Realizability Without Symmetry

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Abstract

In categorical realizability, it is common to construct categories of assemblies and modest sets from applicative structures. In this paper, we introduce several classes of applicative structures and apply the categorical realizability construction to them. Then we obtain closed multicategories, closed categories and skew closed categories, which are more general categorical structures than Cartesian closed categories and symmetric monoidal closed categories. Moreover, we give the necessary and sufficient conditions for obtaining closed multicategories and closed categories of assemblies.

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1 Introduction

In categorical realizability, we construct categories of assemblies and modest sets from applicative structures. The best known usage is applying this construction to partial combinatory algebras (PCAs) which is a class of applicative structures close to the models of the lambda calculus, as in [12]. From PCAs, we obtain Cartesian closed categories of assemblies and use these categories for models of various logics and programming languages like PCF.

The construction of categories of assemblies does not depend on particular structures of applicative structures. Hence we may apply this construction to other classes of applicative structures. Indeed, another usage is introduced in [3] [2], where by applying the construction to **BCI**-algebras, we can obtain symmetric monoidal closed categories (SMCCs) and use them for models of linear logics and languages.

In this paper, by applying the categorical realizability construction to more general classes of applicative structures than PCAs and **BCI**-algebras, we investigate further correspondences between categorical structures of assemblies and classes of applicative structures. To make assemblies on an applicative structure a category, it is sufficient to assume two elements **B** (corresponding to the lambda term $\lambda xyz.x(yz)$ expressing the composition of functions) and **I** (corresponding to the lambda term $\lambda x.x$ expressing the identity function) in the applicative structure. (Here we say such an applicative structure is a **BI**-algebra.) Therefore, classes between **BCI**-algebras and **BI**-algebras may induce some categorical structures more general than SMCCs. For instance, there may exist some classes which induce non-symmetric categorical structures. Indeed, in this paper we introduce such classes of applicative structures, which induce closed multicategories, closed categories and skew closed categories.

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Table 1 Summary of the correspondences.
(Here (†) means that not only the corresponding applicative structures induce the categorical structures, but also the converse hold in some natural conditions, like Proposition 19.).

	Applicative structures	Structure of assemblies and modest sets	Section
Known results	PCA	Cartesian closed category (†)	2.2
	BCI-algebra	symmetric monoidal closed category (†)	2.3
New results	$\mathsf{BK}(-)^\bullet$-algebra	closed category	5.1
	$\mathsf{BII}^\times(-)^\bullet$-algebra	closed category (†)	4
	$\mathsf{BI}(-)^\bullet$-algebra	closed multicategory (†)	3
	$\mathsf{BB'II}^\bullet$-algebra	skew closed category	5.3
	$\mathsf{BII}^\bullet(-)^\circ$-algebra	skew closed category	5.2

Inclusions
PCAs $\subsetneq$ **BCI**-algebras $\subsetneq$ $\mathsf{BII}^\times(-)^\bullet$-algebras $\subsetneq$ $\mathsf{BI}(-)^\bullet$-algebras $\subsetneq$ $\mathsf{BII}^\bullet(-)^\circ$-algebras
PCAs $\subsetneq$ $\mathsf{BK}(-)^\bullet$-algebras $\subsetneq$ $\mathsf{BII}^\times(-)^\bullet$-algebras
BCI-algebras $\subsetneq$ $\mathsf{BB'II}^\bullet$-algebras $\subsetneq$ $\mathsf{BII}^\bullet(-)^\circ$-algebras

In category theory, many closed structures without tensor products have been developed. Closed multicategories, closed categories and skew closed categories are the typical ones. Each of them gives certain axiomatization of internal function spaces without using tensor products and does not have symmetries in general unlike SMCCs.

Closed multicategories introduced in [11] are closed categorical structures for multicategories (extensions of categories, whose maps are allowed to have finitely many arguments) and correspond to planar multiplicative intuitionistic linear logic with only linear implication $\multimap$ and without tensor product nor unit. Closed categories introduced in [5] are something like monoidal closed categories without tensor products, which have internal hom objects defined without using tensor products. In [13], it is shown that closed categories are equivalent to closed multicategories with unit objects. Skew closed categories introduced in [17] are categories with slightly weaker structure than closed categories. There is a categorical structure called skew monoidal categories [18], which have the same components as monoidal categories but the invertibility of unitors and associators are not assumed. Skew closed categories are to skew monoidal categories what closed categories are to monoidal categories.

When we try to obtain these categorical structures by categorical realizability, we face a subtle problem. To exclude symmetries, we have to remove the element C (corresponding to the lambda term $\lambda xyz.xzy$ expressing the swap of arguments) from an applicative structure. However, to obtain closed structures, we need to realize maps sent by internal hom functors $[f, g] : h \mapsto (g \circ h \circ f)$, and realizing them needs some exchange operation. We resolve this problem by introducing restricted exchanges $(-)^\bullet$, $(-)^\circ$ and B'.

For an applicative structure $(|\mathcal{A}|, \cdot)$, $(-)^\bullet$ is a unary operation on $|\mathcal{A}|$ such that $\mathsf{x}^\bullet \cdot \mathsf{y} = \mathsf{y} \cdot \mathsf{x}$ for any x and y in $|\mathcal{A}|$. Since $\mathsf{C} \cdot \mathsf{I} \cdot \mathsf{x}$ satisfies the axiom of $\mathsf{x}^\bullet$, assuming $(-)^\bullet$ is a weaker assumption than assuming C. $\mathsf{BI}(-)^\bullet$-algebras, i.e., **BI**-algebras with $(-)^\bullet$, give rise to (non-symmetric) closed multicategories. Moreover, we show that being a $\mathsf{BI}(-)^\bullet$-algebra is a necessary condition to give a category of assemblies as a closed multicategory under some natural assumptions in Proposition 19.

Other than $\mathsf{BI}(-)^\bullet$-algebras, we also introduce several classes of applicative structures. The correspondences are in table 1. In particular, we also show that $\mathsf{BII}^\times(-)^\bullet$-algebras are necessary to give categories of assemblies as closed categories under some conditions. Table 2 is the summary of elements and constructions assumed in these applicative structures.

Table 2 Summary of the elements and constructions of applicative structures.

Elements and constructions	Axiom	Section
element $\mathbf{S}$	$\mathbf{S}\mathsf{xyz} \simeq \mathsf{xz(yz)}$	2.2
element $\mathbf{K}$	$\mathbf{K}\mathsf{xy} \simeq \mathsf{x}$	2.2
element $\mathbf{B}$	$\mathbf{B}\mathsf{xyz} = \mathsf{x(yz)}$	2.3
element $\mathbf{C}$	$\mathbf{C}\mathsf{xyz} = \mathsf{xzy}$	2.3
element $\mathbf{I}$	$\mathbf{I}\mathsf{x} = \mathsf{x}$	2.3
element $\mathbf{B}'$	$\mathbf{B}'\mathsf{xyz} = \mathsf{y(xz)}$	5.3
element $\mathbf{I}^{\times}$	$\mathbf{I}^{\times}\mathsf{x}\mathbf{I} = \mathsf{x}$	4
unary operation $(-)^{\bullet}$	$\mathsf{x}^{\bullet}\mathsf{y} = \mathsf{yx}$	3
unary operation $(-)^{\circ}$	$\mathsf{x}^{\circ}\mathsf{yz} = \mathsf{y(xz)}$	5.2

To the best of our knowledge, this is the first systematic treatment of realizability semantics for the non-commutative or planar setting. We hope that our analysis brings new insights on categorical realizability and extends its applications to new areas, most notably to non-commutative logics and systems such as Lambek calculus.

The rest of this paper is structured as follows. In Section 2, we recall some basic notions and results in categorical realizability. In Section 3, we introduce $\mathbf{BI}(-)^{\bullet}$-algebras and describe how they correspond to the planar lambda calculus and closed multicategories. Section 4 is about $\mathbf{BII}^{\times}(-)^{\bullet}$-algebras and closed categories, which has a similar story to Section 3. In Section 5, we give three other classes of applicative structures and see categorical structures of assemblies on them. In Section 6, we construct concrete examples of $\mathbf{BI}(-)^{\bullet}$-algebras other than the planar lambda calculus. As an unexpected one, we show that the computational lambda calculus [14] is a $\mathbf{BII}^{\times}(-)^{\bullet}$-algebra. In Section 7, we discuss related work. Finally, in Section 8, we summarize contents of this paper and describe future work.

Basic knowledge of category theory and the lambda calculus is assumed.

2 Background

In this section, we recall some basic concepts and results. All the definitions and propositions in this section are from [12] and [8].

2.1 Applicative structures and categories of assemblies

▶ **Definition 1.** *A* partial applicative structure $\mathcal{A}$ *is a pair of a set* $|\mathcal{A}|$ *and a partial binary operation* $(\mathsf{x},\mathsf{y}) \mapsto \mathsf{x}\cdot\mathsf{y}$ *on* $|\mathcal{A}|$. *Application associates to the left, and we often omit* $\cdot$ *and write it as juxtaposition. For instance,* $\mathsf{xz(yz)}$ *denotes* $(\mathsf{x}\cdot\mathsf{z})\cdot(\mathsf{y}\cdot\mathsf{z})$. *When the binary operation of* $\mathcal{A}$ *is total, we say* $\mathcal{A}$ *is a* total applicative structure.

In the sequel, we use two notations $\downarrow$ and $\simeq$. The down arrow means "defined." For instance, for a partial applicative structure $(|\mathcal{A}|,\cdot)$, $xy\downarrow$ means that $x\cdot y$ is defined. "$\simeq$" denotes the Kleene equality, which means that if the one side of the equation is defined then the other side is also defined and are equal.

▶ **Definition 2.** *Let* $\mathcal{A}$ *be a partial applicative structure.*

(1) *An* assembly *on* $\mathcal{A}$ *is a pair* $X := (|X|, \|\text{-}\|_X)$, *where* $|X|$ *is a set and* $\|\text{-}\|_X$ *is a function sending* $x \in |X|$ *to a non-empty subset* $\|x\|_X$ *of* $|\mathcal{A}|$.

(2) *A* map of assemblies $f : X \to Y$ *is a function* $f : |X| \to |Y|$ *such that there exists an element* $\mathsf{r} \in |\mathcal{A}|$ *realizing* f*, where* "r realizes f" *means that*

$$\forall x \in |X|, \forall \mathsf{a} \in \|x\|_X, \ \mathsf{ra}\downarrow \text{ and } \mathsf{ra} \in \|f(x)\|_Y.$$

We say that r *is a* realizer *of* f *when* r *realizes* f*.*

If we assume two extra conditions on a partial applicative structure, we can construct a category from assemblies and maps of assemblies.

▶ **Definition 3.** *Let* $\mathcal{A}$ *be a partial applicative structure satisfying that:*
(i) $|\mathcal{A}|$ *has an element* $\mathbf{I}$ *such that for any* $\mathsf{x} \in |\mathcal{A}|$, $\mathbf{I}\mathsf{x}\downarrow$ *and* $\mathbf{I}\mathsf{x} = \mathsf{x}$*;*
(ii) *for any* $\mathsf{r}_1, \mathsf{r}_2 \in |\mathcal{A}|$*, there exists* $\mathsf{r}_{1,2} \in |\mathcal{A}|$ *such that for any* $\mathsf{x} \in |\mathcal{A}|$, $\mathsf{r}_{1,2}\mathsf{x} \simeq \mathsf{r}_1(\mathsf{r}_2\mathsf{x})$.
Then we construct categories as follows:
(1) *The* category $\mathbf{Asm}(\mathcal{A})$ of assemblies *on* $\mathcal{A}$ *consists of assemblies on* $\mathcal{A}$ *as its objects and maps of assemblies as its maps. Identity maps and composition maps are the same as those of* $\mathbf{Sets}$*.*
(2) *The* category $\mathbf{Mod}(\mathcal{A})$ of modest sets *on* $\mathcal{A}$ *is the full subcategory of* $\mathbf{Asm}(\mathcal{A})$*, such that each object* $(|X|, \|\text{-}\|_X)$ *has the property:*

$$\forall x, y \in |X|, \ x \neq y \Rightarrow \|x\|_X \cap \|y\|_X = \emptyset.$$

$\mathbf{Asm}(\mathcal{A})$ and $\mathbf{Mod}(\mathcal{A})$ are indeed categories. For any assembly $(|X|, \|\text{-}\|)$ on $\mathcal{A}$, the identity function on $|X|$ is realized by $\mathbf{I}$. Given two maps of assemblies $X \xrightarrow{f} Y \xrightarrow{g} Z$ realized by r_2 and r_1 respectively, the composition function $g \circ f : |X| \to |Z|$ is realized by $\mathsf{r}_{1,2}$.

In particular, a total applicative structure $\mathcal{A}$ which has $\mathbf{I}$ and $\mathbf{B}$ (such that $\mathbf{B}\mathsf{xyz} = \mathsf{x}(\mathsf{yz})$ for all $\mathsf{x}, \mathsf{y}, \mathsf{z} \in |\mathcal{A}|$) induces the category of assemblies and the category of modest sets. We call such a total applicative structure a **BI**-*algebra*.

In the following two subsections, we introduce two well-known classes of partial applicative structures. The categories constructed from partial applicative structures in these classes have useful categorical structures.

▶ Remark. In the rest of this paper, we only deal with categories of assemblies and not with categories of modest sets. However, all propositions hold when we replace the word "assemblies" by "modest sets."

2.2 PCAs and Cartesian closed categories

In this subsection, we recall a well-known class of partial applicative structures called partial combinatory algebras (PCAs). Assemblies on a PCA form a Cartesian closed category.

▶ **Definition 4.** *A* PCA *is a partial applicative structure* $\mathcal{A}$ *which contains two elements* $\mathbf{S}$ *and* $\mathbf{K}$ *satisfying:*
(i) $\forall \mathsf{x}, \mathsf{y} \in |\mathcal{A}|$, $\mathbf{K}\mathsf{x}\downarrow$ *and* $\mathbf{K}\mathsf{xy} \simeq \mathsf{x}$*;*
(ii) $\forall \mathsf{x}, \mathsf{y}, \mathsf{z} \in |\mathcal{A}|$, $\mathbf{S}\mathsf{x}\downarrow$, $\mathbf{S}\mathsf{xy}\downarrow$ *and* $\mathbf{S}\mathsf{xyz} \simeq \mathsf{xz}(\mathsf{yz})$.

▶ **Example 5.** Suppose infinite supply of variables $x, y, z, \ldots$. *Untyped lambda terms* are terms constructed from the following six rules:

$$\frac{}{x \vdash x}\ (\text{identity}) \ ; \qquad \frac{\Gamma \vdash M \qquad \Delta \vdash N}{\Gamma, \Delta \vdash MN}\ (\text{application})$$

where Γ and Δ are sequences of distinct variables and contain no common variables;

$$\frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\text{(abstraction)} \ ; \quad \frac{\Gamma, x, y \vdash M}{\Gamma, y, x \vdash M}\text{(exchange)} \ ; \quad \frac{\Gamma, x, y \vdash M}{\Gamma, x \vdash M[x/y]}\text{(contraction)}$$

where $M[x/y]$ denotes the term obtained by substituting x for all free y in M;

$$\frac{\Gamma \vdash M}{\Gamma, x \vdash M}\text{(weakening)} \quad \text{where } x \text{ is a variable not contained in } \Gamma.$$

Note that abstraction rules are only applied to the rightmost variables. In order to apply the abstraction rule to a variable in a different position, we need to use exchange rules and move the variable to the rightmost place.

We define an equivalence relation $=_\beta$ on lambda terms as the congruence of the relation $(\lambda x.M)N \sim M[N/x]$. Untyped lambda terms form a PCA. The underlying set consists of β-equivalence classes of untyped closed lambda terms (i.e., lambda terms with no free variables) and the application is defined as that of lambda terms. In this example, $\lambda xyz.xz(yz)$ is the representative of **S** and $\lambda xy.x$ is the representative of **K**.

PCAs are closely related to the untyped lambda calculus through the property called *combinatory completeness*.

▶ **Definition 6.** *Let $\mathcal{A}$ be a partial applicative structure. A* polynominal *over $\mathcal{A}$ is a syntactic expression generated by variables, elements of $|\mathcal{A}|$ and applications. For polynominals M and N over $\mathcal{A}$, $M \simeq N$ means that $M[\mathsf{a}_1/x_1, ..., \mathsf{a}_n/x_n] \simeq N[\mathsf{a}_1/x_1, ..., \mathsf{a}_n/x_n]$ holds in $\mathcal{A}$ for any $\mathsf{a}_1, ..., \mathsf{a}_n \in |\mathcal{A}|$, where $\{x_1, ..., x_n\}$ contains all the variables of M and N.*

▶ **Proposition 7.** *(combinatory completeness of PCAs) Let $\mathcal{A}$ be a PCA and M be a polynominal over $|\mathcal{A}|$. For any variable x, there exists a polynominal M' such that the free variables of M' are the free variables of M excluding x and $M'\mathsf{a} \simeq M[\mathsf{a}/x]$ for all $\mathsf{a} \in |\mathcal{A}|$. We write $\lambda^* x.M$ for such M'.*

Proof. We define $\lambda^* x.M$ by induction on the structure of M as follows: $\lambda^* x.x := \mathbf{SKK}$; $\lambda^* x.y := \mathbf{K}y$ (when $x \neq y$); $\lambda^* x.MN := \mathbf{S}(\lambda^* x.M)(\lambda^* x.N)$. ◀

For the special case of the above proposition, any closed lambda term is β-equivalent to some term constructed from $\lambda xyz.xz(yz)$ and $\lambda xy.x$ only using applications.

Although conditions of PCAs are simple, categorical structures induced by these algebras are quite strong and useful.

▶ **Proposition 8.** *Let $\mathcal{A}$ be a PCA. Then $\mathbf{Asm}(\mathcal{A})$ is Cartesian closed and regular.*

▶ Remark. Since all the examples in this paper are total applicative structures, we shall in future only consider total structures, although the definitions and results do generalize smoothly to partial ones. A discussion about "partial **BCI**-algebras" is found in Remark 1 of [8]. From now on, whenever we say "applicative structure", it means total applicative structures.

2.3 BCI-algebras and symmetric monoidal closed categories

In this subsection we recall another class of applicative structures called **BCI**-algebra. **BCI**-algebras are related to linear structures whereas PCAs are not.

▶ **Definition 9.** *A* **BCI**-algebra *is an applicative structure $\mathcal{A}$ which contains three elements* **B**, **C** *and* **I** *satisfying:*

(i) $\forall \mathsf{x}, \mathsf{y}, \mathsf{z} \in |\mathcal{A}|$, $\mathbf{B}\mathsf{xyz} = \mathsf{x}(\mathsf{yz})$;
(ii) $\forall \mathsf{x}, \mathsf{y}, \mathsf{z} \in |\mathcal{A}|$, $\mathbf{C}\mathsf{xyz} = \mathsf{xzy}$;

(iii) $\forall \mathsf{x} \in |\mathcal{A}|$, $\mathbf{I}\mathsf{x} = \mathsf{x}$.

▶ **Example 10.** *Untyped linear lambda terms* are untyped lambda terms constructed without using weakening and contraction rules.

Untyped linear lambda terms form a **BCI**-algebra. The underlying set consists of β-equivalence classes of closed linear lambda terms and the application is defined as that of the lambda calculus. Here $\lambda xyz.x(yz)$, $\lambda xyz.xzy$ and $\lambda x.x$ are the representatives of **B**, **C** and **I** respectively.

▶ **Proposition 11.** *(combinatory completeness of* **BCI***-algebras) Let $\mathcal{A}$ be a* **BCI***-algebra and M be a polynominal over $|\mathcal{A}|$ whose variables appear exactly once in M. For any variable x in M, there exists a polynominal $\lambda^* x.M$ such that the free variables of $\lambda^* x.M$ are the free variables of M excluding x and $(\lambda^* x.M)\mathsf{a} = M[\mathsf{a}/x]$ for all $\mathsf{a} \in |\mathcal{A}|$.*

Proof. We define $\lambda^* x.M$ by induction on the structure of M as follows:

- $\lambda^* x.x := \mathbf{I}$
- $\lambda^* x.MN := \begin{cases} \mathbf{C}(\lambda^* x.M)N & (x \in FV(M)) \\ \mathbf{B}M(\lambda^* x.N) & (x \in FV(N)) \end{cases}$ ◀

For the special case of the above proposition, any closed linear lambda term is β-equivalent to some term constructed from $\lambda xyz.x(yz)$, $\lambda xyz.xzy$ and $\lambda x.x$ only using applications.

Since **BCI**-algebras are related to the linear lambda calculus, categorical structures of assemblies on **BCI**-algebras are also linear.

▶ **Proposition 12.** *Let $\mathcal{A}$ be a* **BCI***-algebra. Then* $\mathbf{Asm}(\mathcal{A})$ *is a symmetric monoidal closed category (SMCC).*

3 $\mathbf{BI}(-)^\bullet$-algebras and closed multicategories

From here, we investigate realizability without symmetry. To obtain $\mathbf{Asm}(\mathcal{A})$ with non-symmetric categorical structures, $\mathcal{A}$ needs to be a structure in between **BI**-algebras and **BCI**-algebras. In this section, we introduce such a new class of applicative structures $\mathbf{BI}(-)^\bullet$-algebra, whose assemblies form closed multicategories.

▶ **Definition 13.** *Let $\mathcal{A}$ be an applicative structure. For x in $|\mathcal{A}|$, we write $\mathsf{x}^\bullet$ for an element of $|\mathcal{A}|$ (whenever it exists) such that $\mathsf{x}^\bullet\mathsf{a} = \mathsf{a}\mathsf{x}$ for all $\mathsf{a} \in |\mathcal{A}|$. We say that $\mathcal{A}$ is a* $\mathbf{BI}(-)^\bullet$-algebra *iff it contains* **B**, **I** *and $\mathsf{x}^\bullet$ for each $\mathsf{x} \in |\mathcal{A}|$.*

A PCA is related to the untyped lambda calculus, which has all term construction rules. A **BCI**-algebra is related to the untyped linear lambda calculus, which does not have weakening nor contraction. Similarly, A $\mathbf{BI}(-)^\bullet$-algebra is related to the untyped planar lambda calculus, which has none of weakening, contraction nor exchange rules.

▶ **Example 14.** *Untyped planar lambda terms* are untyped lambda terms constructed without using weakening, contraction or exchange rules[1]. Untyped closed planar lambda terms form a $\mathbf{BI}(-)^\bullet$-algebra. The underlying set consists of β-equivalence classes of closed planar lambda

[1] The definition of construction rules of planar lambda terms has two different styles. In our definition, the abstraction rule is only allowed for the rightmost variable. Such a term construction is seen in [1]. On the other hand, there is also a definition that the abstraction rule is only allowed for the leftmost variable, as in [20]. Here we choose the former style for preservation the planarity of terms under the $\beta\eta$-conversions.

terms and the application is defined as that of lambda terms. Here $\lambda xyz.x(yz)$ and $\lambda x.x$ are the representatives of B and I. Let M be a representative of x. Then $\lambda x.xM$ is also a closed planar term and is the representative of $\mathsf{x}^\bullet$. We write $\mathcal{L}_{planar}$ for this applicative structure. $\mathcal{L}_{planar}$ does not have a term satisfying the axiom of C. That is intuitively obvious, however, the rigorous proof is a little bit tricky and omitted.

▶ **Proposition 15.** *(combinatory completeness of $\mathsf{BI}(-)^\bullet$-algebras) Let $\mathcal{A}$ be a $\mathsf{BI}(-)^\bullet$-algebra and M be a polynominal over $|\mathcal{A}|$ whose variables appear exactly once in M. For the rightmost variable x of M, there exists $\lambda^* x.M$ such that the free variables of $\lambda^* x.M$ are the free variables of M in the same order excluding x and $(\lambda^* x.M)\mathsf{a} = M[\mathsf{a}/x]$ for all $\mathsf{a} \in |\mathcal{A}|$.*

Proof. We define $\lambda^* x.M$ by induction on the structure of M as follows:

- $\lambda^* x.x := \mathsf{I}$
- $\lambda^* x.MN := \begin{cases} \mathsf{B}N^\bullet(\lambda^* x.M) & (x \in FV(M)) \\ \mathsf{B}M(\lambda^* x.N) & (x \in FV(N)) \end{cases}$

Note that for $\lambda^* x.MN$, x is the rightmost free variable in MN. Therefore, if x is in $FV(M)$, N has no free variables and $N^\bullet$ can be defined. ◀

For the special case of the above proposition, any closed planar lambda term is β-equivalent to some term constructed from $\lambda xyz.x(yz)$ and $\lambda x.x$ using applications and the unary operation $(-)^\bullet : M \mapsto \lambda x.xM$.

Since $\mathsf{CI}\mathsf{x}$ satisfies the axiom of $\mathsf{x}^\bullet$, any BCI-algebra is also a $\mathsf{BI}(-)^\bullet$-algebra. $\mathsf{BI}(-)^\bullet$-algebras are weaker than BCI-algebra, and thus categories of assemblies on $\mathsf{BI}(-)^\bullet$-algebras have weaker categorical structures than those on BCI-algebras. We show that assemblies on a $\mathsf{BI}(-)^\bullet$-algebra form a closed multicategory, which is more general than symmetric monoidal closed categories.

Multicategories are extensions of categories, whose maps are allowed to have finitely many arguments. Closed multicategories are closed categorical structures for multicategories and correspond to planar multiplicative intuitionistic linear logic with only linear implication $\multimap$ and without tensor product nor unit. Here the word "planar" means the planarity of the string diagrams of the modeling categories. The string diagrams do not contain symmetries or braids.

First, we recall the definition of closed multicategories in [13].

▶ **Definition 16.** *A* multicategory $\mathbf{C}$ *consists of the following data:*

1. *a collection $Ob(\mathbf{C})$;*
2. *for each $n \geq 0$ and $X_1, X_2, \dots, X_n, Y \in Ob(\mathbf{C})$, a set $\mathbf{C}(X_1, \dots, X_n; Y)$;*
3. *for each $X \in Ob(\mathbf{C})$, an element $1_X \in \mathbf{C}(X; X)$, called the* identity map*;*
4. *for each $n, k_1, k_2, \dots, k_n \in \mathbb{N}$ and X_{ij}, Y_i, Z $(i \leq n, j \leq k_i)$, a function*

$$\circ : \prod_i^n \mathbf{C}(X_{i1}, \dots, X_{ik_i}; Y_i) \times \mathbf{C}(Y_1, \dots, Y_n; Z) \to \mathbf{C}(X_{11}, \dots, X_{1k_1}, X_{21}, \dots, X_{nk_n}; Z)$$

called the composition*. $g \circ (f_1, \dots, f_n)$ denotes the composition of $g \in \mathbf{C}(Y_1, \dots, Y_n; Z)$ and $f_i \in \mathbf{C}(X_{i1}, \dots, X_{ik_i}; Y_i)$. The compositions satisfy associativity and identity axioms.*

▶ **Definition 17.** *A* closed multicategory *consists of the following data:*

1. *a multicategory $\mathbf{C}$;*
2. *for each $X_1, X_2, \dots, X_n, Y \in Ob(\mathbf{C})$, an object $\underline{\mathbf{C}}(X_1, X_2, \dots, X_n; Y)$, called the* internal hom object*;*

3. *for each* $X_1, \dots, X_n, Y \in Ob(\mathbf{C})$*, a map*

$$ev_{X_1,\dots,X_n;Y} : X_1, \dots, X_n, \underline{\mathbf{C}}(X_1, \dots, X_n; Y) \to Y,$$

called the evaluation map *such that* $\forall Z_1, Z_2, \dots, Z_m \in Ob(\mathbf{C})$*, the function* $\varphi_{X_1,\dots,X_n;Z_1,\dots,Z_m;Y} : \mathbf{C}(Z_1, \dots, Z_m; \underline{\mathbf{C}}(X_1, \dots, X_n; Y)) \to \mathbf{C}(X_1, \dots, X_n, Z_1, \dots, Z_m; Y)$ *sending* f *to* $ev_{X_1,\dots,X_n;Y} \circ (1_{X_1}, \dots, 1_{X_n}, f)$ *is invertible. We write the inverse function* $\Lambda_{X_1,\dots,X_n;Z_1,\dots,Z_m;Y}$.

▶ **Proposition 18.** *Let* $\mathcal{A}$ *be a* $\mathsf{BI}(-)^\bullet$*-algebra. Then* $\mathbf{Asm}(\mathcal{A})$ *is a closed multicategory.*

Proof. Let $\mathbf{C} := \mathbf{Asm}(\mathcal{A})$. It is obvious that $\mathbf{C}$ is a category. We define a bifunctor $[-,-] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$ as follows:

- $[-,-]$ sends $(|X|, \|\text{-}\|_X)$ and $(|Y|, \|\text{-}\|_Y)$ to $(|[X,Y]|, \|\text{-}\|_{[X,Y]})$, where $|[X,Y]|$ is the set of maps from $(|X|, \|\text{-}\|_X)$ to $(|Y|, \|\text{-}\|_Y)$ in $\mathbf{C}$ and $\|f\|_{[X,Y]} := \{\mathsf{r} \mid \mathsf{r} \text{ realizes } f\}$.
- $[-,-]$ sends $f : X' \to X$ and $g : Y \to Y'$ in $\mathbf{C}$ to the function $[f,g] : [X,Y] \to [X',Y']$ which sends $h : X \to Y$ to $g \circ h \circ f$.

We check that $[-,-]$ certainly forms a functor. Let r_f and r_g be the realizers for f and g, then $\mathsf{B}(\mathsf{B}\mathsf{r}_f{}^\bullet\mathsf{B})(\mathsf{B}\mathsf{r}_g)$ is the realizer for $[f,g]$. Therefore, for any $f : X' \to X$ and $g : Y \to Y'$ in $\mathbf{C}$, $[f,g]$ exists in $\mathbf{C}$. It is easy to see that $[-,-]$ preserves identities and compositions.

Next, we show that $\mathbf{C}$ is a multicategory. For $X_1, \dots, X_n, Y \in \mathbf{C}$ $(n > 0)$, the set $\mathbf{C}(X_1, \dots, X_n; Y)$ is defined as the underlying set of the object[2] $[X_n, [X_{n-1}, [\dots[X_1, Y]\dots]]]$. In the case $n = 0$, $\mathbf{C}(;Y)$ is defined as the underlying set $|Y|$. Identity maps and compositions are usual ones in $\mathbf{Sets}$.

To check that composition maps have realizers, we use Proposition 15. Given $g \in \mathbf{C}(Y_1, \dots, Y_m; Z)$ and $f_l \in \mathbf{C}(X^l_1, \dots, X^l_{k_l}; Y_l)$ $(1 \le l \le m)$, whose realizers are $\mathsf{u}, \mathsf{v}_1, \dots, \mathsf{v}_m$ respectively. Then the composition map $g \circ (f_1, \dots, f_m)$ is realized by r such that for any $\mathsf{a}^1_1, \mathsf{a}^1_2, \dots, \mathsf{a}^1_{k_1}, \mathsf{a}^2_1, \dots, \mathsf{a}^m_{k_m}$ in $|\mathcal{A}|$,

$$\mathsf{r}\mathsf{a}^m_{k_m} \dots \mathsf{a}^m_1 \dots \mathsf{a}^1_{k_1} \dots \mathsf{a}^1_1 = \mathsf{u}(\mathsf{v}_m \mathsf{a}^m_{k_m} \dots \mathsf{a}^m_1) \dots (\mathsf{v}_1 \mathsf{a}^1_{k_1} \dots \mathsf{a}^1_1).$$

The existence of such r follows by the combinatory completeness of $\mathsf{BI}(-)^\bullet$-algebras.

Finally, we show that $\mathbf{C}$ is a closed multicategory. We take internal hom objects $\underline{\mathbf{C}}(X_1, \dots, X_n; Y)$ as $[X_n, [X_{n-1}, [\dots[X_1, Y]\dots]]]$ and evaluation maps as the obvious ones, where the evaluation maps are realized by I. φ is invertible as a function and for a map $g : X_1, \dots, X_n, Z_1, \dots, Z_m \to Y$, $\Lambda(g)$ is indeed realized by a realizer of g. ◀

The converse of Proposition 18 holds under some natural conditions.

▶ **Proposition 19.** *Suppose* $\mathcal{A}$ *is an applicative structure and* $\mathbf{C} := \mathbf{Asm}(\mathcal{A})$ *happens to be a closed multicategory of assemblies.* $\mathcal{A}$ *is a* $\mathsf{BI}(-)^\bullet$*-algebra if the following conditions hold:*

(i) $\underline{\mathbf{C}}(;X) = X$ *and* $\mathbf{C}(;X)$ *is the underlying set* $|X|$*;*

(ii) $f \in \mathbf{C}(X;Y)$ *iff* f *is a function from* $|X|$ *to* $|Y|$ *realized by some element of* $|\mathcal{A}|$*;*

(iii) *identity maps are obvious ones;*

(iv) $\underline{\mathbf{C}}(X;Y) = (\mathbf{C}(X;Y), \|\text{-}\|)$ *where* $\|f\| = \{\mathsf{r} \mid \mathsf{r} \text{ realizes } f\}$*;*

[2] Here we reverse the order of arguments due to the difference between closed multicategories and applicative structures. Internal hom objects of a closed multicategory receive arguments from the left side, whereas in an applicative structure, elements receive arguments from the right side. If we employ another definition of closed multicategories with reversing the order of arguments of the compositions and evaluation maps, then it will be suitable for the order of realizers.

(v) $\underline{\mathbf{C}}(X_1,\ldots,X_{n+1};Y) = \underline{\mathbf{C}}(X_{n+1};\underline{\mathbf{C}}(X_1,\ldots,X_n;Y))$ *and* $\mathbf{C}(X_1,\ldots,X_{n+1};Y)$ *is the underlying set of* $\underline{\mathbf{C}}(X_1,\ldots,X_{n+1};Y)$*;*
(vi) *for* $g : Y_1,\ldots,Y_n \to Z$ *and* $f_l : X^l_1,\ldots,X^l_{k_l} \to Y_l$*,* $g \circ (f_1,\ldots,f_n)$ *sends* $x^1_1,\ldots,x^1_{k_1},\ldots,x^n_{k_n}$ *to* $g(f_n(x^n_{k_n},\ldots,x^n_1),\ldots,f_1(x^1_{k_1},\ldots,x^1_1))$*;*
(vii) $ev_{X_1,\ldots,X_n;Y}$ *sends* $x_1,\ldots,x_n,f$ *to* $f(x_n,\ldots,x_1)$*;*
(viii) $\Lambda_{X_1,\ldots,X_n;Z_1,\ldots,Z_m;Y}$ *sends a function* $(x_1,\ldots,x_n,z_1,\ldots,z_m \mapsto f(z_m,\ldots,z_1,x_n,\ldots,x_1))$ *to a function* $(z_1,\ldots,z_m \mapsto f(z_m,\ldots,z_1,-,\ldots,-))$*.*

Proof. Let $X := (|\mathcal{A}|, \|\text{-}\|_X)$, where $\|\mathsf{a}\|_X := \{\mathsf{a}\}$. Suppose $\mathbf{I}_0$ is a realizer of 1_X. Then $\mathbf{I}_0\mathsf{a} \in \{\mathsf{a}\}$ for any $\mathsf{a} \in |\mathcal{A}|$. Therefore $\mathbf{I}_0$ has the property of $\mathbf{I}$.

Let $Y := (|\mathcal{A}| \times |\mathcal{A}|, \|\text{-}\|_Y)$, where $\|(\mathsf{a},\mathsf{a}')\|_Y := \{\mathsf{aa}'\}$. Given arbitrary two element $\mathsf{r},\mathsf{r}' \in |\mathcal{A}|$, define $f : X \to Y$ as $\mathsf{a} \mapsto (\mathsf{r},\mathsf{a})$ and $g : Y \to Y$ as $(\mathsf{a},\mathsf{a}') \mapsto (\mathsf{r}',\mathsf{aa}')$. Let $L^X_{Y,Y} : \underline{\mathbf{C}}(Y;Y) \to \underline{\mathbf{C}}(\underline{\mathbf{C}}(X;Y);\underline{\mathbf{C}}(X;Y))$ be the map

$$\Lambda_{\underline{\mathbf{C}}(X;Y);\underline{\mathbf{C}}(Y;Y);\underline{\mathbf{C}}(X;Y)}(\Lambda_{X;\underline{\mathbf{C}}(X;Y),\underline{\mathbf{C}}(Y;Y);Y}(ev_{Y;Y} \circ (ev_{X;Y}, 1_{\underline{\mathbf{C}}(Y;Y)}))).$$

$L^X_{Y,Y}$ sends g to $(f \mapsto g \circ f)$. Suppose $\mathbf{B}_0$ is a realizer of $L^X_{Y,Y}$. Then $\mathbf{B}_0\mathsf{r}'\mathsf{r}$ realizes $g \circ f$ and thus $\mathbf{B}_0\mathsf{r}'\mathsf{ra} \in \|g(f(\mathsf{a}))\|_Y = \{\mathsf{r}'(\mathsf{ra})\}$ for any $\mathsf{a} \in |\mathcal{A}|$. Therefore $\mathbf{B}_0$ has the property of $\mathbf{B}$.

Given arbitrary $\mathsf{x} \in |\mathcal{A}|$, define $Ev_{\mathsf{x}} : \underline{\mathbf{C}}(X;X) \to X$ as $ev_{X;X} \circ (\mathsf{x}, 1_{[X,X]})$. For any $\mathsf{a} \in |\mathcal{A}|$, Ev_{x} sends $f_{\mathsf{a}} : X \to X$, $\mathsf{a}' \mapsto \mathsf{aa}'$ to $f_{\mathsf{a}}(\mathsf{x})$. Suppose $(\mathsf{x}^\bullet)_0$ is a realizer of Ev_{x}. Then $(\mathsf{x}^\bullet)_0\mathsf{a} \in \{\mathsf{ax}\}$. Therefore $(\mathsf{x}^\bullet)_0$ has the property of $\mathsf{x}^\bullet$. ◀

4 $\mathbf{BII}^\times(-)^\bullet$-algebras and closed categories

In the previous section, we saw that $\mathbf{BI}(-)^\bullet$-algebras correspond to closed multicategories. In this section, we show that a slightly stronger class of applicative structures ($\mathbf{BII}^\times(-)^\bullet$-algebras) corresponds to a slightly stronger categorical structure (closed categories).

Closed categories are something like monoidal closed categories without tensor products, which have internal hom objects defined without using tensor products. It is shown in [13] that closed categories are slightly stronger categorical structures than closed multicategories.

First, we recall the definition of closed categories in [13].

▶ **Definition 20.** *A* closed category *consists of the following data:*
1. *a locally small category* $\mathbf{C}$*;*
2. *a functor* $[-,-] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$*, called the* internal hom functor*;*
3. *an object* I*, called the* unit object*;*
4. *a natural isomorphism* $i_X : [I,X] \to X$*;*
5. *an extranatural transformation* $j_X : I \to [X,X]$*;*
6. *a transformation* $L^X_{Y,Z} : [Y,Z] \to [[X,Y],[X,Z]]$ *natural in* Y,Z *and extranatural in* X*,*

such that the following axioms hold:
- **(i)** $\forall X,Y \in \mathbf{C}$*,* $L^X_{Y,Y} \circ j_Y = j_{[X,Y]}$*;*
- **(ii)** $\forall X,Y \in \mathbf{C}$*,* $i_{[X,Y]} \circ [j_X, 1_{[X,Y]}] \circ L^X_{X,Y} = 1_{[X,Y]}$*;*
- **(iii)** $\forall X,Y,Z,W \in \mathbf{C}$*, the following diagram commutes:*

$$\begin{tikzcd}
{[Z,W]} \arrow[rr, "L^Y_{Z,W}"] \arrow[d, "L^X_{Z,W}"'] & & {[[Y,Z],[Y,W]]} \arrow[dd, "{[1_{[Y,Z]},L^X_{Y,W}]}"] \\
{[[X,Z],[X,W]]} \arrow[d, "L^{[X,Y]}_{[X,Z],[X,W]}"'] & & \\
{[[[X,Y],[X,Z]],[[X,Y],[X,W]]]} \arrow[rr, "{[L^X_{Y,Z},1_{[[X,Y],[X,W]]}]}"'] & & {[[Y,Z],[[X,Y],[X,W]]]}
\end{tikzcd}$$

(iv) $\forall X, Y \in \mathbf{C}$, $L^I_{X,Y} \circ [1_X, i_Y] = [i_X, 1_{[I,Y]}]$;
(v) $\forall X, Y \in \mathbf{C}$, *the function* $\gamma : \mathbf{C}(X, Y) \to \mathbf{C}(I, [X, Y])$ *which sends* $f : X \to Y$ *to* $[1_X, f] \circ j_X$ *is invertible.*

When $\mathcal{A}$ is a $\mathsf{BI}(-)^\bullet$-algebra, the closed multicategory structure of $\mathbf{Asm}(\mathcal{A})$ does not generally extend to a closed category since the natural isomorphism i_X^{-1} is not generally realized. Next, we give the definition of $\mathsf{BII}^\times(-)^\bullet$-algebras, which we assume an extra element $\mathsf{I}^\times$ for the realizer of i_X^{-1}.

▶ **Definition 21.** *Let* $\mathcal{A}$ *be a* $\mathsf{BI}(-)^\bullet$*-algebra.* $\mathsf{I}^\times$ *is defined as an element of* $|\mathcal{A}|$ *(whenever it exists) satisfying* $\mathsf{I}^\times \mathsf{a} \mathsf{I} = \mathsf{a}$ *for all* $\mathsf{a} \in |\mathcal{A}|$*. If* $\mathcal{A}$ *has* $\mathsf{I}^\times$*, we say that* $\mathcal{A}$ *is a* $\mathsf{BII}^\times(-)^\bullet$-algebra.

▶ **Example 22.** Any BCI-algebra is a $\mathsf{BII}^\times(-)^\bullet$-algebra, since CI satisfies the axiom of $\mathsf{I}^\times$.

▶ **Example 23.** $\mathcal{L}_{planar}$ is a $\mathsf{BII}^\times(-)^\bullet$-algebra. For any closed planar term M, M has a β-normal form since it is a linear lambda term. Let $\lambda u.M'$ be the β-normal form of M. Then $(\lambda xyz.x(yz))M(\lambda v.v) =_\beta \lambda z.Mz =_\beta \lambda z.M'[z/u] =_\alpha \lambda u.M' =_\beta M$. Therefore in this case, B satisfies the axiom of $\mathsf{I}^\times$.

▶ Remark. There exists a $\mathsf{BI}(-)^\bullet$-algebra which is not a $\mathsf{BII}^\times(-)^\bullet$-algebra. Add the constant rule:

$$\frac{}{\vdash c}\ (\text{constant})$$

to the construction of planar lambda terms and add no evaluation rules on these constants. We write $\mathcal{L}^c_{planar}$ for the applicative structure which consists of closed planar lambda terms with constants. $\mathcal{L}^c_{planar}$ is a $\mathsf{BI}(-)^\bullet$-algebra and does not contains $\mathsf{I}^\times$.

Note that if we further assume the extensionality (η-equality) on $\mathcal{L}^c_{planar}$, then $\lambda xyz.x(yz)$ satisfies the axiom of $\mathsf{I}^\times$ and the applicative structure forms a $\mathsf{BII}^\times(-)^\bullet$-algebra.

▶ **Proposition 24.** *Let* $\mathcal{A}$ *be a* $\mathsf{BII}^\times(-)^\bullet$*-algebra. Then* $\mathbf{Asm}(\mathcal{A})$ *is a closed category.*

Proof. Since a $\mathsf{BII}^\times(-)^\bullet$-algebra is also a $\mathsf{BI}(-)^\bullet$-algebra, a bifunctor $[-,-]$ can be defined in the same way as in the proof of Proposition 18.

We define the unit object I as $(\{*\}, \|\text{-}\|_I)$, where $\|*\|_I := \{\mathsf{I}\}$. j_X is defined as the function $* \mapsto 1_X$, which is realized by I. i_X is defined as the function sending $(f : * \mapsto x)$ to x, which is realized by $\mathsf{I}^\bullet$. The inverse function of i_X is realized by $\mathsf{I}^\times$. $L^X_{Y,Z}$ is defined as $g \mapsto (f \mapsto g \circ f)$, which is realized by B. It is easy to verify that i,j and L have naturality and satisfy the axioms of closed category.

Finally, we show that γ is invertible. Let $g \in \mathbf{C}(I, [X, Y])$ then $g(*)$ is $\gamma^{-1}(g)$, which is realized by $\mathsf{r}_g\mathsf{I}$, where r_g is a realizer of g. ◀

Like Proposition 19, the converse of the above proposition holds under some conditions.

▶ **Proposition 25.** *Suppose* $\mathcal{A}$ *is an applicative structure and* $\mathbf{C} := \mathbf{Asm}(\mathcal{A})$ *happens to be a closed category. If the following conditions hold, then* $\mathcal{A}$ *is a* $\mathsf{BII}^\times(-)^\bullet$*-algebra.*
(i) $[X, Y] = (\mathbf{C}(X, Y), \|\text{-}\|)$ *where* $\|f\| = \{\mathsf{r} \mid \mathsf{r}\ \text{realizes}\ f\}$;
(ii) $[f, g] : [X, Y] \to [X', Y']$ *is a function which sends* $h : X \to Y$ *to* $g \circ h \circ f$;
(iii) $L^X_{Y,Z}$ *sends* $g : Y \to Z$ *to the function* $(f : X \to Y) \mapsto (g \circ f : X \to Z)$;
(iv) *the underlying set of the unit object* I *is the singleton* $\{*\}$;
(v) i_X *sends a function* $(f : * \mapsto x)$ *to* x.

Proof. Let $X := (|\mathcal{A}|, \|\text{-}\|_X)$, where $\|\mathsf{a}\|_X := \{\mathsf{a}\}$. Suppose $\mathbf{I}_0$ is the realizer of 1_X. Then $\mathbf{I}_0\mathsf{a} \in \{\mathsf{a}\}$ for any $\mathsf{a} \in |\mathcal{A}|$. Therefore $\mathbf{I}_0$ has the property of $\mathbf{I}$.

Let $Y := (|\mathcal{A}| \times |\mathcal{A}|, \|\text{-}\|_Y)$, where $\|(\mathsf{a}, \mathsf{a}')\|_Y := \{\mathsf{a}\mathsf{a}'\}$. Given arbitrary two element $\mathsf{r}, \mathsf{r}' \in |\mathcal{A}|$, define $f : X \to Y$ as $\mathsf{a} \mapsto (\mathsf{r}, \mathsf{a})$ and $g : Y \to Y$ as $(\mathsf{a}, \mathsf{a}') \mapsto (\mathsf{r}', \mathsf{a}\mathsf{a}')$. $L^X_{Y,Y}$ sends g to $(f \mapsto g \circ f)$. Suppose $\mathbf{B}_0$ is the realizer of $L^X_{Y,Y}$. Then $\mathbf{B}_0\mathsf{r}'\mathsf{r}$ realizes $g \circ f$ and thus $\mathbf{B}_0\mathsf{r}'\mathsf{r}\mathsf{a} \in \|g(f(\mathsf{a}))\|_Y = \{\mathsf{r}'(\mathsf{r}\mathsf{a})\}$ for any $\mathsf{a} \in |\mathcal{A}|$. Therefore $\mathbf{B}_0$ has the property of $\mathbf{B}$.

Since $I \cong [I, I]$ and $\mathbf{I} \in \|1_I\|_{[I,I]}$, we can assume $\mathbf{I} \in \|*\|_I$ with loss of generality. Suppose $\mathbf{I}^\times_0$ is the realizer of i^{-1}_X. Then $\mathbf{I}^\times_0\mathsf{a}$ realizes the map $* \mapsto \mathsf{a}$ for any $\mathsf{a} \in |\mathcal{A}|$. Thus $(\mathbf{I}^\times_0\mathsf{a})\mathbf{I} \in \{\mathsf{a}\}$ and $\mathbf{I}^\times_0$ has the property of $\mathbf{I}^\times$.

Given arbitrary $\mathsf{x} \in |\mathcal{A}|$, define $f : I \to X$ as $* \mapsto \mathsf{x}$. For any $\mathsf{a} \in |\mathcal{A}|$, $i_X \circ [f, 1_X]$ sends $f_\mathsf{a} : X \to X$, $\mathsf{a}' \mapsto \mathsf{a}\mathsf{a}'$ to $f_\mathsf{a}(\mathsf{x})$. Suppose $(\mathsf{x}^\bullet)_0$ is the realizer of $i_X \circ [f, 1_X]$. Then $(\mathsf{x}^\bullet)_0\mathsf{a} \in \{\mathsf{a}\mathsf{x}\}$. Therefore $(\mathsf{x}^\bullet)_0$ has the property of $\mathsf{x}^\bullet$. ◀

5 Other cases

In this section, we introduce three classes of applicative structures which are sufficient for inducing some categorical structures on assemblies. Unlike $\mathbf{BI}(-)^\bullet$-algebras for closed multicategories and $\mathbf{BII}^\times(-)^\bullet$-algebras for closed categories, the classes in this section do not provide necessary conditions for inducing such structures.

5.1 $\mathbf{BK}(-)^\bullet$-algebras and closed categories

▶ **Definition 26.** *A* $\mathbf{BK}(-)^\bullet$-algebra *is an applicative structure* $\mathcal{A}$ *which contains* $\mathbf{B}$, $\mathbf{K}$ *and* $\mathsf{x}^\bullet$ *for each* $\mathsf{x} \in |\mathcal{A}|$.

▶ **Example 27.** Consider untyped lambda terms constructed without using contraction or exchange rules. Then β-equivalence classes of these closed terms form a $\mathbf{BK}(-)^\bullet$-algebra.

▶ **Proposition 28.** *(combinatory completeness of* $\mathbf{BK}(-)^\bullet$*-algebras) Let* $\mathcal{A}$ *be a* $\mathbf{BK}(-)^\bullet$*-algebra and* M *be a polynominal over* $|\mathcal{A}|$ *whose variables appear at most once in* M*. For a variable* x *which is the rightmost variable of* M *or not in* M*, there exists a polynominal* $\lambda^* x.M$ *such that the free variables of* $\lambda^* x.M$ *are the free variables of* M *excluding* x *and* $(\lambda^* x.M)\mathsf{a} = M[\mathsf{a}/x]$ *for all* $\mathsf{a} \in |\mathcal{A}|$.

Proof. We define $\lambda x.M$ by induction on the structure of M as follows:

- $\lambda^* x.x := \mathbf{B}\mathbf{B}^\bullet\mathbf{K}$
- $\lambda^* x.y := \mathbf{K}y \quad (x \neq y)$
- $\lambda^* x.MN := \begin{cases} \mathbf{B}N^\bullet(\lambda^* x.M) & (x \in FV(M)) \\ \mathbf{B}M(\lambda^* x.N) & (x \in FV(N)) \\ \mathbf{K}(MN) & (otherwise) \end{cases}$

◀

Since $\mathbf{B}\mathbf{B}^\bullet\mathbf{K}$ satisfies the axiom of $\mathbf{I}$ and $\mathbf{K}$ satisfies the axiom of $\mathbf{I}^\times$, any $\mathbf{BK}(-)^\bullet$-algebra is also a $\mathbf{BII}^\times(-)^\bullet$-algebra. Therefore the next corollary follows by Proposition 24.

▶ **Corollary 29.** *Let* $\mathcal{A}$ *be a* $\mathbf{BK}(-)^\bullet$*-algebra. Then* $\mathbf{Asm}(\mathcal{A})$ *is a closed category.*

5.2 $\mathsf{BII}^\bullet(-)^\circ$-algebras and skew closed categories

▶ **Definition 30.** *Let $\mathcal{A}$ be an applicative structure. For* x *in* $|\mathcal{A}|$*, we write* x° *as an element of* $|\mathcal{A}|$ *(whenever it exists) such that* $\mathsf{x}^\circ\mathsf{a}\mathsf{a}' = \mathsf{a}(\mathsf{x}\mathsf{a}')$ *for all* $\mathsf{a}, \mathsf{a}' \in |\mathcal{A}|$*. We say that $\mathcal{A}$ is a* $\mathsf{BII}^\bullet(-)^\circ$-algebra *iff it contains* B, I, $\mathsf{I}^\bullet$ *and* x° *for each* $\mathsf{x} \in |\mathcal{A}|$.

Since $\mathsf{B}\mathsf{x}^\bullet\mathsf{B}$ satisfies the axiom of x°, any $\mathsf{BI}(-)^\bullet$-algebra is also a $\mathsf{BII}^\bullet(-)^\circ$-algebra. Assemblies on $\mathsf{BII}^\bullet(-)^\circ$-algebras form skew closed categories, which are weaker closed categorical structure than closed categories.

There is a categorical structure called skew monoidal categories [18], which have the same components as monoidal categories but the invertibility of unitors and associators are not assumed. Skew closed categories are to skew monoidal categories what closed categories are to monoidal categories.

First we recall the definition of skew closed categories in [17].

▶ **Definition 31.** *A* (left) skew closed category $\mathbf{C}$ *consists of the following data:*

1. *a locally small category* $\mathbf{C}$*;*
2. *a functor* $[-,-] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$*, called the* internal hom functor*;*
3. *an object I, called the* unit object*;*
4. *a natural transformation* $i_X : [I, X] \to X$*;*
5. *an extranatural transformation* $j_X : I \to [X, X]$*;*
6. *a transformation* $L^X_{Y,Z} : [Y, Z] \to [[X, Y], [X, Z]]$ *natural in Y, Z and extranatural in X,*

such that the following axioms hold:

(i) $\forall X, Y \in \mathbf{C}$, $L^X_{Y,Y} \circ j_Y = j_{[X,Y]}$*;*
(ii) $\forall X, Y \in \mathbf{C}$, $i_{[X,Y]} \circ [j_X, 1_{[X,Y]}] \circ L^X_{X,Y} = 1_{[X,Y]}$*;*
(iii) $\forall X, Y, Z, W \in \mathbf{C}$*, the following diagram commutes:*

$$\begin{tikzcd}
{[Z,W]} \arrow[rr, "L^Y_{Z,W}"] \arrow[d, "L^X_{Z,W}"'] & & {[[Y,Z],[Y,W]]} \arrow[dd, "{[1_{[Y,Z]}, L^X_{Y,W}]}"] \\
{[[X,Z],[X,W]]} \arrow[d, "L^{[X,Y]}_{[X,Z],[X,W]}"'] & & \\
{[[[X,Y],[X,Z]],[[X,Y],[X,W]]]} \arrow[rr, "{[L^X_{Y,Z}, 1_{[[X,Y],[X,W]]}]}"'] & & {[[Y,Z],[[X,Y],[X,W]]]}
\end{tikzcd}$$

(iv) $\forall X, Y \in \mathbf{C}$, $[1_{[I,X]}, i_Y] \circ L^I_{X,Y} = [i_X, 1_Y]$*;*
(v) $i_I \circ j_I = 1_I$*.*

A left skew closed category is called left normal *when the function* $\gamma : \mathbf{C}(X, Y) \to \mathbf{C}(I, [X, Y])$*,* $f \mapsto [1, f] \circ j_X$ *is invertible for any* $X, Y \in \mathbf{C}$.

▶ **Proposition 32.** *Let $\mathcal{A}$ be a* $\mathsf{BII}^\bullet(-)^\circ$*-algebra. Then* $\mathbf{Asm}(\mathcal{A})$ *is a left normal skew closed category.*

Proof. We define the functor $[-,-]$ as in the proof of Proposition 18, where $[f, g]$ is realized by $\mathsf{B}\mathsf{r}_f{}^\circ(\mathsf{B}\mathsf{r}_g)$. The rest of the proof is the same as Proposition 24 except for the existence of i_X^{-1}. ◀

Since any $\mathsf{BI}(-)^\bullet$-algebra is also a $\mathsf{BII}^\bullet(-)^\circ$-algebra, the next corollary follows.

▶ **Corollary 33.** *Let $\mathcal{A}$ be a* $\mathsf{BI}(-)^\bullet$*-algebra. Then* $\mathbf{Asm}(\mathcal{A})$ *is a left normal skew closed category.*

5.3 $\mathbf{BB'II^\bullet}$-algebras and skew closed categories

Unlike PCAs and $\mathbf{BCI}$-algebras, $\mathbf{BI}(-)^\bullet$-algebras, $\mathbf{BK}(-)^\bullet$-algebras and $\mathbf{BII^\bullet}(-)^\circ$-algebras need infinitely many assumptions due to $(-)^\bullet$ or $(-)^\circ$. In this subsection, we introduce a class of applicative structure $\mathbf{BB'II^\bullet}$-algebras, which induces skew closed categories and needs only four assumptions.

▶ **Definition 34.** *A* $\mathbf{BB'II^\bullet}$-algebra *is an applicative structure $\mathcal{A}$ which contains four elements* $\mathbf{B}$, $\mathbf{B}'$, $\mathbf{I}$ *and* $\mathbf{I}^\bullet$, *where* $\mathbf{B}'$ *is a element such that* $\mathbf{B}'\mathsf{xyz} = \mathsf{y}(\mathsf{xz})$ *for all* $\mathsf{x}, \mathsf{y}, \mathsf{z} \in |\mathcal{A}|$.

Since $\mathbf{B}'\mathsf{x}$ satisfies the axiom of x°, any $\mathbf{BB'II^\bullet}$-algebra is also a $\mathbf{BII^\bullet}(-)^\circ$-algebra. Therefore the next corollary follows by Proposition 32.

▶ **Corollary 35.** *Let $\mathcal{A}$ be a $\mathbf{BB'II^\bullet}$-algebra. Then $\mathbf{Asm}(\mathcal{A})$ is a left normal skew closed category.*

▶ Remark. A category of assemblies on a $\mathbf{BB'II^\bullet}$-algebra is not a closed category in general because i_X^{-1} is not realized. If a $\mathbf{BB'II^\bullet}$-algebra has $\mathbf{I}^\times$, then $\mathbf{B}(\mathbf{B}'(\mathbf{B}(\mathbf{BI}^\bullet)(\mathbf{BB}'\mathbf{I}^\times)))\mathbf{B}'$ satisfies the axiom of $\mathbf{C}$, and thus this $\mathbf{BB'II^\bullet}$-algebra becomes a $\mathbf{BCI}$-algebra.

6 Examples

In this section, we introduce three examples of $\mathbf{BI}(-)^\bullet$-algebras.

6.1 Propositions derivable in the planar logic

In this subsection, we construct $\mathcal{F}$ as a $\mathbf{BI}(-)^\bullet$-algebra.

We define the *planar logic* as a sequent calculus whose formulas are constructed from propositional variables and an implication symbol $\multimap$, and whose derivation rules are the following ones:

$$\frac{}{A \vdash_p A}\ (\text{identity}) \quad \text{where } A \text{ is a formula;} \quad \frac{A, \Gamma \vdash_p B}{\Gamma \vdash_p A \multimap B}\ (\multimap\text{-introduction})\ ;$$

$$\frac{\Gamma \vdash_p A \qquad \Delta \vdash_p A \multimap B}{\Gamma, \Delta \vdash_p B}\ (\multimap\text{-elimination})\ ,$$

where Γ and Δ are sequences of distinct formulas.

Let $\mathcal{F}$ be the powerset of $\{A \mid \vdash_p A \text{ is derivable in the planar logic}\}$. Then $\mathcal{F}$ gives rise to a $\mathbf{BI}(-)^\bullet$-algebra. Indeed, we can define the applicative structure on $\mathcal{F}$ as follows:

- For $M, N \in \mathcal{F}$, the application $MN := \{A_1 \mid \exists A_2 \in N, (A_2 \multimap A_1) \in M\}$.
- $\mathbf{B} := \{(A_1 \multimap A_2) \multimap ((A_3 \multimap A_1) \multimap (A_3 \multimap A_2)) \mid A_1, A_2, A_3 \text{ are formulas}\}$.
- $\mathbf{I} := \{A_1 \multimap A_1 \mid A_1 \text{ is a formula}\}$.
- For $M \in \mathcal{F}$, $M^\bullet := \{(A_1 \multimap A_2) \multimap A_2 \mid A_2 \text{ is a formula and } A_1 \in M\}$.

6.2 Binary trees from ordered groups

In this subsection, we construct $\mathcal{T}$ as a $\mathbf{BII}^\times(-)^\bullet$-algebra.

Take an ordered group $(G, \cdot, e, \leq)$. Let T be a set of binary trees whose leaves are labeled by elements of G:

$$t ::= g \mid t \multimap t \quad (g \in G).$$

We define a function $|\text{-}| : T \to G$ by induction: $|g| := g$ and $|t_1 \multimap t_2| := |t_1|^{-1} \cdot |t_2|$.

Let $\mathcal{T}$ be the powerset of $\{t \in T \mid e \leq |t|\}$. Then $\mathcal{T}$ gives rise to a $\mathbf{BII}^\times(-)^\bullet$-algebra. Indeed, we can define the applicative structure on $\mathcal{T}$ as follows:

- For $M, N \in \mathcal{T}$, $MN := \{t_1 \mid \exists t_2 \in N, (t_2 \multimap t_1) \in M\}$.
- $\mathbf{B} := \{(t_1 \multimap t_2) \multimap ((t_3 \multimap t_1) \multimap (t_3 \multimap t_2)) \mid t_1, t_2, t_3 \in T\}$.
- $\mathbf{I} := \{t_1 \multimap t_1 \mid t_1 \in T\}$.
- $\mathbf{I}^{\times} := \{t_1 \multimap ((t_2 \multimap t_2) \multimap t_1) \mid t_1, t_2 \in T\}$
- For $M \in \mathcal{T}$, $M^{\bullet} := \{(t_1 \multimap t_2) \multimap t_2 \mid t_2 \in T, t_1 \in M\}$.

▶ Remark. Whether $\mathcal{T}$ includes $\mathbf{C}$ or not depends on G. For instance, when G is Abelian, $\mathcal{T}$ has $\mathbf{C}$ as $\{(t_1 \multimap (t_2 \multimap t_3)) \multimap (t_2 \multimap (t_1 \multimap t_3)) \mid t_1, t_2, t_3 \in T\}$.

The example in this subsection is based on $Comod(\overline{G})$ introduced in [6]. $Comod(\overline{G})$ is a category constructed from a group G, whose objects are sets equipped with G valued functions and whose maps are relations between the objects compatible to the valuations. For any (not necessarily ordered) group G, $Comod(\overline{G})$ is a pivotal category. $\mathcal{T}$ is a set of maps from the unit object to a reflexive object of $Comod(\overline{G})$ with alteration for order structures.

6.3 Computational lambda calculus and its models

In this subsection, we show the *untyped computational lambda calculus* [14] is a $\mathbf{BII}^{\times}(-)^{\bullet}$-algebra. The following axiomatization is from [15].

Suppose infinite supply of variables $x, y, z, \ldots$. The values, terms and evaluation contexts are defined as follows:

- $V \in \text{Values} ::= x \mid (\lambda x.M)$;
- $M \in \text{Terms} ::= V \mid M(M')$;
- $E \in \text{EvalCtx} ::= [-] \mid EM \mid VE$.

An equivalence relation $=_R$ on Terms is defined as the congruence of the following equations:

1. $\beta_v : \quad (\lambda x.M)V = M[V/x]$;
2. $\eta_v : \quad \lambda x.Vx = V \quad (x \notin FV(V))$;
3. $\beta_\Omega : \quad (\lambda x.E[x])M = E[M] \quad (x \notin FV(E))$,
   where $E[N]$ denotes the term obtained by substituting N for $[-]$ in E.

Then, (Terms$/\!=_R$) forms a $\mathbf{BII}^{\times}(-)^{\bullet}$-algebra. Here the application is that of the computational lambda calculus. $\lambda xyz.x(yz)$, $\lambda x.x$, $\lambda xy.yx$ and $\lambda x.xM$ are representatives of $\mathbf{B}$, $\mathbf{I}$, $\mathbf{I}^{\times}$ and $M^{\bullet}$ respectively.

Although the computational lambda calculus consists of the same terms as the ordinary lambda calculus, this example is not a PCA nor a **BCI**-algebra. Intuitively, the computational lambda calculus is sound for reasoning about effectful programs, which cannot be discarded, duplicated nor exchanged in general; hence it cannot have **S**, **K** nor **C**.

The syntactical proof is as follows. Assume that the computational lambda calculus is a **BCI**-algebra. Then there exists a term $\mathbf{C}'$ such that $\mathbf{C}'MN =_R NM$ in the computational lambda calculus for any term M and N. Take two different variables u and v which are not free in $\mathbf{C}'$. Then $\mathbf{C}'(uu)(vv) =_R (vv)(uu)$ holds in the computational lambda calculus. Since the CPS-translation $[\![-]\!]$ is sound [16], $[\![-]\!]$ sends $\mathbf{C}'(uu)(vv)$ and $(vv)(uu)$ to the $\beta\eta$-equal terms.

$$[\![\mathbf{C}'(uu)(vv)]\!] =_{\beta\eta} \lambda k.[\![\mathbf{C}']\!](\lambda z.uu(\lambda w.zw(\lambda x.vv(\lambda y.xyk))))$$
$$[\![(vv)(uu)]\!] =_{\beta\eta} \lambda k.vv(\lambda x.uu(\lambda y.xyk))$$

The former contains a subterm $uu(...vv...)$, whereas the latter contains a subterm $vv(...uu...)$. Therefore, these two terms are not $\beta\eta$-equal, and it yields a contradiction.

We can also obtain $\mathbf{BII}^{\times}(-)^{\bullet}$-algebras from models of the computational lambda calculus. Take a Cartesian closed category $\mathbf{C}$ and a strong monad T on $\mathbf{C}$ with an object X such that $X \cong (X \to TX)$. Then $\mathbf{C}(1, TX)$ is a model of the computational lambda calculus and forms a $\mathbf{BII}^{\times}(-)^{\bullet}$-algebra.

7 Related work

In [19], Zeilberger showed certain kind of planar maps with orientations are generated by combining several "imploid moves" including those corresponding to **B** and **I**. In particular for an untyped case, his work gives the combinatory completeness of $\mathbf{BI}(-)^{\bullet}$-algebras. Moreover, this paper suggests that we can obtain models of $\mathbf{BI}(-)^{\bullet}$-algebras from reflexive objects of skew closed categories.

In this paper, we deal with several classes of applicative structures other than PCAs nor **BCI**-algebras. [7] is a textbook covering basic facts about combinatory algebras. In [10], Komori investigated $\mathbf{BB'I}$ logic, one of combinatory logics with restricted exchanges. Our $\mathbf{BB'II}^{\bullet}$-algebra is inspired by it and assemblies on $\mathbf{BB'II}^{\bullet}$-algebras are models of $\mathbf{BB'I}$ logics with an extra axiom $\vdash ((\phi \to \phi) \to \psi) \to \psi$. Futhermore, using "guarded merge" introduced in that paper, we can construct lambda terms which form a $\mathbf{BB'II}^{\bullet}$-algebra.

$\mathbf{BI}(-)^{\bullet}$-algebras give rise to models of certain fragments of the Lambek calculus. Correspondences about Lambek calulus and closed multicategories are in [11]. A basic Lambek calculus is $L(\bullet, I, \backslash, /)$ which has products, a unit and implications for both sides, whose models are monoidal biclosed category. Closed multicategories are models of $L(\backslash)$ or $L(/)$. Closed categories are models of $L(I, \backslash)$ or $L(I, /)$.

Realizability models for exponential modalities ! are introduced in [3] [2] as "linear combinatory algebras (LCAs)." In [8], linear/non-linear realizability models are constructed from adjoint pairs between **BCI**-algebras and PCAs. We are currently developing the construction of realizability models for exchange modalities in the same way as exponential modalities. Exchange modalities are modalities associating the Lambek calculus to the commutative linear logic. The characterization of an exchange modality on the Lambek calculus is in [4] and categorical models of exchange modalities are introduced in [9]. A categorical model of exchange modality on $L(\bullet, I, \backslash, /)$ is given as a monoidal adjunction between a monoidal biclosed category and an SMCC. With some appropriate conditions, adjoint pairs between $\mathbf{BI}(-)^{\bullet}$-algebras and **BCI**-algebras may give rise to realizability models of exchange modalities on $L(\backslash)$ or $L(/)$.

8 Conclusion

In this paper, we have presented several classes of applicative structures and identified categorical structures of assemblies on them. In particular, we have shown that $\mathbf{BI}(-)^{\bullet}$-algebras are the necessary and sufficient conditions for obtaining closed multicategories under some conditions, and that $\mathbf{BII}^{\times}(-)^{\bullet}$-algebras are those for closed categories.

There are several directions for further investigation of this paper. We conclude this paper by describing three of future tasks.

First, we may investigate more correspondences between applicative structures and categorical structures of assemblies. We have not given appropriate classes of applicative structures inducing other important categorical structures such as non-symmetric monoidal closed categories nor monoidal biclosed categories.

Second, as mentioned in Section 7, we may characterize realizability models of exchange modalities. Furthermore, if we obtain a class of applicative structures for monoidal biclosed categories, adjoint pairs between such applicative structures and **BCI**-algebras could give rise to categorical models of exchange modalities on $L(\bullet, I, \backslash, /)$.

Third, more examples of $\mathbf{BI}(-)^{\bullet}$-algebras are desired. Finding interesting examples, we may get new perspectives for analyzing non-commutative logical systems.

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