

# Sparse Hop Spanners for Unit Disk Graphs

Adrian Dumitrescu 

Department of Computer Science, University of Wisconsin–Milwaukee, WI, USA  
dumitres@uwm.edu

Anirban Ghosh 

School of Computing, University of North Florida, Jacksonville, FL, USA  
anirban.ghosh@unf.edu

Csaba D. Tóth 

Department of Mathematics, California State University Northridge, Los Angeles, CA, USA  
Department of Computer Science, Tufts University, Medford, MA, USA  
csaba.toth@csun.edu

---

## Abstract

A unit disk graph  $G$  on a given set of points  $P$  in the plane is a geometric graph where an edge exists between two points  $p, q \in P$  if and only if  $|pq| \leq 1$ . A subgraph  $G'$  of  $G$  is a  $k$ -hop spanner if and only if for every edge  $pq \in G$ , the topological shortest path between  $p, q$  in  $G'$  has at most  $k$  edges. We obtain the following results for unit disk graphs.

- I Every  $n$ -vertex unit disk graph has a 5-hop spanner with at most  $5.5n$  edges. We analyze the family of spanners constructed by Biniáz (2020) and improve the upper bound on the number of edges from  $9n$  to  $5.5n$ .
- II Using a new construction, we show that every  $n$ -vertex unit disk graph has a 3-hop spanner with at most  $11n$  edges.
- III Every  $n$ -vertex unit disk graph has a 2-hop spanner with  $O(n \log n)$  edges. This is the first nontrivial construction of 2-hop spanners.
- IV For every sufficiently large  $n$ , there exists a set  $P$  of  $n$  points on a circle, such that every plane hop spanner on  $P$  has hop stretch factor at least 4. Previously, no lower bound greater than 2 was known.
- V For every point set on a circle, there exists a plane 4-hop spanner. As such, this provides a tight bound for points on a circle.
- VI The maximum degree of  $k$ -hop spanners cannot be bounded from above by a function of  $k$ .

**2012 ACM Subject Classification** Theory of computation → Design and analysis of algorithms

**Keywords and phrases** graph approximation,  $\varepsilon$ -net, hop-spanner, unit disk graph, lower bound

**Digital Object Identifier** 10.4230/LIPIcs.ISAAC.2020.57

**Related Version** A full version of this paper is available at <https://arxiv.org/abs/2002.07840>.

**Funding** *Anirban Ghosh*: Research on this paper was partially supported by the University of North Florida Academic Technology Grant and by the NSF award CCF-1947887.

*Csaba D. Tóth*: Research on this paper was partially supported by the NSF award DMS-1800734.

## 1 Introduction

A  $k$ -spanner (or  $k$ -hop spanner) of a connected graph  $G = (V, E)$  is a subgraph  $G' = (V, E')$ , where  $E' \subseteq E$ , with the additional property that the distance between any two vertices in  $G'$  is at most  $k$  times the distance in  $G$  [24, 38]. (The *distance* between two vertices is the minimum number of edges on a path between them.) The graph  $G$  itself is a 1-hop spanner. The minimum  $k$  for which  $G'$  is a  $k$ -spanner of  $G$  is referred to as the *hop stretch factor* (or *hop number*) of  $G'$ . An alternative characterization of  $k$ -spanners is given in the following lemma [38].



© Adrian Dumitrescu, Anirban Ghosh, and Csaba D. Tóth;  
licensed under Creative Commons License CC-BY

31st International Symposium on Algorithms and Computation (ISAAC 2020).

Editors: Yixin Cao, Siu-Wing Cheng, and Minming Li; Article No. 57; pp. 57:1–57:17



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

► **Lemma 1** (Peleg and Schäffer [38]). *The subgraph  $G' = (V, E')$  is a  $k$ -spanner of the graph  $G = (V, E)$  if and only if the distance between  $u$  and  $v$  in  $G'$  is at most  $k$  for every edge  $uv \in E$ .*

If the subgraph  $G'$  has only  $O(|V|)$  edges, then  $G'$  is called a *sparse spanner*. In this paper we are concerned with constructing sparse  $k$ -spanners (with small  $k$ ) for unit disk graphs in the plane. Given a set  $P$  of  $n$  points  $p_1, \dots, p_n$  in the plane, the unit disk graph (UDG) is a geometric graph  $G = G(P)$  on the vertex set  $P$  whose edges connect points that are at most unit distance apart. A *spanner of a point set  $P$*  is a spanner of its UDG.

Recognizing UDGs was shown to be NP-Hard by Breu and Kirkpatrick [9]. Unit disk graphs are commonly used to model network topology in ad hoc wireless networks and sensor networks. They are also used in multi-robot systems for practical purposes such as planning, routing, power assignment, search-and-rescue, information collection, and patrolling; refer to [2, 18, 23, 28, 34] for some applications of UDGs. For packet routing and other applications, a bounded-degree plane geometric spanner of the wireless network is often desired but not always feasible [7]. Since a UDG on  $n$  points can have a quadratic number of edges, a common desideratum is finding sparse subgraphs that approximate the respective UDG with respect to various criteria. *Plane spanners*, in which no two edges cross, are desirable for applications where edge crossings may cause interference.

Obviously, for every  $k \geq 1$ , every graph  $G = (V, E)$  on  $n$  vertices has a  $k$ -spanner with  $|E| = O(n^2)$  edges. If  $G$  is the complete graph, a star rooted at any vertex is a 2-hop spanner with  $n - 1$  edges. However, the  $O(n^2)$  bound on the size of a 2-hop spanner cannot be improved; a classic example [24] is that of a complete bipartite graph with  $n/2$  vertices on each side. In general, if  $G$  has girth  $k + 2$  or higher, then its only  $k$ -spanner is  $G$  itself. According to Erdős' girth conjecture [21], the maximum size of a graph with  $n$  vertices and girth  $k + 2$  is  $\Theta(n^{1+1/\lceil k/2 \rceil})$  for  $k \geq 2$ . The conjecture has been confirmed for some small values of  $k$ , but remains open for  $k > 9$ . For any graph  $G$  with  $n$  vertices, a  $k$ -spanner with  $O(n^{1+1/\lceil k/2 \rceil})$  edges can be constructed in linear time [4, 5]. We show that for unit disk graphs, we can do much better in terms of the number of edges for every  $k \geq 2$ .

Spanners in general and unit disk graph spanners in particular are used to reduce the size of a network and the amount of routing information. They are also used for maintaining network connectivity, improving throughput, and optimizing network lifetime [6, 22, 23, 27, 39].

Spanners for UDGs with hop stretch factors bounded by a constant were introduced by Catusse, Chepoi, and Vaxès in [11]. They constructed (i) 5-hop spanners with at most  $10n$  edges for  $n$ -vertex UDGs; and (ii) plane 449-hop spanners with less than  $3n$  edges. Recently, Biniáz [6] improved both these results, and showed that for every  $n$ -vertex unit disk graph, there exists a 5-hop spanner with at most  $9n$  edges. The author also showed how to construct a plane 341-hop spanner for a  $n$ -vertex unit disk graph. It is straightforward to verify that the algorithms presented in [6, 11] run in time that is polynomial in  $n$ .

**Our results.** The following are shown for unit disk graphs.

- I Every  $n$ -vertex unit disk graph has a 5-hop spanner with at most  $5.5n$  edges (Theorem 4 in Section 2). We carefully analyze the construction proposed by Biniáz [6] and improve the upper bound on the number of edges from the  $9n$  to  $5.5n$ .
- II Using a new construction, we show that every  $n$ -vertex unit disk graph has a 3-hop spanner with at most  $11n$  edges (Theorem 5 in Section 2). Previously, no 3-hop spanner construction algorithm was known.
- III Every  $n$ -vertex unit disk graph has a 2-hop spanner with  $O(n \log n)$  edges. This is the first construction with a subquadratic number of edges (Theorem 10 in Section 3) and our main result.

- IV For every  $n \geq 8$ , there exists an  $n$ -element point set  $S$  such that every plane hop spanner on  $S$  has hop stretch factor at least 3. If  $n$  is sufficiently large, the lower bound can be raised to 4 (Theorems 11 and 12 in Section 4). A trivial lower bound of 2 can be easily obtained by placing four points at the four corners of a square of side-length  $1/2$ .
- V For every point set  $S$  on a circle  $C$ , there exists a plane 4-hop spanner (Theorem 13 in Section 4). The lower bound of 4 holds for some point-set on a circle.
- VI For every pair of integers  $k \geq 2$  and  $\Delta \geq 2$ , there exists a set  $S$  of  $n = O(\Delta^k)$  points such that the unit disk graph  $G = (V, E)$  on  $S$  has no  $k$ -spanner whose maximum degree is at most  $\Delta$  (Theorem 14 in Section 5). An extension to dense graphs is given by Theorem 15 in Section 5. In contrast, Kanj and Perković [23] showed that UDGs admit bounded-degree *geometric* spanners.

**Related work.** Peleg and Schäffer [38] have shown that for a given graph  $G$  (not necessarily a UDG) and a positive integer  $m$ , it is NP-complete to decide whether there exists a 2-spanner of  $G$  with at most  $m$  edges. They also showed that for every graph on  $n$  vertices, a  $(4k + 1)$ -spanner with  $O(n^{1+1/k})$  edges can be constructed in polynomial time. In particular, every graph on  $n$  vertices has a  $O(\log n)$ -spanner with  $O(n)$  edges. Their result was improved by Althöfer et al. [1], who showed that a  $(2k - 1)$ -spanner with  $O(n^{1+1/k})$  edges can be constructed in polynomial time; the run-time was later improved to linear [4, 8]. Kortsarz and Peleg obtained approximation algorithms for the problem of finding, in a given graph, a 2-spanner of minimum size [24] or minimum maximum degree [25].

In the geometric setting, where the vertices are embedded in a metric space, spanners have been studied in [3, 10, 12, 13, 26, 28] and many other papers. In particular, plane geometric spanners were studied in [7, 8, 17, 16]. The reader is also referred to the surveys [8, 20, 30] and the monograph [33] dedicated to this subject.

**Notation and terminology.** For two points  $p, q \in \mathbb{R}^2$ , we denote the Euclidean distance by  $d(p, q)$  or sometimes by  $|pq|$ . The distance between two sets,  $A, B \subset \mathbb{R}^2$ , is defined by  $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ . For a set  $A$ , its boundary and interior are denoted by  $\partial A$  and  $\text{int}(A)$ , respectively. The diameter of a set  $A$ , denoted  $\text{diam}(A)$ , is defined by  $\text{diam}(A) = \sup\{d(a, b) : a, b \in A\}$ .

Given a graph,  $N(u)$  denotes the set of vertices adjacent to  $u$ . For  $p, q \in V$ , let  $\rho(p, q)$  denote a shortest path in  $G'$ , i.e., a path containing the fewest edges; and  $h(p, q)$  denote the corresponding hop distance (number of edges). For brevity, a hop spanner for a point set  $P \subset \mathbb{R}^2$  is a hop spanner for the UDG on  $P$ .

A geometric graph is *plane* if any two distinct edges are either disjoint or only share a common endpoint. Whenever we discuss plane graphs (plane spanners in particular), we assume that the points (vertices) are in *general position*, i.e., no three points are collinear.

A *unit disk* (resp., *circle*) is a disk (resp., circle) of unit radius. The complete bipartite graph with parts of size  $m$  and  $n$  is denoted by  $K_{m,n}$ ; in particular,  $K_{1,n}$  is a star on  $n + 1$  vertices. We use the shorthand notation  $[n]$  for the set  $\{1, 2, \dots, n\}$ .

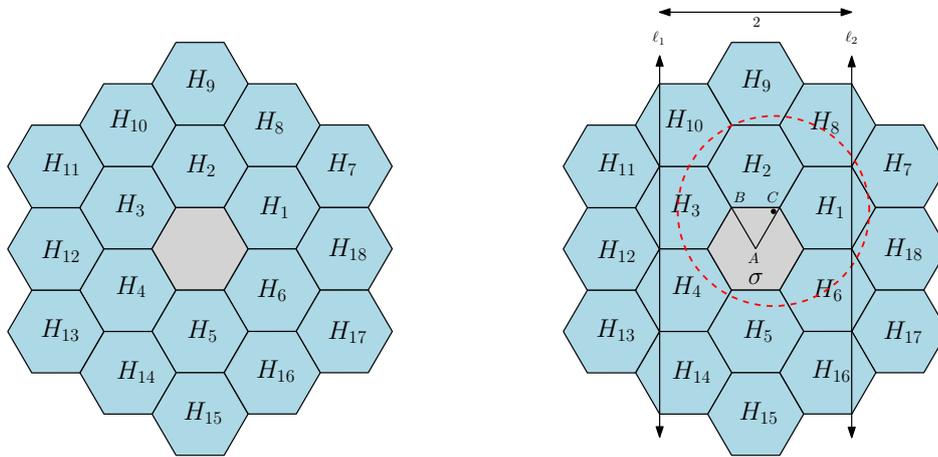
## 2 Sparse (possibly nonplane) hop spanners

In this section we construct hop spanners with a linear number of edges that provide various trade-offs between the two parameters of interest: number of hops and number of edges.

### 2.1 Construction of 5-hop spanners

We start with a short outline of the 5-hop spanner constructed by Biniáz [6, Theorem 3]; it is based on a hexagonal tiling with cells of unit diameter. Note that the UDG contains every edge between points in the same cell. In every nonempty cell, a star is formed rooted at an arbitrarily chosen point in the cell. Then, for every pair of cells, exactly one edge of the UDG is chosen, if such an edge exists. The author showed that the resulting graph is a 5-hop spanner with at most  $9n$  edges.

We next provide a more detailed description and an improved analysis of the construction. Consider a regular hexagonal tiling  $\mathcal{T}$  in the plane with cells of unit diameter; refer to Fig. 1 (left). We may assume that no point lies on a cell boundary. Every point in  $P$  lies in the interior of some cell of  $\mathcal{T}$  (and so the distance between any two points inside a cell is less than 1). Let  $p \in P$  be a point in a cell  $\sigma$ . Denote by  $H_1, \dots, H_6$  the six cells adjacent to  $\sigma$  in counterclockwise order; these cells form the *first layer* around  $\sigma$ . Let  $H_7, \dots, H_{18}$  be the twelve cells at distance two from  $\sigma$  in counterclockwise order, forming the *second layer* around  $\sigma$ , such that  $H_7$  is adjacent to only  $H_1$  in the first layer.



■ **Figure 1** Left: Hexagonal grid with cells of unit diameter; the figure shows the two layers of cells around the lightly shaded cell in the middle. Right: The unit disk centered at  $p$  intersects 11 cells  $H_1, \dots, H_{10}, H_{18}$ .

For every two distinct cells  $\sigma, \tau \in \mathcal{T}$ , take an arbitrary edge  $pq \in E$ ,  $p \in \sigma$ ,  $q \in \tau$ , if such an edge exists; we call such an edge a *bridge*. Each cell  $\sigma$  can have bridges to at most 18 other cells, namely those in the two layers around  $\sigma$ . A bridge is *short* if it connects points in adjacent cells and *long* otherwise.

► **Lemma 2.** *Let  $p \in P$  be a point that lies in cell  $\sigma$ . The unit disk  $D$  centered at  $p$  intersects at most five cells from the second layer around  $\sigma$ .*

**Proof.** Let  $A$  be the center of  $\sigma$  (shaded gray in Fig. 1 (right)). Subdivide  $\sigma$  into six regular triangles incident to  $A$ . By symmetry, we can assume that  $p \in \Delta ABC$ , where  $BC = \sigma \cap H_2$ .

Note that  $d(\Delta ABC, H_i) > 1$  for  $i \in \{13, 14, 15, 16, 17\}$ , and  $D$  is disjoint from the five cells  $H_{13}, H_{14}, H_{15}, H_{16}$ , and  $H_{17}$ . Now, observe that  $d(H_7 \cup H_{18}, H_{11} \cup H_{12}) = 2$ . Hence,  $D$  intersects at most one of  $H_7 \cup H_{18}$  and  $H_{11} \cup H_{12}$ . Consequently,  $D$  intersects at most  $12 - 5 - 2 = 5$  cells from the second layer around  $\sigma$ . ◀

Obviously, any two points in a cell  $\sigma$  are at most unit distance apart. Further, observe that the unit disk  $D$  centered at  $p$  intersects all six cells  $H_1, \dots, H_6$ . As such, Lemma 2 immediately yields the following.

► **Corollary 3.** *All neighbors of each point  $p \in \sigma$  lies in  $\sigma$  and at most 11 cells around  $\sigma$ .*

► **Theorem 4.** *The (possibly nonplane) 5-hop spanner constructed by Biniaz [6, Theorem 3] has at most  $5.5n$  edges.*

**Proof.** Let  $P$  be a set of  $n$  points and  $G = (V, E)$  be the corresponding UDG. Let  $x \geq 1$  be the number of points in a hexagonal cell  $\sigma \in \mathcal{T}$ . The construction has  $x - 1$  inner edges that make a star and at most 18 outer edges (bridges) connecting points in  $\sigma$  with points in other cells. We analyze the situation depending on  $x$ .

If  $x = 1$ , there are no inner edges and at most 11 outer edges by Corollary 3. As such, the degree of the (unique) point in  $\sigma$  is at most 11.

If  $x = 2$ , there is one inner edge and at most 16 outer edges. Indeed, by Lemma 2, each point  $p \in P \cap \sigma$  has neighbors in at most five cells from the second layer around  $\sigma$  (besides points in  $P$  in the six cells in the first layer). Two points in  $P \cap \sigma$  can jointly have neighbors in at most  $6 + 5 + 5 = 16$  other cells. As such, the average degree for points in  $\sigma$  is at most  $(2 + 16)/2 = 9$ .

If  $x \geq 3$ , there are  $x - 1$  inner edges and at most 18 outer edges. As such, the average degree for points in  $\sigma$  is at most

$$\frac{2(x - 1) + 18}{x} = \frac{2x + 16}{x} \leq \frac{22}{3}.$$

Summation over all cells implies that the average degree in the resulting 5-hop spanner  $G'$  is at most 11, thus  $G'$  has at most  $5.5n$  edges. ◀

## 2.2 Construction of 3-hop spanners

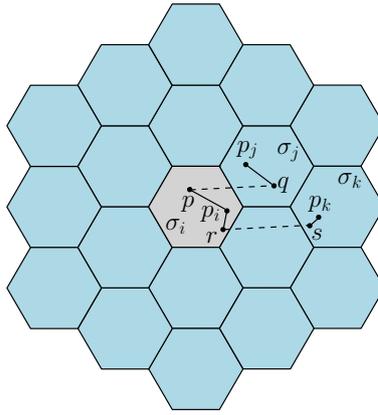
Here we show that every point set in the the plane has a 3-hop spanners of linear size. This brings down the hop-stretch factor of Biniaz's construction from 5 to 3 at the expense of increasing in the number of edges (from  $5.5n$  to  $11n$ ).

► **Theorem 5.** *Every  $n$ -vertex unit disk graph has a (possibly nonplane) 3-hop spanner with at most  $11n$  edges.*

**Proof.** Our construction is based on a hexagonal tiling  $\mathcal{T}$  with cells of unit diameter (as in Subsection 2.1). Let  $G'$  be the 5-hop spanner described in Section 2.1. We construct a new graph  $G''$  that consists of all bridges from  $G'$  and, for each nonempty cell  $\sigma \in \mathcal{T}$ , a spanning star of the points in  $\sigma$  defined as follows.

Let  $\sigma \in \mathcal{T}$  be a nonempty cell. Let  $p_i \in P \cap \sigma$ . For every cell  $\tau \in \mathcal{T}$  in the two layers around  $\sigma$ , if  $d(p_i, \tau) \leq 1$  and  $G'$  contains a bridge  $pq$ , where  $p \in \sigma \setminus \{p_i\}$  and  $q \in \tau$ , then we add the edge  $p_i p$  to  $G''$ . Since  $\text{diam}(\sigma) = 1$ , if  $pq$  is a short bridge, then  $p$  is the center of a spanning star on  $P \cap \sigma$ . In addition, if no short bridge is incident to any point in  $\sigma$ , then we add a spanning star of  $P \cap \sigma$  (centered at the endpoint of a long bridge, if any) to  $G''$ .

It is easy to see that the hop distance between any two points within a cell is at most 2. Indeed, by construction, the points in each nonempty cell are connected by a spanning star. Consider now a pair of points  $p_i \in \sigma_i$ ,  $p_j \in \sigma_j$ ,  $i \neq j$ , where  $p_i p_j \in E$ . By construction, there is a bridge  $pq \in G''$  between the cells  $\sigma_i$  and  $\sigma_j$ . As such,  $p_i$  is connected to  $p_j$  by a 3-hop path  $p_i, p, q, p_j$ . Refer to Fig. 2 for an illustration.



■ **Figure 2** Three points in  $P$ ,  $p_i \in \sigma_i$ ,  $p_j \in \sigma_j$ , and  $p_k \in \sigma_k$  where  $p_i p_j, p_i p_k \in E$ . Edge  $pq$  is a short bridge connecting  $\sigma_i$  and  $\sigma_j$  and edge  $rs$  is a long bridge connecting  $\sigma_i$  and  $\sigma_k$ .

We can bound the average degree of the points in  $\sigma$  as follows. Let  $x$  be the number of points in  $\sigma$ . By Corollary 3, the neighbors of each point  $p_i \in \sigma$  lie in  $\sigma$  and at most 11 cells around  $\sigma$ . If  $p_i$  is not incident to any bridge, we add at most 11 edges between  $p_i$  and other points in  $\sigma$ ; these edges increase the sum of degrees in  $\sigma$  by  $2 \cdot 11 = 22$ . Otherwise assume that  $p_i$  is incident to  $b_i$  bridges, for some  $1 \leq b_i \leq 11$ . Then we add edges from  $p_i$  to at most  $11 - b_i$  other points in  $\sigma$ . The  $b_i$  bridges each have only one endpoint in  $\sigma$ . Overall, these edges contribute  $2(11 - b_i) + b_i = 22 - b_i < 22$  to the sum of degrees in  $\sigma$ .

If no short bridge has an endpoint in  $\sigma$ , then by Lemma 2 we add at most 5 edges between each point  $p_i \in \sigma$  and endpoints of long bridges; these edges increase the sum of degrees in  $\sigma$  by  $2 \cdot 5 = 10$ . However, we also add a spanning star that contributes  $2(x - 1)$  to the same sum. Overall, the sum of degrees in  $\sigma$  is bounded from above by

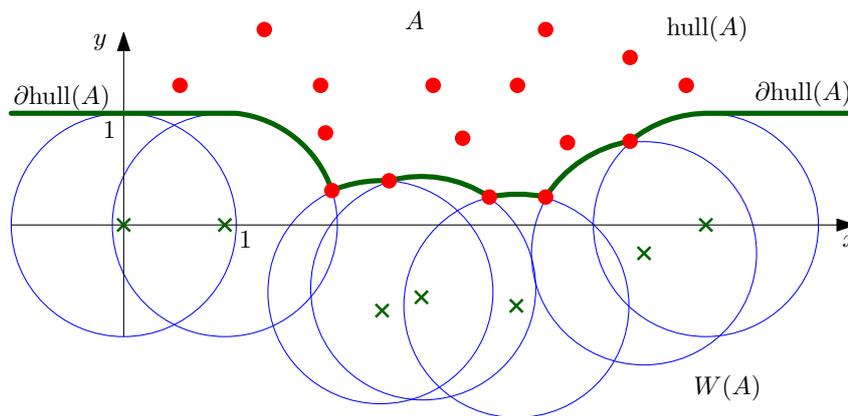
$$\begin{cases} 2 \cdot 11x = 22x, & \text{if some short bridge has an endpoint in } \sigma \\ 2(x - 1) + 10x < 12x, & \text{otherwise.} \end{cases}$$

Thus, the average vertex degree is at most 22 in all  $\sigma \in \mathcal{T}$ . Consequently, the 3-hop spanner  $G''$  has at most  $11n$  edges. ◀

### 3 Construction of 2-hop spanners

In this section, we construct a 2-hop spanner with  $O(n \log n)$  edges for a set of  $n$  points in the plane. We begin with a construction in a bipartite setting (cf. Lemma 9), and then extend it to the general setup.

We briefly review the concept of  $\varepsilon$ -nets [32], which is crucial for our construction. Let  $(P, \mathcal{R})$  be a set system (a.k.a. *range space*), where  $P$  is a finite set in an ambient space and  $\mathcal{R}$  is a collection of subsets of that space (called *ranges*). For  $\varepsilon > 0$ , an  $\varepsilon$ -net for  $(P, \mathcal{R})$  is a set  $N \subset P$  such that for every  $R \in \mathcal{R}$ ,  $|P \cap R| \geq \varepsilon \cdot |P|$  implies  $N \cap R \neq \emptyset$ . When the ambient space is  $\mathbb{R}^d$  for some  $d \in \mathbb{N}$ , and  $\mathcal{R}$  is a collection of semi-algebraic sets, there exists an  $\varepsilon$ -net of size  $O(\frac{d}{\varepsilon} \log \frac{d}{\varepsilon})$ , and this bound is best possible in many cases [36]. However, for some geometric set systems,  $\varepsilon$ -nets of size  $O(\frac{1}{\varepsilon})$  are possible. For example, if  $P$  is a set of points in the plane and  $\mathcal{R}$  consists of halfplanes, then there exists an  $\varepsilon$ -net of size  $O(\frac{1}{\varepsilon})$  [37]. We adapt this results to unit disks in a somewhat stronger form (cf. Lemma 8).



■ **Figure 3** A set  $A$  of 16 points above the  $x$ -axis,  $W(A)$ , and  $\text{hull}(A)$ . The boundary  $\partial\text{hull}(A)$  is an  $x$ -monotone curve, which consists of horizontal segments and arcs of unit circles centered on or below the  $x$ -axis (the centers are marked with crosses).

**Alpha-shapes.** As a generalization of convex hulls of a set of points, Edelsbrunner, Kirkpatrick and Seidel [19] introduced  $\alpha$ -shapes, using balls of radius  $1/\alpha$  instead of halfplanes. We introduce a similar concept, in the bipartite setting, as follows; see Fig. 3 for an illustration. We consider the set system  $(A, \mathcal{D})$ , where  $A$  is a finite set of points in the plane above the  $x$ -axis and  $\mathcal{D}$  is the set of all unit disks centered on or below the  $x$ -axis. Let  $W(A)$  be the union of all unit disks  $D \in \mathcal{D}$  such that  $A \cap \text{int}(D) = \emptyset$ ; and let  $\text{hull}(A) = \mathbb{R}^2 \setminus \text{int}(W(A))$ .

The following easy observation shows that disks in  $\mathcal{D}$ , restricted to the upper halfplane  $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ , behave similarly to halfplanes in  $\mathbb{R}^2$ .

► **Lemma 6.** *For any two points  $p_1, p_2 \in \mathbb{R}^2$  above the  $x$ -axis, there is at most one unit circle centered at a point on or below the  $x$ -axis that is incident to both  $p_1$  and  $p_2$ . Consequently, for any two unit disks  $D_1, D_2 \in \mathcal{D}$ , at most one point in  $\partial D_1 \cap \partial D_2$  lies above the  $x$ -axis.*

**Proof.** Suppose that two unit circles,  $c_1$  and  $c_2$ , are incident to both  $p_1$  and  $p_2$ . Then the centers of  $c_1$  and  $c_2$  are on the orthogonal bisector of segment  $p_1p_2$ , on opposite sides of the line through  $p_1p_2$ . Hence one of the circle centers is above the  $x$ -axis. Therefore at most one of the circles is centered at a point on or below the  $x$ -axis. ◀

We continue with a few basic properties of the boundary of  $\text{hull}(A)$ , which exhibit the same behavior as convex hulls with respect to lines in the plane.

► **Lemma 7.** *The set system  $(A, \mathcal{D})$  defined above has the following properties:*

1.  $\partial\text{hull}(A)$  lies above the  $x$ -axis;
2. every vertical line intersects  $\partial\text{hull}(A)$  in one point, thus  $\partial\text{hull}(A)$  is an  $x$ -monotone curve;
3. for every unit disk  $D \in \mathcal{D}$ , the intersection  $D \cap (\partial\text{hull}(A))$  is connected (possibly empty);
4. for every unit disk  $D \in \mathcal{D}$ , if  $A \cap D \neq \emptyset$ , then  $A \cap D$  contains a point in  $\partial\text{hull}(A)$ .

**Proof.** Let  $h$  be the minimum of the  $y$ -coordinates of the points in  $A$ . If  $h \geq 1$ , then  $W(A) = \{(x, y) : y \leq 1\}$  is a halfplane bounded by the line  $y = 1$ , so the lemma trivially holds. In the remainder of the proof, assume that  $0 < h < 1$ .

- (1) Since  $0 < h < 1$ , the halfplane below the horizontal line  $y = h$  lies in the interior of  $W(A)$  (as every point below this line is in the interior of a unit disk whose center is below the  $x$ -axis and whose interior is disjoint from  $A$ ). Property 1 follows.

- (2) Let  $p \in \partial\text{hull}(A)$ . Then  $p$  lies on the boundary of a unit disk  $D_p$  whose center is below the  $x$ -axis (and whose interior is disjoint from  $A$ ). In particular  $D_p \subset W(A)$ . The vertical line segment from  $p$  to the  $x$ -axis lies in  $D_p$ , hence in  $W(A)$ . Consequently,  $W(A)$  contains the vertical downward ray emanating from  $p$ . Property 2 follows.
- (3) Let  $D \in \mathcal{D}$ . Suppose, to the contrary, that the intersection  $D \cap (\partial\text{hull}(A))$  has two or more components. By property 1, the  $x$ -coordinates of the components are disjoint intervals, and the components have a natural left-to-right ordering. Let  $p_1$  be the rightmost point in the first component, and let  $p_2$  be the leftmost point in the second component. Clearly  $p_1, p_2 \in \partial D$ . Let  $q$  be an arbitrary point in  $\partial\text{hull}(A)$  between  $p_1$  and  $p_2$ . Then  $q$  lies on the boundary of a unit disk  $D_q$  whose center is below the  $x$ -axis (and whose interior is disjoint from  $A$ ). Since  $D_q \subset W(A)$ , neither  $p_1$  nor  $p_2$  is in the interior of  $D_q$ . Since the center of  $D_q$  is below the  $x$ -axis,  $\partial D_q$  contains two interior-disjoint circular arcs between  $q$  and the  $x$ -axis; and both arcs must cross  $\partial D$ . We have found two intersection points in  $\partial D \cap \partial D_q$  above the  $x$ -axis, contradicting Lemma 6. This completes the proof of Property 3.
- (4) Let  $D \in \mathcal{D}$  such that  $A \cap D \neq \emptyset$ . By continuously translating  $D$  vertically down until its interior is disjoint from  $A$ , we obtain a unit disk  $D'$  such that  $A \cap \text{int}(D') = \emptyset$  but  $A \cap \partial D' \neq \emptyset$ . Since the center of  $D'$  is vertically below the center of  $D$ , we have  $A \cap \partial D' \subset A \cap D$  and  $D' \subset W(A)$ . This implies that  $A \cap \partial D' \subset \partial\text{hull}(A)$ , as required.  $\blacktriangleleft$

► **Lemma 8.** *Consider the set system  $(A, \mathcal{D})$  defined above. For every  $\varepsilon \in (0, \frac{2}{3})$ , we can construct an  $\varepsilon$ -net  $N = \{v_1, \dots, v_k\} \subset A$ , labeled by increasing  $x$ -coordinates, such that*

- (i)  $|N| \leq \lfloor 2/\varepsilon \rfloor$ ;
- (ii)  $N \subset \partial\text{hull}(A)$ ;
- (iii) for every  $D \in \mathcal{D}$ , the points in  $D \cap N$  are consecutive in  $N$ ; and
- (iv) for every  $D \in \mathcal{D}$ ,  $|N \cap D| \geq 5$  implies  $|A \cap D| \geq 2\varepsilon|A|$ .

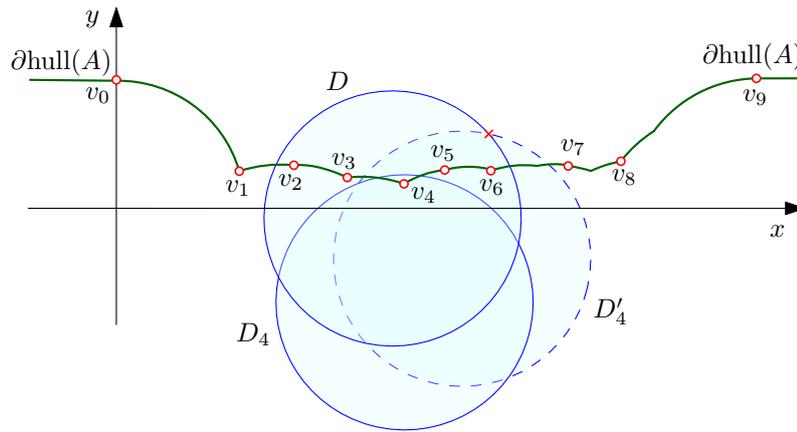
**Proof.** Let  $M = A \cap \partial\text{hull}(A)$  be the set of points in  $A$  lying on the boundary of  $\text{hull}(A)$ . By Lemma 7(4), if a unit disk  $D \in \mathcal{D}$  contains any point in  $A$ , it contains a point from  $M$ . Consequently  $M$  is an  $\varepsilon$ -net for  $(A, \mathcal{D})$  for every  $\varepsilon > 0$ . For a given  $\varepsilon > 0$ , let  $N = N_\varepsilon$  be a minimal subset of  $M$  that is an  $\varepsilon$ -net for  $(A, \mathcal{D})$  (obtained, for example, by successively deleting points from  $M$  while we maintain an  $\varepsilon$ -net).

Let  $N = \{v_1, \dots, v_k\}$ , where we label the elements in  $N$  by increasing  $x$ -coordinates. For notational convenience, we introduce a point  $v_0 \in \partial\text{hull}(A)$  on a vertical line one unit left of  $v_1$ , and  $v_{k+1} \in \partial\text{hull}(A)$  on a vertical line one unit right of  $v_k$ . For  $i = 1, \dots, k$ , the minimality of  $N$  implies that  $N \setminus \{v_i\}$  is not an  $\varepsilon$ -net, and so there exists a unit disk  $D \in \mathcal{D}$  such that  $|A \cap D| \geq \varepsilon|A|$  and  $D \cap N = \{v_i\}$ . Let  $D_i \in \mathcal{D}$  be such a disk, with  $|A \cap D_i| \geq \varepsilon|A|$  and  $D_i \cap N = \{v_i\}$ . By Lemma 7(3),  $D_i$  contains a connected arc of the  $x$ -monotone curve  $\partial\text{hull}(A)$ , but  $D_i$  contains neither  $v_{i-1}$  nor  $v_{i+1}$ . In particular, the  $x$ -coordinate of every point in  $A \cap D_i$  lies between that of  $v_{i-1}$  and  $v_{i+1}$ . Consequently, every point in  $A$  lies in at most two disks  $D_i$ ,  $1 \leq i \leq k$ . It follows that

$$k \cdot \varepsilon|A| = \sum_{i=1}^k \varepsilon|A| \leq \sum_{i=1}^k |A \cap D_i| \leq 2|A|,$$

hence  $k \leq \lfloor 2/\varepsilon \rfloor$ . This proves (i).

By construction, we have  $N \subset M \subset \partial\text{hull}(A)$ , which confirms (ii), and (iii) follows from Lemma 7(3). It remains to prove (iv); refer to Fig. 4. Assume that  $D \in \mathcal{D}$  and  $|N \cap D| \geq 5$ . By (iii), we may assume that  $D$  contains five consecutive points in  $N$ , say,  $v_i, \dots, v_{i+4}$ . For  $j \in \{i+1, i+2, i+3\}$ , consider the disk  $D_j \in \mathcal{D}$  defined above,



■ **Figure 4** Illustration for the proof of Lemma 8(iv) with  $i = 2$  and  $j = 4$ . A unit disk  $D$  with  $D \cap N = \{v_2, v_3, v_4, v_5, v_6\}$ , and a unit disk  $D_4$  with  $v_4 \in D_4$  and  $v_3, v_5 \notin D_4$ . A hypothetical unit disk  $D'_4$  (dashed) such that  $v_4 \in D'_4$ , and  $\partial D'_4 \cap \text{hull}(A)$  crosses  $\partial D \cap \text{hull}(A)$ .

where  $v_j \in D_j$  but  $v_{j-1}, v_{j+1} \notin D_j$ . In particular,  $D_j \cap (\partial \text{hull}(A))$  lies between  $v_{j-1}$  and  $v_{j+1}$ . By Lemma 6, the circular arcs  $\partial D \cap \text{hull}(A)$  and  $\partial D_j \cap \text{hull}(A)$  cross at most once. However, if they cross once, then  $D_j$  contains one of the endpoints of  $D \cap (\partial \text{hull}(A))$ , and by Lemma 7(3) it contains  $\{v_i, \dots, v_j\}$  or  $\{v_j, \dots, v_{i+4}\}$ , which is a contradiction. We conclude that  $\partial D \cap \text{hull}(A)$  and  $\partial D_j \cap \text{hull}(A)$  do not cross. Consequently,  $D_j \cap \text{hull}(A) \subset D \cap \text{hull}(A)$ , hence  $A \cap D_j \subset A \cap D$ . As noted above,  $|A \cap D_j| \geq \varepsilon |A|$ . Furthermore,  $A \cap D_{i+1}$  and  $A \cap D_{i+3}$  are disjoint as they are on opposite sides of the vertical line passing through  $v_{i+2}$ . Thus we obtain  $|A \cap D| \geq |A \cap (D_{i+1} \cup D_{i+3})| \geq |A \cap D_{i+1}| + |A \cap D_{i+3}| \geq 2\varepsilon |A|$ , as claimed. ◀

Let  $A$  and  $B$  be two disjoint point sets above and below the  $x$ -axis, respectively. Denote by  $U(A, B)$  the unit disk graph on  $A \cup B$  and by  $G(A, B)$  the bipartite subgraph of  $U(A, B)$  consisting of all edges between  $A$  and  $B$ .

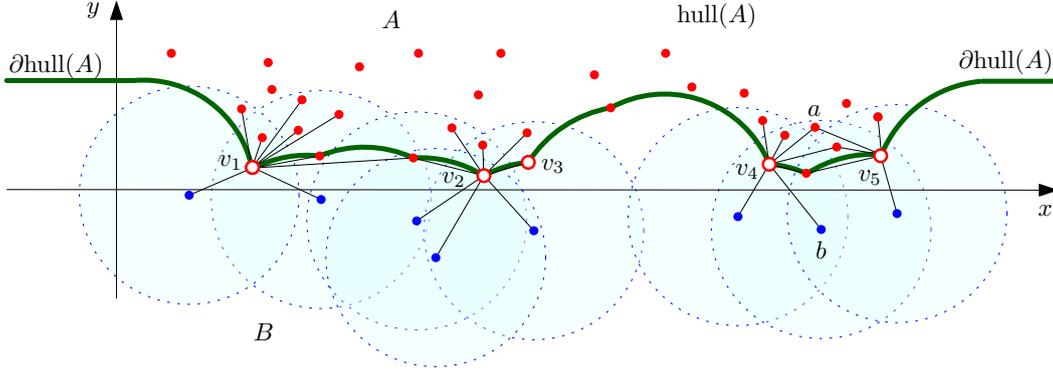
► **Lemma 9.** *Let  $P = A \cup B$  be a set of  $n$  points in the plane such that  $\text{diam}(A) \leq 1$ ,  $\text{diam}(B) \leq 1$ , and  $A$  (resp.,  $B$ ) is above (resp., below) the  $x$ -axis. Then there is a subgraph  $H$  of  $U(A, B)$  with at most  $O(n \log n)$  edges such that for every edge  $ab$  of  $G(A, B)$ ,  $H$  contains a path of length at most 2 between  $a$  and  $b$ .*

**Proof.** Our proof is constructive. For every point  $b \in B$ , let  $D_b$  be the unit disk centered at  $b$ . Consider the set system  $(A, \mathcal{B})$ , where  $\mathcal{B} = \{D_b : b \in B\}$ . We partition the set of disks  $\mathcal{B}$  into  $O(\log n)$  subsets based on the number of points of  $A$  contained in the disks. For every  $i = 1, \dots, \lceil \log n \rceil$ , let

$$\mathcal{B}_i = \left\{ D \in \mathcal{B} : \frac{|A|}{2^i} \leq |A \cap D| < \frac{|A|}{2^{i-1}} \right\}.$$

For every  $i = 1, \dots, \lceil \log n \rceil$ , let  $\varepsilon_i = \frac{1}{2^i}$ . Lemma 8 yields an  $\varepsilon_i$ -net  $N_i \subset A$  of size at most  $\lceil 2/\varepsilon_i \rceil = 2^{i+1}$  for  $(A, \mathcal{B}_i)$ .

We construct the graph  $H$  as a union of stars; see Fig. 5 for an illustration. For every  $i = 1, \dots, \lceil \log n \rceil$  and every  $v \in N_i$ , we create a star centered at  $v$  as follows. Let  $B_i(v)$  be the set of points  $b \in B$  such that  $D_b \in \mathcal{B}_i$  (that is,  $|A|/2^i \leq |A \cap D_b| < |A|/2^{i-1}$ ),  $v \in D_b$ , and  $v$  is the leftmost point in  $N_i \cap D_b$ . Let  $A_i(v)$  be the set of points  $a \in A$  contained in unit



■ **Figure 5** Set  $A$  (resp.,  $B$ ) is above (resp., below) the  $x$ -axis. The points in an  $\varepsilon_i$ -net  $N_i = \{v_1, \dots, v_5\}$  are marked with hollow dots. The graph  $H_i$  is a union of stars centered at  $v_1, \dots, v_5$ . (To avoid clutter, the depicted point set does not meet conditions  $\text{diam}(A) \leq 1$  and  $\text{diam}(B) \leq 1$  of Lemma 9.)

disks centered at some point in  $B_i(v)$ . Let  $S_i(v)$  be the star on  $A_i(v) \cup B_i(v)$  centered at  $v$ . By construction, every point in  $B_i(v)$  is at distance at most 1 from  $v$ , and  $\text{diam}(A) \leq 1$ ; this implies that  $S_i(v)$  is a subgraph of  $U(A, B)$ . Let  $H_i$  be the union of all stars centered at vertices in  $N_i$ ; and let  $H$  be the union of the graphs  $H_i$  for  $i = 1, \dots, \lceil \log n \rceil$ . Note that  $H$  is a union of stars in  $U(A, B)$ , hence a subgraph of  $U(A, B)$ .

To prove correctness, we show that for every edge  $ab$  of  $G(A, B)$  (with  $a \in A$ ,  $b \in B$ ),  $H$  contains a path of length 2 between  $a$  and  $b$ . Since  $ab$  is an edge of  $G(A, B)$ , we have  $|ab| \leq 1$  hence  $a \in D_b$ . There exists an index  $i \in \{1, \dots, \lceil \log n \rceil\}$  for which  $D_b \in \mathcal{B}_i$ . As  $|A \cap D_b| \geq |A|/2^i = \varepsilon_i |A|$ , and  $N_i$  is an  $\varepsilon_i$ -net for  $(A, \mathcal{B}_i)$ , we have  $D_b \cap N_i \neq \emptyset$ . Let  $v$  be the leftmost point in  $D_b \cap N_i$ . Then by construction  $a \in A_i(v)$  and  $b \in B_i(v)$ . If  $a = v$ , then the star  $S_i(v)$  contains the edge  $ab$ , otherwise  $S_i(v)$  contains the path  $a, v, b$  of length 2.

It remains to derive an upper bound on the number of edges in  $H$ . We claim that  $H_i$  has  $O(n)$  edges for all  $i = 1, \dots, \lceil \log n \rceil$ , which implies that  $H$  has  $O(n \log n)$  edges overall.

Let  $b \in B$ . There is a unique index  $i$  such that  $|A|/2^i \leq |A \cap D_b| < |A|/2^{i-1}$ ; and there is a unique leftmost point  $v$  in  $N_i \cap D_b$ . Therefore,  $b$  is a leaf of only one star  $S_i(v)$ , and so its degree is at most 1 in  $H_i$ , hence in  $H$ .

Let  $i \in \{1, \dots, \lceil \log n \rceil\}$ . Assume that  $N_i = \{v_1, \dots, v_k\}$  is sorted by increasing  $x$ -coordinates. We also introduce points  $v_0$  and  $v_{k+1}$  on  $\partial\text{hull}(A)$  as specified previously.

Let  $a \in A$ ; refer to Fig. 5. Assume that  $a$  is in a star  $S_i(v_j)$  for some  $v_j \in N_i$ . Assume further that the  $x$ -coordinate of  $a$  is between that of  $v_{\ell-1}$  and  $v_\ell$  for some  $\ell \in \{1, \dots, k+1\}$ . Since  $a$  is in  $S_i(v_j)$ , there exists a point  $b \in B$  such that  $a \in D_b$ ,  $D_b \in \mathcal{B}_i$ , and  $v_j$  is the leftmost point in  $D_b \cap N_i$ . Since  $D_b \in \mathcal{B}_i$ , we have  $|A \cap D_b| < 2\varepsilon_i |A|$ .

By Lemma 8(iv),  $D_b$  contains at most 4 points from the net  $N_i$ . In particular, the unit circle  $\partial D_b$  intersects  $\partial\text{hull}(A)$  in two points: once between  $v_{j-1}$  and  $v_j$ , and once between  $v_j$  and  $v_{j+4}$ . Consequently,  $0 \leq \ell - j \leq 4$ , thus  $a$  is in at most 5 possible stars  $S_i(v_j)$ ,  $v_j \in N_i$ . It follows that  $H_i$  has at most  $5|A| + |B| \leq 5n$  edges, as required.  $\blacktriangleleft$

We now consider the general case.

► **Theorem 10.** *Every  $n$ -vertex unit disk graph has a (possibly nonplane) 2-hop spanner with  $O(n \log n)$  edges.*

**Proof.** Let  $P$  be a set of  $n$  points in the plane. Consider a tiling of the plane with regular hexagons of unit diameter; and assume that no point in  $P$  lies on the boundary of any hexagon. Let  $\mathcal{T}$  be the set of nonempty hexagons. Then  $P$  is partitioned into  $O(n)$  sets  $\{P \cap \sigma : \sigma \in \mathcal{T}\}$ . As noted in Section 2.1, for every  $\sigma \in \mathcal{T}$ , there are 18 other cells within unit distance; see Fig. 1 (left).

For each cell  $\sigma \in \mathcal{T}$ , choose an arbitrary vertex  $v_\sigma \in P \cap \sigma$ , and create a star  $S_\sigma$  centered at  $v_\sigma$  on the vertex set  $P \cap \sigma$ . The overall number of edges in all stars  $S_\sigma, \sigma \in \mathcal{T}$ , is

$$\sum_{\sigma \in \mathcal{T}} (|P \cap \sigma| - 1) = n - |\mathcal{T}| \leq n.$$

For every pair of cells  $\sigma_i, \sigma_j \in \mathcal{T}$ , where  $d(\sigma_i, \sigma_j) \leq 1$ , consider the bipartite graph  $G_{i,j} = G(P \cap \sigma_i, P \cap \sigma_j)$ . By Lemma 9, there is a graph  $H_{i,j}$  of size

$$O((|P \cap \sigma_i| + |P \cap \sigma_j|) \log(|P \cap \sigma_i| + |P \cap \sigma_j|)) = O((|P \cap \sigma_i| + |P \cap \sigma_j|) \log n).$$

Since every vertex appears in at most 18 such bipartite graphs, the total number of edges in these graphs is at most  $O(\sum_{\sigma \in \mathcal{T}} |P \cap \sigma| \log n) = O(n \log n)$ .

We show that the union of the stars  $S_\sigma, \sigma \in \mathcal{T}$ , and the graphs  $H_{i,j}$  is a 2-hop spanner. Let  $ab$  be an edge of the unit disk graph. If both  $a$  and  $b$  are in the same cell, say  $\sigma \in \mathcal{T}$ , then  $ab$  is an edge in the star or the star  $S_\sigma$  contains the path  $a, v_\sigma, b$ . Otherwise,  $a$  and  $b$  lie in two distinct cells, say  $\sigma_i, \sigma_j \in \mathcal{T}$ , such that  $d(\sigma_i, \sigma_j) \leq |ab| \leq 1$ . By Lemma 9 (where the role of the  $x$ -axis is taken by any separating line),  $H_{i,j}$  contains a path of length at most 2 between  $a$  and  $b$ , as required. ◀

#### 4 Lower bounds for plane hop spanners

A trivial lower bound of 2 for the hop stretch factor of plane subgraphs of UDGs can be easily obtained by taking the four corners of a square of side-length  $\frac{1}{2}$ . In this case, the UDG is the complete graph but a plane spanner cannot contain both diagonals of the square. Our main result in this section is a lower bound of 4 for sufficiently large  $n$  (cf. Theorem 12). We begin with a lower bound of 3 that holds already for  $n = 8$ . Due to space constraints, the proof of Theorem 11 is omitted and can be found in the full version.

► **Theorem 11.** *For every  $n \geq 8$ , there exists an  $n$ -element point set  $S$  on a circle such that every plane hop spanner on  $S$  has hop stretch factor at least 3.*

We next derive a better bound assuming that  $n$  is sufficiently large.

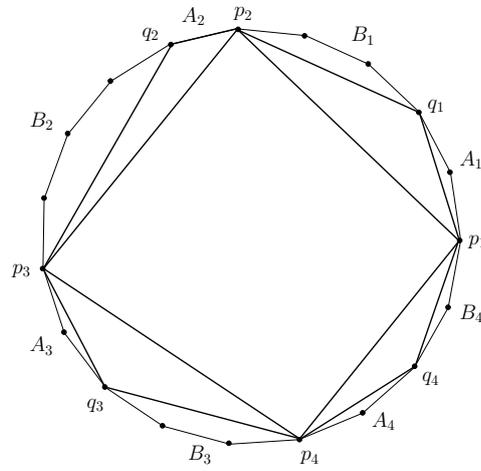
► **Theorem 12.** *For every sufficiently large  $n$ , there exists an  $n$ -element point set  $S$  on a circle such that every plane hop spanner on  $S$  has hop stretch factor at least 4.*

**Proof.** Consider a set  $S$  of  $n$  points that form the vertices of regular  $n$ -gon  $R$  inscribed in a circle  $C$ , where the circle is just a bit larger than the circumscribed circle of an equilateral triangle of unit edge length. Formally, for a given  $\varepsilon \in (0, 1/50)$ , set  $n = \lceil 2\varepsilon^{-1} \rceil$  and choose the radius of  $C$  such that every sequence of  $(\frac{1}{3} - \varepsilon)n$  consecutive points from  $S$  makes a subset of diameter at most 1; and any larger sequence makes a subset of diameter larger than 1. Note that  $\varepsilon n \geq 2$ . (We may set  $\varepsilon = 0.02$ , which yields  $n = 100$ .)

The short circular arc between two consecutive vertices of  $R$  is referred to as an *elementary arc*. (Its center angle is  $2\pi/n$ .) If  $A$  is a set of elementary arcs,  $X(A)$  denotes its set of endpoints; obviously  $|X(A)| \geq |A|$ , with equality when  $A$  covers the entire circle  $C$ .

Suppose, for the sake of contradiction, that the unit disk graph  $G$  has a plane subgraph  $G'$  with hop number at most 3. First, augment  $G'$  to a maximal noncrossing subgraph of  $G$ , by successively adding edges from  $G \setminus G'$  that do not introduce crossings. Adding edges does not increase the hop number of  $G'$ , which remains at most 3.

We define *maximal* edges in  $G'$  as follows. Associate every edge of  $G'$  with the shorter circular arc between its endpoints. Observe that containment between arcs is a partial order (poset). An edge of  $G'$  is *maximal* if the associated arc is maximal in this poset. Due to planarity, if two arcs overlap, then one of the arcs contains the other. Hence the maximal edges correspond to nonoverlapping arcs. As such, the maximal edges form a convex cycle, i.e., a convex polygon  $P = p_1, p_2, \dots, p_k$ . Refer to Fig. 6. By the choice of  $C$ , we have  $k \geq 4$ . Each edge of the polygon  $P$  determines a set of points, called *block*, that lie on the associated circular arc (both endpoints of the edge are included). Since the length of each edge of  $P$  is at most 1, the restriction of  $G'$  to the vertices in a block is a triangulation.



■ **Figure 6** The partition induced by the blocks for  $n = 19$  and  $k = 4$ . The edges  $p_i p_{i+1}$  are maximal edges of  $G'$  and  $\Delta p_i p_{i+1} q_i$  is the unique triangle adjacent to  $p_i p_{i+1}$  in the triangulation of the  $i$ th block. Since  $n = 19$  is small, the figure only illustrates the notation used in the proof of Theorem 12;  $|A_1| = 2$ ,  $|B_1| = 3$ ,  $|A_2| = 1$ ,  $|B_2| = 4$ , etc.

Let  $A_i \cup B_i$  be the sets of elementary arcs in counterclockwise order covering the  $i$ th block such that  $A_i$  and  $B_i$  are separated by a common vertex  $q_i$ , where the triangle  $\Delta p_i p_{i+1} q_i$  is the (unique) triangle adjacent to the chord  $p_i p_{i+1}$  in the triangulation of the  $i$ th block (where addition is modulo  $k$ , so that  $k + 1 = 1$ ). In particular,  $q_i$  is the last endpoint of an elementary arc in  $A_i$  and the first endpoint of an elementary arc in  $B_i$ , in counterclockwise order. As such, we have

$$\sum_{i=1}^k (|A_i| + |B_i|) = n. \quad (1)$$

By definition, we have

$$|A_i| + |B_i| \leq \left(\frac{1}{3} - \varepsilon\right) n, \quad \text{for } i = 1, \dots, k. \quad (2)$$

By the maximality of the blocks in  $G'$ , we have

$$(|A_i| + |B_i|) + (|A_{i+1}| + |B_{i+1}|) \geq \left(\frac{1}{3} - \varepsilon\right) n, \quad \text{for } i = 1, \dots, k. \quad (3)$$

By the maximality of  $G'$ , we also have  $k \leq 6$ , since otherwise an averaging argument would yield two adjacent blocks, say,  $i$  and  $i + 1$ , that can be merged by adding one chord of length at most 1 and so that the merged sequence of points has size at most

$$|A_i| + |B_i| + |A_{i+1}| + |B_{i+1}| \leq \frac{2n}{7} < \left(\frac{1}{3} - \varepsilon\right) n,$$

which would be a contradiction. We claim that

$$|B_i| + |A_{i+1}| \geq \left(\frac{1}{3} - 3\varepsilon\right) n, \quad \text{for } i = 1, \dots, k. \tag{4}$$

Suppose for contradiction that  $|B_i| + |A_{i+1}| \leq \left(\frac{1}{3} - 3\varepsilon\right) n$  holds for some  $i$ . Consider the  $\varepsilon n$  elementary arcs preceding the arcs in  $B_i$  and the  $\varepsilon n$  elementary arcs following the arcs in  $A_{i+1}$ , in counterclockwise order. Denote these sets of arcs by  $U_i$  and  $V_i$ , respectively ( $|U_i| = |V_i| = \varepsilon n$ ). Recall that  $\varepsilon n \geq 2$  and thus  $|X(U_i)|, |X(V_i)| \geq |U_i| = \varepsilon n \geq 2$ .

We claim that there exist  $u \in X(U_i)$  and  $v \in X(V_i)$  such that  $|uv| \leq 1$  and  $h(u, v) \geq 4$ . Indeed,  $\text{diam}(X(U_i \cup B_i \cup A_{i+1} \cup V_i)) \leq 1$  since  $X(U_i \cup B_i \cup A_{i+1} \cup V_i)$  contains at most

$$\left(\frac{1}{3} - 3\varepsilon\right) n + 2\varepsilon n \leq \left(\frac{1}{3} - \varepsilon\right) n$$

consecutive points. This proves the first part of the claim for any  $u \in X(U_i)$  and  $v \in X(V_i)$ . For the second part, we can take  $u$  as one of the two vertices preceding  $q_i$  that is not  $p_i$ , and similarly we can take  $v$  as one of the two vertices following  $q_{i+1}$  that is not  $p_{i+2}$ . With this choice, we have  $h(u, p_{i+1}) \geq 2$  and  $h(p_{i+1}, v) \geq 2$ , and  $\rho(u, v)$  passes through  $p_{i+1}$ . Consequently,

$$h(u, v) \geq h(u, p_{i+1}) + h(p_{i+1}, v) \geq 2 + 2 = 4.$$

We have reached a contradiction, which proves (4). The summation of (4) over all  $i = 1, \dots, k$ , in combination with (1) and the inequality  $k \geq 4$  yields

$$n = \sum_{i=1}^k (|A_i| + |B_i|) = \sum_{i=1}^k (|B_i| + |A_{i+1}|) \geq k \left(\frac{1}{3} - 3\varepsilon\right) n \geq 0.27 kn \geq 1.08 n.$$

This last contradiction completes the proof. ◀

**An upper bound for points on a circle.** For many problems dealing with finite point configurations in the plane, points in convex position or on a circle may allow for tighter bounds; see, e.g., [14, 15, 31, 40]. We show that the lower bound of 4 for points on a circle is tight in this case. Due to space constraints, the proof of Theorem 13 is omitted and can be found in the full version.

► **Theorem 13.** *For every point set  $S$  on a circle  $C$ , there exists a plane 4-hop spanner.*

## 5 The maximum degree of hop spanners cannot be bounded

A standard counting argument shows that dense (abstract) graphs do not admit constant bounded degree hop spanners (irrespective of planarity). We start with an observation regarding the complete UDG  $K_n$  and then extend it and show that the maximum degree of hop spanners of sparse UDGs is also unbounded.

► **Theorem 14.** *For every pair of integers  $k \geq 2$  and  $\Delta \geq 2$ , there exists a set  $S$  of  $n = O(\Delta^k)$  points such that the unit disk graph  $G = (V, E)$  on  $S$  has no  $k$ -spanner whose maximum degree is at most  $\Delta$ .*

**Proof.** Consider a set  $S$  of  $n$  points so that the unit disk graph  $G = (V, E)$  on  $S$  is the complete graph  $K_n$  (e.g.,  $n$  points in a disk of unit diameter). Choose a point  $p \in S$ . Let  $S_0 = \{p\}$ . Let  $N(u)$  denote the set of vertices adjacent to  $u$  in  $G'$ . Since the degree of  $p$  in  $G'$  is at most  $\Delta$ ,  $|N(p)| \leq \Delta$ . Let  $S_1 := N(p)$ . The points in  $S_1$  have edges to a set  $S_2$  of at most  $\Delta(\Delta - 1)$  points in  $S \setminus (S_0 \cup S_1)$ . In general, the set  $S_i$  contains at most  $\Delta(\Delta - 1)^{i-1}$  points in  $S \setminus \cup_{j=0}^{i-1} S_j$ . Consider the sets  $S_0, \dots, S_k$ . Now it is easy to check that

$$\sum_{i=0}^k |S_i| \leq 1 + \Delta \frac{(\Delta - 1)^k - 1}{\Delta - 2} = O(\Delta^k).$$

Let  $M$  denote the above expression in  $\Delta$ ,  $q$  be a point in  $S \setminus (\cup_{i=0}^k S_i)$  and  $n = M + 1$ . Observe that  $h(p, q) \geq k + 1$ , whereas  $pq \in E$ , and so  $G'$  is not a  $k$ -spanner for  $S$ . ◀

An alternative argument is included below – in a form that we use later in this section. We arrange  $n$  points so that the unit disk graph  $G = (V, E)$  on  $S$  is the complete graph  $K_n$  (e.g.,  $n$  points in a disk of unit diameter). Assume that the points are labeled from 1 to  $n$ ; and assume there exists a subgraph  $G'$  whose maximum degree is at most  $\Delta$  that is a  $k$ -spanner of  $G$ . For each vertex  $v \in V$ , label the elements in  $N(v)$  by  $1, 2, \dots$  (the maximum label is  $\leq \Delta$ ), in some arbitrary fashion. For every edge  $uv \in E$ ,  $u < v$ , there is a connecting path of at most  $k$  edges in  $G'$ . Such a path can be uniquely encoded by a string of length  $k + 1$  over the alphabet  $[n] \cup \{0, 1, 2, \dots, \Delta\}$ : by specifying the start vertex  $u$  followed by an encoding of the edges in the path. There are at most  $\Delta$  choices for the first edge in the path; and at most  $\Delta - 1$  choices for any subsequent edge and zero for indicating the end of a path whose length is shorter than  $k$ ; the encoding of a path whose length  $\ell$  is shorter than  $k$  has  $k - \ell$  trailing zeros at the end. As such, there are at most  $(n - 1)\Delta^k$  encodings. If

$$(n - 1)\Delta^k < \binom{n}{2},$$

i.e., if  $n > 2\Delta^k$ , some edge has no encoding, which is a contradiction, and this completes the proof.

► **Theorem 15.** *Let  $t: \mathbb{N} \rightarrow \mathbb{N}$ ,  $t(n) \leq (n - 1)/2$ , be an integer function that tends to  $\infty$  with  $n$ . For every pair of integers  $k \geq 2$  and  $\Delta \geq 2$ , there exists  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$ , there is a set  $S$  of  $n$  points in the plane such that*

- (i) *the unit disk graph  $G = (V, E)$  on  $S$  has  $\Theta(n \cdot t(n))$  edges, and*
- (ii)  *$G$  has no  $k$ -spanner whose maximum degree is at most  $\Delta$ .*

**Proof.** Observe that  $2t(n) + 1 \leq n$ . Let  $n$  be large enough so that  $t(n) > 2\Delta^k$  (we can choose infinitely many  $n$  with this property). Write  $t = t(n)$ . For a given  $t$ , arrange  $n$  points into  $\lfloor \frac{n}{2t+1} \rfloor$  groups of size  $2t + 1$  and a remaining group (if any) of size  $n - \lfloor \frac{n}{2t+1} \rfloor (2t + 1)$ . Place the groups in disjoint disks of unit diameter in the plane, so that the UDG of each group is a complete graph; and arrange the disks along a line such that the UDG has exactly one edge between any two consecutive groups. Assume that there exists a subgraph  $G'$  whose maximum degree is at most  $\Delta$  that is a  $k$ -spanner of  $G$ . Encode paths in  $G'$  as in the proof of Theorem 14. The number of edges in  $G$  is bounded from above and from below as follows:

$$\begin{aligned}
 |E| &\leq \left\lfloor \frac{n}{2t+1} \right\rfloor \binom{2t+1}{2} + \binom{2t}{2} + \left\lfloor \frac{n}{2t+1} \right\rfloor \\
 &\leq \frac{n}{2t+1}(2t+1)t + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor = nt + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor, \tag{5}
 \end{aligned}$$

$$|E| \geq \left\lfloor \frac{n}{2t+1} \right\rfloor \binom{2t+1}{2} \geq \binom{2t+1}{2} = t(2t+1), \tag{6}$$

$$|E| \geq \left\lfloor \frac{n}{2t+1} \right\rfloor \binom{2t+1}{2} \geq \left( \frac{n}{2t+1} - 1 \right) (2t+1)t = nt - (2t+1)t. \tag{7}$$

We distinguish two cases:  $t$  is large or  $t$  is small as specified below.

If  $\frac{n-1}{4} \leq t \leq \frac{n-1}{2}$ , by (5) and (6) we have

$$|E| \leq nt + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor \leq nt + (n-2)t + 1 \leq 2nt.$$

$$|E| \geq t(2t+1) \geq \frac{n-1}{4} (2t+1) > (n-1)\Delta^k.$$

If  $t \leq \frac{n-1}{4}$ , by (5) and (7) we have

$$|E| \leq nt + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor \leq nt + \frac{(n-3)t}{2} + \frac{n}{3}.$$

$$|E| \geq nt - (2t+1)t = t(n - (2t+1)) \geq t \left( n - \frac{n+1}{2} \right) = \frac{n-1}{2}t > (n-1)\Delta^k.$$

Note that  $|E| = O(nt)$  and  $|E| = \Omega(nt)$  in both cases; consequently,  $|E| = \Theta(nt)$ . Since  $|E|$  exceeds the number of encodings (analogous to the proof of Theorem 14) in both cases, we reach a contradiction and this completes the proof. ◀

## 6 Conclusion

Observe that if  $G$  is a UDG that is triangle-free, then the only 2-hop spanner of  $G$  is the graph itself; recall the bipartite case mentioned in Section 1. Thus if  $G$  has a superlinear number of edges and is triangle-free, then by the above observation, every 2-hop spanner of  $G$  (and there is only one,  $G$ ) has a superlinear number of edges. This direction does not materialize in a superlinear lower bound for 2-hop spanners because of the following.

► **Proposition 16.** *Let  $G$  be a UDG that is triangle-free. Then  $G$  has at most  $2.5n$  edges.*

**Proof.** It suffices to show that the degree of every vertex is at most five. Assume for contradiction that a point  $p$  has degree at least six and let  $q$  and  $r$  be two consecutive neighbors of  $p$  in order of visibility, where  $|pq|, |pr| \leq 1$ . Put  $\alpha = \angle qpr$ ; we have  $\alpha \leq \pi/3$ . Since at least another interior angle of the triangle  $\Delta pqr$  is at least  $\pi/3$ , the Law of Sines implies  $|qr| \leq \max\{|pq|, |pr|\} \leq 1$  and thus  $\Delta pqr$  is a triangle in  $G$ , a contradiction. ◀

We conclude with two remaining open problems:

1. Are there point sets for which every plane hop-spanner has hop stretch factor at least 5?
2. Can our  $O(n \log n)$  upper bound on the size of 2-hop spanners on  $n$  points in the plane be improved? Are there  $n$ -element point sets for which every 2-hop spanner has  $\omega(n)$  edges? Recent results show that unit disks may exhibit surprising behavior [29, 35].

## References

- 1 Ingo Althöfer, Gautam Das, David P. Dobkin, Deborah Joseph, and José Soares. On sparse spanners of weighted graphs. *Discret. Comput. Geom.*, 9:81–100, 1993. doi:10.1007/BF02189308.
- 2 Khaled M. Alzoubi, Xiang-Yang Li, Yu Wang, Peng-Jun Wan, and Ophir Frieder. Geometric spanners for wireless ad hoc networks. *IEEE Trans. Parallel Distrib. Syst.*, 14(4):408–421, 2003. doi:10.1109/TPDS.2003.1195412.
- 3 Boris Aronov, Mark de Berg, Otfried Cheong, Joachim Gudmundsson, Herman J. Haverkort, Michiel H. M. Smid, and Antoine Vigneron. Sparse geometric graphs with small dilation. *Comput. Geom.*, 40(3):207–219, 2008. doi:10.1016/j.comgeo.2007.07.004.
- 4 Surender Baswana and Sandeep Sen. A simple and linear time randomized algorithm for computing sparse spanners in weighted graphs. *Random Struct. Algorithms*, 30(4):532–563, 2007. doi:10.1002/rsa.20130.
- 5 Surender Baswana and Sandeep Sen. *Simple Algorithms for Spanners in Weighted Graphs*, pages 1981–1986. Springer, New York, 2016. doi:10.1007/978-1-4939-2864-4\_10.
- 6 Ahmad Biniiaz. Plane hop spanners for unit disk graphs: Simpler and better. *Comput. Geom.*, 89:101622, 2020. doi:10.1016/j.comgeo.2020.101622.
- 7 Prosenjit Bose, Joachim Gudmundsson, and Michiel H. M. Smid. Constructing plane spanners of bounded degree and low weight. *Algorithmica*, 42(3-4):249–264, 2005. doi:10.1007/s00453-005-1168-8.
- 8 Prosenjit Bose and Michiel H. M. Smid. On plane geometric spanners: A survey and open problems. *Comput. Geom.*, 46(7):818–830, 2013. doi:10.1016/j.comgeo.2013.04.002.
- 9 Heinz Breu and David G. Kirkpatrick. Unit disk graph recognition is NP-hard. *Comput. Geom.*, 9(1-2):3–24, 1998. doi:10.1016/S0925-7721(97)00014-X.
- 10 Gruia Călinescu, Sanjiv Kapoor, and Mohammad Sarwat. Bounded-hops power assignment in ad hoc wireless networks. *Discret. Appl. Math.*, 154(9):1358–1371, 2006. doi:10.1016/j.dam.2005.05.034.
- 11 Nicolas Catusse, Victor Chepoi, and Yann Vaxès. Planar hop spanners for unit disk graphs. In *Proc. 6th Workshop on Algorithms for Sensor Systems, Wireless Ad Hoc Networks, and Autonomous Mobile Entities (ALGOSENSORS)*, volume 6451 of LNCS, pages 16–30. Springer, 2010. doi:10.1007/978-3-642-16988-5\_2.
- 12 Gautam Das and Deborah Joseph. Which triangulations approximate the complete graph? In *Proc. International Symposium on Optimal Algorithms*, volume 401 of LNCS, pages 168–192. Springer, 1989. doi:10.1007/3-540-51859-2\_15.
- 13 David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discret. Comput. Geom.*, 5:399–407, 1990. doi:10.1007/BF02187801.
- 14 Ding-Zhu Du, Frank K. Hwang, and S. C. Chao. Steiner minimal tree for points on a circle. *Proceedings of the American Mathematical Society*, 95(4):613–618, 1985. doi:10.1090/S0002-9939-1985-0810173-6.
- 15 Ding-Zhu Du, Frank K. Hwang, and J. F. Weng. Steiner minimal trees for regular polygons. *Discret. Comput. Geom.*, 2:65–84, 1987. doi:10.1007/BF02187871.
- 16 Adrian Dumitrescu and Anirban Ghosh. Lattice spanners of low degree. *Discret. Math. Algorithms Appl.*, 8(3):1650051:1–1650051:19, 2016. doi:10.1142/S1793830916500518.
- 17 Adrian Dumitrescu and Anirban Ghosh. Lower bounds on the dilation of plane spanners. *Int. J. Comput. Geom. Appl.*, 26(2):89–110, 2016. doi:10.1142/S0218195916500059.
- 18 Ayan Dutta, Anirban Ghosh, and O. Patrick Kreidl. Multi-robot informative path planning with continuous connectivity constraints. In *Proc. IEEE International Conference on Robotics and Automation (ICRA)*, pages 3245–3251, 2019. doi:10.1109/ICRA.2019.8794090.
- 19 Herbert Edelsbrunner, David G. Kirkpatrick, and Raimund Seidel. On the shape of a set of points in the plane. *IEEE Trans. Inf. Theory*, 29(4):551–558, 1983. doi:10.1109/TIT.1983.1056714.

- 20 David Eppstein. Spanning trees and spanners. In Jörg-Rüdiger Sack and Jorge Urrutia, editors, *Handbook of Computational Geometry*, pages 425–461. North Holland, 2000. doi:10.1016/b978-044482537-7/50010-3.
- 21 Paul Erdős. Extremal problems in graph theory. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, pages 29–36, Prague, 1964. Publishing House of the Czechoslovak Academy of Sciences.
- 22 Jie Gao, Leonidas J. Guibas, John Hershberger, Li Zhang, and An Zhu. Geometric spanners for routing in mobile networks. *IEEE J. Sel. Areas Commun.*, 23(1):174–185, 2005. doi:10.1109/JSAC.2004.837364.
- 23 Iyad A. Kanj and Ljubomir Perkovic. On geometric spanners of euclidean and unit disk graphs. In *Proc. 25th Symposium on Theoretical Aspects of Computer Science (STACS)*, volume 1 of *LIPICs*, pages 409–420. Schloss Dagstuhl, 2008. doi:10.4230/LIPICs.STACS.2008.1320.
- 24 Guy Kortsarz and David Peleg. Generating sparse 2-spanners. *J. Algorithms*, 17(2):222–236, 1994. doi:10.1006/jagm.1994.1032.
- 25 Guy Kortsarz and David Peleg. Generating low-degree 2-spanners. *SIAM J. Comput.*, 27(5):1438–1456, 1998. doi:10.1137/S0097539794268753.
- 26 Christos Levcopoulos and Andrzej Lingas. There are planar graphs almost as good as the complete graphs and almost as cheap as minimum spanning trees. *Algorithmica*, 8(3):251–256, 1992. doi:10.1007/BF01758846.
- 27 Xiang-Yang Li. Algorithmic, geometric and graphs issues in wireless networks. *Wirel. Commun. Mob. Comput.*, 3(2):119–140, 2003. doi:10.1002/wcm.107.
- 28 Xiang-Yang Li and Yu Wang. Efficient construction of low weighted bounded degree planar spanner. *Int. J. Comput. Geom. Appl.*, 14(1-2):69–84, 2004. doi:10.1142/S0218195904001366.
- 29 Colin McDiarmid and Tobias Müller. The number of disk graphs. *Eur. J. Comb.*, 35:413–431, 2014. doi:10.1016/j.ejc.2013.06.037.
- 30 Joseph S. B. Mitchell. Geometric shortest paths and network optimization. In Jörg-Rüdiger Sack and Jorge Urrutia, editors, *Handbook of Computational Geometry*, pages 633–701. North Holland, 2000. doi:10.1016/b978-044482537-7/50016-4.
- 31 Wolfgang Mulzer. *Minimum dilation triangulations for the regular n-gon*. Master’s thesis, Freie Universität Berlin, 2004.
- 32 Nabil H. Mustafa and Kasturi R. Varadarajan. Epsilon-approximations and epsilon-nets. In Jacob E. Goodman, Joseph O’Rourke, and Csaba D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, chapter 47. CRC Press, Boca Raton, FL, 3 edition, 2017.
- 33 Giri Narasimhan and Michiel H. M. Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007. doi:10.1201/9781315119601.
- 34 Tim Nieberg, Johann L. Hurink, and Walter Kern. Approximation schemes for wireless networks. *ACM Trans. Algorithms*, 4(4):49:1–49:17, 2008. doi:10.1145/1383369.1383380.
- 35 János Pach and Dömötör Pálvölgyi. Unsplittable coverings in the plane. *Adv. Math.*, 302:433–457, 2016. doi:10.1016/j.aim.2016.07.011.
- 36 János Pach and Gábor Tardos. Tight lower bounds for the size of epsilon-nets. *J. Amer. Math. Soc.*, 26(3):645–658, 2013. doi:10.1090/S0894-0347-2012-00759-0.
- 37 János Pach and Gerhard J. Woeginger. Some new bounds for epsilon-nets. In *Proc. 6th ACM Symposium on Computational Geometry (SoCG)*, pages 10–15, 1990. doi:10.1145/98524.98529.
- 38 David Peleg and Alejandro A. Schäffer. Graph spanners. *Journal of Graph Theory*, 13(1):99–116, 1989. doi:10.1002/jgt.3190130114.
- 39 Rajmohan Rajaraman. Topology control and routing in ad hoc networks: a survey. *SIGACT News*, 33(2):60–73, 2002. doi:10.1145/564585.564602.
- 40 Sattar Sattari and Mohammad Izadi. An improved upper bound on dilation of regular polygons. *Comput. Geom.*, 80:53–68, 2019. doi:10.1016/j.comgeo.2019.01.009.