Diverse Collections in Matroids and Graphs

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Abstract

We investigate the parameterized complexity of finding diverse sets of solutions to three fundamental combinatorial problems, two from the theory of matroids and the third from graph theory. The input to the Weighted Diverse Bases problem consists of a matroid M, a weight function $\omega : E(M) \to \mathbb{N}$, and integers $k \geq 1, d \geq 0$. The task is to decide if there is a collection of k *bases* $B_1, \ldots, B_k$ of M such that the weight of the symmetric difference of any pair of these bases is at least d. This is a diverse variant of the classical matroid base packing problem. The input to the Weighted Diverse Common Independent Sets problem consists of two matroids M_1, M_2 defined on the same ground set E, a weight function $\omega : E \to \mathbb{N}$, and integers $k \geq 1, d \geq 0$. The task is to decide if there is a collection of k *common independent sets* $I_1, \ldots, I_k$ of M_1 and M_2 such that the weight of the symmetric difference of any pair of these sets is at least d. This is motivated by the classical weighted matroid intersection problem. The input to the Diverse Perfect Matchings problem consists of a graph G and integers $k \geq 1, d \geq 0$. The task is to decide if G contains k *perfect matchings* $M_1, \ldots, M_k$ such that the symmetric difference of any two of these matchings is at least d.

The underlying problem of finding *one* solution (basis, common independent set, or perfect matching) is known to be doable in polynomial time for each of these problems, and Diverse Perfect Matchings is known to be NP-hard for $k = 2$. We show that Weighted Diverse Bases and Weighted Diverse Common Independent Sets are both NP-hard. We show also that Diverse Perfect Matchings cannot be solved in polynomial time (unless P = NP) even for the case $d = 1$. We derive fixed-parameter tractable (FPT) algorithms for all three problems with (k, d) as the parameter.

The above results on matroids are derived under the assumption that the input matroids are given as *independence oracles*. For Weighted Diverse Bases we present a polynomial-time algorithm that takes a representation of the input matroid over a finite field and computes a $poly(k, d)$-sized kernel for the problem.

2012 ACM Subject Classification Theory of computation → Fixed parameter tractability

Keywords and phrases Matroids, Diverse solutions, Fixed-parameter tractable algorithms

Digital Object Identifier 10.4230/LIPIcs.STACS.2021.31

Related Version *Full Version*: https://arxiv.org/abs/2101.04633

Funding *Fedor V. Fomin*: Supported by the Research Council of Norway via the project "MULTIVAL" (grant no. 263317).
Petr A. Golovach: Supported by the Research Council of Norway via the project "MULTIVAL" (grant no. 263317).

Fahad Panolan: Seed grant, IIT Hyderabad (SG/IITH/F224/2020-21/SG-79).
Saket Saurabh: European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant no. 819416), and Swarnajayanti Fellowship grant DST/SJF/MSA-01/2017-18.

1 Introduction

In this work we study the parameterized complexity of finding *diverse collections of solutions* to three basic algorithmic problems. Two of these problems arise in the theory of matroids. The third problem belongs to the domain of graph theory, and its restriction to bipartite graphs can be rephrased as a question about matroids. Each of these is a fundamental algorithmic problem in its respective domain.

Diverse FPT Algorithms

Nearly every existing approach to solving algorithmic problems focuses on finding *one solution of good quality* for a given input. For algorithmic problems which are – eventually – motivated by problems from the real world, finding "one good solution" may not be of much use for practitioners of the real-world discipline from which the problem was originally drawn. This is primarily because the process of abstracting out a "nice" algorithmic problem from a "messy" real-world problem invariably involves throwing out a lot of "side information" which is very relevant to the real-world problem, but is inconvenient, difficult, or even impossible to model mathematically.

The other extreme of enumerating *all* (or even all minimal or maximal) solutions to an input instance is also usually not a viable solution. A third approach is to look for *a few solutions of good quality* which are "far away" from one another according to an appropriate notion of distance. The intuition is that given such a collection of "diverse" solutions, an end-user can choose one of the solutions by factoring in the "side information" which is absent from the algorithmic model.

These and other considerations led Fellows to propose *the Diverse X Paradigm* [9]. Here "X" is a placeholder for an optimization problem, and the goal is to study the fixed-parameter tractability of finding a diverse collection of good-quality solutions for X. Recall that the *Hamming distance* of two sets is the size of their symmetric difference. A natural measure of diversity for problems whose solutions are subsets of some kind is the *minimum* Hamming distance of any pair of solutions. In this work we study the parameterized complexity of finding diverse collections of solutions for three fundamental problems with this diversity measure and its weighted variant.

Our problems

Let M be a matroid on ground set $E(M)$ and with rank function $r()$. The departure point of our work is the classical theorem of Edmonds from 1965 [6] about matroid partition. This theorem states that a matroid M has k *pairwise disjoint* bases if and only if, for every subset X of $E(M)$,

$$k \cdot r(X) + |E(M) - X| \geq k \cdot r(M).$$

An important algorithmic consequence of this result is that given access to an independence oracle for a matroid M, one can find a maximum number of *pairwise disjoint bases* of M in polynomial time (See, e.g., [18, Theorem 42.5]). This in turn implies, for instance, that the maximum number of pairwise edge-disjoint spanning trees of a connected graph can be found in polynomial time.

We take a fresh look at this fundamental result of Edmonds: what happens if we don't insist that the bases be pairwise disjoint, and instead allow them to have some pairwise intersection? We work in the weighted setting where each element e of the ground set $E(M)$ has a positive integral weight $\omega(e)$ associated with it, and the weight of a subset X of $E(M)$ is the sum of the weights of the elements in X. The relaxed version of the pairwise disjoint bases problem is then: Given an independence oracle for a matroid M and integers k, d as input, find if M has k bases $B_1, \ldots, B_k$ such that for every pair of bases B_i, B_j ; $i \neq j$ the weight $\omega(B_i \triangle B_j)$ of their symmetric difference is at least d. We call this the Weighted Diverse Bases problem:

Weighted Diverse Bases

Input: A matroid M, a weight function $\omega\colon E(M) \to \mathbb{N}$, and integers $k \geq 1$ and $d \geq 0$.

Task: Decide whether there are bases $B_1, \ldots, B_k$ of M such that $\omega(B_i \triangle B_j) \geq d$ holds for all distinct $i, j \in \{1, \ldots, k\}$.

Due to the expressive power of matroids Weighted Diverse Bases captures many interesting computational problems. We list a few examples; in each case the weight function assigns positive integral weights, $k \geq 1$ and $d \geq 0$ are integers, and we say that a collection of objects is *diverse* if the weight of the symmetric difference of each pair of objects in the collection is at least d. When M is a graphic matroid Weighted Diverse Bases corresponds to finding diverse *spanning trees* in an edge-weighted graph. When M is a vector matroid then this is the problem of finding diverse *column (or row) bases* of a matrix with column (or row) weights. And when M is a transversal matroid on a weighted ground set then this problem corresponds to finding diverse *systems of distinct representatives*.

Another celebrated result of Edmonds is the *Matroid Intersection Theorem* [7] which states that if M_1, M_2 are matroids on a common ground set E and with rank functions r_1, r_2, respectively, then the size of a largest subset of E which is independent in both M_1 and M_2 (a *common independent set*) is given by

$$\min_{T \subseteq E}(r_1(T) + r_2(E - T)).$$

Edmonds showed that given access to independence oracles for M_1 and M_2, a maximum-size common independent set of M_1 and M_2 can be found in polynomial time [7]. This is called the Matroid Intersection problem. Frank [12] found a polynomial-time algorithm for the more general Weighted Matroid Intersection problem where the input has an additional weight function $\omega : E \to \mathbb{N}$ and the goal is to find a common independent set of the maximum *weight*. The second problem that we address in this work is a "diverse" take on Weighted Matroid Intersection where we replace the maximality requirement on individual sets with a lower bound on the weight of their symmetric difference. Given M_1, M_2, ω as above and integers k, d, we ask if there are k common independent sets whose pairwise symmetric differences have weight at least d each; this is the Weighted Diverse Common Independent Sets problem.

Weighted Diverse Common Independent Sets

Input: Matroids M_1 and M_2 with a common ground set E, a weight function $\omega\colon E \to \mathbb{N}$, and integers $k \geq 1$ and $d \geq 0$.

Task: Decide whether there are sets $I_1, \ldots, I_k \subseteq E$ such that I_i is independent in both M_1 and M_2 for every $i \in \{1, \ldots, k\}$ and $\omega(I_i \triangle I_j) \geq d$ for all distinct $i, j \in \{1, \ldots, k\}$.

Weighted Diverse Common Independent Sets also captures many interesting algorithmic problems. We give a few examples (*cf.* [18, Section 41.1a]). We use "diverse" here in the sense defined above. Given a bipartite graph G with edge weights, Weighted Diverse Common Independent Sets can be used to ask if there is a diverse collection of k *matchings* in G. A *partial orientation* of an undirected graph G is a directed graph obtained by (i) assigning directions to some subset of edges of G and (ii) deleting the remaining edges. Given an undirected graph $G = (V, E)$ with edge weights and a function $\iota : V \to \mathbb{N}$, we say that a partial orientation $\mathcal{O}$ of G *respects* ι if the in-degree of every vertex v in $\mathcal{O}$ is at most $\iota(v)$. We can use Weighted Diverse Common Independent Sets to ask if there is a diverse collection of k partial orientations of G, all of which respect ι. For a third example, let $G = (V, E)$ be an undirected graph with edge weights, in which each edge is assigned a – not necessarily distinct – *color*. A *colorful forest* in G is any subgraph of G which is a forest in which no two edges have the same color. We can use Weighted Diverse Common Independent Sets to ask if there is a diverse collection of k colorful forests in G.

Finding whether a bipartite graph has a perfect matching or not is a well-known application of Matroid Intersection ([18, Section 41.1a]). The third problem that we study in this work is a diverse version of the former problem, extended to general graphs. Note that there is no known interpretation of the problem of finding perfect matchings in (general) undirected graphs in terms of Matroid Intersection.

Diverse Perfect Matchings

Input: An undirected graph G on n vertices, and integers $k \geq 1$ and $d \geq 0$.

Task: Decide whether there are perfect matchings $M_1, \ldots, M_k$ of G such that $|M_i \triangle M_j| \geq d$ for all distinct $i, j \in \{1, \ldots, k\}$.

Our results

We assume throughout that matroids in the input are given in terms of an *independence oracle*. Recall that with this assumption, we can find *one* basis of the largest weight and *one* common independent set (of two matroids) of the largest weight, both in polynomial time. In contrast, we show that the diverse versions Weighted Diverse Bases and Weighted Diverse Common Independent Sets are both NP-hard, even when the weights are expressed in unary[1].

▶ **Theorem 1.** *Both Weighted Diverse Bases and Weighted Diverse Common Independent Sets are strongly* NP*-complete, even on the uniform matroids* U_n^3.

Given this hardness, we analyze the parameterized complexity of these problems with d, k as the parameters. Our first result is that Weighted Diverse Bases is fixed-parameter tractable (FPT) under this parameterization:

▶ **Theorem 2.** *Weighted Diverse Bases can be solved in* $2^{\mathcal{O}(dk^2(\log k + \log d))} \cdot |E(M)|^{\mathcal{O}(1)}$ *time.*

We have a stronger result if the input matroid is given as a representation over a finite field (and not just as a "black box" independence oracle): in this case we show that Weighted Diverse Bases admits a *polynomial kernel* with this parameterization.

[1] See Theorem 7 for an alternative hardness result for Weighted Diverse Bases.

▶ **Theorem 3.** *Given a representation of the matroid M over a finite field GF(q) as input, we can compute a kernel of* Weighted Diverse Bases *of size* $\mathcal{O}(k^6 d^4 \log q)$.

We then show that our second matroid-related diverse problem is also FPT under the same parameterization.

▶ **Theorem 4.** *Weighted Diverse Common Independent Sets can be solved in time* $2^{\mathcal{O}(k^3 d^2 \log(kd))} \cdot |E|^{\mathcal{O}(1)}$.

We now turn to the problem of finding diverse perfect matchings. Diverse Perfect Matchings is known to be NP-hard already when $k = 2$ and G is a 3-regular graph [16, 10]. Since all perfect matchings of a graph have the same size the symmetric difference of two distinct perfect matchings is at least 2. Setting $d = 1$ in Diverse Perfect Matchings is thus equivalent to asking whether G has at least k distinct perfect matchings. Since a bipartite graph on n vertices has at most $\frac{n}{2}!$ perfect matchings and since $\log(\frac{n}{2}!) = \mathcal{O}(n \log n)$ we get – using binary search – that there is a polynomial-time Turing reduction from the problem of *counting* the number of perfect matchings in a bipartite graph to Diverse Perfect Matchings instances with $d = 1$. Since the former problem is #P-complete [19] we get

▶ **Theorem 5.** *Diverse Perfect Matchings with $d = 1$ cannot be solved in time polynomial in $n = |V(G)|$ even when graph G is bipartite, unless* P = NP.

Thus we get that Diverse Perfect Matchings is unlikely to have a polynomial-time algorithm even if *one* of the two numbers k, d is a small constant. We show that the problem *does* have a (randomized) polynomial-time algorithm when *both* these parameters are bounded; Diverse Perfect Matchings is (randomized) FPT with k and d as parameters:

▶ **Theorem 6.** *There is an algorithm that given an instance of Diverse Perfect Matchings, runs in time $2^{2^{\mathcal{O}(kd)}} n^{\mathcal{O}(1)}$ and outputs the following: If the input is a No-instance then the algorithm outputs No. Otherwise the algorithm outputs Yes with probability at least $1 - \frac{1}{e}$.*

Note that Theorem 6 implies, in particular, that Diverse Perfect Matchings can be solved in (randomized) polynomial time when $kd \leq c_1 + \frac{\log \log n}{c_2}$ holds for some constants c_1, c_2 which depend on the constant hidden by the $\mathcal{O}()$ notation.

Our methods

We prove the NP-hardness results (Theorem 1) by reduction from the 3-Partition problem. To show that Weighted Diverse Bases is FPT (Theorem 2) we observe first that if the input matroid M contains a set of size $\Omega(kd)$ which is *both* independent *and* co-independent in M then the input is a Yes instance of Weighted Diverse Bases (Lemma 14). We can check for the existence of such a set in time polynomial in $|E(M)|$, so we assume without loss of generality that no such set exists. We then show that starting with an arbitrary basis of M and repeatedly applying the greedy algorithm (Proposition 9) *poly*(k, d)-many times we can find, in time polynomial in $(|E(M)| + k + d)$, (i) a subset $S^* \subseteq E(M)$ of size *poly*(k, d) and (ii) a matroid $\widetilde{M}$ on the ground set S^* such that $(\widetilde{M}, \omega, k, d)$ is *equivalent* to the input instance (M, ω, k, d) (Lemma 15). We also show how to compute a useful partition of $E(\widetilde{M}) = S^*$ which speeds up the subsequent FPT-time search for a diverse set of bases in $\widetilde{M}$. The kernelization result for Weighted Diverse Bases (Theorem 3) follows directly from Lemma 15. This "compression lemma" is thus the main technical component of our algorithms for Weighted Diverse Bases.

To show that WEIGHTED DIVERSE COMMON INDEPENDENT SETS is FPT (Theorem 4) we observe first that if the two input matroids M_1, M_2 have a *common* independent set of size $\Omega(kd)$ then the input is a YES instance of WEIGHTED DIVERSE COMMON INDEPENDENT SETS (Lemma 16). So we assume that this is not the case, and then show (Lemma 17) that we can construct, in $f(k, d)$ time, a collection $\mathcal{F}$ of common independent sets of M_1 and M_2 of size $g(k, d)$ such that if the input is a YES-instance then it has a solution $I_1, \ldots, I_k$ with $I_i \in \mathcal{F}$ for $i \in \{1, \ldots, k\}$. The FPT algorithm for WEIGHTED DIVERSE COMMON INDEPENDENT SETS follows by a simple search in the collection $\mathcal{F}$.

Our algorithm for DIVERSE PERFECT MATCHINGS is based on two procedures.

P1 Given an undirected graph G on n vertices, perfect matchings $M_1, \ldots, M_r$ of G, and a non-negative integer s as input, this procedure (Lemma 18) runs in time $2^{\mathcal{O}(rs)}n^{\mathcal{O}(1)}$ and outputs a perfect matching M of G such that $|M \triangle M_i| \geq 2s$ holds for all $i \in \{1, \ldots, r\}$ (if such a matching exists), with probability at least $\frac{2}{3}e^{-rs}$.

P2 Given an undirected graph G on n vertices, a perfect matching M of G, and non-negative integers r, d, s, this procedure(Lemma 19) runs in time $2^{\mathcal{O}(r^2 s)}n^{\mathcal{O}(1)}$, and outputs r perfect matchings $M_1^\star, \ldots, M_r^\star$ of G such that $|M \triangle M_i^\star| \leq s$ holds for all $i \in \{1, \ldots, r\}$ and $|M_i^\star \triangle M_j^\star| \geq d$ holds for all distinct $i, j \in [r]$ (if such matchings exist), with probability at least e^{-rs}. If no such perfect matchings exist, then the algorithm outputs NO.

Let (G, k, d) be the input instance of DIVERSE PERFECT MATCHINGS. We use procedure **P1** to greedily compute a collection of matchings which are "far apart": We start with an arbitrary perfect matching M_1. In step i, we have a collection of perfect matchings $M_1, \ldots, M_{i-1}$ such that $|M_j \triangle M_{j'}| \geq 2^{k-i}d$ holds for any two distinct $j, j' \in \{1, \ldots, i-1\}$. We now run procedure **P1** with $r = i - 1$ and $s = 2^{k-i}d$ to find – if it exists – a matching M_i such that $|M_i \triangle M_j| \geq 2^{k-i+1}d$ holds for all $j \in \{1, \ldots, i\}$. By exhaustively applying **P1** we get a collection of perfect matchings $M_1, \ldots, M_q$ such that

(a) for any two distinct integers $i, j \in \{1, \ldots, q\}$, $|M_i \triangle M_j| \geq 2^{k-q+1}d$, and

(b) for any other perfect matching $M \notin \{M_1, \ldots, M_q\}$, $|M \triangle M_j| \leq 2^{k-q}d$.

Thus, if $k \leq q$, then clearly $\{M_1, \ldots, M_k\}$ is a solution. Otherwise, let $\mathcal{M} = \{M_1^\star, \ldots, M_k^\star\}$ be a hypothetical solution. Then for each $M_i^\star$ there is a *unique* matching M_j in $\{M_1, \ldots, M_q\}$ such that $|M_j \triangle M_i^\star| < 2^{(k-q)}d$ holds (Claim 20). For each $i \in \{1, \ldots, q\}$ we guess the number r_i of perfect matchings from $\mathcal{M}$ that are *close* to M_i, and use procedure **P2** to compute a set of r_i diverse perfect matchings that are close to M_i. The union of all the matchings computed for all $i \in \{1, \ldots, q\}$ form a solution.

We use algebraic methods and color coding to design procedure **P1**. The Tutte matrix $\mathbf{A}$ of an undirected graph G over the field $\mathbb{F}_2[X]$ is defined as follows, where $\mathbb{F}_2$ is the Galois field on $\{0, 1\}$ and $X = \{x_e : e \in E(G)\}$. The rows and columns of $\mathbf{A}$ are labeled with $V(G)$ and for each $e = \{u, v\} \in E(G)$, $\mathbf{A}[u, v] = \mathbf{A}[v, u] = x_e$. All other entries in $\mathbf{A}$ are zeros. There is a bijective correspondence between the set of monomials of $det(\mathbf{A})$ and the set of perfect matchings of G. Procedure **P1** extracts the required matching from $det(\mathbf{A})$ using color coding. Procedure **P2** is realized using color coding and dynamic programming.

Related work

Recall that all bases of a matroid have the same size, and that the number of bases of a matroid on ground set E is at most $2^{|E|}$. So using the same argument as for Theorem 5 we get that WEIGHTED DIVERSE BASES generalizes – via Turing reductions – the problem of *counting the number of bases* of a matroid. Each of these reduced WEIGHTED DIVERSE

Bases instances will have $d = 1$, and a weight function which assigns the weight 1 to each element in the ground set. Counting the number of bases of a matroid is known to be #P-complete even for restricted classes of matroids such as transversal [3], bircircular [14], and binary matroids [20]. Hence we have the following alternative[2] hardness result for Weighted Diverse Bases

▶ **Theorem 7.** *Weighted Diverse Bases cannot be solved in time polynomial in* $|E(M)|$ *unless* P = NP, *even when* $d = 1$ *and every element of the ground set* $E(M)$ *has weight* 1.

The study of the parameterized complexity of finding diverse sets of solutions is a very recent development, and only a handful of results are currently known. In the work which introduced this notion Baste et al. [1] showed that diverse variants of a large class of graph problems which are FPT when parameterized by the *treewidth* of the input graph, are also FPT when parameterized by the treewidth and the number of solutions in the collection. In a second article [2] the authors show that for each fixed positive integer d, two diverse variants – one with the *minimum* Hamming distance of any pair of solutions, and the other with the *sum* of all pairwise Hamming distances of solutions – of the d-Hitting Set problem are FPT when parameterized by the size of the hitting set and the number of solutions. In a recent manuscript on diverse FPT algorithms [10] the authors show that the problem of finding *two* maximum-sized matchings in an undirected graph such that their symmetric difference is at least d, is FPT when parameterized by d. Note that our result on Diverse Perfect Matchings generalizes this to $k \geq 2$ matchings, *provided* the input graph has a perfect matching.

In a very recent manuscript Hanaka et al. [15] propose a number of results about finding diverse solutions. We briefly summarize their results which are germane to our work. For a collection of sets $X_1, \ldots, X_k$ let $d_{sum}(X_1, \ldots, X_k)$ denote the sum of all pairwise Hamming distances of these sets and let $d_{min}(X_1, \ldots, X_k)$ denote the smallest Hamming distance of any pair of sets in the collection. Hanaka et al. show that there is an algorithm which takes an independence oracle for a matroid M and an integer k as input, runs in time polynomial in $(|E(M)| + k)$, and finds a collection $B_1, B_2, \ldots, B_k$ of k bases of M which maximizes $d_{sum}(B_1, B_2, \ldots, B_k)$. This result differs from our work on Weighted Diverse Bases in two key aspects. They deal with the unweighted (counting) case, and their diversity measure is the *sum* of the pairwise symmetric differences, whereas we look at the *minimum* (weight of the) symmetric difference. These two measures are, in general, not comparable.

Hanaka et al. also look at the complexity of finding k matchings $M_1, \ldots, M_k$ in a graph G where each M_i is of size t. They show that such collections of matchings maximizing $d_{min}(M_1, \ldots, M_k)$ and $d_{sum}(M_1, \ldots, M_k)$ can be found in time $2^{\mathcal{O}(kt \log(kt))} \cdot |V(G)|^{\mathcal{O}(1)}$. The key difference with our work is that their algorithm looks for matchings of a specified size t whereas ours looks for perfect matchings, of size $t = \frac{|V(G)|}{2}$; note that this t does not appear in the exponential part of the running time of our algorithm (Theorem 6). The manuscript [15] has a variety of other interesting results on diverse FPT algorithms as well.

Organization of the rest of the paper

In the next section we collect together some preliminary results. In Section 3 we prove that Weighted Diverse Bases and Weighted Diverse Common Independent Sets are strongly NP-hard. In Section 4 we derive our FPT and kernelization algorithms for

[2] Compare with Theorem 1.

Weighted Diverse Bases, and in Section 5 we show that Weighted Diverse Common Independent Sets is FPT. We derive our results for Diverse Perfect Matchings in Section 6. We conclude in Section 7.

All the proofs omitted in this Extended Abstract can be found in the full version on arXiv at `https://arxiv.org/abs/2101.04633`.

2 Preliminaries

We use $X \triangle Y$ to denote the *symmetric difference* $(X \setminus Y) \cup (Y \setminus X)$ of sets X and Y. We use $\mathbb{N}$ to denote the set of positive integers.

A parameterized problem Π is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. We say that a parameterized problem Π is *fixed parameter tractable (*FPT*)*, if there is an algorithm that given an instance (x, k) of Π as input, solves in time $f(k)|x|^{\mathcal{O}(1)}$, where f is an arbitrary function and $|x|$ is the length of x. A kernelization algorithm for a parameterized problem Π is a polynomial time algorithm (computable function) $\mathcal{A} : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ such that $(x, k) \in \Pi$ if and only if $(x', k') = \mathcal{A}((x, k)) \in \Pi$ and $|x'| + k' \leq g(k)$ for some computable function g. When g is a polynomial function, we say that Π admits a polynomial kernel. For a detailed overview about parameterized complexity we refer to the monographs [5, 4, 11]

A pair $M = (E, \mathcal{I})$, where E is a finite *ground* set and $\mathcal{I}$ is a family of subsets of the ground set, called *independent sets* of E, is a *matroid* if it satisfies the following conditions, called *independence axioms*: **(I1)** $\emptyset \in \mathcal{I}$; **(I2)** If $A \subseteq B \subseteq E(M)$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and **(I3)** If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there is $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$. We use $E(M)$ and $\mathcal{I}(M)$ to denote the ground set and the set of independent sets, respectively. As is standard for matroid problems, we assume that each matroid M that appears in the input is given by an *independence oracle*, that is, an oracle that in constant (or polynomial) time replies whether a given $A \subseteq E(M)$ is independent in M or not. An inclusion-wise maximal independent set B is called a *basis* of M. All the bases of M have the same size that is called the *rank* of M, denoted $\mathsf{rank}(M)$. The *rank* of a subset $A \subseteq E(M)$, denoted $\mathsf{rank}(A)$, is the maximum size of an independent set $X \subseteq A$; the function $\mathsf{rank}\colon 2^{E(M)} \to \mathbb{Z}$ is the *rank* function of M.

The *dual* of a matroid $M = (E, \mathcal{I})$, denoted M^*, is the matroid whose ground set is E and whose set of bases is $\mathcal{B}^* = \{\overline{B} \mid B \in \mathcal{B}(M)\}$. That is, the bases of M^* are exactly the complements of the bases of M. A basis (independent set, rank, respectively) of M^* is a *cobasis* (*coindependent set*, *corank*, respectively) of M. Given an independence oracle for M we can construct – with an overhead which is polynomial in $|E|$ – a *rank* oracle for M, and thence *corank* and *coindependence* oracles for M.

Let M be a matroid and let $\mathbb{F}$ be a field. An $n \times m$-matrix $\mathbf{A}$ over $\mathbb{F}$ is a *representation of M over* $\mathbb{F}$ if there is one-to-one correspondence f between $E(M)$ and the set of columns of $\mathbf{A}$ such that for any $X \subseteq E(M)$, $X \in \mathcal{I}(M)$ if and only if the columns $f(X)$ are linearly independent (as vectors of $\mathbb{F}^n$); if M has such a representation, then it is said that M has a *representation over* $\mathbb{F}$. In other words, $\mathbf{A}$ is a representation of M if M is isomorphic to the *linear matroid* of $\mathbf{A}$, i.e., the matroid whose ground set is the set of columns of $\mathbf{A}$ and a set of columns is independent if and only if these columns are linearly independent.

Let $1 \leq r \leq n$ be integers. We use U_n^r to denote the *uniform* matroid, that is, the matroid with the ground set of size n such that the bases are all r-element subsets of the ground set.

We use the classical results of Edmonds [7] and Frank [12] about the Weighed Matroid Intersection problem. The task of this problem is, given two matroids M_1 and M_2 with the same ground set E and a weight function $\omega\colon E \to \mathbb{N}$, find a set X of maximum weight such that X is independent in both matroids. Edmonds [7] proved that the problem can be solved in polynomial time for the unweighted case (that is, the task is to find a common independent set of maximum size; we refer to this variant as MATROID INTERSECTION) and the result was generalized for the variant with the weights by Frank in [12].

▶ **Proposition 8** ([7, 12]). *WEIGHTED MATROID INTERSECTION can be solved in polynomial time.*

We also need another classical result of Edmonds [8] that a basis of maximum weight can be found by the greedy algorithm. Recall that, given a matroid M with a weight function $\omega\colon E(M) \to \mathbb{N}$, the greedy algorithm finds a basis B of maximum weight as follows. Initially, $B := \emptyset$. Then at each iteration, the algorithm finds an element of $x \in E(M) \setminus B$ of maximum weight such that $B \cup \{x\}$ is independent and sets $B := B \cup \{x\}$. The algorithms stops when there is no element that can be added to B.

▶ **Proposition 9** ([8]). *The greedy algorithm finds a basis of maximum weight of a weighted matroid in polynomial time.*

We need the following observation(See [17, Lemma 2.1.10]).

▶ **Observation 10.** *Let X and Y be disjoint sets such that X is independent and Y is coindependent in a matroid M. Then there is a basis B of M such that $X \subseteq B$ and $Y \cap B = \emptyset$.*

Observe that for any sets X and Y that are subsets of the same universe, $X \triangle Y = \overline{X} \triangle \overline{Y}$. This implies the following.

▶ **Observation 11.** *For every matroid M, every weight function $\omega\colon E(M) \to \mathbb{N}$, and all integers $k \geq 1$ and $d \geq 0$, the instances (M, ω, k, d) and (M^*, ω, k, d) of WEIGHTED DIVERSE BASES are equivalent.*

We need the following simple observations about the symmetric differences of perfect matchings.

▶ **Observation 12.** *The cardinality of symmetric differences of perfect matchings in a graph obeys the triangle inequality. That is, for a graph G and perfect matchings M_1, M_2, M_3 in G, $|M_1 \triangle M_2| + |M_2 \triangle M_3| \geq |M_1 \triangle M_3|$.*

▶ **Observation 13.** *Let G be a graph and M_1 and M_2 be two perfect matchings in G. Then $|M_1 \triangle M_2| = 2 \cdot |M_1 \setminus M_2| = 2 \cdot |M_2 \setminus M_1|$.*

3 Hardness of Weighted Diverse Bases and Weighted Diverse Common Independent Sets

Both WEIGHTED DIVERSE BASES and WEIGHTED DIVERSE COMMON INDEPENDENT SETS are NP-complete in the strong sense even for uniform matroids. Both reductions are from the 3-PARTITION problem which is known to be NP-complete in the strong sense, i.e., it is NP-complete even if the various integers in the input are encoded in unary [13, SP15].

▶ **Theorem 1.** *Both WEIGHTED DIVERSE BASES and WEIGHTED DIVERSE COMMON INDEPENDENT SETS are strongly NP-complete, even on the uniform matroids U_n^3.*

4 An FPT algorithm and kernelization for Weighted Diverse Bases

In this section we show that Weighted Diverse Bases is FPT when parameterized by k and d. Moreover, if the input matroid is representable over a finite field and is given by such a representation, then Weighted Diverse Bases admits a polynomial kernel. We start with the observation that if the input matroid has a sufficiently big set that is simultaneously independent and coindependent, then diverse bases always exist.

▶ **Lemma 14.** *Let M be a matroid, and let $k \geq 1$ and $d \geq 0$ be integers. If there is $X \subseteq E(M)$ of size at least $k\lceil \frac{d}{2} \rceil$ such that X is simultaneously independent and coindependent, then (M, ω, k, d) is a yes-instance of Weighted Diverse Bases for any weight function ω.*

Proof. Let $X \subseteq E(M)$ be a set of size at least $k\lceil \frac{d}{2} \rceil$ such that X is simultaneously independent and coindependent. Then there is a partition $X_1, \ldots, X_k$ of X such that $|X_i| \geq \lceil \frac{d}{2} \rceil$ for every $i \in \{1, \ldots, k\}$. Let $i \in \{1, \ldots, k\}$. Since X is independent, X_i is independent, and since X is coindependent, then $X \setminus X_i$ is coindependent. Then by Observation 10, there is a basis B_i of M such that $X_i \subseteq B_i$ and $B_i \cap (X \setminus X_i) = \emptyset$. The latter property means that $B_i \cap X_j = \emptyset$ for every $j \in \{1, \ldots, k\}$ such that $j \neq i$. We consider the bases B_i defined in this manner for all $i \in \{1, \ldots, k\}$. Then for every distinct $i, j \in \{1, \ldots, k\}$, $X_i \cup X_j \subseteq B_i \triangle B_j$. Therefore, $\omega(B_i \triangle B_j) \geq \omega(X_i \cup X_j) \geq |X_i \cup X_j| = |X_i| + |X_j| \geq 2\lceil \frac{d}{2} \rceil \geq d$ for any $\omega\colon E(M) \to \mathbb{N}$. Hence, (M, ω, k, d) is a yes-instance of Weighted Diverse Bases. ◀

Using Proposition 8 we can check, in polynomial time, whether the conditions of Lemma 14 are satisfied. If they are, then we correctly return Yes. We show that if there is no such large set X as in Lemma 14 then there is a way to repeatedly apply the greedy algorithm of Proposition 9 to obtain an equivalent small instance of the problem, as stated in the following "compression" lemma.

▶ **Lemma 15.** *There is an algorithm that, given an instance (M, ω, k, d) of Weighted Diverse Bases, runs in time polynomial in $(|E(M)| + k + d)$ and either correctly decides that (M, ω, k, d) is a yes-instance or outputs an equivalent instance $(\widetilde{M}, \omega, k, d)$ of Weighted Diverse Bases such that $E(\widetilde{M}) \subseteq E(M)$ and $|E(\widetilde{M})| \leq 2\lceil \frac{d}{2} \rceil^2 k^3$. In the latter case, the algorithm also computes a partition (L, L^*) of $E(\widetilde{M})$ with the property that for every basis B of $\widetilde{M}$, $|B \cap L| \leq \lceil \frac{d}{2} \rceil k$ and $|L^* \setminus B| \leq \lceil \frac{d}{2} \rceil k$, and the algorithm outputs an independence oracle for $\widetilde{M}$ that answers queries for $\widetilde{M}$ in time polynomial in $|E(M)|$. Moreover, if M is representable over a finite field $\mathbb{F}$ and is given by such a representation, then the algorithm outputs a representation of $\widetilde{M}$ over $\mathbb{F}$.*

Given Lemma 15 it is easy to show that Weighted Diverse Bases is FPT when parameterized by k and d.

▶ **Theorem 2.** *Weighted Diverse Bases can be solved in $2^{\mathcal{O}(dk^2(\log k + \log d))} \cdot |E(M)|^{\mathcal{O}(1)}$ time.*

Proof. Let (M, ω, k, d) be an instance of Weighted Diverse Bases. We run the algorithm from Lemma 15. If the algorithm solves the problem, then we are done. Otherwise, the algorithm outputs an equivalent instance $(\widetilde{M}, \omega, k, d)$ of Weighted Diverse Bases such that $E(\widetilde{M}) \subseteq E(M)$ and $|E(\widetilde{M})| \leq 2\lceil \frac{d}{2} \rceil^2 k^3$. Moreover, the algorithm computes the partition (L, L^*) of $E(\widetilde{M})$ with the property that for every basis B of $\widetilde{M}$, $|B \cap L| \leq \lceil \frac{d}{2} \rceil k$ and $|L^* \setminus B| \leq \lceil \frac{d}{2} \rceil k$. Then we check all possible k-tuples of bases by brute force and verify whether there are k bases forming a solution. By the properties of L and L^*, $\widetilde{M}$ has

$(d^2k^3)^{\mathcal{O}(dk)}$ distinct bases. Therefore, we check at most $(d^2k^3)^{\mathcal{O}(dk^2)}$ k-tuples of bases. We conclude that this checking can be done in $2^{\mathcal{O}(dk^2(\log k+\log d))} \cdot |E(M)|^{\mathcal{O}(1)}$ time, and the claim follows. ◀

If the input matroid is given by a representation over a finite field, then Weighted Diverse Bases also admits a polynomial kernel when parameterized by k and d.

▶ **Theorem 3.** *Given a representation of the matroid M over a finite field $\mathsf{GF}(q)$ as input, we can compute a kernel of Weighted Diverse Bases of size $\mathcal{O}(k^6d^4\log q)$.*

Proof. Let (M, ω, k, d) be an instance of Weighted Diverse Bases. Let also $\mathbf{A}$ be its representation over $\mathsf{GF}(q)$. We run the algorithm from Lemma 15. If the algorithm solves the problem and reports that (M, ω, k, d) is a yes-instance, we return a trivial yes-instance of the problem. Otherwise, the algorithm outputs an equivalent instance $(\widetilde{M}, \omega, k, d)$ of Weighted Diverse Bases such that $E(\widetilde{M}) \subseteq E(M)$ and $|E(\widetilde{M})| \leq 2\lceil \frac{d}{2}\rceil^2k^3$. Moreover, the algorithm computes a representation $\tilde{\mathbf{A}}$ of $\widetilde{M}$ over $\mathsf{GF}(q)$. Clearly, it can be assumed that the number of rows of the matrix $\tilde{\mathbf{A}}$ equals $\mathsf{rank}(\widetilde{M})$. Since $\mathsf{rank}(\widetilde{M}) \leq |E(\widetilde{M})|$, the matrix $\tilde{\mathbf{A}}$ has $\mathcal{O}(k^6d^4)$ elements. Because $\tilde{\mathbf{A}}$ is a matrix over $\mathsf{GF}(q)$, it can be encoded by $\mathcal{O}(k^6d^4\log q)$ bits. Finally, note that the weights of the elements can be truncated by d, that is, we can set $\omega(e) := \min\{\omega(e), d\}$ for every $e \in E(\widetilde{M})$. Then the weights can be encoded using $\mathcal{O}(d^2k^3\log d)$ bits. This concludes the construction of our kernel. ◀

5 An FPT algorithm for Weighted Diverse Common Independent Sets

In this section we show that Weighted Diverse Common Independent Sets is FPT when parameterized by k and d. We use a similar win-win approach as for Weighted Diverse Bases and observe that if the two matroids from an instance of Weighted Diverse Common Independent Sets have a sufficiently big common independent set, then we have a yes-instance of Weighted Diverse Common Independent Sets.

▶ **Lemma 16.** *Let M_1 and M_2 be matroids with a common ground set E, and let $k \geq 1$ and $d \geq 0$ be integers. If there is an $X \subseteq E$ of size at least $k\lceil\frac{d}{2}\rceil$ such that X is a common independent set of M_1 and M_2, then (M_1, M_2, ω, k, d) is a yes-instance of Weighted Diverse Common Independent Sets for any weight function $\omega : E \to \mathbb{N}$.*

Lemma 16 implies that we can assume that the maximum size of a common independent set of the input matroids is bounded. We use this to prove the following crucial lemma.

▶ **Lemma 17.** *Let (M_1, M_2, ω, k, d) be an instance of Weighted Diverse Common Independent Sets such that the maximum size of a common independent set of M_1 and M_2 is at most s. Then there is a set $\mathcal{F}$ of common independent sets of M_1 and M_2, of size $|\mathcal{F}| = 2^{\mathcal{O}(s^2\log(ks))} \cdot d$, such that if (M_1, M_2, ω, k, d) is a yes-instance of Weighted Diverse Common Independent Sets then the instance has a solution $I_1, \ldots, I_k$ with $I_i \in \mathcal{F}$ for $i \in \{1, \ldots, k\}$. Moreover, $\mathcal{F}$ can be constructed in $2^{\mathcal{O}(s^2\log(ks))} \cdot d \cdot |E|^{\mathcal{O}(1)}$ time where E is the (common) ground set of M_1 and M_2.*

Combining Lemma 16 and Lemma 17, we obtain the main result of this section.

▶ **Theorem 4.** *Weighted Diverse Common Independent Sets can be solved in time $2^{\mathcal{O}(k^3d^2\log(kd))} \cdot |E|^{\mathcal{O}(1)}$.*

Proof. Let (M_1, M_2, ω, k, d) be an instance of WEIGHTED DIVERSE COMMON INDEPENDENT SETS. First, we use Proposition 8 to solve MATROID INTERSECTION for M_1 and M_2 and find a common independent set X of maximum size. If $|X| \geq k\lceil \frac{d}{2} \rceil$, then by Lemma 16, we conclude that (M_1, M_2, ω, k, d) is a yes-instance. Assume that this is not the case. Then the maximum size of a common independent set of M_1 and M_2 is $s < k\lceil \frac{d}{2} \rceil$. We apply Lemma 17 and construct the set $\mathcal{F}$ of size $2^{\mathcal{O}((kd)^2 \log(kd))}$ in $2^{\mathcal{O}((kd)^2 \log(kd))} \cdot |E|^{\mathcal{O}(1)}$ time. By this lemma, if (M_1, M_2, ω, k, d) is a yes-instance, it has a solution $I_1, \ldots, I_k$ such that $I_i \in \mathcal{F}$ for $i \in \{1, \ldots, k\}$. Hence, to solve the problem we go over all k-tuples of the elements of $\mathcal{F}$, and for each k-tuple, we verify whether these common independent sets of M_1 and M_2 give a solution. Clearly, we have to consider $2^{\mathcal{O}(k^3 d^2 \log(kd))}$ tuples. Hence, the total running time is $2^{\mathcal{O}(k^3 d^2 \log(kd))} \cdot |E|^{\mathcal{O}(1)}$. ◀

6 Perfect Matchings

In this section we prove that DIVERSE PERFECT MATCHINGS is fixed parameter tractable when parameterized by k and d. There are two main ingredients to our algorithm. The first ingredient is an algorithm that helps us greedily compute a collection of matchings which are "far apart".

▶ **Lemma 18.** *There is an algorithm that given an undirected graph G, perfect matchings $M_1, \ldots, M_r$, and a non-negative integer s, runs in time $2^{\mathcal{O}(rs)} n^{\mathcal{O}(1)}$, and outputs a perfect matching M such that $|M \setminus M_i| \geq s$ for all $i \in \{1, \ldots, r\}$ (if such a matching exists) with probability at least $\frac{2}{3} e^{-rs}$.*

The second ingredient is an algorithm which lets us compute matchings which are "close" to each matching in the "spread-out" collection computed using Lemma 18.

▶ **Lemma 19.** *There is an algorithm that given an undirected graph G, a perfect matching M, and non-negative integers r, d, s, runs in time $2^{\mathcal{O}(r^2 s)} n^{\mathcal{O}(1)}$, and outputs r perfect matchings $M_1^\star, \ldots, M_r^\star$ such that $|M \triangle M_i^\star| \leq s$ for all $i \in \{1, \ldots, r\}$ and $|M_i^\star \triangle M_j^\star| \geq d$ for all distinct $i, j \in [r]$ (if such matchings exist) with probability at least e^{-rs}. If no such perfect matchings exist, then the algorithm outputs No*

Putting these two ingredients together we get

▶ **Theorem 6.** *There is an algorithm that given an instance of DIVERSE PERFECT MATCHINGS, runs in time $2^{2^{\mathcal{O}(kd)}} n^{\mathcal{O}(1)}$ and outputs the following: If the input is a NO-instance then the algorithm outputs NO. Otherwise the algorithm outputs YES with probability at least $1 - \frac{1}{e}$.*

Proof. Let (G, k, d) be the input instance. Our algorithm $\mathcal{A}$ has two steps. In the first step of $\mathcal{A}$ we compute a collection of matchings greedily such that they are far apart using Lemma 18. Towards that first we run an algorithm to compute a maximum matching in G and let M_1 be the output. If M_1 is not a perfect matching we output No and stop. Next we iteratively apply Lemma 18 to compute a collection of perfect matchings that are far apart. Formally, at the beginning of step i, where $\leq 1 \leq i < k$, we have perfect matchings $M_1, \ldots, M_i$ such that $|M_j \setminus M_{j'}| \geq 2^{k-i} d$ for any two distinct $j, j' \in \{1, \ldots, i\}$. Now, we apply Lemma 18 with $r = i$ and $s = 2^{k-i-1} d$ and it will either output a matching M_{i+1} such that $|M_{i+1} \setminus M_j| \geq 2^{k-i-1} d$ for all $j \in \{1, \ldots, i\}$, or not. If no such matching exists, then the first step of the algorithm $\mathcal{A}$ is complete. So at the end of the first step of the algorithm $\mathcal{A}$, we have perfect matchings $M_1, \ldots, M_q$, where $q \in \{1, \ldots, k\}$ such that

(i) for any two distinct integers $i, j \in \{1, \ldots, q\}$, $|M_i \setminus M_j| \geq 2^{k-q}d$, and
(ii) if $q \neq k$, then for any other perfect matching $M \notin \{M_1, \ldots, M_q\}$, $|M \setminus M_j| \leq 2^{k-q-1}d$.

If $q = k$, then $\{M_1, \ldots, M_k\}$ is a solution to the instance (G, k, d), and hence our algorithm $\mathcal{A}$ outputs Yes. Now on, we assume that $q \in \{1, \ldots, k-1\}$. Statements (i) and (ii), and Observation 13 imply that

(iii) for any two distinct integers $i, j \in \{1, \ldots, q\}$, $|M_i \triangle M_j| \geq 2^{k-q+1}d$, and
(iv) for any perfect matching $M \notin \{M_1, \ldots, M_q\}$, $|M \triangle M_j| < 2^{k-q}d$.

Statements (ii) and (iv), and Observation 12 imply the following claim.

▷ Claim 20. For any perfect matching M, there exists a unique $i \in \{1, \ldots, q\}$ such that $|M \triangle M_i| < 2^{k-q}d$.

Let $\mathcal{M} = \{M_1^\star, \ldots, M_k^\star\}$ is a solution to the instance (G, k, d). Then, by Claim 20, there is a partition of $\mathcal{M}$ into $\mathcal{M}_1 \uplus \ldots \uplus \mathcal{M}_q$ (with some blocks possibly being empty) such that for each $i \in \{1, \ldots, q\}$, and each $M \in \mathcal{M}_i$, $|M \triangle M_i| \leq 2^{k-q}d$. Thus, in the second step of our algorithm $\mathcal{A}$, we guess $r_1 = |\mathcal{M}_1|, \ldots, r_q = |\mathcal{M}_q|$ and apply Lemma 19. That is, for each $i \in \{1, \ldots, q\}$ such that $r_i \neq 0$, we apply Lemma 19 with $M = M_i$, $r = r_i$, and $s = 2^{k-q}d$. Then for each $i \in 1, \ldots, q$, let the output of Lemma 19 be $N_{i,1}, \ldots, N_{r_i}$. Clearly $|N_{i,j} \triangle N_{i,j'}| \geq d$ for any two distinct $j, j' \in \{1, \ldots, r_i\}$. Observation 12 and statement (iii) implies that for any two distinct $i, j \in \{1, \ldots, q\}$, the cardinality of the symmetric difference between a matching in $\{N_{i,1}, \ldots, N_{i,r_i}\}$ and a matching in $\{N_{j,1,\ldots,N_{j,r_j}}\}$ is at least d.

If algorithm $\mathcal{A}$ computes a solution in any of the guesses for $r_1, \ldots, r_d$, then we output Yes. Otherwise we output No. As the number of choices for $r_1, \ldots r_k$ is upper bounded by $k^{\mathcal{O}(k)}$, from Lemmas 18 and 19 we get that the running time of $\mathcal{A}$ is $2^{2^{\mathcal{O}(kd)}} n^{\mathcal{O}(1)}$ and the success probability is at least $2^{-2^{ckd}}$ for some constant c. To get success probability $1 - 1/e$, we do $2^{2^{ckd}}$ many executions of $\mathcal{A}$ and output Yes if we succeed in at least one of the iterations and output No otherwise. Thus, running time of the overall algorithm is $2^{2^{\mathcal{O}(kd)}} n^{\mathcal{O}(1)}$. ◀

7 Conclusion

We took up weighted diverse variants of two classical matroid problems and the unweighted diverse variant of a classical graph problem. We showed that the two diverse matroid problems are NP-hard, and that the diverse graph problem cannot be solved in polynomial time even for the smallest sensible measure of diversity. We then showed that all three problems are FPT with the combined parameter (k, d) where k is the number of solutions and d is the diversity measure.

We conclude with a list of open questions:

- We showed that the unweighted, counting variant of Weighted Diverse Bases does not have a polynomial-time algorithm unless P = NP (Theorem 7). This is the case when all the weights are 1 and $d = 1$ or $d = 2$. Both the weighted and unweighted variants can be solved in polynomial time when $k = 1$ (the greedy algorithm) and $k = 2$ ((weighted) matroid intersection). What happens for larger, constant values of d and/or k? Till what values of d, k does the problem remain solvable in polynomial time? These questions are interesting also for special types of matroids. For instance, is there a polynomial-time algorithm that checks if an input graph has *three* spanning trees whose edge sets have pairwise symmetric difference at least d, or is this already NP-hard?

- A potentially easier question along the same vein would be: we know from Theorem 7 that Weighted Diverse Bases is unlikely to have an FPT algorithm parameterized by d alone. Is Weighted Diverse Bases FPT parameterized by k alone?
- Unlike for the other two problems, we don't have hardness results for Weighted Diverse Common Independent Sets for small values of k or d. Is Weighted Diverse Common Independent Sets FPT when parameterized by either d or k? Is this problem in P when all the weights are 1?

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