Exploiting c-Closure in Kernelization Algorithms for Graph Problems

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Abstract

A graph is c-closed if every pair of vertices with at least c common neighbors is adjacent. The c-closure of a graph G is the smallest number c such that G is c-closed. Fox et al. [SIAM J. Comput. '20] defined c-closure and investigated it in the context of clique enumeration. We show that c-closure can be applied in kernelization algorithms for several classic graph problems. We show that Dominating Set admits a kernel of size $k^{\mathcal{O}(c)}$, that Induced Matching admits a kernel with $\mathcal{O}(c^7k^8)$ vertices, and that Irredundant Set admits a kernel with $\mathcal{O}(c^{5/2}k^3)$ vertices. Our kernelization exploits the fact that c-closed graphs have polynomially-bounded Ramsey numbers, as we show.

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1 Introduction

Parameterized complexity [10, 15] aims at understanding which properties of input data can be used in the design of efficient algorithms for problems that are hard in general. The properties of input data are encapsulated in the notion of a *parameter*, a numerical value that can be attributed to each input instance I. For a given hard problem and parameter k, the first aim is to find a *fixed-parameter algorithm*, an algorithm that solves the problem in $f(k) \cdot |I|^{\mathcal{O}(1)}$ time. Such an algorithm is efficient when f grows moderately and k takes on small values. A second aim is to provide a *kernelization*. This is an algorithm that given any instance (I, k) of a parameterized problem computes in polynomial time an equivalent instance of size $g(k)$. If g grows not too much and k takes on small values, then a kernelization

Table 1 A comparison of the c-closure with the number n of vertices, number m of edges, and the maximum degree Δ in social and biological networks.

Instance name	n	m	Δ	c
adjnoun-adjacency	112	425	49	14
arenas-jazz	198	2742	100	42
ca-netscience	379	914	34	5
bio-celegans	453	2025	237	26
bio-diseasome	516	1188	50	9
soc-wiki-Vote	889	2914	102	18
arenas-email	1133	5451	71	19
bio-yeast	1458	1948	56	8
ca-CSphd	1882	1740	46	3
soc-hamsterster	2426	16630	273	77
ca-GrQc	4158	13422	81	43
soc-advogato	5167	39432	807	218
bio-dmela	7393	25569	190	72
ca-HepPh	11204	117619	491	90
ca-AstroPh	17903	196972	504	61
soc-brightkite	56739	212945	1134	184

provably shrinks large input instances and thus gives a guarantee for the efficacy of data reduction rules. A central part of the design of good parameterized algorithms is thus the identification of suitable parameters.

A good parameter should have the following advantageous traits. Ideally, it should be easy to understand and compute.[1] It should take on small values in real-world input data. It should describe input properties that are not captured by other parameters. Finally, many problems should be amenable to parameterization using this parameter. In other words, the parameter should help when designing fixed-parameter algorithms or kernelizations.

Fox et al. [19] recently introduced the graph parameter *c-closure* which describes a structural feature of many real-world graphs: When two vertices have many common neighbors, it is likely that they are adjacent. More precisely, the c-closure of a graph is defined as follows.

▶ **Definition 1.1** ([19]). *A graph $G = (V, E)$ is c-*closed* if every pair of vertices $u \in V$ and $v \in V$ with at least c common neighbors is adjacent. The c-closure of a graph is the smallest number c such that G is c-closed.*

The parameter has many of the desirable traits mentioned above: It is easy to understand and easy to compute. Moreover, social networks are c-closed for relatively small values of c [19] (see also Table 1). In addition, the c-closure of a graph gives a new class of graphs which are not captured by other measures. This follows from the observation that every complete graph is 1-closed. Hence, a graph can have bounded c-closure but, for example, unbounded degeneracy and thus unbounded treewidth. Conversely, the graph consisting of two vertices u and v and many vertex-disjoint u-v-paths of length two is 2-degenerate,

[1] This cannot always be guaranteed. For example, the important parameter treewidth is hard to compute and not as easily understood as simpler parameters.

has treewidth two, and unbounded c-closure. Generally, one may observe that c-closure is different from many common parameterizations which measure, in different ways, the sparseness of the input graph. In this sense, the structure described by the c-closure is novel. The aim of this work is to show that c-closure also has the final, most important trait: it helps when designing fixed-parameter algorithms.

Fox et al. [19] applied c-closure to the enumeration of maximal cliques, showing that a c-closed graph may have at most $3^{(c-1)/3} \cdot n^2$ maximal cliques. In combination with known clique enumeration algorithms this implies that all maximal cliques of a graph can be enumerated in $\mathcal{O}^*(3^{c/3})$ time. We are not aware of any further fixed-parameter algorithms that make use of the c-closure parameter.

An easy example for how parameterization by c-closure helps can be seen for the Independent Set problem. In Independent Set we are given an undirected graph $G = (V, E)$ and an integer k and want to determine whether G contains a set of k vertices that are pairwise nonadjacent. Independent Set is W[1]-hard when parameterized by k [15, 10]. When one uses the maximum degree of G as an additional parameter, then Independent Set has a trivial kernelization: Any graph with at least $(\Delta+1)k$ vertices has an independent set of size at least k. With the following data reduction rule, we can obtain a kernelization for the combination of c and k.

▶ **Reduction Rule 1.2.** *If G contains a vertex v of degree at least $(c-1)(k-1)+1$, then remove v from G.*

To see that Reduction Rule 1.2 is correct, we need to show that the resulting graph G' has an independent set I of size k if and only if the original graph G has one. The nontrivial direction to show is that if G has an independent set I of size k, then so does G'. Since this clearly holds for $v \notin I$, we assume that $v \in I$. To replace v by some other vertex, we make use of the c-closure: Every vertex u in $I \setminus \{v\}$ has at most $c-1$ neighbors in common with v since u and v are nonadjacent. Thus, at most $(c-1)(k-1)$ neighbors of v are also neighbors of some vertex in $I \setminus \{v\}$. Consequently, some neighbor w of v has no neighbors in $I \setminus \{v\}$ and, therefore, $(I \setminus \{v\}) \cup \{w\}$ is an independent set of size k in G'.

Applying Reduction Rule 1.2 exhaustively results in an instance with maximum degree less than ck which, due to the discussion above, directly gives the following.

▶ **Proposition 1.3.** Independent Set *admits a kernel with at most ck^2 vertices.*

Motivated by this simple result for a famous graph problem, we study how c-closure can be useful for further classic graph problems when they are parameterized by a combination of c and the solution size parameter k. We obtain the following positive results. In Section 4, we show that Dominating Set admits a kernel of size $k^{\mathcal{O}(c)}$ computable in $\mathcal{O}^*(2^c)$ time and show that this kernelization is asymptotically optimal with respect to the dependence of the exponent on c. Our results also hold for the more general Threshold Dominating Set problem where each vertex needs to be dominated r times. In Section 5, we show that Induced Matching admits a kernel with $\mathcal{O}(c^7k^8)$ vertices by means of LP relaxation of Vertex Cover. Finally in Section 6, we show that Irredundant Set admits a kernel with $\mathcal{O}(c^{5/2}k^3)$ vertices. All kernelizations exploit a bound on Ramsey numbers for c-closed graphs, which we prove in Section 3. This bound is – in contrast to Ramsey numbers of general graphs – polynomial in the size of a sought clique and independent set. We believe that this bound on the Ramsey numbers is of independent interest and that it provides a useful tool in the design of fixed-parameter algorithms for more problems on c-closed graphs.

2 Preliminaries

For $m \leq n \in \mathbb{N}$, we write $[m,n]$ to denote the set $\{m, m+1, \ldots, n\}$ and $[n]$ for $[1,n]$. For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. Let $X, Y \subseteq V(G)$ be vertex subsets. We use $G[X]$ to denote the subgraph induced by X. We also use $G[X,Y] := (X \cup Y, \{xy \in E(G) \mid x \in X, y \in Y\})$ to denote the bipartite subgraph induced by X, Y for $X \cap Y = \emptyset$. We let $G - X$ denote the graph obtained by removing vertices in X. We denote by $N_G(X) := \{y \in V(G) \setminus X \mid xy \in E(G), x \in X\}$ and $N_G[X] := N_G(X) \cup X$, the open and closed neighborhood of X, respectively. For all these notations, when X is a singleton $\{x\}$ we may write x instead of $\{x\}$. Let $v \in V(G)$. We denote the degree of v by $\deg_G(v)$. We call v *isolated* if $\deg_G(v) = 0$ and *non-isolated* otherwise. We also say that v is a *leaf vertex* if $\deg_G(v) = 1$ and a *non-leaf vertex* if $\deg_G(v) \geq 2$. Moreover, we say that v is *simplicial* if $N_G(v)$ is a clique. The maximum and minimum degree of G are $\Delta_G := \max_{v \in V(G)} \deg_G(v)$ and $\delta_G := \min_{v \in V(G)} \deg_G(v)$, respectively. The degeneracy of G is $d_G := \max_{S \subseteq V(G)} \delta_{G[S]}$. We say that G is c-closed for $c = \max(\{0\} \cup \{|N_G(u) \cap N_G(v)| \mid uv \notin E(G)\}) + 1$. In particular, any cluster graph (a disjoint union of complete graphs) is 1-closed. We drop the subscript $\cdot_G$ when it is clear from context. A graph G has girth g if the shortest cycle in G has length g.

In this paper, we investigate the parameterized complexity of various problems whose input comprises of a graph G and an integer k. A problem is *fixed-parameter tractable* if it can be solved in $f(k) \cdot n^{\mathcal{O}(1)}$ time where $n := |V(G)|$ and f is some computable function. Instances (G,k) and (G',k') are *equivalent* if (G,k) is a Yes-instance if and only if (G',k') is a Yes-instance. A *kernelization algorithm* is a polynomial-time algorithm which transforms an instance (G,k) into an equivalent instance (G',k') such that $|V(G')| + k' \leq g(k)$, where g is some computable function. It is well-known that a problem is fixed-parameter tractable if and only if it admits a kernelization algorithm. Our kernelization algorithms consist of a sequence of *reduction rules*. Given an instance (G,k), a reduction rule computes an instance (G',k'). We will develop kernelization algorithms for c-closed graphs. For our purposes, we say that a reduction rule is *correct* if the input instance (G,k) for a c-closed graph G is equivalent to the resulting instance (G',k') and G' is also c-closed. For more information on parameterized complexity, we refer to the standard monographs [10, 15].

We will make use of the following observations throughout this work.

▶ **Observation 2.1.** *If G is c-closed, then $G - v$ is also c-closed for any $v \in V(G)$.*

▶ **Observation 2.2.** *Let C be a maximal clique in a c-closed graph G. Then $|C \cap N(v)| < c$ for every $v \in V(G) \setminus C$.*

▶ **Observation 2.3.** *Let G be a c-closed graph and let C be a*

- *clique of size at most $c-1$ in G or*
- *a maximal clique in G.*

Then, the graph G' obtained by attaching a simplicial vertex v to C (that is, $N_{G'}(v) = C$) is c-closed.

Some proofs are deferred to a full version of this work.

3 On Ramsey Numbers of c-Closed Graphs

Ramsey's theorem states that there is a function R such that any graph G with at least $R(a,b)$ vertices contains a clique of size a or an independent set of size b, for any $a, b \in \mathbb{N}$. The numbers $R(a,b)$ are referred to as Ramsey numbers. It is known that $R(t,t) > 2^{t/2}$ for any $t \geq 3$ [17, 23] and hence $R(t,t)$ grows exponentially with t. Here, we show that the Ramsey number $R(a,b)$ is actually polynomial in a and b in c-closed graphs. Let $R_c(a,b) := (c-1) \cdot \binom{b-1}{2} + (a-1)(b-1) + 1$.

▶ **Lemma 3.1.** *Any c-closed graph G on at least $R_c(a,b)$ vertices contains a clique of size a or an independent set of size b.*

Proof. Assume to the contrary that G has no clique of size a and no independent set of size b. Let $I = \{v_1, \dots, v_{|I|}\}$ be a maximum independent set of G. Also let C_i be the set of vertices adjacent to v_i (including v_i) and nonadjacent to any other vertex in I (that is, $C_i = N[v_i] \setminus N(I \setminus \{v_i\})$) for each $i \in [|I|]$. Suppose that there exist $u \neq u' \in C_i$ with $uu' \notin E(G)$. Then, $(I \setminus \{v_i\}) \cup \{u, u'\}$ is an independent set of size $|I| + 1$, which contradicts the choice of I. Hence, we see that C_i is a clique. Note that every vertex of G is adjacent to some vertex in I due to the maximality of I. It follows that

$$|V(G)| \leq \sum_{i \in [|I|]} |C_i| + \sum_{i < j \in [|I|]} |N(v_i) \cap N(v_j)|.$$

Note that $|C_i| \leq a - 1$ for each $i \in [|I|]$ and $|N(v_i) \cap N(v_j)| \leq c - 1$ for $i < j \in [|I|]$ by the c-closure of G. Since $|I| < b$, we have a contradiction on $|V(G)|$. ◀

The bound in Lemma 3.1 is essentially tight: Consider a graph G consisting of a disjoint union of $b-1$ complete graphs, each of order $a-1$. Note that G is c-closed for any $c \in \mathbb{N}$ and that G has no clique of size a or independent set of size b. Thus, we have a tight bound for $c = 1$. This example also suggests that the bound in Lemma 3.1 cannot be asymptotically improved for $a \geq cb$.

4 (Threshold) Dominating Set

In this section we show that Threshold Dominating Set admits a kernel with $k^{\mathcal{O}(cr)}$ vertices. The problem is defined as follows.

Threshold Dominating Set
Input: A graph G and $r, k \in \mathbb{N}$.
Question: Is there a vertex set $D \subseteq V(G)$ such that $|D| \leq k$ and each vertex $v \in V(G)$ is dominated by D at least r times, that is, $|N[v] \cap D| \geq r$?

Dominating Set is the special case of Threshold Dominating Set when $r = 1$.

Dominating Set is W[2]-hard when parameterized by k even in bipartite or split graphs [30]. Furthermore, Dominating Set was shown to remain NP-hard on graphs with girth at least t for any constant t [2]. Hence, Dominating Set is NP-hard even on 2-closed graphs.

There are several fixed-parameter tractability results in restricted graph classes: When the graph G contains no induced C_3 or C_4, Dominating Set admits a kernel of $\mathcal{O}(k^3)$ vertices and Threshold Dominating Set is fixed-parameter tractable [30]. Furthermore, Dominating Set in d-degenerate graphs can be solved in $k^{\mathcal{O}(dk)}n$ time [3]. This result was extended to an algorithm with running time $\mathcal{O}^*(k^{\mathcal{O}(dkr)})$ for Threshold Dominating Set in d-degenerate graphs [21].

When the graph G does not contain the complete bipartite graph $K_{i,j}$ for fixed $j \leq i$ as a (not necessarily induced) subgraph, Dominating Set admits a kernel of $\mathcal{O}((j+1)^{i+1}k^{i^2})$ vertices which can be computed in $\mathcal{O}(n^i)$ time [29]. Since d-degenerate graphs do not contain a $K_{d+1,d+1}$ as a subgraph, Dominating Set admits a kernelization of $\mathcal{O}(k^{(d+1)^2})$ vertices computable in $\mathcal{O}^*(2^d)$ time [29]. This kernel size is essentially optimal since Dominating Set in d-degenerate graphs admits no kernel of size $\mathcal{O}(k^{(d-3)(d-1)-\epsilon})$ for any $\epsilon > 0$ unless NP $\subseteq$ coNP/poly [11]. When G does not contain the complete bipartite graph $K_{t,t}$ as a (not necessarily induced) subgraph, Dominating Set can be solved in $2^{\mathcal{O}(tk^2(4k)^t)}$ time [31]. Since each d-degenerate graph does not contain a $K_{d+1,d+1}$ as a subgraph, this extends the result of [3]. None of the above kernelizations and fixed-parameter algorithms implies a tractability result on c-closed graphs, since the respective structural restrictions on G all exclude cliques of some size. Moreover, since any graph without induced C_3 or C_4 is 2-closed, our results extend the kernelization algorithms for these graphs to a more general class of graphs.

To obtain a kernel for Threshold Dominating Set in c-closed graphs we first provide a kernelization for a more general, colored variant defined as follows. The input graph is a *bw-graph*, where the vertex set $V(G)$ is partitioned into black vertices B and white vertices W. We only require to dominate black vertices r times. The problem is defined as follows.

BW-Threshold Dominating Set
Input: A bw-graph G and $r, k \in \mathbb{N}$.
Question: Does G contain a *bw-threshold dominating set* $D \subseteq V(G)$, that is, a set such that $|N[v] \cap D| \geq r$ for each vertex $v \in B$, of size at most k?

Clearly, each instance (G, k) of Threshold Dominating Set is equivalent to the instance (G, k) of BW-Threshold Dominating Set where each vertex is black.

4.1 Polynomial Kernel in c-closed Graphs

We first develop a kernelization algorithm for BW-Threshold Dominating Set and then we will remove colors at the end. Before we present our reduction rules, we prove the following lemma, which will simplify some proofs later in this section.

▶ **Lemma 4.1.** *Let (G, k) be a Yes-instance of* BW-Threshold Dominating Set *and let v be a simplicial vertex with at least r neighbors. Then, there exists a bw-threshold dominating set D of size at most k such that $v \notin D$.*

Proof. Suppose that G has a bw-threshold dominating set D of size at most k. We are immediately done if $v \notin D$, so we can assume that $v \in D$. If $N[v] \subseteq D$, then $D \setminus \{v\}$ is a bw-threshold dominating set of size at most k. Otherwise, there is a vertex $u \in N(v) \setminus D$ and $(D \setminus \{v\}) \cup \{u\}$ is a bw-threshold dominating set of size at most k not containing v. ◀

We first aim to bound the number of black vertices. Our first reduction rule exploits the fact that any bw-threshold dominating set includes at least r vertices in C, where C is a maximal clique containing sufficiently many black vertices.

▶ **Reduction Rule 4.2.** *Let C be a maximal clique containing at least ck black vertices. Then,*
1. *add a vertex u and add an edge uv for each $v \in C$,*
2. *color u black, and*
3. *color all the vertices in C white.*

Note that Reduction Rule 4.2 does not add new maximal cliques. Hence, Reduction Rule 4.2 can be applied exhaustively in $\mathcal{O}^*(3^{c/3})$ time, because all maximal cliques can be enumerated in $\mathcal{O}^*(3^{c/3})$ time [19].

▶ **Lemma 4.3.** *Reduction Rule 4.2 is correct.*

Proof. Let D be a bw-threshold dominating set of G of size at most k. We claim that $|D \cap C| \geq r$. Assume to the contrary that $|D \cap C| \leq r - 1$. By Observation 2.2, each vertex in $D \setminus C$ dominates at most $c-1$ vertices in C. Since C contains at least ck black vertices, there is a black vertex in C that is not dominated r times by D, a contradiction. Thus, $|D \cap C| \geq r$. Let G' be the graph obtained as a result of Reduction Rule 4.2. Since $uv \in E(G)$ for each $v \in C$, we see that $|N_{G'}(u) \cap D| \geq r$ and thus D is also a bw-threshold dominating set of the graph G'. The other direction of the equivalence follows from Lemma 4.1. Finally, note that Reduction Rule 4.2 maintains the c-closure by Observation 2.3. ◀

We will assume henceforth that Reduction Rule 4.2 has been applied exhaustively. Recall that each c-closed graph on at least $R_c(a, b) = (c-1)\binom{b-1}{2} + (a-1)(b-1) + 1$ vertices contains a clique of size a or an independent set of size b by Lemma 3.1. Since G does not contain any black clique of size at least ck, each subgraph of G with at least $\rho := R_c(ck, k+1)$ black vertices contains an independent set of at least $k+1$ black vertices. We take advantage of this observation in the following two reduction rules.

▶ **Reduction Rule 4.4.** *Suppose that $r \leq c - 1$. We define Reduction Rule 4.4.i for each $i \in [1, c-r]$ as follows: Let C be a clique of size exactly $c - i$ and let $P := B \cap \{v \in V(G) \mid C \subseteq N(v)\}$ be the set of common black neighbors of C. If $|P| > k^{i-1}\rho$, then*

1. *add a vertex u and add an edge uv for each $v \in C$,*
2. *color u black, and*
3. *color all the vertices in C and P white.*

Apply Reduction Rule 4.4.i in increasing order of i.

Since $|P| > k^{i-1}\rho$, the common black neighbors P of C include an independent set I of size $k + 1$. To apply Reduction Rule 4.4.i exhaustively, we consider each pair of vertices v, v' from V and then we consider each clique in the common neighborhood $N(v) \cap N(v')$. Since $|N(v) \cap N(v')| < c$, Reduction Rule 4.4 can be applied exhaustively in $\mathcal{O}^*(2^c)$ time.

▶ **Lemma 4.5.** *Reduction Rule 4.4 is correct.*

Proof. Let C be a clique of size exactly $c - i$ which has more than $k^{i-1}\rho$ common black neighbors P. We prove the following claim for increasing $i \in [1, c-r]$.

▷ Claim. If Reduction Rule 4.4.j has been applied exhaustively for each $j \in [i-1]$, then any bw-threshold dominating set D of size at most k includes at least r vertices of C.

Proof. Suppose that $i = 1$. We assume to the contrary that $|D \cap C| \leq r-1$. Recall that there is no clique of at least ck black vertices by Reduction Rule 4.2. Since $|P| > \rho$, we see from Lemma 3.1 that P contains an independent set I of at least $k+1$ vertices. By pigeon-hole principle, there exists a vertex $w \in D \setminus C$ which is adjacent to at least two vertices x and y in I. Hence, x and y have at least c common neighbors $C \cup \{w\}$, contradicting the c-closure of G. It follows that D contains at least r vertices of C.

Suppose that $i \in [2, c-r]$. Again we assume to the contrary that $|D \cap C| \leq r - 1$. Since $|P| > k^{i-1}\rho$, there exists a vertex $w \in D \setminus C$ that dominates at least $k^{i-2}\rho$ vertices of P. Observe that w and each vertex in C have at least $k^{i-2}\rho > c$ common neighbors in P. Hence, we have $vw \in E(G)$ for each $v \in C$. Thus, $C \cup \{w\}$ is a clique of size $c - i + 1$ with at least $k^{i-2}\rho$ common black neighbors. However, this contradicts the fact that Reduction Rule 4.4.$(i-1)$ has been applied exhaustively. Therefore, we obtain $|D \cap C| \geq r$. ◁

Let G' be the graph obtained as a result of Reduction Rule 4.4. By the above claim, any bw-threshold dominating set in G is also a bw-threshold dominating set in G'. The other direction follows from Lemma 4.1. Finally, note that G' is c-closed by Observation 2.3. ◀

▶ **Reduction Rule 4.6.** *Suppose that $r \geq c$. Let C be a clique of size exactly $c-1$ and let $P := B \cap \{v \in V(G) \mid C \subseteq N(v)\}$ be the set of common black neighbors of C. If $|P| > \rho$, then return No.*

▶ **Lemma 4.7.** *Reduction Rule 4.6 is correct.*

Proof. Suppose that G has a bw-threshold dominating set D of size at most k. We show that for each clique C of size $c-1$, there are at most ρ common black neighbors. Assume to the contrary that $|P| > \rho$ for $P := B \cap \{v \in V(G) \mid C \subseteq N(v)\}$. Then, there is an independent set $I \subseteq P$ of size $k+1$ by Lemma 3.1. Since $r \geq c$, there are two vertices $x, y \in I$ that are adjacent to a vertex $v \in D \setminus C$. Now, we have a contradiction to the c-closure of G, because x and y have $|C \cup \{v\}| = c$ neighbors. ◀

Note that Reduction Rule 4.6 also can be applied exhaustively in $\mathcal{O}^*(2^c)$ time. Hereafter, we will assume that Reduction Rules 4.4 and 4.6 have been applied exhaustively. In the next lemma, we show that the number of black neighbors is upper-bounded for each vertex.

▶ **Lemma 4.8.** *Each vertex has at most $k^{c-1}\rho$ black neighbors for any Yes-instance (G, k).*

Proof. First, suppose that $r \leq c-1$. To prove the lemma, we prove the following more general claim:

▷ Claim. Let $i \in [r]$ and let C be a clique of size exactly i with the set P of common black neighbors. Then, $|P| \leq k^{c-i}\rho$.

Proof. We prove the claim by induction on decreasing i. By Reduction Rule 4.4, the claim holds for the base case $i = r$. Suppose that $i < r$. Since $|C| = i \leq r-1$, for any bw-threshold dominating set D of size at most k, there is a vertex $v \in D \setminus C$ that dominates at least $|P|/k$ vertices of P. As $|P|/k > c$, the set $C \cup \{v\}$ is a clique with $|P|/k$ common black neighbors. By induction hypothesis, we obtain $|P|/k \leq k^{c-i-1}\rho$ and equivalently, $|P| \leq k^{c-i}\rho$. ◁

Observe that the lemma follows from the above claim for $i = 1$. Using Reduction Rule 4.6, the lemma can be proven analogously for the case $r \geq c$ as well. ◀

By Lemma 4.8, there are at most $k^c\rho$ black vertices for any Yes-instance (G, k):

▶ **Reduction Rule 4.9.** *If G contains more than $k^c\rho$ black vertices, then return No.*

To compute a kernel it remains to upper-bound the number of white vertices in G.

▶ **Reduction Rule 4.10.** *Let w be a white vertex in G. If there exist at least r vertices $v_1, \ldots, v_r$ such that $N(w) \cap B \subseteq N[v_i] \cap B$ for each $i \in [r]$, then remove w.*

It is easy to see that Reduction Rule 4.10 can be applied exhaustively in polynomial time.

▶ **Lemma 4.11.** *Reduction Rule 4.10 is correct.*

Proof. Let $G' := G - w$. Suppose that G has a bw-threshold dominating set D of size at most k. If $w \notin D$, then D is also a bw-threshold dominating set of G'. Hence, we can assume that $w \in D$. If $v_i \in D$ for all $i \in [r]$, then $D \setminus \{w\}$ is a bw-threshold dominating set for G and hence also for G'. Otherwise, there exists some $i \in [r]$ with $v_i \notin D$. Since $N(w) \cap B \subseteq N[v_i] \cap B$, the set $(D \setminus \{w\}) \cup \{v_i\}$ is a bw-threshold dominating set of size at most k of G and G'. The other direction follows trivially. Since Reduction Rule 4.10 only deletes white vertices the c-closure is maintained. ◀

In the following, we will assume that Reduction Rule 4.10 has been applied exhaustively. Now, we obtain a bound on the number of white vertices in G.

▶ **Lemma 4.12.** *The graph G contains $\mathcal{O}(c|B|^2 + |B|^{r-1})$ white vertices.*

Proof. Since Reduction Rule 4.10 has been applied exhaustively, G contains at most r white vertices w such that $N(w) \subseteq W$. Hence, it remains to bound the number of white vertices with at least one black neighbor. Observe that by the c-closure of G, there are $\mathcal{O}(c\,|B|^2)$ white vertices that are neighbors of two nonadjacent vertices $u, v \in B$.

Note that for all remaining white vertices w, the set $B_w := N(w) \cap B$ of black neighbors is a clique. Since Reduction Rule 4.10 has been applied exhaustively, we have $|B_w| < r$. Moreover, for each clique $C \subseteq B$ of size $i \in [r-1]$, there are at most $r - i$ white vertices with $B_w = C$. Thus, the number of white vertices w such that B_w is a clique is

$$\sum_{i=1}^{r-1} i\,|B|^{r-i} = \frac{|B|(|B|^r - 1)}{(|B|-1)^2} - \frac{|B|}{|B|-1} r \in \mathcal{O}(|B|^{r-1}).$$

Overall, there are $\mathcal{O}(c\,|B|^2 + |B|^{r-1})$ white vertices. ◀

Recall that there are $k^c\rho \in \mathcal{O}(ck^{c+2})$ black vertices by Reduction Rule 4.9. Hence, the overall number of vertices is $\mathcal{O}(c^3k^{2c+4} + c^{r-1}k^{(c+2)(r-1)})$, resulting in the following theorem:

▶ **Theorem 4.13.** BW-Threshold Dominating Set *has a kernel with $k^{\mathcal{O}(cr)}$ vertices computable in $\mathcal{O}^*(2^c)$ time.*

To obtain a kernel for Threshold Dominating Set, it remains to show that any BW-Threshold Dominating Set instance can be transformed into an equivalent instance of Threshold Dominating Set.

▶ **Theorem 4.14.** Threshold Dominating Set *has a kernel with $k^{\mathcal{O}(cr)}$ vertices computable in $\mathcal{O}^*(2^c)$ time.*

Proof. To obtain an $k^{\mathcal{O}(cr)}$-vertex kernel for Threshold Dominating Set, we first construct an equivalent instance (G, k) of BW-Threshold Dominating Set using Theorem 4.13. Then, we transform (G, k) into an equivalent instance (G', k') of Threshold Dominating Set in $k^{\mathcal{O}(cr)}$-closed graphs as follows.

We start with a copy of G. We add a clique $Q := \{w_1, \ldots, w_{r+1}\}$ of $r+1$ vertices. Then, for each white vertex w we add edges $ww_1, \ldots, ww_r$. Then, we remove all vertex colors. We call the resulting graph G'. Let $C = \{w_1, \ldots w_r\}$ and let $k' = k + r$. We show that (G, k) is a Yes-instance if and only if (G', k') is a Yes-instance.

Let D be a bw-threshold dominating set of G. By construction, $D \cup C$ is a threshold dominating set of size at most k' of G'. Conversely, suppose that G' has a threshold dominating set D' of size at most k'. By Lemma 4.1, we can assume that $w_{r+1} \notin D'$. Since $\deg_{G'}(w_{r+1}) = r$, it holds that $N_{G'}(w_{r+1}) = C \subseteq D'$. Hence, all white vertices of G are dominated r times by C in G'. Thus, $D := D' \setminus C$ is a bw-threshold dominating set of size at most k for G. ◀

Since the kernelization does not change the parameter r, it also gives a kernelization for Dominating Set.

▶ **Corollary 4.15.** Dominating Set *has a kernel with $k^{\mathcal{O}(c)}$ vertices which is computable in $\mathcal{O}^*(2^c)$ time.*

To complement this result, we show that there is no kernel for DOMINATING SET significantly smaller than that of Corollary 4.15 under a widely believed assumption.

▶ **Theorem 4.16.** *For $c \geq 3$,* DOMINATING SET *has no kernel of size $\mathcal{O}(k^{c-1-\epsilon})$ unless* coNP $\subseteq$ NP/poly.

Proof. We will show the theorem by a reduction from λ-HITTING SET.

λ-HITTING SET
Input: A set family $\mathcal{F}$ over an universe U, where each $S \in \mathcal{F}$ has size λ, and $k \in \mathbb{N}$.
Question: Is there a subset $X \subseteq U$ of size at most k such that for each $S \in \mathcal{F}$ we have $X \cap S \neq \emptyset$?

For any $\lambda \geq 2$, λ-HITTING SET does not have a kernel of size $\mathcal{O}(k^{\lambda-\epsilon})$ unless coNP $\subseteq$ NP/poly [13, 14]. Let $(U, \mathcal{F}, k)$ be an instance of λ-HITTING SET. We will construct a $\lambda+1$-closed graph G as follows: The vertex set $V(G)$ is $U \cup \mathcal{F}$. We add edges such that U forms a clique in G. We also add an edge between $u \in U$ and $S \in \mathcal{F}$ if and only if $u \in S$. Finally, we set $k' = k$. Since $\deg_G(S) = \lambda$ for each $S \in \mathcal{F}$ the graph G is $\lambda+1$-closed.

By construction, each hitting set X of size at most k is also a dominating set of size at most k of G. For the converse direction, we may assume by Lemma 4.1 that there is a dominating set D of size at most k for G not containing any vertex from $\mathcal{F}$. Thus, D is also a hitting set of $(U, \mathcal{F}, k)$.

Observe that our reduction preserves the parameter (that is, $k = k'$). Thus, it follows from the result of Hermelin and Wu [22] that if DOMINATING SET admits a kernel of size $\mathcal{O}(k^{\lambda-1-\epsilon})$ for some $\epsilon > 0$, then λ-HITTING SET admits a kernel of size $\mathcal{O}(k^{\lambda-\epsilon})$, implying that coNP $\subseteq$ NP/poly [13, 14]. ◀

We also obtain an algorithm for THRESHOLD DOMINATING SET which is faster than brute-force search on the kernel of Theorem 4.14 and an improved kernel on bipartite graphs.

▶ **Theorem 4.17.** THRESHOLD DOMINATING SET *can be solved in $\mathcal{O}^*(3^{c/3} + (ck)^{\mathcal{O}(rk)})$ time and* DOMINATING SET *can be solved in $\mathcal{O}^*((ck)^{\mathcal{O}(k)})$ time.*

▶ **Theorem 4.18.** DOMINATING SET *in bipartite graphs has a kernel with $\mathcal{O}(c^3k^4)$ vertices.*

5 Induced Matching

In this section, we develop kernelizations for INDUCED MATCHING in c-closed graphs.

INDUCED MATCHING
Input: A graph G and $k \in \mathbb{N}$.
Question: Is there a set M of at least k edges such that endpoints of distinct edges in M are pairwise nonadjacent?

INDUCED MATCHING is W[1]-hard when parameterized by k, even in bipartite graphs [26]. In terms of kernelizations, INDUCED MATCHING admits a kernel with $\mathcal{O}(\Delta^2 k)$ vertices [26] and $\mathcal{O}(k^d)$ vertices [18, 24]. The latter kernelization result is essentially tight: Unless coNP $\subseteq$ NP/poly, INDUCED MATCHING has no kernel of size $\mathcal{O}(k^{d-3-\varepsilon})$ for any $\varepsilon > 0$ [11]. Despite the lower bound in degenerate graphs, we discover in this section that INDUCED MATCHING in c-closed graphs has a polynomial kernel when parameterized by $k + c$.

5.1 Ramsey-like Bounds for Induced Matchings

Dabrowski et al. [12] derived fixed-parameter tractability for Induced Matching in $(K_a, K_{b,b})$-free graphs. At the heart of their algorithm lies a Ramsey-type result for induced matchings: For $a, b \in \mathbb{N}$, there exists an integer $Q_{a,b}$ such that any bipartite graph with a matching of size at least $Q_{a,b}$ contains a biclique $K_{a,a}$ or an induced matching of size b. In this subsection, we present analogous results for c-closed graphs where the number $Q_{a,b}$ is polynomial in a and b. We begin with two preliminary lemmas.

▶ **Lemma 5.1.** *Any graph G with a matching M of size at least $2\Delta b$ has an induced matching of size b.*

Proof. We prove by induction on b. The lemma clearly holds for the base case $b = 0$. For $b > 0$, let uv be a matched edge in M and let $G' := G - N[\{u, v\}]$. Since $|N[\{u, v\}]| \leq 2\Delta_G$, there is a matching of size at least $2\Delta_G b - 2\Delta_G \geq 2\Delta_{G'}(b - 1)$ in G'. Consequently, there is an induced matching M' of size $b - 1$ in G' by induction hypothesis. Thus, G has an induced matching $M' \cup \{uv\}$ of size b. ◀

▶ **Lemma 5.2.** *Suppose that G is a c-closed bipartite graph. If there are at least $2b$ vertices of degree at least cb, then G contains an induced matching of size at least b.*

Proof. Let A, B be a bipartition of G. Without loss of generality, assume that A contains a set A' of exactly b vertices of degree at least cb. Since G is c-closed, $|N(v) \cap N(v')| < c$ for all $v, v' \in A'$. It follows that each $v \in A'$ has a neighbor $u \in N(v)$ such that $u \notin N(v')$ for all $v' \in A' \setminus \{v\}$. Thus, G contains an induced matching of size b. ◀

In the following lemma, we obtain a Ramsey-type result for induced matchings in c-closed bipartite graphs.

▶ **Lemma 5.3.** *Let $Q_c(b) := 2cb^2 + 2b \in \mathcal{O}(cb^2)$. Let G be a c-closed bipartite graph. If G has a matching M of size at least $Q_c(b)$, then G contains an induced matching of size at least b.*

Proof. If there are at least $2b$ vertices of degree at least cb in G, then Lemma 5.2 yields an induced matching of size b. Thus, we can assume that $|S| < 2b$ for the set S of vertices of degree at least cb. Observe that $G - S$ has a matching of size $2cb^2$ and that $\Delta_{G-S} \leq cb$. Thus, $G - S$ has an induced matching of size b by Lemma 5.1. ◀

We extend Lemma 5.3 to non-bipartite c-closed graphs in the subsequent two lemmas. Recall that each c-closed graph G with at least $R_c(a, b) \in \mathcal{O}(cb^2 + ab)$ vertices contains a clique of a vertices or an independent set of b vertices by Lemma 3.1. Our proofs for Lemmas 5.4 and 5.5 put Lemmas 3.1 and 5.3 together.

▶ **Lemma 5.4.** *Let $Q'_c(a, b) := R_c(a, Q_c(b)) \in \mathcal{O}(cab^2 + c^3b^4)$. Any c-closed graph G with an independent set I of size at least $Q'_c(a, b)$ and a matching M saturating I contains a clique of size a or an induced matching of size b.*

Proof. Suppose that G contains no clique of size a. We show that there is an induced matching of size b in G. Let $H := V(M) \setminus I$ be the set of vertices matched to I in M. Since $|H| \geq R_c(a, Q_c(b))$, it follows from Lemma 3.1 that there is an independent set $H' \subseteq H$ of size at least $Q_c(b)$ in G'. Let $I' \subseteq I$ be the set of vertices matched to H' in M. Then, there is an induced matching of size at least b in $G[H' \cup I']$ by Lemma 5.3. Thus, G contains an induced matching of size b. ◀

▶ **Lemma 5.5.** *Let $Q''_c(a,b) := R_c(a, Q'_c(b)) \in \mathcal{O}(c^3a^2b^4 + c^7b^8)$. Any c-closed graph G with a matching M of size at least $Q''_c(a,b)$ contains a clique of size a or an induced matching of size b.*

Proof. Suppose that G contains no clique of size a. We will show that there is an induced matching of size b in G. Let I and H be disjoint vertex sets such that I and H consist of distinct endpoints of each edge in M. Since $|I| \geq R_c(a, Q'_c(b))$, it follows from Lemma 3.1 that there is an independent set $I' \subseteq I$ of size $Q'_c(b)$. Let $H' \subseteq H$ be the set of vertices matched to I' and let $G' := G[H' \cup I']$. Since I is an independent set of size at least $Q'_c(b)$, it follows from Lemma 5.3 that there is an induced matching M' of size at least b in G'. Consequently, G contains an induced matching of size b. ◀

5.2 Polynomial Kernel in c-closed Graphs

In this subsection, we prove that Induced Matching in c-closed graphs admits a kernel with $\mathcal{O}(c^7k^8)$ vertices. Our kernelization is based on Lemmas 5.4 and 5.5. To utilize these lemmas, we start with a reduction rule that destroys large cliques.

▶ **Reduction Rule 5.6.** *Let $v \in V(G)$ and let M_v be a maximum matching in $G[N(v)]$. If $|M_v| \geq 2ck$, then remove v.*

▶ **Lemma 5.7.** *Reduction Rule 5.6 is correct.*

Proof. Let $v \in V(G)$, let M_v be a maximum matching in $G[N_G(v)]$ of size at least $2ck$, and let $G' := G - v$. Suppose that G has an induced matching M of size at least k. We show that G' contains an induced matching of size at least k as well. We are done if M does not use v, because M is also an induced matching in G'. So we can assume that M uses v. Let $v_1v_2, \dots, v_{2k-1}v_{2k}$ be k edges of M such that $v_{2k} = v$. By the definition of induced matching, $v_i \notin N_G(v)$ holds for each $i \in [2k-2]$. Thus, the c-closure of G yields that $|N_G(v) \cap N_G(v_i)| < c$ for each $i \in [2k-2]$. Since M_v is of size at least $2ck$, there is an edge e in M_v neither whose endpoint is adjacent to any vertex v_i for $i \in [2k-2]$. Hence, the edges $v_1v_2, \dots, v_{2k-3}v_{2k-2}, e$ form an induced matching of size k in G'. The other direction follows trivially. Note that the c-closure is maintained by Observation 2.1. ◀

Henceforth, we assume that Reduction Rule 5.6 has been applied for each vertex. In the next lemma, we verify that there is no large clique.

▶ **Lemma 5.8.** *There is no clique of size $4ck + 1$ in G.*

Proof. Suppose that G contains a clique C of size at least $4ck + 1$ and let $v \in C$. Note that $C \subseteq N[v]$. Let M_v be a maximum matching in $G[N(v)]$. Also, let $N_v^1 \subseteq N(v)$ be the set of vertices incident with M_v and let $N_v^0 := N(v) \setminus N_v^1$. Since M_v is a maximum matching, N_v^0 is an independent set in $G[N(v)]$. Thus, C includes at most one vertex of N_v^0, that is, $|C \cap N_v^0| \leq 1$. Moreover, it follows from Reduction Rule 5.6 that $|M_v| \leq 2ck - 1$ and hence $|N_v^1| \leq 4ck - 2$. Now, we have a contradiction because $|C| = |C \cap N_v^0| + |C \cap N_v^1| + 1 \leq 4ck$. ◀

Once we show that the graph has a sufficiently large matching, Lemma 5.5 tells us that we can find a sufficiently large induced matching as well. Note, however, that a graph may not have a sufficiently large matching, even if it contains sufficiently many vertices (consider a star $K_{1,n}$ with n leaves). Our way around this obstruction is the LP (Linear Programming) relaxation of Vertex Cover (henceforth, we will abbreviate it as VCLP). It is well-known in the theory of kernelization that VCLP almost trivially yields a linear-vertex kernel for

Vertex Cover [7] due to the Nemhauser-Trotter theorem [28]. Here, we will exploit VCLP to ensure that after we apply some reduction rules, either the size of G is upper-bounded or the minimum vertex cover size (or equivalently the maximum matching size) of G is sufficiently large.

Recall that Vertex Cover can be formulated as an integer linear program as follows, using a variable x_v for each $v \in V(G)$:

$$\min \sum_{v \in V(G)} x_v \qquad \text{subject to} \quad x_u + x_v \geq 1 \quad \forall uv \in E(G), \quad x_v \in \{0,1\} \quad \forall v \in V(G).$$

In VCLP, the last integral constraint is relaxed to $0 \leq x_v \leq 1$ for each $v \in V(G)$. It is known that VCLP admits a half-integral optimal solution (that is, $x_v \in \{0, 1/2, 1\}$ for each $v \in V(G)$) and such a solution can be computed in $\mathcal{O}(m\sqrt{n})$ time via a reduction to Maximum Matching (see, for instance, [4] or [10, Section 2.5]). Suppose that we have a half-integral optimal solution $(x_v)_{v \in V(G)}$. Let $V_0 := \{v \in V(G) \mid x_v = 0\}$, $V_1 := \{v \in V(G) \mid x_v = 1\}$, and $V_{1/2} := \{v \in V(G) \mid x_v = 1/2\}$.

We will bound the sizes of V_0, V_1, and $V_{1/2}$ in the upcoming rules. We begin with $V_{1/2}$. We use the bound Q''_c as specified in Lemma 5.5.

▶ **Reduction Rule 5.9.** *If $|V_{1/2}| \geq 3Q''_c(4ck + 1, k)$, then return Yes.*

To show the correctness, we will use the fact that $VC + MM \geq 2LP$ for any graph G [20, Lemma 2.1]. Here, VC, MM, and LP refer to the minimum vertex cover size, the maximum matching size, and the optimal VCLP cost of G.

▶ **Lemma 5.10.** *Reduction Rule 5.9 is correct.*

Proof. Observe that the optimal cost of VCLP for $G[V_{1/2}]$ is $|V_{1/2}|/2$. Let X be a minimum vertex cover and M be a maximum matching in $G[V_{1/2}]$. Then, it follows that $|X| + |M| \geq |V_{1/2}|$ [20, Lemma 2.1]. Since $V(M)$ is a vertex cover in $G[V_{1/2}]$, we also have $2|M| \geq |X|$. Thus, $|M| \geq |V_{1/2}|/3 \geq Q''_c(4ck, k)$. Recall that there is no clique of size $4ck+1$ by Lemma 5.8. Hence, Lemma 5.5 yields that G contains an induced matching of size at least k. ◀

We next upper-bound the size of V_1. See Lemma 5.4 for the definition of Q'_c.

▶ **Reduction Rule 5.11.** *If $|V_1| \geq Q'_c(4ck + 1, k)$, then return Yes.*

To prove the correctness of Reduction Rule 5.11, let us introduce the notion of *crowns* [9]. For a graph G, a crown is an ordered pair (I, H) of vertex sets of G with the following properties:

1. $I \neq \emptyset$ is an independent set in G,
2. $H = N(I)$, and
3. there is a matching saturating H in $G[H, I]$.

Crowns are closely related to VCLP – in fact, (V_0, V_1) is a crown [1, 8].

▶ **Lemma 5.12.** *Reduction Rule 5.11 is correct.*

Proof. Since (V_0, V_1) is a crown in G, there is a matching M saturating V_1 in $G[V_0, V_1]$. By definition, $I := V_0 \cap V(M)$ is an independent set of size $|V_1| \geq Q'_c(4ck + 1, k)$ in G. Now, it follows from Lemma 5.4 that $G[I \cup V_1]$ contains an induced matching of size k. ◀

To deal with V_0, we introduce some additional rules which may add or remove vertices. Let us start with a simple rule. Basically, if there are multiple leaf vertices with the same neighborhood, then only one of them is relevant.

▶ **Reduction Rule 5.13.** *If $v_1 \in V_1$ has more than one leaf neighbor, then remove all but one of them.*

The correctness of Reduction Rule 5.13 is obvious and thus we omit the proof.

▶ **Reduction Rule 5.14.** *Let $v_0 \in V_0$ and let $v_1 \in V_1$. If $N_G[v_0] \subseteq N_G[v_1]$ and there is no leaf vertex attached to v_1, then attach a leaf vertex ℓ to v_1.*

▶ **Lemma 5.15.** *Reduction Rule 5.14 is correct.*

Proof. Let G' be the graph obtained by adding a leaf vertex ℓ to v_1. The forward direction is trivial. For the other direction, note that any induced matching M' in G' is an induced matching in G if M' does not include $v_1\ell$. Hence, it suffices to show that if there is an induced matching M' in G' such that $|M'| \geq k$ and $v_1\ell \in M'$, then there is an induced matching of size k in G as well. By the definition of induced matching, $M' \setminus \{v_1\ell\}$ includes no edge incident with a neighbor of v_1. Since $N_G[v_0] \subseteq N_G[v_1]$, the same holds for v_0. Thus, $(M' \setminus \{v_1\ell\}) \cup \{v_0v_1\}$ is an induced matching of size at least k in G.

For $c > 1$, Reduction Rule 5.14 maintains the c-closure by Observation 2.3. Note that Induced Matching can be solved in linear time when G is 1-closed: Since G is a disjoint union of complete graphs, (G, k) is a Yes-instance if and only if G contains at least k cliques of size at least two. ◀

▶ **Reduction Rule 5.16.** *Let $v_0 \in V_0$ be a non-leaf vertex. If each vertex $v_1 \in N_G(v_0)$ has a leaf neighbor, then remove v_0.*

▶ **Lemma 5.17.** *Reduction Rule 5.16 is correct.*

Proof. Let $G' = G - v_0$. Suppose that G has an induced matching M of size at least k. If M does not use v_0 we are done. So assume that M includes v_0v_1 for $v_1 \in N_G(v_0)$. Since there is a leaf vertex ℓ attached to v_1, the set $(M \setminus \{v_0v_1\}) \cup \{v_1\ell\}$ is an induced matching of size at least k in G'. The other direction follows trivially. The c-closure is maintained by Observation 2.1. ◀

▶ **Theorem 5.18.** Induced Matching *has a kernel with $\mathcal{O}(c^7k^8)$ vertices.*

Proof. We apply Reduction Rules 5.6, 5.9, 5.11, 5.13, 5.14 and 5.16 exhaustively. We also remove all isolated vertices. It is easy to verify that all these rules can be exhaustively applied in polynomial time.

Note that $|V_{1/2}| \in \mathcal{O}(c^7k^8)$ and $|V_1| \in \mathcal{O}(c^3k^4)$ by Reduction Rules 5.9 and 5.11. We show that $|V_0| \in \mathcal{O}(c|V_1|^2) = \mathcal{O}(c^7k^8)$. Note that there are at most $|V_1|$ leaf vertices in V_0 by Reduction Rule 5.13. All other vertices in V_0 are adjacent to at least two nonadjacent vertices in V_1: If there exists a vertex $v_0 \in V_0$ such that $N_G(v_0)$ is a clique of size at least two, then Reduction Rule 5.14 adds a leaf vertex to each vertex in $N_G(v_0)$ and Reduction Rule 5.16 removes v_0. Since G is c-closed, there are $c\binom{|V_1|}{2}$ non-leaf vertices in V_0. It follows that $|V_0| < |V_1| + c\binom{|V_1|}{2} \in \mathcal{O}(c^7k^8)$. ◀

We also obtain smaller kernels in bipartite graphs. Our kernelization is based on the following lemma, proven by a meet-in-the-middle approach on vertex degrees. Interestingly, this lemma will also play a central role in the kernelization for Irredundant Set in Section 6.

▶ **Lemma 5.19.** *Any bipartite graph G with at least $6\Delta^{3/2}b + 2\Delta b$ non-isolated vertices has an induced matching of size b.*

▶ **Theorem 5.20.** Induced Matching *in bipartite graphs has a kernel with $\mathcal{O}(\Delta^{3/2}k)$ vertices and a kernel with $\mathcal{O}(c^{3/2}k^{5/2})$ vertices.*

6 Irredundant Set

A vertex set $S \subseteq V(G)$ is *irredundant* if there is a private neighbor for each vertex v in S. Here, a *private neighbor* of $v \in S$ is a vertex $v' \in N[v]$ (possibly $v' = v$) such that $v' \notin N(u)$ for each $u \in S \setminus \{v\}$.

Irredundant Set
Input: A graph G and $k \in \mathbb{N}$.
Question: Is there an irredundant set S of at least k vertices in G?

Irredundant Set is W[1]-hard in general [16] but it admits a kernel with at most $(d+1)k$ vertices in d-degenerate graphs. This is because any d-degenerate graph on at least $(d+1)k$ vertices contains an independent set and thus an irredundant set of at least k vertices. In this section, we show that Irredundant Set admits a kernel with $\mathcal{O}(c^{5/2}k^3)$ vertices. Our kernelization relies on the Ramsey bound (Lemma 3.1) and the bound on induced matchings (Lemma 5.19). We show that the following reduction rule suffices to obtain a polynomial kernel.

▶ **Reduction Rule 6.1.** *If $u, v \in V(G)$ are simplicial vertices such that $N_G[u] = N_G[v]$, then remove v.*

▶ **Lemma 6.2.** *Reduction Rule 6.1 is correct.*

Proof. Let $u, v \in V(G)$ be vertices such that $N_G[u] = N_G[v]$. Let G' be the graph obtained by removing v as specified in Reduction Rule 6.1. Suppose that (G, k) is a Yes-instance with a solution S. It must hold that $u \notin S$ or $v \notin S$ by the definition of irredundant sets. Without loss of generality, assume that $v \notin S$. If v is a private neighbor of $w \in S$ (possibly $w = u$), then u is also a private neighbor of w. Thus, (G', k) is also a Yes-instance. The other direction follows trivially. The c-closure is maintained by Observation 2.1. ◀

We prove that Reduction Rule 6.1 yields a kernelization of the claimed size.

▶ **Theorem 6.3.** Irredundant Set *in c-closed graphs has a kernel with $\mathcal{O}(c^{5/2}k^3)$ vertices.*

Proof. We assume that Reduction Rule 6.1 has been applied exhaustively.

To simplify notation, let $\alpha' := 6c^{3/2}k + 2ck + 1 \in \mathcal{O}(c^{3/2}k)$ and $\alpha := R_c(\alpha', k) \in \mathcal{O}(c^{3/2}k^2)$. We claim that any instance (G, k) with at least $R_c(c\alpha + 1, k) \in \mathcal{O}(c^{5/2}k^3)$ vertices is a Yes-instance. By Lemma 3.1, G has a clique of size $c\alpha + 1$ or an independent set of size k. Since any independent set is also an irredundant set, (G, k) is a Yes-instance when G contains an independent set of size k. Thus, we assume that there is no independent set of size k in G.

It remains to show that if G has a maximal clique C of size greater than $c\alpha$, then (G, k) is a Yes-instance. Let $C' = \{v \in C \mid N_G(v) \setminus C \neq \emptyset\}$ be the set of vertices in C that have at least one neighbor outside C. There exists at most one vertex v with $N_G[v] = C$ by Reduction Rule 6.1 and thus $|C'| \geq |C| - 1 \geq c\alpha$. Let $G' = G - (C \setminus C')$. That is, G' is a graph obtained by removing a vertex adjacent to all vertices in C, if such a vertex exists. For each $i \in [\alpha]$, we will choose vertices $x_i \in C'$ and $y_i \in N_{G'}(C')$ as follows: Let x_i be an arbitrary vertex in $C' \setminus \bigcup_{j \in [i-1]} N_{G'}(y_j)$ and let y_i be an arbitrary vertex in $N_{G'}(x_i)$. Note that $C' \setminus \bigcup_{j \in [i-1]} N_{G'}(y_j) \neq \emptyset$ for each $i \in [\alpha]$, because $|C'| \geq c\alpha$ and y_j has less than c neighbors in C' for all $j \in [i-1]$ by Observation 2.2.

Since G has no independent set of size k, Lemma 3.1 gives us a clique of size α' among $y_1, \dots, y_\alpha$. Without loss of generality, let $Y = \{y_1, \dots, y_{\alpha'}\}$ be a clique of size α' and let $X = \{x_1, \dots, x_{\alpha'}\}$. For $X' = X \setminus \{x_1\}$ and $Y' = Y \setminus \{y_1\}$, we prove that the bipartite graph $G[X', Y']$ has an induced matching of size k, using Lemma 5.19. First we show that $\Delta_{G[X',Y']} < c$. All vertices in Y' have less than c neighbors in X' by Observation 2.2.

By the choice of x_i and y_i, we have $x_i \notin N_G(y_1)$ for all $i \in [2, \alpha']$. It follows from the c-closure of G that x_i has less than c neighbors in Y' for each $i \in [2, \alpha']$. Thus, we have $\Delta_{G[X',Y']} < c$. Note that we choose x_i and y_i such that there is an edge $x_i y_i \in E(G)$ for each $i \in [2, \alpha']$. So $G[X', Y']$ has no isolated vertices. Therefore, it follows from Lemma 5.19 that there is an induced matching $\{x_{i_1} y_{i_1}, \ldots, x_{i_k} y_{i_k}\}$ of size k in $G[X', Y']$. Now, the set $\{x_{i_1}, \ldots, x_{i_k}\}$ is an irredundant set in G, where y_{i_j} is a private neighbor of x_{i_j} for each $j \in [k]$. ◀

7 Conclusion

We have demonstrated that the c-closure of a graph can be exploited in the design of parameterized algorithms for well-studied graph problems. We believe that the c-closure could become a standard secondary parameter just as the maximum degree Δ or the degeneracy d of the input graph and that studying problems with respect to this parameter may often lead to useful tractability results. In essence, whenever one obtains a fixed-parameter algorithm that uses Δ as one of its parameters, one should ask whether Δ can be replaced by the c-closure of the input graph. As concrete applications of the c-closure parameterization, one could consider further graph problems that are hard with respect to the solution size. For example, is Perfect Code [6] fixed-parameter tractable with respect to $c + k$ where k is the size of the code and does it admit a polynomial kernelization for this parameter? Further problems to investigate could be r-Regular Induced Subgraph which is W[1]-hard when parameterized by the subgraph size [27] or cardinality constrained optimization problems in graphs such as computing a maximum cut where the number of vertices in one part is constrained to be k [5]. These problems are often fixed-parameter tractable for the combination of the cardinality constraint k and the maximum degree Δ [5, 25]. Which of these problems is also fixed-parameter tractable for the combination of k and c? For answering such questions, the Ramsey bound of Lemma 3.1 could prove useful.

References

1 Faisal N. Abu-Khzam, Michael R. Fellows, Michael A. Langston, and W. Henry Suters. Crown structures for vertex cover kernelization. *Theory of Computing Systems*, 41(3):411–430, 2007.

2 Vladimir E. Alekseev, Dmitry V. Korobitsyn, and Vadim V. Lozin. Boundary classes of graphs for the dominating set problem. *Discrete Mathematics*, 285(1-3):1–6, 2004.

3 Noga Alon and Shai Gutner. Linear time algorithms for finding a dominating set of fixed size in degenerated graphs. *Algorithmica*, 54(4):544–556, 2009.

4 Reuven Bar-Yehuda and Shimon Even. A local-ratio theorem for approximating the weighted vertex cover problem. In *Proceedings of the 9th International Workshop Graph-Theoretic Concepts in Computer Science (WG '83)*, pages 17–28. Universitätsverlag Rudolf Trauner, Linz, 1983.

5 Leizhen Cai. Parameterized complexity of cardinality constrained optimization problems. *The Computer Journal*, 51(1):102–121, 2008.

6 Marco Cesati. Perfect code is W[1]-complete. *Information Processing Letters*, 81(3):163–168, 2002.

7 Jianer Chen, Iyad A. Kanj, and Weijia Jia. Vertex cover: Further observations and further improvements. *Journal of Algorithms*, 41(2):280–301, 2001.

8 Miroslav Chlebík and Janka Chlebíková. Crown reductions for the minimum weighted vertex cover problem. *Discrete Applied Mathematics*, 156(3):292–312, 2008.

9 Benny Chor, Mike Fellows, and David W. Juedes. Linear kernels in linear time, or how to save k colors in $O(n^2)$ steps. In *Proceedings of the 30th International Workshop Graph-Theoretic Concepts in Computer Science (WG '04)*, volume 3353 of *Lecture Notes in Computer Science*, pages 257–269. Springer, 2004.

10 Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.

11 Marek Cygan, Fabrizio Grandoni, and Danny Hermelin. Tight kernel bounds for problems on graphs with small degeneracy. *ACM Transactions on Algorithms*, 13(3):43:1–43:22, 2017.

12 Konrad Dabrowski, Marc Demange, and Vadim V. Lozin. New results on maximum induced matchings in bipartite graphs and beyond. *Theoretical Computer Science*, 478:33–40, 2013.

13 Holger Dell and Dániel Marx. Kernelization of packing problems. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '12)*, pages 68–81. SIAM, 2012.

14 Holger Dell and Dieter van Melkebeek. Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. *Journal of the ACM*, 61(4):23:1–23:27, 2014.

15 Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

16 Rodney G. Downey, Michael R. Fellows, and Venkatesh Raman. The complexity of irredundant sets parameterized by size. *Discrete Applied Mathematics*, 100(3):155–167, 2000.

17 Paul Erdös. Some remarks on the theory of graphs. *Bulletin of the American Mathematical Society*, 53(4):292–294, 1947.

18 Rok Erman, Łukasz Kowalik, Matjaž Krnc, and Tomasz Waleń. Improved induced matchings in sparse graphs. *Discrete Applied Mathematics*, 158(18):1994–2003, 2010.

19 Jacob Fox, Tim Roughgarden, C. Seshadhri, Fan Wei, and Nicole Wein. Finding cliques in social networks: A new distribution-free model. *SIAM Journal on Computing*, 49(2):448–464, 2020.

20 Shivam Garg and Geevarghese Philip. Raising the bar for vertex cover: Fixed-parameter tractability above a higher guarantee. In *Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '16)*, pages 1152–1166. SIAM, 2016.

21 Petr A. Golovach and Yngve Villanger. Parameterized complexity for domination problems on degenerate graphs. In *Proceedings of the 34th International Workshop Graph-Theoretic Concepts in Computer Science (WG '08)*, volume 5344 of *Lecture Notes in Computer Science*, pages 195–205, 2008.

22 Danny Hermelin and Xi Wu. Weak compositions and their applications to polynomial lower bounds for kernelization. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '12)*, pages 104–113. SIAM, 2012.

23 Stasys Jukna. *Extremal Combinatorics - With Applications in Computer Science*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2011.

24 Iyad A. Kanj, Michael J. Pelsmajer, Marcus Schaefer, and Ge Xia. On the induced matching problem. *Journal of Computer and System Sciences*, 77(6):1058–1070, 2011.

25 Christian Komusiewicz and Manuel Sorge. An algorithmic framework for fixed-cardinality optimization in sparse graphs applied to dense subgraph problems. *Discrete Applied Mathematics*, 193:145–161, 2015.

26 Hannes Moser and Somnath Sikdar. The parameterized complexity of the induced matching problem. *Discrete Applied Mathematics*, 157(4):715–727, 2009.

27 Hannes Moser and Dimitrios M. Thilikos. Parameterized complexity of finding regular induced subgraphs. *Journal of Discrete Algorithms*, 7(2):181–190, 2009.

28 George L. Nemhauser and Leslie E. Trotter. Vertex packings: Structural properties and algorithms. *Mathematical Programming*, 8:232–248, 1975.

29 Geevarghese Philip, Venkatesh Raman, and Somnath Sikdar. Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond. *ACM Transactions on Algorithms*, 9(1):11:1–11:23, 2012.

30 Venkatesh Raman and Saket Saurabh. Short cycles make W-hard problems hard: FPT algorithms for W-hard problems in graphs with no short cycles. *Algorithmica*, 52(2):203–225, 2008.

31 Jan Arne Telle and Yngve Villanger. FPT algorithms for domination in biclique-free graphs. In *Proceedings of the 20th Annual European Symposium on Algorithms (ESA '12)*, volume 7501 of *Lecture Notes in Computer Science*, pages 802–812. Springer, 2012.