ell]$. If $v \in L_i$, then MinND$(v) = d_i$, since each vertex in L_i has degree d_i, and is adjacent to only vertices in R which have degree at least $d_\ell \geq d_i$. If $v \in R_i$, then also MinND$(v) = d_i$, since each vertex in R_i is adjacent to at least one vertex in L_i, and $N[v]$ is contained in the set $R \cup (\cup_{j \geq i} L_j)$, whose vertices have degree at least d_i.

The leader function over V is as follows. For each $v \in \cup_{i=1}^{\ell} L_i$, we set *leader*$(v) = v$, and for each $v \in R_i$, we set *leader*(v) to any arbitrary neighbor of v in L_i. Since each vertex in $L = \cup_{i=1}^{\ell} L_i = \{leader(v) \mid v \in V\}$ is a leader of itself, the set L of leaders and the set F of followers must be mutually disjoint. ◀

As a corollary of the above results, the following is immediate.

▶ **Theorem 8.** *The sequence $\sigma = (d_2^{n_2}, d_1^{n_1})$ is MinND-realizable if and only if $d_1 \leq \lfloor \frac{n_1 d_1}{d_1+1} \rfloor$ and $d_2 \leq \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$.*

Proof. Suppose $\sigma = (d_2^{n_2}, d_1^{n_1})$ is realizable. Then Theorem 6 implies (i) $n_1 \geq d_1 + 1$ which implies $d_1 \leq \lfloor \frac{n_1 d_1}{d_1+1} \rfloor$, and (ii) $d_\ell = d_2 \leq \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$. The converse follows from Theorem 7. ◀

▶ Remark 9. In Appendix A, we additionally solve the more involved case of sequences of length three. That is, we provide a complete characterization of the realizability of sequences of the form $\sigma = (d_3^{n_3}, d_2^{n_2}, d_1^{n_1})$ over general graphs.

For a sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$, let $\gamma = (d_1+1)/d_1$. As $\lfloor \frac{\gamma n_1 d_1}{d_1+1} \rfloor + \ldots + \lfloor \frac{\gamma n_i d_i}{d_i+1} \rfloor \geq n_1 + \cdots + n_i \geq d_i$, we also have the following result providing a γ (≤ 2) approximation.

▶ **Corollary 10.** *For any sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ satisfying the first necessary condition* (NC1)*, the sequence $\sigma^\gamma = (d_\ell^{\lceil \gamma n_\ell \rceil}, \ldots, d_1^{\lceil \gamma n_1 \rceil})$ satisfies the sufficient condition* (SC)*.*

4.1 Sequences admitting Disjoint-Leader-Follower Sets

Finally, we state our results on sequences admitting disjoint Leader-Follower sets.

▶ **Theorem 11.** *A sequence $\sigma = (n_\ell^{d_\ell} \ldots d_1^{n_1})$ is* MinND*-realizable by a graph G having disjoint leader-set (L) and follower-set (F) with respect to some leader function, if and only if, for each $i \in [1,\ell]$, $d_i \leq \sum_{j=1}^{i} \lfloor \frac{n_j d_j}{d_j+1} \rfloor$.*

Proof. Let us suppose there exists a leader function over G for which $L \cap F = \emptyset$, then for each $i \in [1,\ell]$, $L_i \subseteq V_i$. This is because if for some i, there exists $w \in L_i \setminus V_i$, then $deg(w) = d_i \neq \text{MinND}(d_i)$, which implies that w is a leader as well as a follower. Since $L_i \subseteq V_i$, by Theorem 6, $d_i \leq \sum_{j=1}^{i} \lfloor \frac{n_j d_j}{d_j+1} \rfloor$, for each $i \in [1,\ell]$. The converse claim follows from Theorem 7. ◀

References

1 Martin Aigner and Eberhard Triesch. Realizability and uniqueness in graphs. *Discrete Mathematics*, 136:3–20, 1994.

2 Amotz Bar-Noy, Keerti Choudhary, David Peleg, and Dror Rawitz. Realizability of graph specifications: Characterizations and algorithms. In *25th SIROCCO*, LNCS, pages 3–13, 2018.

3 Amotz Bar-Noy, Keerti Choudhary, David Peleg, and Dror Rawitz. Efficiently realizing interval sequences. In *30th ISAAC*, pages 47:1–47:15, 2019.

4 Amotz Bar-Noy, Keerti Choudhary, David Peleg, and Dror Rawitz. Graph profile realizations and applications to social networks. In *13th WALCOM*, LNCS, pages 1–12, 2019.

5 Amotz Bar-Noy, Keerti Choudhary, David Peleg, and Dror Rawitz. Graph realizations: Maximum degree in vertex neighborhoods. In *Proc. SWAT*, 2020.

6 Michael D. Barrus and Elizabeth A. Donovan. Neighborhood degree lists of graphs. *Discrete Mathematics*, 341(1):175–183, 2018.

7 Joseph K. Blitzstein and Persi Diaconis. A sequential importance sampling algorithm for generating random graphs with prescribed degrees. *Internet Mathematics*, 6(4):489–522, 2011.

8 Sheshayya A. Choudum. A simple proof of the Erdös-Gallai theorem on graph sequences. *Bull. Austral. Math. Soc.*, 33(1):67–70, 1991.

9 Brian Cloteaux. Fast sequential creation of random realizations of degree sequences. *Internet Mathematics*, 12(3):205–219, 2016.

10 Paul Erdös and Tibor Gallai. Graphs with prescribed degrees of vertices [hungarian]. *Matematikai Lapok*, 11:264–274, 1960.

11 S.L. Feld. Why your friends have more friends than you do. *Amer. J. Sociology*, 96:1464–1477, 1991.

12 S. Louis Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph – I. *SIAM J. Appl. Math.*, 10(3):496–506, 1962.

13 Peter L. Hammer, Toshihide Ibaraki, and Bruno Simeone. Threshold sequences. *SIAM J. Algebraic Discrete Methods*, 2(1):39–49, 1981.

14 V. Havel. A remark on the existence of finite graphs. *Casopis Pest. Mat.*, 80:477–480, 1955.

15 P.J. Kelly. A congruence theorem for trees. *Pacific J. Math.*, 7:961–968, 1957.

16 Milena Mihail and Nisheeth Vishnoi. On generating graphs with prescribed degree sequences for complex network modeling applications. *3rd ARACNE*, 2002.

17 Elchanan Mossel and Nathan Ross. Shotgun assembly of labeled graphs. *CoRR*, abs/1504.07682, 2015. `arXiv:1504.07682`.
18 Peter V. O'Neil. Ulam's conjecture and graph reconstructions. *Amer. Math. Monthly*, 77:35–43, 1970.
19 Gerard Sierksma and Han Hoogeveen. Seven criteria for integer sequences being graphic. *J. Graph Theory*, 15(2):223–231, 1991.
20 Amitabha Tripathi and Himanshu Tyagi. A simple criterion on degree sequences of graphs. *Discrete Applied Mathematics*, 156(18):3513–3517, 2008.
21 Regina Tyshkevich. Decomposition of graphical sequences and unigraphs. *Discrete Mathematics*, 220(1-3):201–238, 2000.
22 S.M. Ulam. *A collection of mathematical problems*. Wiley, 1960.
23 D.L. Wang and D.J. Kleitman. On the existence of n-connected graphs with prescribed degrees ($n > 2$). *Networks*, 3:225–239, 1973.
24 N.C. Wormald. Models of random regular graphs. *Surveys in Combin.*, 267:239–298, 1999.

A MinND realization of tri-sequences in General Graphs

Here we consider the scenario when a sequence has only three distinct degrees.

We now provide a complete characterization of sequence $\sigma = (d_3^{n_3}, d_2^{n_2}, d_1^{n_1})$.

▶ **Theorem 12.** *The necessary and sufficient conditions for MinND-realizability of the sequence $\sigma = (d_\ell^{n_\ell}, \cdots, d_1^{n_1})$ when $\ell = 3$ is*

1. $d_1 + 1 \le n_1$,
2. $d_2 + 1 \le n_1 + n_2$,
3. $d_3 \le \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor + \lfloor \frac{n_3 d_3}{d_3+1} \rfloor$, *and*
4. *either* $d_2 \le \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$, *or* $d_3 + 1 \le n_1 + n_2 + n_3 - \left(1 + \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil\right)$.

Proof. Suppose $\sigma = (d_3^{n_3}, d_2^{n_2}, d_1^{n_1})$ is realizable, then by Theorem 6, it follows that the first three conditions stated above are necessary.

To prove that all four conditions are necessary, we are left to show that if $d_2 > \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$, then $d_3 + 1 \le n_1 + n_2 + n_3 - \left(1 + \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil\right)$. We consider a graph G that realizes σ. Let V_1, V_2, V_3 be the partition of $V(G)$ as defined in Section 4. Consider a vertex $w \in V_2$. Observe that $leader(w)$ must lie in V_1, because if $L_2 \cap V_2$ is non-empty, then Theorem 6 implies $d_2 \le \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$. We first show that $|L_1| \ge \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil$. The set $N(w) \cap V_1$ has size at least $d_2 - n_2$. Each vertex $x \in L_1$ can serve as a leader of at most d_1 vertices in open-neighborhood of w. Indeed, if $x \in N(w)$ then it can not count w (lying outside $N(w)$), and if $x \notin N(w)$ then it can not count itself (again lying outside $N(w)$). Thus to cover the set $N(w) \cap V_1$ at least $\left\lceil \frac{d_2 - n_2}{d_1} \right\rceil$ leaders are required, thereby showing $|L_1| \ge \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil$. Now consider a vertex $y \in V_3$, note that $N[y]$ excludes w (as degree of w is d_2), as well as L_1 (as vertices in L_1 have degree d_1). Therefore, we obtain the following relation.

$$d_3 + 1 = |\ N[y]\ | \le |V_1 \setminus L_1| + |V_2 \setminus w| + |V_3| \le n_1 + n_2 + n_3 - \left(1 + \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil\right)$$

We now prove the sufficiency claims. If $d_2 \le \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$, then the conditions 1-4 are sufficient by Theorem 7. So let us focus on the scenario when $d_2 > \lfloor \frac{n_1 d_1}{d_1+1} \rfloor + \lfloor \frac{n_2 d_2}{d_2+1} \rfloor$. Let $N = n_1 + n_2 + n_3 - \left(1 + \left\lceil \frac{d_2 - n_2}{d_1} \right\rceil\right)$. The vertex-set of our realized graph $G = (V, E)$ will be a union of three disjoint sets $L_1, L_2 = \{w\}$, and Z of size respectively $\left\lceil \frac{d_2 - n_2}{d_1} \right\rceil$, 1, and N. Initially, the edge-set E is an empty-set. Between vertex pairs in Z, we add edges so that the induced graph $G[Z]$ is identical to $\textsc{graph}(N, d_3, \left\lceil \frac{N}{d_3+1} \right\rceil, \left\lfloor \frac{Nd_3}{d_3+1} \right\rfloor)$. This step is possible since $d_3 + 1 \leq N$, and ensures that $\textsc{MinND}_{G[Z]}(z) = d_3$, for $z \in Z$. Let L_3 denote the set of those vertices in Z whose degree is equal to d_3. We connect w to arbitrary $N - n_3 = n_2 + (n_1 - |L_1 \cup L_2|)$ vertices in $Z \setminus L_3$, and any arbitrary $\alpha := d_2 - (n_1 + n_2 - |L_1 \cup L_2|)$ vertices in L_1. Since $deg_G(w) = d_2$, this step ensures that $\textsc{MinND}$ of exactly n_2 vertices in Z decreases to d_2. Let Y be a subset of arbitrary $(n_1 - |L_1 \cup L_2|)$ neighbors of w in Z. Finally, we connect each $x \in L_1 \cap N[w]$ to arbitrary $d_1 - 1$ vertices in Y, and each $x' \in L_1 \setminus N[w]$ to arbitrary d_1 vertices in Y, so as to ensure each vertex in Y is adjacent to at least one leader in L_1. Since vertices in L_1 have degree d_1, this ensures $\textsc{MinND}_G(x) = d_1$, for each $x \in \{w\} \cup Y \cup L_1$. This completes the construction of G. ◀