# Grundy Distinguishes Treewidth from Pathwidth


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## Abstract

Structural graph parameters, such as treewidth, pathwidth, and clique-width, are a central topic of study in parameterized complexity. A main aim of research in this area is to understand the "price of generality" of these widths: as we transition from more restrictive to more general notions, which are the problems that see their complexity status deteriorate from fixed-parameter tractable to intractable? This type of question is by now very well-studied, but, somewhat strikingly, the algorithmic frontier between the two (arguably) most central width notions, treewidth and pathwidth, is still not understood: currently, no natural graph problem is known to be W-hard for one but FPT for the other. Indeed, a surprising development of the last few years has been the observation that for many of the most paradigmatic problems, their complexities for the two parameters actually coincide exactly, despite the fact that treewidth is a much more general parameter. It would thus appear that the extra generality of treewidth over pathwidth often comes "for free".

Our main contribution in this paper is to uncover the first natural example where this generality comes with a high price. We consider Grundy Coloring, a variation of coloring where one seeks to calculate the worst possible coloring that could be assigned to a graph by a greedy First-Fit algorithm. We show that this well-studied problem is FPT parameterized by pathwidth; however, it becomes significantly harder (W[1]-hard) when parameterized by treewidth. Furthermore, we show that Grundy Coloring makes a second complexity jump for more general widths, as it becomes para-NP-hard for clique-width. Hence, Grundy Coloring nicely captures the complexity trade-offs between the three most well-studied parameters. Completing the picture, we show that Grundy Coloring is FPT parameterized by modular-width.



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## 1 Introduction

The study of the algorithmic properties of *structural graph parameters* has been one of the most vibrant research areas of parameterized complexity in the last few years. In this area we consider graph complexity measures ("graph width parameters"), such as treewidth, and attempt to characterize the class of problems which become tractable for each notion of width. The most important graph widths are often comparable to each other in terms of their generality. Hence, one of the main goals of this area is to understand which problems separate two comparable parameters, that is, which problems transition from being FPT for a more restrictive parameter to W-hard for a more general one[1]. This endeavor is sometimes referred to as determining the "price of generality" of the more general parameter.

The two most widely studied graph widths are probably treewidth and pathwidth, which have an obvious containment relationship to each other. Despite this, to the best of our knowledge, no natural problem is currently known to delineate their complexity border in the sense we just described. Our main contribution is exactly to uncover a natural, well-known problem which fills this gap. Specifically, we show that Grundy Coloring, the problem of ordering the vertices of a graph to maximize the number of colors used by the First-Fit coloring algorithm, is FPT parameterized by pathwidth, but W[1]-hard parameterized by treewidth. We then show that Grundy Coloring makes a further complexity jump if one considers clique-width, as in this case the problem is para-NP-complete. Hence, Grundy Coloring turns out to be an interesting specimen, nicely demonstrating the algorithmic trade-offs involved among the three most central graph widths.

**Graph widths and the price of generality.** Much of modern parameterized complexity theory is centered around studying graph widths, especially treewidth and its variants. In this paper we focus on the parameters summarized in Figure 1, and especially the parameters that form a linear hierarchy, from vertex cover, to tree-depth, pathwidth, treewidth, and clique-width. Each of these parameters is a strict generalization of the previous ones in this list. On the algorithmic level we would expect this relation to manifest itself by the appearance of more and more problems which become *intractable* as we move towards the more general parameters. Indeed, a search through the literature reveals that for each step in this list of parameters, several *natural* problems have been discovered which distinguish the two consecutive parameters (we give more details below). The one glaring exception to this rule seems to be the relation between treewidth and pathwidth.

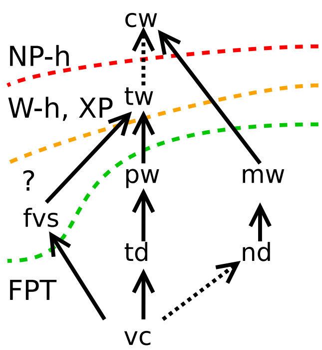
Treewidth is a parameter of central importance to parameterized algorithmics, in part because wide classes of problems (notably all $MSO_2$-expressible problems [18]) are FPT for this parameter. Treewidth is usually defined in terms of tree decompositions of graphs, which naturally leads to the equally well-known notion of pathwidth, defined by forcing the decomposition to be a path. On a graph-theoretic level, the difference between the two

[1] We assume the reader is familiar with the basics of parameterized complexity theory, such as the classes FPT and W[1], as given in standard textbooks [21].

notions is well-understood and treewidth is known to describe a much richer class of graphs. In particular, while all graphs of pathwidth $k$ have treewidth at most $k$, there exist graphs of constant treewidth (in fact, even trees) of unbounded pathwidth. Naturally, one would expect this added richness of treewidth to come with some negative algorithmic consequences in the form of problems which are FPT for pathwidth but W-hard for treewidth. Furthermore, since treewidth and pathwidth are probably the most studied parameters in our list, one might expect the problems that distinguish the two to be the first ones to be discovered.

Nevertheless, so far this (surprisingly) does not seem to have been the case: on the one hand, FPT algorithms for pathwidth are DPs which also extend to treewidth; on the other hand, we give (in Section 1.1) a semi-exhaustive list of dozens of natural problems which are W[1]-hard for treewidth and turn out without exception to also be hard for pathwidth. In fact, even when this is sometimes not explicitly stated in the literature, the same reduction that establishes W-hardness by treewidth also does so for pathwidth. Intuitively, an explanation for this phenomenon is that the basic structure of such reductions typically resembles a $k \times n$ (or smaller) grid, which has both treewidth and pathwidth bounded by $k$.

Our main motivation in this paper is to take a closer look at the algorithmic barrier between pathwidth and treewidth and try to locate a natural (that is, not artificially contrived) problem whose complexity transitions from FPT to W-hard at this barrier. Our main result is the proof that Grundy Coloring is such a problem. This puts in the picture the last missing piece of the puzzle, as we now have natural problems that distinguish the parameterized complexity of any two consecutive parameters in our main hierarchy.




| Parameter | Result | Ref |
|---|---|---|
| Clique-width | para-NP-hard | Theorem 25 |
| Treewidth | W[1]-hard | Theorem 16 |
| Pathwidth | FPT | Theorem 20 |
| Modular-width | FPT | Theorem 26 |

In the figure, clique-width, treewidth, pathwidth, tree-depth, vertex cover, feedback vertex set, neighborhood diversity, and modular-width are indicated as cw, tw, pw, td, vc, fvs, nd, and mw respectively. Arrows indicate more general parameters. Dotted arrows indicate that the parameter may increase exponentially, (e.g. graphs of vc $k$ have nd at most $2^k + k$).

**Figure 1** Summary of considered graph parameters and results.

**Grundy Coloring.** In the Grundy Coloring problem we are given a graph $G = (V, E)$ and are asked to order $V$ in a way that maximizes the number of colors used by the greedy (First-Fit) coloring algorithm. The notion of Grundy coloring was first introduced by Grundy in the 1930s, and later formalized in [17]. Since then, the complexity of Grundy Coloring has been very well-studied (see [1, 3, 14, 30, 44, 46, 52, 55, 73, 74, 76, 77, 78] and the references therein). For the natural parameter, namely the number of colors to be used, Grundy coloring was recently proved to be W[1]-hard in [1]. An XP algorithm for Grundy Coloring parameterized by treewidth was given in [74], using the fact that the Grundy number of any graph is at most $\log n$ times its treewidth. In [13] Bonnet et al. explicitly asked whether this can be improved to an FPT algorithm. They also observed that the problem is FPT parameterized by vertex cover. It appears that the complexity of Grundy Coloring parameterized by pathwidth was never explicitly posed as a question and it was not suspected that it may differ from that for treewidth. We note that, since the problem (as given in Definition 1) is easily seen to be $MSO_1$ expressible for a fixed Grundy number, it is FPT for all considered parameters if the Grundy number is also a parameter [19], so we intuitively want to concentrate on cases where the Grundy number is large.

**Our results.** Our results illuminate the complexity of GRUNDY COLORING parameterized by pathwidth and treewidth, as well as clique-width and modular-width. More specifically:

1. We show that GRUNDY COLORING is W[1]-hard parameterized by treewidth via a reduction from $k$-MULTI-COLORED CLIQUE. The main building block of our reduction is the structure of binomial trees, which have treewidth one but unbounded pathwidth, which explains the complexity jump between the two parameters. As mentioned, an XP algorithm is known in this case [74], so this result is in a sense tight.
2. We show that GRUNDY COLORING is FPT parameterized by pathwidth. Our main tool here is a combinatorial lemma, which draws heavily from known combinatorial bounds on the performance of First-Fit coloring on intervals graphs [53, 65]. We use this lemma to show that on any graph the Grundy number is at most a linear function of the pathwidth.
3. We show that GRUNDY COLORING is para-NP-complete parameterized by clique-width, that is, NP-complete for graphs of constant clique-width (specifically, clique-width 6).
4. We show that GRUNDY COLORING is FPT parameterized by neighborhood diversity (which is defined in [56]) and leverage this result to obtain an FPT algorithm parameterized by modular-width (which is defined in [38]).

Our main interest is concentrated in the first two results, which achieve our goal of finding a natural problem distinguishing pathwidth from treewidth. The result for clique-width nicely fills out the picture by giving an intuitive view of the evolution of the complexity of the problem and showing that in a case where no non-trivial bound can be shown on the optimal value, the problem becomes hopelessly hard from the parameterized point of view.

**Other related work.** Let us now give a brief survey of "price of generality" results involving our considered parameters, that is, results showing that a problem is efficient for one parameter but hard for a more general one. In this area, the results of Fomin et al. [35], introducing the term "price of generality", have been particularly impactful. This work and its follow-ups [36, 37], were the first to show that four natural graph problems (COLORING, EDGE DOMINATING SET, MAX CUT, HAMILTONICITY) which are FPT for treewidth, become W[1]-hard for clique-width. In this sense, these problems, as well as problems discovered later such as counting perfect matchings [20], SAT [68, 23], ∃∀-SAT [59], ORIENTABLE DELETION [45], and $d$-REGULAR INDUCED SUBGRAPH [16], form part of the "price" we have to pay for considering a more general parameter. This line of research has thus helped to illuminate the complexity border between the two most important sparse and dense parameters (treewidth and clique-width), by giving a list of *natural* problems distinguishing the two. (An artificial $MSO_2$-expressible such problem was already known much earlier [19, 58]).

Let us now focus in the area below treewidth in Figure 1 by considering problems which are in XP but W[1]-hard parameterized by treewidth. By now, there is a small number of problems in this category which are known to be W[1]-hard even for vertex cover: LIST COLORING [31] was the first such problem, followed by CSP (for the vertex cover of the dual graph) [70], and more recently by $(k, r)$-CENTER, $d$-SCATTERED SET, and MIN POWER STEINER TREE [49, 48, 50] on weighted graphs. Intuitively, it is not surprising that problems W[1]-hard by vertex cover are few and far between, since this is a very restricted parameter.

Indeed, for most problems in the literature which are W[1]-hard by treewidth, vertex cover is the only parameter (among the ones considered here) for which the problem becomes FPT.

A second interesting category are problems which are FPT for tree-depth ([66]) but W[1]-hard for pathwidth. Mixed Chinese Postman Problem was the first discovered problem of this type [43], followed by Min Bounded-Length Cut [25, 10], ILP [40], Geodetic Set [51] and unweighted $(k, r)$-Center and $d$-Scattered Set [49, 48].

To the best of our knowledge, for all remaining problems which are known to be W[1]-hard by treewidth, the reductions that exist in the literature also establish W[1]-hardness for pathwidth. Below we give a (semi-exhaustive) list of problems which are known to be W[1]-hard by treewidth. After reviewing the relevant works we have verified that all of the following problems are in fact shown to be W[1]-hard parameterized by pathwidth (and in many case by feedback vertex set and tree-depth), even if this is not explicitly claimed.

## 1.1 Known problems which are W-hard for treewidth and for pathwidth

- Precoloring Extension and Equitable Coloring are shown to be W[1]-hard for both tree-depth and feedback vertex set in [31] (though the result is claimed only for treewidth). This is important, because Equitable Coloring often serves as a starting point for reductions to other problems. A second hardness proof for this problem was recently given in [22]. These two problems are FPT by vertex cover [33].
- Capacitated Dominating Set and Capacitated Vertex Cover are W[1]-hard for both tree-depth and feedback vertex set [24] (though again the result is claimed for treewidth).
- Min Maximum Out-degree on weighted graphs is W[1]-hard by tree-depth and feedback vertex set [72].
- General Factors is W[1]-hard by tree-depth and feedback vertex set [71].
- Target Set Selection is W[1]-hard by tree-depth and feedback vertex set [9] but FPT for vertex cover [67].
- Bounded Degree Deletion is W[1]-hard by tree-depth and feedback vertex set, but FPT for vertex cover [11, 39].
- Fair Vertex Cover is W[1]-hard by tree-depth and feedback vertex set [54].
- Fixing Corrupted Colorings is W[1]-hard by tree-depth and feedback vertex set [12] (reduction from Precoloring Extension).
- Max Node Disjoint Paths is W[1]-hard by tree-depth and feedback vertex set [29, 34].
- Defective Coloring is W[1]-hard by tree-depth and feedback vertex set [8].
- Power Vertex Cover is W[1]-hard by tree-depth but open for feedback vertex set [2].
- Majority CSP is W[1]-hard parameterized by the tree-depth of the incidence graph [23].
- List Hamiltonian Path is W[1]-hard for pathwidth [62].
- L(1,1)-Coloring is W[1]-hard for pathwidth, FPT for vertex cover [33].
- Counting Linear Extensions of a poset is W[1]-hard (under Turing reductions) for pathwidth [26].
- Equitable Connected Partition is W[1]-hard by pathwidth and feedback vertex set, FPT by vertex cover [28].
- Safe Set is W[1]-hard parameterized by pathwidth, FPT by vertex cover [7].
- Matching with Lower Quotas is W[1]-hard parameterized by pathwidth [4].
- Subgraph Isomorphism is W[1]-hard parameterized by the pathwidth of $G$, even when $G, H$ are connected planar graphs of maximum degree 3 and $H$ is a tree [61].
- Metric Dimension is W[1]-hard by pathwidth [15].
- Simple Comprehensive Activity Selection is W[1]-hard by pathwidth [27].

- Defensive Stackelberg Game for IGL is W[1]-hard by pathwidth (reduction from Equitable Coloring) [5].
- Directed $(p, q)$-Edge Dominating Set is W[1]-hard parameterized by pathwidth [6].
- Maximum Path Coloring is W[1]-hard for pathwidth [57].
- Unweighted $k$-Sparsest Cut is W[1]-hard parameterized by the three combined parameters tree-depth, feedback vertex set, and $k$ [47].
- Graph Modularity is W[1]-hard parameterized by pathwidth plus feedback vertex set [63].

Let us also mention in passing that the algorithmic differences of pathwidth and treewidth may also be studied in the context of problems which are hard for constant treewidth. Such problems also generally remain hard for constant pathwidth (examples are Steiner Forest [42], Bandwidth [64], Minimum mcut [41]). One could also potentially try to distinguish between pathwidth and treewidth by considering the parameter dependence of a problem that is FPT for both. Indeed, for a long time the best-known algorithm for Dominating Set had complexity $3^k$ for pathwidth, but $4^k$ for treewidth. Nevertheless, the advent of fast subset convolution techniques [75], together with tight SETH-based lower bounds [60] has, for most problems, shown that the complexities on the two parameters coincide exactly.

Finally, let us mention a case where pathwidth and treewidth have been shown to be quite different in a sense similar to our framework. In [69] Razgon showed that a CNF can be compiled into an OBDD (Ordered Binary Decision Diagram) of size FPT in the pathwidth of its incidence graphs, but there exist formulas that always need OBDDs of size XP in the treewidth. Although this result does separate the two parameters, it is somewhat adjacent to what we are looking for, as it does not speak about the complexity of a decision problem, but rather shows that an OBDD-producing algorithm parameterized by treewidth would need XP time simply because it would have to produce a huge output in some cases.

## 2 Definitions and Preliminaries

For non-negative integers $i, j$, we use $[i, j]$ to denote the set $\{k \mid i \leq k \leq j\}$. Note that if $j < i$, then the set $[i, j]$ is empty. We will also write simply $[i]$ to denote the set $[1, i]$.

We give two equivalent definitions of our main problem.

▶ **Definition 1.** *A $k$-Grundy Coloring of a graph $G = (V, E)$ is a partition of $V$ into $k$ non-empty sets $V_1, \ldots, V_k$ such that: (i) for each $i \in [k]$ the set $V_i$ induces an independent set; (ii) for each $i \in [k-1]$ the set $V_i$ dominates the set $\bigcup_{i<j\leq k} V_j$.*

▶ **Definition 2.** *A $k$-Grundy Coloring of a graph $G = (V, E)$ is a proper $k$-coloring $c : V \to [k]$ that results by applying the First-Fit algorithm on an ordering of $V$; the First-Fit algorithm colors one by one the vertices in the given ordering, assigning to a vertex the minimum color that is not already assigned to one of its preceding neighbors.*

The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the maximum $k$ such that $G$ admits a $k$-Grundy Coloring. In a given Grundy Coloring, if $u \in V_i$ (equiv. if $c(u) = i$) we will say that $u$ was given color $i$. The Grundy Coloring problem is the problem of determining the maximum $k$ for which a graph $G$ admits a $k$-Grundy Coloring. It is not hard to see that a proper coloring is a Grundy coloring if and only if every vertex assigned color $i$ has at least one neighbor assigned color $j$, for each $j < i$.

## 3 W[1]-Hardness for Treewidth

In this section we prove that Grundy Coloring parameterized by treewidth is W[1]-hard (Theorem 16). Our proof relies on a reduction from $k$-Multi-Colored Clique and initially establishes W[1]-hardness for a more general problem where we are given a target color for a set of vertices (Lemma 8); we then reduce this to Grundy Coloring. Interestingly, this intermediate problem turns out to be W[1]-hard even for pathwidth (Lemma 12), since our reduction uses the standard strategy of constructing a grid-like structure of dimensions $k \times n$. The reason this reduction fails to prove that Grundy Coloring is W[1]-hard by pathwidth is that we use some gadgets to implement the targets and a support operation (which "pre-colors" some vertices) and for these gadgets we use trees of unbounded pathwidth. The results of Section 4 show that this is essential: our reduction *needs* some part that causes it to have high pathwidth, otherwise the Grundy number of the constructed graph would be bounded by the parameter, resulting in an instance that can be solved in FPT time.

Let us now present the different parts of our construction. We will make use of the structure of binomial trees $T_i$.

▶ **Definition 3.** *The binomial tree $T_i$ with root $r_i$ is a rooted tree defined recursively in the following way: $T_1$ consists simply of its root $r_1$; in order to construct $T_i$ for $i > 1$, we construct one copy of $T_j$ for all $j < i$ and a special vertex $r_i$, then we connect $r_j$ with $r_i$. An alternative equivalent definition of the binomial tree $T_i$, $i \geq 2$ is that we construct two trees $T_{i-1}$ , $T'_{i-1}$, we connect their roots $r_{i-1}$, $r'_{i-1}$ and select one of them as the new root $r_i$.*

▶ **Proposition 4.** *Let $i \geq 2$, $T_i$ be a binomial tree and $1 \leq t < i$. There exist $2^{i-t-1}$ binomial trees $T_t$ which are vertex-disjoint and non-adjacent subtrees in $T_i$, where no $T_t$ contains the root $r_i$ of $T_i$.*

▶ **Proposition 5.** $\Gamma(T_i) \leq i$. *Furthermore, for all $j \leq i$ there exists a Grundy coloring which assigns color $j$ to the root of $T_i$.*

The proofs of Propositions 4 and 5 can be found in the full version of this paper.

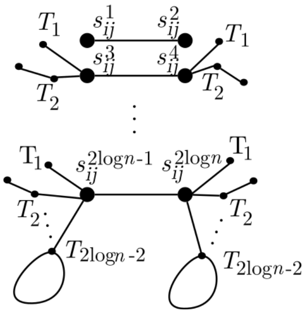
A Grundy coloring of $T_i$ that assigns color $i$ to $r_i$ is called *optimal*. If $r_i$ is assigned color $j < i$ then we call the Grundy coloring *sub-optimal*.

We now define a generalization of the Grundy coloring problem with target colors and show that it is W[1]-hard parameterized by treewidth. We later describe how to reduce this problem to Grundy Coloring such that the treewidth does not increase by a lot.

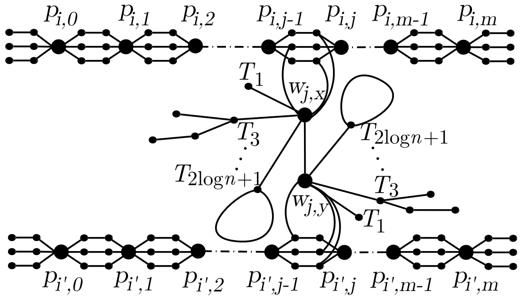
▶ **Definition 6** (Grundy Coloring with Targets). *We are given a graph $G(V, E)$, an integer $t \in \mathbb{N}$ called the target and a subset $S \subset V$. (For simplicity we will say that vertices of $S$ have target $t$.) If $G$ admits a Grundy Coloring which assigns color $t$ to some vertex $s \in S$ we say that, for this coloring, vertex $s$* achieves its target. *If there exists a Grundy Coloring of $G$ which assigns to all vertices of $S$ color $t$, then we say that $G$ admits a* Target-achieving Grundy Coloring. *Grundy Coloring with Targets is the decision problem associated to the question "given $G, S, t$ as defined above, does $G$ admit a Target-achieving Grundy Coloring ?".*

We will also make use of the following operation:

▶ **Definition 7** (Tree-support). *Given a graph $G = (V, E)$, a vertex $u \in V$ and a set $N$ of positive integers, we define the* tree-support *operation as follows: (a) for all $i \in N$ we add a copy of $T_i$ in the graph; (b) we connect $u$ to the root $r_i$ of each of the $T_i$. We say that we* add supports *$N$ on $u$. The trees $T_i$ will be called the* supporting trees *or* supports *of $u$. Slightly abusing notation, we also call* supports *the numbers $i \in N$.*




**(a)** Vertex Selection gadget $S_{i,j}$.




**(b)** Propagators $p_{i,j}$ and Edge Selection gadget $W_j$. The edge selection checkers and the supports of the $p_{i,j}$ and $s_{i,j}^l$ are not depicted. In the example $B_x = 010$ and $B_y = 100$.

**Figure 2** The gadgets. Figure 2a is an enlargment of Figure 2b between $p_{i,j-1}$ and $p_{i,j}$.

Intuitively, the tree-support operation ensures that vertex $u$ may have at least one neighbor of color $i$ for each $i \in N$ in a Grundy coloring, and thus increase the color $u$ can take. Observe that adding supporting trees to a vertex does not increase the treewidth, but does increase the pathwidth (binomial trees have unbounded pathwidth).

Our reduction is from $k$-Multi-Colored Clique, proven to be W[1]-hard in [32]: given a $k$-multipartite graph $G = (V_1, V_2, \dots, V_k, E)$, decide if for every $i \in [k]$ we can pick $u_i \in V_i$ forming a clique, where $k$ is the parameter. We can also assume that $\forall i \in [k], |V_i| = n$, that $n$ is a power of 2, and that $V_i = \{v_{i,0}, v_{i,1}, \dots, v_{i,n-1}\}$. Furthermore, let $|E| = m$. We construct an instance of Grundy Coloring with Targets $G' = (V', E')$ and $t = 2\log n + 4$ (where all logarithms are base two) using the following gadgets:

**Vertex selection $S_{i,j}$.** See Figure 2a. This gadget consists of $2\log n$ vertices $S_{i,j}^1 \cup S_{i,j}^2 = \bigcup_{l\in[\log n]}\{s_{i,j}^{2l-1}\} \cup \bigcup_{l\in[\log n]}\{s_{i,j}^{2l}\}$, where for each $l \in [\log n]$ we connect vertex $s_{i,j}^{2l-1}$ to $s_{i,j}^{2l}$ thus forming a matching. Furthermore, for each $l \in [2, \log n]$, we add supports $[2l-2]$ to vertices $s_{i,j}^{2l-1}$ and $s_{i,j}^{2l}$. Observe that the vertices $s_{i,j}^{2l-1}$ and $s_{i,j}^{2l}$ together with their supports form a binomial tree $T_{2l}$ with either of these vertices as the root. We construct $k(m+2)$ gadgets $S_{i,j}$, one for each $i \in [k], j \in [0, m+1]$.

The vertex selection gadget $S_{i,1}$ encodes in binary the vertex that is selected in the clique from $V_i$. In particular, for each pair $s_{i,1}^{2l-1}, s_{i,1}^{2l}$, $l \in [\log n]$ either of these vertices can take the maximum color in an optimal grundy coloring of the binomial tree $T_{2l}$ (that is, a coloring that gives the root of the binomial tree $T_{2l}$ color $2l$). A selection corresponds to bit 0 or 1 for the $l^{th}$ binary position. In order to ensure that for each $j \in [m]$ all (middle) $S_{i,j}$ encode the same vertex, we use propagators.

**Propagators $p_{i,j}$.** See Figure 2b. For $i \in [k]$ and $j \in [0, m]$, a propagator $p_{i,j}$ is a single vertex connected to all vertices of $S_{i,j}^2 \cup S_{i,j+1}^1$. To each $p_{i,j}$, we also add supports $\{2\log n + 1, 2\log n + 2, 2\log n + 3\}$. The propagators have target $t = 2\log n + 4$.

**Edge selection $W_j$.** See Figure 2b. Let $j = (v_{i,x}, v_{i',y}) \in E$, where $v_{i,x} \in V_i$ and $v_{i',y} \in V_{i'}$. The gadget $W_j$ consists of four vertices $w_{j,x}, w_{j,y}, w'_{j,x}, w'_{j,y}$. We call $w'_{j,x}, w'_{j,y}$ the *edge selection checkers*. We have the edges $(w_{j,x}, w_{j,y}), (w'_{j,x}, w_{j,x}), (w'_{j,y}, w_{j,y})$. Let us now describe the connections of these vertices with the rest of the graph. Let $B_x = b_1 b_2 \dots b_{\log n}$ be the binary representation of $x$. We connect $w_{j,x}$ to each vertex $s_{ij}^{2l-b_l}$, $l \in [\log n]$ (we do similarly for $w_{j,y}, S_{i',j}$, and $B_y$). We add to each of $w_{j,x}, w_{j,y}$ supports $\bigcup_{l\in[\log n+1]}\{2l-1\}$. We add to each of $w'_{j,x}, w'_{j,y}$ supports $[2\log n + 3] \setminus \{2\log n + 1\}$ and set the target $t = 2\log n + 4$ for these two vertices. We construct $m$ such gadgets, one for each edge. We say that $W_j$ *is activated* if at least one of $w_{j,x}, w_{j,y}$ receives color $2\log n + 3$.

**Edge validators $q_{i,i'}$.** We construct $\binom{k}{2}$ of them, one for each pair $(i, i'), i < i' \in [k]$. The edge validator is a single vertex that is connected to all vertices $w_{j,x}$ for which $j$ is an edge between $V_i$ and $V_{i'}$. We add supports $[2 \log n + 1]$ and a target of $t = 2 \log n + 4$.

The edge validator plays the role of an "or" gadget: in order for it to achieve its target, at least one of its neighboring edge selection gadgets should be activated.

▶ **Lemma 8.** *$G$ has a clique of size $k$ if and only if $G'$ has a target-achieving Grundy coloring.*

**Proof.** ⇒) Suppose that $G$ has a clique. We color the vertices of $G'$ in the following order: First, we color the vertex selection gadget $S_{i,j}$. We start from the supports which we color optimally. We then color the matchings as follows: let $v_{i,x}$ be the vertex that was selected in the clique from $V_i$ and $b_1 b_2 \dots b_{\log n}$ be the binary representation of $x$; we color vertices $s_{i,j}^{2l-(1-b_l)}$, $l \in [\log n]$ with color $2l - 1$ and vertices $s_{i,j}^{2l-b_l}$, $l \in [\log n]$ will receive color $2l$. For the propagators, we color their supports optimally. Propagators have $2 \log n + 3$ neighbors each, all with different colors, so they receive color $2 \log n + 4$, thus achieving the targets.

Then, we color the edge validators $q_{i,i'}$ and the edge selection gadgets $W_j$ that correspond to edges of the clique (that is, $j = (v_{i,x}, v_{i',y}) \in E$ and $v_{i,x} \in V_i$, $v_{i',y} \in V_{i'}$ are selected in the clique). We first color the supports of $q_{i,i'}, w_{j,x}, w_{j,y}$ optimally. From the construction, vertex $w_{j,x}$ is connected with vertices $s_{i,j}^{2l-b_l}$ which have already been colored $2l$, $l \in [\log n]$ and with supports $\bigcup_{l \in [\log n+1]} \{2l - 1\}$, thus $w_{j,x}$ will receive color $2 \log n + 2$. Similarly $w_{j,y}$ already has neighbors which are colored $[2 \log n + 1]$, but also $w_{j,x}$, thus it will receive color $2 \log n + 3$. These $W_j$ will be activated. Since both $w_{j,x}, w_{j,y}$ connect to $q_{i,i'}$, the latter will be assigned color $2 \log n + 4$, thus achieving its target. As for $w'_{j,x}$ and $w'_{j,y}$, these vertices have one neighbor colored $c$, where $c = 2 \log n + 2$ or $c = 2 \log n + 3$. We color their support $T_c$ sub-optimally so that the root receives color $2 \log n + 1$; we color their remaining supports optimally. This way, vertices $w'_{j,x}, w'_{j,y}$ can be assigned color $t = 2 \log n + 4$, achieving the target.

Finally, for the remaining $W_j$, we claim that we can assign to both $w_{j,x}, w_{j,y}$ a color that is at least as high as $2 \log n + 1$. Indeed, we assign to each supporting tree $T_r$ of $w_{j,x}$ a coloring that gives its root the maximum color that is $\leq r$ and does not appear in any neighbor of $w_{j,x}$ in the vertex selection gadget. We claim that in this case $w_{j,x}$ will have neighbors with all colors in $[2 \log n]$, because in every interval $[2l - 1, 2l]$ for $l \in [\log n]$, $w_{j,x}$ has a neighbor with a color in that interval and a support tree $T_{2l+1}$. If $w_{j,x}$ has color $2 \log n + 1$ then we color the supports of $w'_{j,x}$ optimally and achieve its target, while if $w_{j,x}$ has color higher than $2 \log n + 1$, we achieve the target of $w'_{j,x}$ as in the previous paragraph.

⇐) Suppose that $G'$ admits a coloring that achieves the target for all propagators, edge selection checkers, and edge validators. We will prove the following three claims:

▷ Claim 9. The coloring of the vertex selection gadgets is consistent throughout. This corresponds to a selection of $k$ vertices of $G$.

▷ Claim 10. $\binom{k}{2}$ edge selection gadgets have been activated. That correspond to $\binom{k}{2}$ edges of $G$ being selected.

▷ Claim 11. If an edge selection gadget $W_j = \{w_{j,x}, w_{j,y}\}$ with $j = (v_{i,x}, v_{i',y})$ has been activated then the coloring of the vertex selection gadgets $S_{i,j}$ and $S_{i',j}$ corresponds to the selection of vertices $v_{i,x}$ and $v_{i',y}$. In other words, selected vertices and edges form indeed a clique of size $k$ in $G$.

Proof of Claim 9. Suppose that an edge selection checker $w'_{j,x}$ achieved its target. We claim that this implies that $w_{j,x}$ has color at least $2\log n + 1$. Indeed, $w'_{j,x}$ has degree $2\log n + 3$, so its neighbors must have all distinct colors in $[2\log n + 3]$, but among the supports there are only 2 neighbors which may have colors in $[2\log n + 1, 2\log n + 3]$. Therefore, the missing color must come from $w_{j,x}$. We now observe that vertices from the vertex selection gadgets have color at most $2\log n$, because if we exclude from their neighbors the vertices $w_{j,x}$ (which we argued have color at least $2\log n + 1$) and the propagators (which have target $2\log n + 4$), these vertices have degree at most $2\log n - 1$.

Suppose that a propagator $p_{i,j}$ achieves its target of $2\log n + 4$. Since this vertex has a degree of $2\log n + 3$, that means that all of its neighbors should receive all the colors in $[2\log n + 3]$. As argued, colors $[2\log n + 1, 2\log n + 3]$ must come from the supports. Therefore, the colors $[2\log n]$ come from the neighbors of $p_{i,j}$ in the vertex selection gadgets.

We now note that, because of the degrees of vertices in vertex selection gadgets, only vertices $s_{i,j}^{2\log n}, s_{i,j+1}^{2\log n-1}$ can receive colors $2\log n, 2\log n - 1$; from the rest, only $s_{i,j}^{2\log n-2}, s_{i,j+1}^{2\log n-3}$ can receive colors $2\log n - 2, 2\log n - 3$ etc. Thus, for each $l \in [\log n]$, if $s_{i,j}^{2l}$ receives color $2l-1$ then $s_{i,j+1}^{2l-1}$ should receive color $2l$ and vice versa. With similar reasoning, in all vertex selection gadgets we have that $s_{i,j}^{2l-1}, s_{i,j}^{2l}$ received the two colors $\{2l-1, 2l\}$ since they are neighbors. As a result, the colors of $s_{i,j+1}^{2l-1}$, $s_{i,j}^{2l-1}$ (and thus the colors of $s_{i,j+1}^{2l}$, $s_{i,j}^{2l}$) are the same, therefore, the coloring is consistent, for all values of $j \in [m]$. ◁

Proof of Claim 10. If an edge validator achieves its target of $2\log n + 4$, then at least one of its neighbors from an edge selection gadget has received color $2\log n + 3$. We know that each edge selection gadget only connects to a unique edge validator, so there should be $\binom{k}{2}$ edge selection gadgets which have been activated in order for all edge validators to achieve the target. ◁

Proof of Claim 11. Suppose that an edge validator $q_{i,i'}$ achieves its target. That means that there exists an edge selection gadget $W_j = \{w_{j,x}, w_{j,y}, w'_{j,x}, w'_{j,y}\}$ for which at least one of its vertices $\{w_{j,x}, w_{j,y}\}$, say vertex $w_{j,x}$, has received color $2\log n + 3$. Let $j$ be an edge connecting $v_{i,x} \in V_i$ to $v_{i',y} \in V_{i'}$. Since the degree of $w_{j,x}$ is $2\log n + 4$ and we have already assumed that two of its neighbors ($q_{i,i'}$ and $w'_{j,x}$) have color $2\log n + 4$, in order for it to receive color $2\log n + 3$ all its other neighbors should receive all colors in $[2\log n + 2]$. The only possible assignment is to give colors $2l$, $l \in [\log n]$ to its neighbors from $S_{i,j}$ and color $2\log n + 2$ to $w_{j,y}$. The latter is, in turn, only possible if the neighbors of $w_{j,y}$ from $S_{i',j}$ receive all colors $2l$, $l \in [\log n]$. The above corresponds to selecting vertex $v_{i,x}$ from $V_i$ and $v_{i',y}$ from $V_{i'}$. ◁

◀

▶ **Lemma 12.** *Let $G''$ be the graph that results from $G'$ if we remove all the tree-supports. Then $G''$ has pathwidth at most $\binom{k}{2} + 2k + 3$.*

The proof of Lemma 12 can be found in the full version of the paper.

We will now show how to implement the targets using the tree-filling operation below.

▶ **Definition 13** (Tree-filling). *Let $G = (V, E)$ be a graph and $S = \{s_1, s_2, \ldots, s_j\} \subset V$ a set of vertices with target $t$. The* tree-filling *operation is the following. First, we add in $G$ a binomial tree $T_i$, where $i = \lceil \log j \rceil + t + 1$. Observe that, by Proposition 4, there exist $2^{i-t-1} > j$ vertex-disjoint and non-adjacent sub-trees $T_t$ in $T_i$. For each $s \in S$, we find such a copy of $T_t$ in $T_i$, identify $s$ with its root $r_t$, and delete all other vertices of the sub-tree $T_t$.*

The tree-filling operation might in general increase treewidth, but we will do it in a way that it only increases by a constant factor in regards to the pathwidth of $G$.

▶ **Lemma 14.** *Let $G = (V, E)$ be a graph of pathwidth $w$ and $S = \{s_1, \ldots, s_j\} \subset V$ a subset of vertices having target $t$. Then there is a way to apply the tree-filling operation such that the resulting graph $H$ has $tw(H) \leq 4w + 5$.*

**Proof. Construction of $H$.** Let $(\mathcal{P}, \mathcal{B})$ be a path-decomposition of $G$ whose largest bag has size $w + 1$ and $B_1, B_2, \ldots, B_j \in \mathcal{B}$ distinct bags where $\forall a, s_a \in B_a$ (assigning a distinct bag to each $s_a$ is always possible, as we can duplicate bags if necessary). We call those bags *important*. We define an ordering $o : S \to \mathbb{N}$ of the vertices of $S$ that follows the order of the important bags from left to right, that is $o(s_a) < o(s_b)$ if $B_a$ is on the left of $B_b$ in $\mathcal{P}$. For simplicity, let us assume that $o(s_a) = a$ and that $B_a$ is to the left of $B_b$ if $a < b$.

We describe a recursive way to do the substitution of the trees in the tree-filling operation. Crucially, when $j > 2$ we will have to select an appropriate mapping between the vertices of $S$ and the disjoint subtrees $T_t$ in the added binomial tree $T_i$, so that we will be able to keep the treewidth of the new graph bounded.

- If $j = 1$ then $i = t + 1$. We add to the graph a copy of $T_i$, arbitrarily select the root of a copy of $T_t$ contained in $T_i$, and perform the tree-filling operation as described.
- Suppose that we know how to perform the substitution for sets of size at most $\lceil j/2 \rceil$, we will describe the substitution process for a set of size $j$. We have $i = \lceil \log j \rceil + t + 1$ and for all $j$ we have $\lceil \log \lceil j/2 \rceil \rceil = \lceil \log j \rceil - 1$. Split the set $S$ into two (almost) equal disjoint sets $S^L$ and $S^R$ of size at most $\lceil j/2 \rceil$, where for all $s_a \in S^L$ and for all $s_b \in S^R$, $a < b$. We perform the tree-filling on each of these sets by constructing two binomial trees $T^L_{i-1}, T^R_{i-1}$ and doing the substitution; then, we connect their roots and set the root of the left tree as the root $r_i$ of $T_i$, thus creating the substitution of a tree $T_i$.

**Small treewidth.** We now prove that the new graph $H$ that results from applying the tree-filling operation on $G$ and $S$ as described above has a tree decomposition $(\mathcal{T}, \mathcal{B}')$ of width $4w + 5$; in fact we prove by induction on $j$ a stronger statement: if $A, Z \in \mathcal{B}$ are the left-most and right-most bags of $\mathcal{P}$, then there exists a tree decomposition $(\mathcal{T}, \mathcal{B}')$ of $H$ of width $4w + 5$ with the added property that there exists $R \in \mathcal{B}'$ such that $A \cup Z \cup \{r_i\} \subset R$, where $r_i$ is the root of the tree $T_i$.

For the base case, if $j = 1$ we have added to our graph a $T_i$ of which we have selected an arbitrary sub-tree $T_t$, and identified the root $r_t$ of $T_t$ with the unique vertex of $S$ that has a target. Take the path decomposition $(\mathcal{P}, \mathcal{B})$ of the initial graph and add all vertices of $A$ (its first bag) and the vertex $r_i$ (the root of $T_i$) to all bags. Take an optimal tree decomposition of $T_i$ of width 1 and add $r_i$ to each bag, obtaining a decomposition of width 2. We add an edge between the bag of $\mathcal{P}$ that contains the unique vertex of $S$, and a bag of the decomposition of $T_i$ that contains the selected $r_t$. We now have a tree decomposition of the new graph of width $2w + 2 < 4w + 5$. Observe that the last bag of $\mathcal{P}$ now contains all of $A, Z$ and $r_i$.

For the inductive step, suppose we applied the tree-filling operation for a set $S$ of size $j > 1$. Furthermore, suppose we know how to construct a tree decomposition with the desired properties (width $4w + 5$, one bag contains the first and last bags of the path decomposition $\mathcal{P}$ and $r_i$), if we apply the tree-filling operation on a target set of size at most $j - 1$. We show how to obtain a tree decompostition with the desired properties if the target set has size $j$.

By construction, we have split the set $S$ into two sets $S^L, S^R$ and have applied the tree-filling operation to each set separately. Then, we connected the roots of the two added trees to obtain a larger binomial tree. Observe that for $|S| = j > 1$ we have $|S^L|, |S^R| < j$.

Let us first cut $\mathcal{P}$ in two parts, in such a way that the important bags of $S^L$ are on the left and the important bags of $S^R$ are on the right. We call $A^L = A$ and $Z^L$ the leftmost and rightmost bags of the left part and $A^R$, $Z^R = Z$ the leftmost and rightmost bags of the right part. We define as $G^L$ (respectively $G^R$) the graph that contains all the vertices of the left (respectively right) part. Let $r_i$ be the root of $T_i$ and $r_{i-1}$ the root of its subtree $T_{i-1}$. From the inductive hypothesis, we can construct tree decompositions $(\mathcal{T}^{\mathcal{L}}, \mathcal{B}^{\mathcal{L}})$, $(\mathcal{T}^{\mathcal{R}}, \mathcal{B}^{\mathcal{R}})$ of width $4w + 5$ for the graphs $H^L$, $H^R$ that occur after applying tree-filling on $G^L, S^L$ and $G^R, S^R$; furthermore, there exist $R^L \in \mathcal{B}^{\mathcal{L}}, R^R \in \mathcal{B}^{\mathcal{R}}$ such that $R^L \supseteq A \cup Z^L \cup \{r_i\}$ and $R^R \supseteq A^R \cup Z \cup \{r_{i-1}\}$.

We construct a new bag $R' = A \cup A^R \cup Z^L \cup Z \cup \{r_{i-1}, r_i\}$, and we connect $R'$ to both $R^L$ and $R^R$, thus combining the two tree-decompositions into one. Last we create a bag $R = A \cup Z \cup \{r_i\}$ and attach it to $R'$. This completes the construction of $(\mathcal{T}, \mathcal{B}')$.

Observe that $(\mathcal{T}, \mathcal{B}')$ is a valid tree-decomposition for $H$:

- $V(H) = V(H^L) \cup V(H^R)$, thus $\forall v \in V(H), v \in \mathcal{B}^{\mathcal{L}} \cup \mathcal{B}^{\mathcal{R}} \subset \mathcal{B}$.
- $E(H) = E(H^L) \cup E(H^R) \cup \{(r_{i-1}, r_i)\}$. We have that $r_{i-1}, r_i \in R' \in \mathcal{B}$. All other edges were dealt with in $\mathcal{T}^{\mathcal{L}}, \mathcal{T}^{\mathcal{R}}$.
- Each vertex $v \in V(H)$ that belongs in exactly one of $H^L, H^R$ trivially satisfied the connectivity requirement: bags that contain $v$ are either fully contained in $\mathcal{T}^{\mathcal{L}}$ or $\mathcal{T}^{\mathcal{R}}$. A vertex $v$ that is in both $H^L$ and $H^R$ is also in $Z^L \cap A^R$ due to the properties of path-decompositions, hence in $R'$. Therefore, the sub-trees of bags that contain $v$ in $\mathcal{T}^{\mathcal{L}}, \mathcal{T}^{\mathcal{R}}$, form a connected sub-tree in $\mathcal{T}$.

The width of $\mathcal{T}$ is $\max\{tw(H^L), tw(H^R), |R'| - 1\} = 4w + 5$. ◀

The last thing that remains to do in order to complete the proof is to show the equivalence between achieving the targets and finding a Grundy coloring.

▶ **Lemma 15.** *Let $G$ and $G'$ be two graphs as described in Lemma 8 and let $H$ be constructed from $G'$ by using the tree-filling operation. Then $G$ has a clique of size $k$ iff $\Gamma(H) \geq \lceil \log(k(m+1) + \binom{k}{2} + 2m) \rceil + 2\log n + 5$. Furthermore, $tw(H) \leq 4\binom{k}{2} + 8k + 17$.*

The proof of Lemma 15 can be found in the full version of the paper.

▶ **Theorem 16.** *Grundy Coloring parameterized by treewidth is W[1]-hard.*

## 4 FPT for pathwidth

In this section, we show that, in contrast to treewidth, Grundy Coloring is FPT parameterized by pathwidth. We achieve this by providing an upper bound on the Grundy number of any graph as a function of its pathwidth. Pipelining this with the algorithm of [74], we obtain a dependency on pathwidth alone. In order to obtain our bound, we rely on the following result on the performance ratio of the first-fit coloring algorithm on interval graphs.

▶ **Theorem 17** ([65]). *First-Fit is 8-competitive for online coloring interval graphs.*

In other words, interval graphs satisfy $\Gamma(G) \leq 8 \cdot \chi(G)$. Since for any interval graph $G$ we have $\chi(G) = pw(G) + 1$, we immediately obtain the following:

▶ **Corollary 18.** *For every interval graph $G$, $\Gamma(G) \leq 8 \cdot (pw(G) + 1)$.*

▶ **Lemma 19.** *For every graph $G$, $\Gamma(G) \leq 8 \cdot (pw(G) + 1)$.*

**Proof.** For a contradiction, suppose there exists $G$ such that $\Gamma(G) > 8 \cdot (pw(G)+1)$, and let $c : V(G) \to \{1, \ldots, \Gamma(G)\}$ be a Grundy coloring using $\Gamma(G)$ colors. In addition, let $G$ have the smallest possible number of vertices, i.e., there is no $G'$ satisfying those conditions with $|V(G')| < |V(G)|$. This implies that, for every optimal path decomposition of $G$, there is no bag $B$ and vertices $u, v \in B$ such that $c(u) = c(v)$.

Indeed, if such vertices exist, adding the edge $uv$ to $G$ and contracting $uv$ yields a new graph $G'$ such that $pw(G') \le pw(G)$ (edge contraction does not increase the pathwidth), $\Gamma(G') \ge \Gamma(G)$ (since $c$ when limited to V(G') is a valid Grundy coloring of $G'$) and $|V(G')| < |V(G)|$, contradicting the assumption that $G$ is smallest possible.

In addition, for any $u, v$ such that $c(u) \ne c(v)$ and $v \notin N(u)$, adding edge $uv$ to $G$ does not decrease the Grundy number of $G$ since $c$ remains a valid Grundy coloring of the new graph. In particular, since, as previously observed, vertices in any bag of an optimal path decomposition of $G$ all have pairwise different colors, turning every bag of such a decomposition into a clique does not decrease the Grundy number of $G$. More precisely, this yields a graph $G'$ such that $pw(G') = pw(G)$ and $\Gamma(G') \ge \Gamma(G)$, where $G'$ is an interval graph. Applying Corollary 18 we obtain $\Gamma(G) \le \Gamma(G') \le 8 \cdot (pw(G')+1)$, contradiction. ◀

Combining Lemma 19 with the $O^*(2^{O(tw(G)\cdot\Gamma(G))})$ algorithm of [74], we have:

▶ **Theorem 20.** *Grundy Coloring can be solved in time* $O^*(2^{O(pw(G)^2)})$.

Finally, note that there exist interval graphs that satisfy $\Gamma(G) \ge r \cdot pw(G)$, for any $r < 5$ [53], therefore, the constant in Lemma 19 cannot be improved below 5.

## 5 NP-hardness for Constant Clique-width

In this section we prove that Grundy Coloring is NP-hard even for constant clique-width via a reduction from 3-SAT. We use a similar idea of adding supports as in Section 3, but supports now will be cliques instead of binomial trees. The support operation is defined as:

▶ **Definition 21.** *Given a graph $G = (V, E)$, a vertex $u \in V$ and a set of positive integers $S$, we define the **support** operation as follows: for each $i \in S$, we add to $G$ a clique of size $i$ (using new vertices) and we connect one arbitrary vertex of each such clique to $u$.*

When applying the support operation we will say that we support vertex $u$ with set $S$ and we will call the vertices introduced supporting vertices. Intuitively, the support operation ensures that the vertex $u$ may have at least one neighbor with color $i$ for each $i \in S$.

We are now ready to describe our construction. Suppose we are given a 3CNF formula $\phi$ with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $c_1, \ldots, c_m$. We assume without loss of generality that each clause contains exactly three variables. We construct a graph $G(\phi)$ as follows:

1. For each $i \in [n]$ we construct two vertices $x_i^P, x_i^N$ and the edge $(x_i^P, x_i^N)$.
2. For each $i \in [n]$ we support the vertices $x_i^P, x_i^N$ with the set $[2i-2]$. (Note that $x_1^P, x_1^N$ have empty support).
3. For each $i \in [n], j \in [m]$, if variable $x_i$ appears in clause $c_j$ then we construct a vertex $x_{i,j}$. Furthermore, if $x_i$ appears positive in $c_j$, we connect $x_{i,j}$ to $x_{i'}^P$ for all $i' \in [n]$; otherwise we connect $x_{i,j}$ to $x_{i'}^N$ for all $i' \in [n]$.
4. For each $i \in [n], j \in [m]$ for which we constructed a vertex $x_{i,j}$ in the previous step, we support that vertex with the set $(\{2k \mid k \in [n]\} \cup \{2i-1, 2n+1, 2n+2\}) \setminus \{2i\}$.
5. For each $j \in [m]$ we construct a vertex $c_j$ and connect to all (three) vertices $x_{i,j}$ already constructed. We support the vertex $c_j$ with the set $[2n]$.

6. For each $j \in [m]$ we construct a vertex $d_j$ and connect it to $c_j$. We support $d_j$ with the set $[2n+3] \cup [2n+5, 2n+3+j]$.
7. We construct a vertex $u$ and connect it to $d_j$ for all $j \in [m]$. We support $u$ with the set $[2n+4] \cup [2n+5+m, 10n+10m]$.

This completes the construction. Before we proceed, let us give some intuition. Observe that we have constructed two vertices $x_i^P, x_i^N$ for each variable. The support of these vertices and the fact that they are adjacent, allow us to give them colors $\{2i-1, 2i\}$. The choice of which gets the higher color encodes an assignment to variable $x_i$. The vertices $x_{i,j}$ are now supported in such a way that they can "ignore" the values of all variables except $x_i$; for $x_i$, however, $x_{i,j}$ "prefers" to be connected to a vertex with color $2i$ (since $2i-1$ appears in the support of $x_{i,j}$, but $2i$ does not). Now, the idea is that $c_j$ will be able to get color $2n+4$ if and only if one of its literal vertices $x_{i,j}$ was "satisfied" (has a neighbor with color $2i$). The rest of the construction checks if all clause vertices are satisfied in this way.

We now state the lemmata that certify the correctness of our reduction. Their proofs appear in the full version of the paper.

▶ **Lemma 22.** *If $\phi$ is satisfiable then $G(\phi)$ has a Grundy coloring with $10n+10m+1$ colors.*

▶ **Lemma 23.** *If $G(\phi)$ has a Grundy coloring with $10n+10m+1$ colors, then $\phi$ is satisfiable.*

▶ **Lemma 24.** *The graph $G(\phi)$ has constant clique-width.*

▶ **Theorem 25.** *Given graph $G = (V, E)$, $k$-Grundy Coloring is NP-hard even when the clique-width of the graph $cw(G)$ is a constant.*

## 6 FPT for modular-width

In this section we show that Grundy Coloring is FPT parameterized by modular-width. Recall that $G = (V, E)$ has modular-width $w$ if $V$ can be partitioned into at most $w$ modules, such that each module is a singleton or induces a graph of modular-width $w$. Neighborhood diversity is the restricted version of this measure where modules are required to be cliques or independent sets. We sketch the main ideas of the algorithm (a full proof is in the full version of the paper).

The first step is to show that Grundy Coloring is FPT parameterized by neighborhood diversity. Similarly to the standard Coloring algorithm for this parameter [56], we observe that, without loss of generality, all modules can be assumed to be cliques, and hence any color class has one of $2^w$ possible types. We would like to use this to reduce the problem to an ILP with $2^w$ variables, but unlike Coloring, the ordering of color classes matters. We thus prove that the optimal solution can be assumed to have a "canonical" structure where each color type only appears in consecutive colors. We then extend the neighborhood diversity algorithm to modular-width using the idea that we can calculate the Grundy number of each module separately, and then replace it with an appropriately-sized clique.

▶ **Theorem 26.** *Let $G = (V, E)$ be a graph of modular-width $w$. The Grundy number of $G$ can be computed in time $2^{O(w2^w)} n^{O(1)}$.*

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