

Strong Hanani-Tutte for the Torus

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Abstract

If a graph can be drawn on the torus so that every two independent edges cross an even number of times, then the graph can be embedded on the torus.

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1 Introduction

Two edges in a graph are *independent* if they do not share a vertex. A drawing of a graph on a surface is *independently even*, or *iocr-0* for short, if every two independent edges cross¹ an even number of times in the drawing.

In the plane, there is a beautiful characterization of planar graphs known as the Hanani-Tutte theorem which says that a graph has an embedding in the plane (i.e., is planar) if and only if it has an independently even drawing in the plane. Equivalently, any drawing of a non-planar graph in the plane must contain two independent edges that cross oddly. The Hanani-Tutte theorem has a wide array of applications in computational and combinatorial geometry [32], some of which we discuss below.

There are several proofs of the Hanani-Tutte theorem, including the original 1934 proof by Hanani and the 1970 proof by Tutte; see [26] for more references. We also know that the result remains true for the projective plane² [4, 25]. On the other hand, counterexamples were found recently which show that the Hanani-Tutte theorem does not extend to orientable surfaces of genus 4 and higher [8]. This opened up a path to the study of approximate versions of the Hanani-Tutte theorem [9, 11].

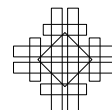
We complement these results by proving that the Hanani-Tutte theorem does extend to the torus.

► **Theorem 1.** *Let G be a graph. Suppose that G can be drawn on the torus so that every two independent edges cross evenly. Then G can be embedded on the torus.*

Among orientable surfaces, this leaves only the double and triple torus for which we do not yet know whether the Hanani-Tutte theorem holds. For non-orientable surfaces, all cases starting with the Klein bottle are open. Our approach extends and refines techniques developed in [4, 25] and other papers.

¹ Crossings for us are proper crossings, not shared endpoints or touching points.

² Equivalently, a sphere with a crosscap. We assume that the reader is familiar with the basic terminology of drawings and embeddings in surfaces. For background see [6, 21].



The proof of Theorem 1 is inductive, and for the induction to work we need to strengthen the result; this strengthened version is Theorem 4 in Section 3. For the induction, we carefully define a partial order on drawings of graphs on the torus, then consider a minimal counterexample to Theorem 4 with respect to the partial order. Our reductions are mostly, but not always, minor preserving.

In the base case of the proof, we work with drawings of subdivisions of $K_{3,t}$, $t \leq 6$ and K_5 , possibly with some added paths in the case of $K_{3,t}$. Reducing to the base case is a relatively smooth procedure using natural redrawing tools, up to the point when the graph is a subdivision of $K_{3,t}$ or K_5 with additional simple “bridges”. In this case an extensive case analysis seems to us the only way to proceed. The bulk of the full version of the paper deals with these cases. The main difficulty then lies in performing the reduction steps so that the case analysis is manageable.

Applications

Smorodinsky and Sharir [33] showed that if P is a collection of n points, and C a collection of m pseudo-disks in the plane such that every pseudo-disk in C passes through a distinct pair of points in P , and no pseudo-disk contains a point of P in its interior, then $m \leq 3n - 6$, where $3n - 6$ is just the maximal number of edges in a planar graph on n vertices. For pseudodisks in the projective plane, we have $m \leq 3n - 3$ [32]. Using Theorem 1, one can now obtain $m \leq 3n$ for pseudodisks in the torus.

A family of simply connected regions is *k-admissible* if each pair of region boundaries intersect in an even number of points, not exceeding k . Whitesides and Zhao [36] showed that the union of $n \geq 3$ planar *k-admissible* sets is bounded by at most $k(3n - 6)$ arcs, where $3n - 6$ is again the maximum number of edges in an n -vertex planar graph. This result was reproved by Pach and Sharir [23] using the Hanani-Tutte theorem. Consequently, we can get a bound of $k(3n - 3)$ for the projective plane [32] and $3kn$ for the torus.

Keszegh [17] gave a more uniform treatment of these types of results based on hypergraphs, which at the core again uses the Hanani-Tutte theorem. While he only states results for the plane, all of his material should lift to the projective plane and the torus based on the availability of the Hanani-Tutte theorem on those surfaces.

Known Results

The Hanani-Tutte theorem [2, 35] for the plane has been known for a while, with many proofs; for a survey of known results see [32]. It was shown to be true for the projective plane by Pelsmajer et al. [25] using the excluded minors for the projective plane, and later directly, without recourse to excluded minors, by É. Colin de Verdière et al. [4]. Excluded minors stop being useful at this point, since we do not know the complete list of excluded minors for the torus or any higher-order surfaces.

If we strengthen the assumption of the Hanani-Tutte theorem to also require adjacent edges to cross each other evenly, then embeddability follows, for any surfaces. This is known as the weak Hanani-Tutte theorem.³

► **Theorem 2** (Weak Hanani-Tutte for Surfaces [1, 28]). *If a graph can be drawn in a surface S (orientable or not) so that every two edges cross an even number of times, then the graph can be embedded in S with the same rotation system (if S is orientable), or the same embedding scheme (if S is non-orientable).*

³ Naming this variant “weak” is somewhat misleading, as it has a strengthened conclusion.

Unfortunately, the available proofs of Theorem 2 do not give any insight on how to establish the strong version on a surface (when it is possible). It does not even settle the seemingly easy question of whether a graph which can be drawn in a surface so that the only crossings are between adjacent edges, can always be embedded in that surface.

For work on applying Hanani-Tutte to different planarity variants, see [5, 12, 13, 14, 16, 31]. A variant of the (strong) Hanani-Tutte theorem in the context of approximating maps of graphs, which works on any surface, was announced in a paper co-authored by the first author [7].

Organization

We introduce the terminology and basic redrawing tools in Section 2 and 3, respectively. As a sample application, in Section 4 we extend the Hanani-Tutte theorem to the 1-spindle (a pseudosurface). In Section 5, we discuss the base case of our inductive proof of Theorem 1. In Section 6, we discuss the properties of a hypothetical minimal counterexample to Theorem 1. In Section 7 we outline the proof of Theorem 1. We conclude with open problems in Section 8.

2 Terminology, Definitions, and Basic Properties

For the purposes of this paper, graphs are simple (no multiple edges or loops), and surfaces are compact 2-manifolds (with or without boundary).

For a particular drawing D of a graph G on a surface S , we make the following definitions: The *crossing parity* of a pair of edges is the number of times the two edges cross modulo 2. Two edges form an *odd pair* if their crossing parity is 1. A subgraph (or single edge) is *even* if none of its edges belong to an odd pair in G .

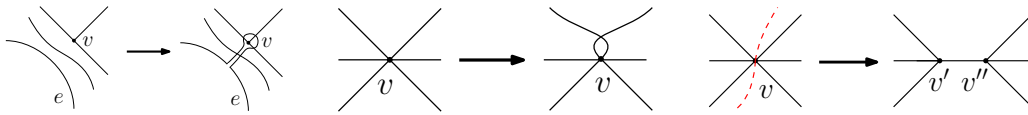
A closed curve γ on a surface S is *non-essential* if it forms the boundary of a (not necessarily connected) closed sub-manifold of S , or equivalently, if its complement in S can be two-colored so that path-connected components sharing a non-trivial part of γ receive opposite colors. In homological terms, a closed curve γ on S is *essential* if its homology class does not vanish over \mathbb{Z}_2 . A closed curve *separates the surface* if removing the curve from the surface disconnects the surface. A curve is *simple* if it is free of self-intersections. A closed simple curve γ separates the surface if and only if γ is non-essential. An essential simple closed curve is therefore *non-separating*.

We apply the same terminology to cycles in graph drawings, since a cycle determines a closed curve. We say a subgraph in a drawing is *essential* if it contains an essential cycle.

Closed curves in the plane are non-essential. Non-essential curves, on any surface, tend to be easier to handle when proving results similar to ours, because they cross every other closed curve an even number of times. (This is easy to see by two-coloring of the complement of the non-essential curve as described above.) The greater difficulty in proving Hanani-Tutte type results lies in the presence of essential curves.

Edge-vertex move

An *edge-vertex* (e, v)-*move* (also known as *van Kampen's finger-move*, or *edge-vertex switch*) is a generic deformation of the edge e in a drawing of G changing the crossing parity between e and all the edges incident to v , without changing any other crossing parities; see the left illustration in Figure 1. An edge-vertex move is performed as follows. Connect an interior point of e to v via a curve γ that does not pass through any vertices (and does not cross e). Then reroute e close to γ and around v (as shown in the illustration). Since e traverses γ twice, only the crossing parities of e with edges incident to v change.



■ **Figure 1** *Left:* An (e, v) -move. *Middle:* An edge-flip. *Right:* A vertex-split.

Edge-flip

An *edge-flip*, or *flip*, (at v) in a drawing is a redrawing operation that happens near a vertex v , and which takes two consecutive edges in the rotation at v and (locally) exchanges their position in the rotation at v ; see the middle illustration in Figure 1. As a result, the crossing parity between the two flipped edges changes, and no other crossing parities are affected.

Edge contraction and vertex split

A *contraction* of an edge $e = uv$ in a drawing of a graph is an operation that turns e into a vertex by moving v along e towards u while dragging all the other edges incident to v along e . If e is even, then contracting e in this fashion does not change the crossing parity of any pair of edges. Contraction may introduce multi-edges or loops at the vertices; we avoid this by only contracting partially, just enough to make uv crossing-free.

We will also often use the following operation which can be thought of as the inverse of a contraction in a drawing. To *split* a vertex v , we split its rotation into two contiguous parts, and then cut through the vertex to separate those two parts. This results in two vertices v' and v'' which we connect by a crossing-free edge $v'v''$ so that contracting $v'v''$ recovers the original rotation at v ; see the right illustration in Figure 1. Vertex splits are not unique.

Edge-vertex moves and contractions may introduce self-crossings of edges. Such self-crossings are easily resolved [15, Section 3.1]: remove the crossing, and reconnect the four severed ends so that the edge consists of a single curve. In this redrawing, essential cycles remain essential and non-essential cycles remain non-essential. (This is easy to see by two-coloring of the complement of the curve corresponding to a non-essential cycle.)

3 Redrawing iocr-0-Drawings on the Torus

In this section we establish some of the basic redrawing tools for iocr-0-drawings. We start with some results which (mostly) work on all surfaces, and may be useful for establishing Hanani-Tutte type results for surfaces other than the torus. We focus on orientable surfaces.

We will say that a vertex v in a drawing D of a graph on a surface is *even* if every two edges incident to v cross each other an even number of times; otherwise, v is *odd*.⁴

A drawing D_2 of a graph G in an orientable surface S is *compatible* with a drawing D_1 of G in S if every even vertex in D_1 is even in D_2 and the rotation at each even vertex in D_1 is preserved in D_2 . Note that the compatibility relation is transitive, but not necessarily symmetric. We define a notion of connectivity on even vertices: two even vertices u and v in a drawing of a graph are *evenly connected* if there exists a path connecting u and v consisting only of even vertices. An *evenly connected component* in a drawing is a maximal connected subgraph in the underlying abstract graph induced by a set of even vertices.

⁴ This will not conflict with the usual degree-based definition of even and odd vertices, since we will not be using that terminology.

A drawing D_2 of a graph G in an orientable surface S is *weakly compatible* with a drawing D_1 of G in S if every even vertex in D_1 is even in D_2 and, for every evenly connected component K in D_1 , either the rotation in D_2 at every vertex $v \in V(K)$ is the same as the rotation of v in D_1 , or the rotation in D_2 at every vertex $v \in V(K)$ is the reverse of the rotation of v in D_1 . Note that the weak compatibility relation is transitive, but not necessarily symmetric, just like compatibility.

We can now state a version of the Hanani-Tutte theorem on the plane that implies both the weak and the strong version. While the result follows from the proof of the Hanani-Tutte theorem in [26], it was first explicitly stated, and given a new proof, in [10].

► **Theorem 3** (The Unified Hanani-Tutte Theorem [10]). *If G has an iocr-0-drawing D in the plane, then G has an embedding in the plane compatible with D .*

Theorem 3 does not hold on any surface other than the plane [8, Theorem 7]. The counterexample requires compatibility; it fails for weak compatibility. If we assume that the graph is 3-connected, then weak compatibility can be achieved on the torus.

► **Theorem 4.** *If a 3-connected graph G has an iocr-0-drawing D in the torus, then G has an embedding in the torus that is weakly compatible with D .*

The next lemma is one of our main redrawing tools. It shows that we can always clear an essential cycle of crossings, in any surface. A precursor of this lemma, for the projective plane can be found in [25].

► **Lemma 5.** *Let D be an iocr-0-drawing of a graph G on a surface S . Suppose that C is an essential cycle in D . Then there exists an iocr-0-drawing D' of G compatible with D on S in which C is crossing free, and each cycle is essential in D' if and only if it is essential in D .*

As a first application of Lemma 5 we obtain the following corollary.

► **Corollary 6.** *If G has an iocr-0-drawing D on the torus containing an essential cycle C consisting of even vertices only, then G has an embedding on the torus that is compatible with D .*

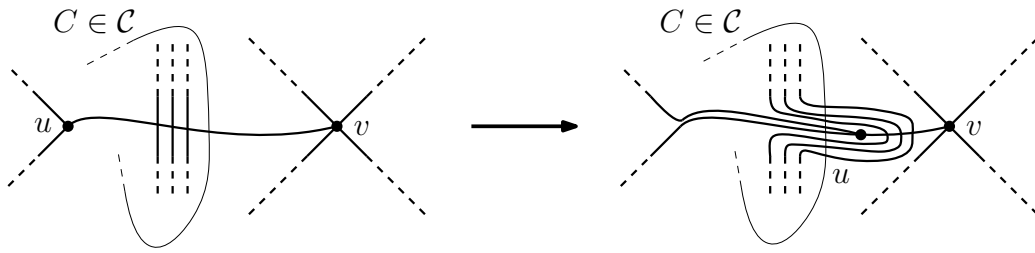
The following lemma shows, roughly speaking, that in an iocr-0-drawing, one vertex alone cannot make the difference between planarity and non-planarity. (It appeared, with a slightly different proof, for the projective plane in [25].)

► **Lemma 7.** *Let $x \in V(G)$ and $H = G - x$. Suppose there is an iocr-0-drawing D of G in a surface S such that H is non-essential. Then G has a plane embedding. If S is orientable, then the plane embedding of G restricted to H is compatible with D restricted to H .*

Lemma 7 implies that if a graph H has a non-essential embedding on a surface, then H has a compatible plane embedding.

Proof. We consider the surface S as a sphere with handles and crosscaps. We can choose a set \mathcal{C} of 1-sided and 2-sided essential curves so that cutting the surface along these curves in \mathcal{C} results in a planar surface, with a single hole for each crosscap and handle⁵: for each crosscap we choose a 1-sided closed curve cutting through the crosscap, and for each handle we pick two 2-sided closed curves (sharing a single point) so that cutting the surface along the two curves results in a single “square” boundary hole.

⁵ For non-orientable surfaces Mohar [20] calls these “planarizing system of disjoint curves”. If we deformed the curves of \mathcal{C} so that they all shared a single point, we’d get a set of generators of the fundamental group of the surface.



■ **Figure 2** Contracting the edge uv by pulling u along uv to v .

Given an edge uv , we may contract the edge by pulling u toward v until uv no longer crosses any curve of \mathcal{C} . For any edge e that crosses uv , when u reaches e then e will be deformed so that instead of u crossing e , e will get pulled along to just stay in front of u , see Figure 2. This may cause e to add new crossings with curves of \mathcal{C} , two crossings at a time, whenever u crosses a curve of \mathcal{C} . Thus the crossing parity between each edge of G not incident to u and each curve of \mathcal{C} will be unchanged by this operation.

Let F be a rooted maximum spanning forest in $H := G - x$. For each component of F , perform a breadth-first search transversal, contracting each edge as described towards the root of the component. After an edge e of F is contracted it has zero crossings with every curve of \mathcal{C} ; later contractions may add crossings between e and curves of \mathcal{C} but without affecting the crossing parity. Thus, at the end of this process, every edge of F crosses every curve of \mathcal{C} evenly.

Consider any edge $e \in E(H) - E(F)$. Since we assumed that H is non-essential (and that did not change during the redrawing we did), e must cross each $C \in \mathcal{C}$ evenly (if it did not, the (unique) cycle in $F \cup \{e\}$ is essential, a contradiction). In summary, every edge of H crosses each $C \in \mathcal{C}$ evenly.

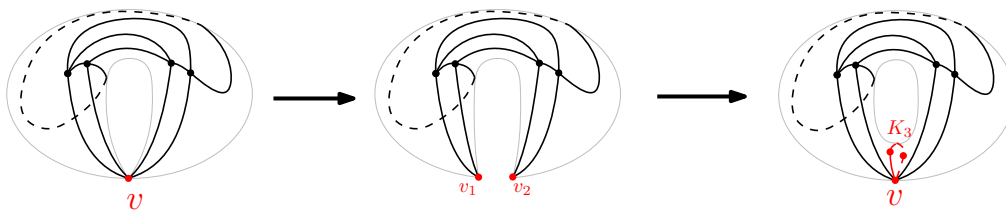
Now cut the surface along the curves in \mathcal{C} , and fill in each boundary hole with a disk, resulting in a sphere. For each edge-crosscap intersection, reconnect the ends with a curve passing straight through the disk; for each edge broken at a handle, reconnect it straight across the disk. We remove any resulting self-crossings using the standard method [15, Section 3.1]. Since each edge of H intersects each curve of \mathcal{C} evenly, its crossing parity with other edges does not change. In particular: if two edges did not cross oddly before the redrawing, but they do now, then both edges are incident to x , and even vertices, except possibly x , remain even.

We can then apply Theorem 3 to obtain a plane drawing of G . If S is orientable, then the subdrawing of H is compatible to the original drawing of H on S . ◀

The following result shows that an even tree (all its edges are even) can always be cleaned of crossings. The result is true for all surfaces, and it remains true for forests, but we will not need these stronger versions.

► **Lemma 8.** *If G has an iocr-0-drawing D that contains an even tree T , then there exists an iocr-0-drawing of G that is compatible with D in which T is free of crossings. Only edges incident to T are redrawn.*

A pair of edge-disjoint cycles C_1 and C_2 *touch* at a vertex $v \in V(C_1) \cap V(C_2)$ if in the rotation at v the edges of C_1 and C_2 do not interleave; otherwise, we say the cycles *cross* at v . A pair of *nearly disjoint* cycles is a pair of edge-disjoint cycles for which any shared vertices are even and touching.



■ **Figure 3** Surgery in the proof of Theorem 10. Eliminating the pinched point splits the vertex v into two vertices v_1 and v_2 that are subsequently merged using the handle. A 3-cycle K_3 is added around the handle passing through v .

► **Lemma 9.** *If G has an iocr-0-drawing on the torus containing two nearly disjoint essential cycles, then G has an embedding on the torus that is compatible with the iocr-0-drawing, and the two nearly disjoint cycles remain essential.*

This lemma implies that we can assume that a counterexample to the Hanani-Tutte theorem on the torus does not contain two vertex-disjoint essential cycles.

4 The 1-Spindle

Before we move on to the torus, we show how to apply our tools so far in a much simpler context than the torus (or the projective plane).

We show that there is a strong Hanani-Tutte theorem for the 1-spindle. The *1-spindle* is a pseudosurface obtained from the sphere by identifying two distinct points (the *pinchpoint*). Embeddings are defined as usual; for drawings, we allow at most one edge to pass through the pinchpoint.

► **Theorem 10** (Hanani-Tutte for 1-Spindle). *If a graph has an iocr-0-drawing on the 1-spindle, it can be embedded on the 1-spindle. The embedding is compatible with the original drawing, except for (possibly) the vertex at the pinchpoint.*

One might suppose that this result should follow directly from a Hanani-Tutte theorem for the projective plane or the torus, but if true, it does not seem to be immediately obvious.

Proof. Fix an iocr-0-drawing of G on the 1-spindle. We can assume that there is a vertex v at the pinchpoint. If not, there must be an edge e passing through the pinchpoint (otherwise, we have an iocr-0-drawing on the sphere, and we are done by the Hanani-Tutte theorem in the plane). Consider the subcurve γ of e between the pinchpoint and a vertex v . For any edge f which crosses γ oddly, we perform an (f, u) -move for every vertex u in G . This does not change the crossing parity of any pair of edges, but it does ensure that γ is an even curve. We can then partially contract e by pulling v along γ onto the pinchpoint. Since γ is even, the drawing remains iocr-0.

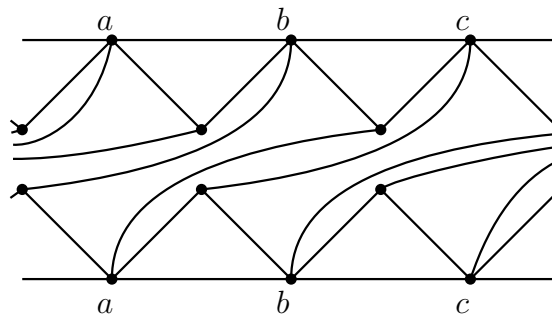
Refer to Figure 3. Remove the pinchpoint (splitting v into two copies), giving us a drawing on a sphere. Add a handle close to the two copies (close to where they were split), and move them back together, merging them into a single vertex. This gives us an iocr-0-drawing of G on the torus. Add a crossing-free essential cycle, say a 3-cycle K_3 , through v on the handle; let G^* denote the new graph. If the drawing of $G - v$ (as part of G^*) does not contain an essential cycle, then G is planar, by Lemma 7, and we are done. So the drawing of $G - v$ contains an essential cycle, implying that the drawing of G^* contains two disjoint essential cycles. By Lemma 9 we can find a compatible embedding of G^* in the torus. Note that in that embedding the essential cycle K_3 is still essential (and free of crossings), so we can contract it until it becomes a point, which yields an embedding of G on the 1-spindle. ◀

5 Hanani-Tutte for Some Kuratowski Minors

Fulek and Kynčl [11] showed that the Hanani-Tutte theorem is true for any surface if we restrict ourselves to the graphs known as Kuratowski minors, which include $K_{3,t}$ for any $t \geq 3$.

► **Lemma 11** ([11]). *For $t \geq 7$, $K_{3,t}$ does not admit an iocr-0-drawing on the torus.*

Since $K_{3,7}$ has no toroidal embedding, Lemma 11 implies that the Hanani-Tutte theorem on the torus is true for all $K_{3,t}$, $t \geq 7$ (and their subdivisions). Figure 4 shows a toroidal embedding of $K_{3,6}$ with an 3-cycle added to the vertices of degree 6, so Lemma 11 is sharp even if with that added 3-cycle.



■ **Figure 4** A torus embedding of $K_{3,6} \cup K_3$, with the union taken over the three vertices of degree 6.

As a base case for our proof of Theorem 1, we need a unified Hanani-Tutte result like Theorem 3 for Kuratowski minors on the torus. Fulek and Kynčl [8] showed that this is not possible, even for a $K_{3,4}$: there is an iocr-0-drawing of $K_{3,4}$ on the torus which does not have a compatible embedding. We can show, however, that there always is a *weakly* compatible embedding of $K_{3,n}$ up to $n = 6$ which is sharp, since $K_{3,7}$ has no embedding on the torus.

We establish a slightly stronger result. Let a, b , and c be the three vertices of degree n in $K_{3,n}$. We call a graph a $K_{3,n}$ with *bracers* (at a, b, c) if it is the union of the $K_{3,n}$ with (any number of) internally-disjoint paths of length at most two added between any two of $\{a, b, c\}$ (no multiple edges). These paths are the *bracers*.

► **Lemma 12.** *Let G be a subgraph of a subdivision of a $K_{3,n}$ with bracers at $\{a, b, c\}$. For every iocr-0-drawing of G on the torus, there exists a weakly compatible embedding of G .*

The result remains true for bracers of arbitrary length, but we will not need that.

Proof. We can assume that G contains no vertex v of degree 0 or 1, since removing such a vertex cannot make another even vertex odd, and any embedding of $G - v$ can easily be extended to G . If v is a vertex of degree 2 with neighbors u, w with $uw \notin E(G)$, then we can suppress v by similar reasoning. Thus we may assume that G is an induced subgraph of a $K_{3,n}$ with bracers.

Let $S = \{a, b, c\}$. We want to replace bracers of length 1 with bracers of length 2. So suppose there is an edge uw with $u, w \in S$. If necessary, we use edge-flips at u to ensure that uw crosses every edge incident to u evenly. We can then subdivide uw with a vertex v placed very close to u ; then even vertices stay even, and after embedding we can suppress v . Thus we may assume that all bracers are paths of length 2. Finally, we may assume that all the vertices of degree at most 3 are even, by performing edge-flips if necessary. So any odd vertices must belong to S .

If S contains no even vertices, then each even vertex of G is its own evenly connected component. Since any two rotations of a degree-3 vertex are weakly compatible, any embedding of G on the torus suffices, which we have from Figure 4. If S contains only even vertices, we are done by Theorem 2.

Suppose that S contains exactly one odd vertex. If there is an essential cycle that avoids the odd vertex in S , we are done by Corollary 6. Otherwise, all essential cycles pass through the odd vertex in S , in which case we are done by Lemma 7.

This leaves us with the case that S contains two odd vertices and one even vertex. This case is more complicated and its proof can be found in the full version of the paper. ◀

We also need a unified Hanani-Tutte theorem for K_5 .

► **Lemma 13.** *For every iocr-0-drawing D of a subgraph G of a subdivision of K_5 on the torus, there exists an embedding of G on the torus that is compatible with D .*

6 Minimal Counterexamples

In this section we collect properties of a minimal counterexample to the Hanani-Tutte theorem on the torus. We order graphs first by their number of isolated vertices (**increasing**), then by the number of edges (**increasing**), and finally by the number of vertices (**decreasing**). We denote by \prec the strict partial ordering based on these three numbers, and refer to a \prec -minimal graph (with certain properties). We write *minimal*, rather than \prec -minimal.

Since isolated vertices do not affect embeddability on a surface, nor the Hanani-Tutte criterion, a minimal counterexample contains no isolated vertices. Graphs without isolated vertices are then ordered by number of edges (increasing), and number of vertices (decreasing); equivalently, we use the (strict, partial) lexicographic order on $(|E(G)|, 2|E(G)| - |V(G)|)$; since graphs without isolated vertices satisfy $|V(G)| \leq 2|E(G)|$, this order is well-founded. Note that if H is a proper subgraph of G , then $H \prec G$, simply because it has fewer edges (since G has no isolated vertices).

Section 6.1 presents some basic properties of minimal counterexamples with respect to cycles and cuts. In Section 6.2 we identify what we call an X -configuration, which must occur in a minimal counterexample to Theorem 1, the Hanani-Tutte theorem on the torus. One issue we will not further discuss in this extended abstract, is that the proof of Theorem 1 requires a strengthened assumption on (weakly) compatible embeddings, for which we do not know how to show that an X -configuration occurs.

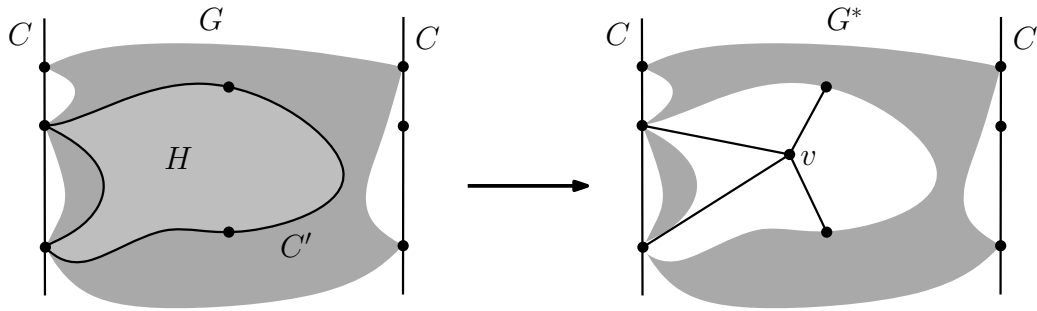
6.1 Nearly Disjoint Cycles and Cuts

The next lemmas are true whether “weakly” is included or omitted.⁶

► **Lemma 14.** *If G is a minimal graph that has an iocr-0-drawing D on the torus, but does not have a [weakly] compatible embedding on the torus, then D does not contain a pair of nearly-disjoint cycles at least one of which is essential.*

We use the previous lemma to repeatedly reduce a minimal counterexample G to the Hanani-Tutte theorem on the torus, to show that it does not exist.

⁶ It is tempting to assume that the weakly compatible version of the lemma implies the compatible version, but this is not necessarily the case, since the minimal counterexample in the two cases may be different, and therefore have different properties.



■ **Figure 5** Construction of a reduced graph G^* drawn on a cylinder, illustrating a case in the omitted proof of Lemma 15. C and C' are weakly disjoint cycles. C is essential so we are able to clear it of crossings, then cut along it, creating a cylinder bounded by two copies of C . Replacing the subgraph H interior to C' with a vertex v constructs G^* from G . Note that $G^* \prec G$.

The following lemma allows us to focus on 2-connected (for compatibility and weak compatibility) or 3-connected counterexamples (in general).

► **Lemma 15.** *If G is a minimal graph D that has an iocr-0-drawing D on the torus, but does not have a [weakly] compatible embedding on the torus, then G is 2-connected and it has no 2-cut consists of two odd vertices of D .*

If we do not require the embedding to be [weakly] compatible, then a minimal counterexample must be 3-connected.

6.2 The X -Configuration

The Hanani-Tutte theorem holds for cubic graphs on arbitrary surfaces; this is because vertices of degree 3 can be made even by edge-flips, at which point Theorem 2 can be applied.

Consider a vertex v incident to four edges e_1, e_2, e_3, e_4 which occur in (cyclic) order e_1, e_2, e_3, e_4 in the rotation at v . Suppose that e_1 and e_3 cross oddly, and every other pair of (these four) edges crosses evenly. No matter what edge-flips are performed at v , there will always remain at least one pair of edges crossing oddly. Moreover, this configuration is the unique obstacle to a vertex being even, up to edge-flips: suppose v is incident to four edges. Using edge-flips we can make three consecutive edges at v cross evenly with each other. Say the edges are e_1, e_2, e_3, e_4 , in this order, and every two of e_1, e_2 and e_3 cross evenly. If e_4 crosses exactly e_2 oddly, we are done. If it crosses e_3 and e_1 oddly, we move the end of e_4 once around v , so that e_4 crosses exactly e_2 oddly. In all other cases, e_4 and the edges that it crosses oddly form a consecutive subset of the cyclic order, so e_4 can be made to cross all of e_1, e_2 and e_3 evenly using edge-flips.

Hence, the edges incident to a vertex v of degree 4 are equivalent via edge flips to one of two configurations, either with no odd crossings or with a single odd crossing between a pair of non-consecutive edges in the rotation at v . The next lemma shows that four edges are always the obstacle to making a vertex even by flips, independent of its degree.

► **Lemma 16.** *If a vertex in a drawing cannot be made even by flips, then it is incident to four edges which cannot be made to cross each other evenly by flips.*

Lemma 16 can also be found in the full version of [7, Claim 8].

The core of the inductive proof will be working with a specific configuration in iocr-0-drawings: Two essential cycles C_1 and C_2 which intersect in a single vertex v so that the edges of C_1 and C_2 incident to v cannot be made to cross each other evenly by edge-flips at v . We call such a pair (C_1, C_2) an X -configuration, see Figure 6.



■ **Figure 6** An X -configuration (left). An illustration of the construction of an X -configuration (C_1, C_2) in the proof of Lemma 17; crossings are not shown.

► **Lemma 17.** *If G is a minimal 3-connected graph that has an iocr-0-drawing D on the torus, but does not have an embedding on the torus, then D contains an X -configuration (C_1, C_2) .*

We emphasize that Lemma 17 does not require the embedding to be compatible, or weakly compatible.

Proof. If every vertex of G in an iocr-0-drawing of G on the torus can be made even by flips, we can obtain an embedding of G by Theorem 2, the weak version of the Hanani-Tutte theorem. Therefore, there exists a vertex v for which this is not the case. By Lemma 16 there are four edges $e_i = vu_i$, $1 \leq i \leq 4$ incident to v which cannot all be made to cross each other evenly by edge-flips.

Refer to Figure 6 (on the right). Using the 3-connectedness of G , we will find edge-disjoint cycles C_1, C_2 intersecting in v as follows: Consider a minimal path in G connecting two u_i vertices that avoid v ; without loss of generality this path is P_{12} between u_1 and u_2 in $G - \{v, u_3, u_4\}$. Let C_{12} denote the cycle obtained as the edge union of $E(P_{12}), \{e_1\}$ and $\{e_2\}$. Let P_3 denote a path in $G - \{v, u_4\}$ between u_3 and $V(C_{12}) - \{v\}$. Let w denote the end vertex of P_3 on $C_{12} - \{v\}$. Finally, let P_4 denote a path in $G - \{v, w\}$ between u_4 and $(V(P_3) \cup V(C_{12})) - \{v, w\}$. No matter where P_4 ends, the edge union of $E(C_{12}), E(P_3), E(P_4), \{e_3\}$ and $\{e_4\}$ contains a pair of cycles C_1 and C_2 meeting exactly in v .

As explained before Lemma 16, using edge-flips we can assume that the e_i cross each other evenly, with the exception of one pair of edges which are not consecutive at v . If the ends of C_1 and C_2 do not alternate at v , then the two edges crossing oddly cannot both belong to the same C_i , since they would then be consecutive; therefore one belongs to C_1 and the other to C_2 , which implies that C_1 and C_2 cross each other oddly (all edges of the cycles not incident to v cross evenly, and the two cycles do not cross at v), hence both cycles are essential. If the ends of C_1 and C_2 do alternate at v , then the two edges crossing oddly must belong to the same cycle (since otherwise they would be consecutive), so again C_1 and C_2 cross each other oddly (at v , no other odd crossings), and both are essential. ◀

7 Proof of Theorem 1

Assume, for a contradiction, that Theorem 1 fails. Then there is a graph G with an iocr-0-drawing D on the torus, such that G cannot be embedded on the torus. Let G be a minimal such counterexample, with iocr-0-drawing D . By Lemma 15 we know that G is 3-connected, and by Lemma 17 that D contains an X -configuration (C, C_0) . So there is a counterexample to the following statement, a strengthened version of Theorem 1:

Let D be an iocr-0-drawing of a graph G containing an X -configuration (C, C_0) , then G has a weakly compatible embedding on the torus.

If the statement is true, we say that G – or more precisely, (G, D, C, C_0) – satisfies *HT-XWC*, acronym for Hanani-Tutte–X-Weakly-Compatibility. So we know that there is a (G, D, C, C_0) which does not satisfy HT-XWC, and therefore a minimal such counterexample.

The omitted technical parts of the proof proceed by reducing the 4-tuple (G, D, C, C_0) using Lemma 14 and similar technical tools. In the end, an application of Lemma 12 or Lemma 13 shows that no counterexample violating HT-XWC exists, which is the desired contradiction.

8 Open Questions

8.1 Crossing Elimination

Theorem 1 implies that if a graph can be drawn on the torus so that its only crossings are between adjacent edges, then it is toroidal. This might appear to be intuitively clear, but for the plane, for the projective plane, and now for the torus, we only know it to be true by virtue of their respective Hanani-Tutte theorems; even for the plane no simpler proof is known. Since we know that the Hanani-Tutte theorem does not hold for orientable surfaces of genus at least 4 [8], this begs the question whether we can always remove adjacent crossings.

► **Conjecture 18.** *If a graph can be drawn in a surface without any independent crossings, then the graph is embeddable in that surface.*

The counterexample from [8] requires independent edges to cross (evenly), so it does not resolve this conjecture.

The Hanani-Tutte theorem is related to the *independent odd crossing number* $\text{iocr}(G)$ of a graph G , the fewest number of pairs of independent edges that have to cross oddly in a drawing of G (see [24, Section 6]). The Hanani-Tutte theorem then states that $\text{iocr}(G) = 0$ implies that $\text{cr}(G) = 0$, where $\text{cr}(G)$ is the traditional crossing number of G . It is even true that $\text{iocr}(G) = \text{cr}(G)$ for $\text{iocr}(G) \leq 2$ [29], though we know that there are graphs G for which $\text{iocr}(G) < \text{cr}(G)$ [27]. The two crossing numbers cannot be arbitrarily far apart though, one can show that $\text{cr}(G) \leq \binom{2 \text{iocr}(G)}{2}$ [29]. It is not known whether similar bounds, or any bounds, for that matter, hold for other surfaces, not even the projective plane. (The subscript in crossing number variants indicates the surface we work on.)

► **Conjecture 19.** $\text{cr}_\Sigma(G) \leq \binom{2 \text{iocr}_\Sigma(G)}{2}$, where Σ is the projective plane, or the torus.

We also now have a crossing lemma for iocr on the torus.

► **Corollary 20 (Crossing Lemma).** $\text{iocr}_T(G) \geq cm^3/n^2$, where T is the torus, $c = 1/64$, $m = |E(G)|$, and $n = |V(G)|$.

The proof follows from the standard crossing lemma argument done carefully combined with Theorem 1 (see the section on crossing lemma variants in [30]). Since iocr lower bounds several other crossing number variants, such as iacr , pcr , and pcr_- this also implies a crossing lemma for these variants. Also see [18] for a sketch of an argument that shows a crossing lemma for iocr for arbitrary surfaces, but without explicit constants c .

The proof of Theorem 1 is algorithmic in the sense that given an iocr -0-drawing of G we can find an embedding of G in the torus (all reductions in the lemmas are constructive in that sense, so the minimal counterexample assumption can be turned into a recursion). This does not imply that testing whether a graph can be embedded in the torus lies in polynomial time. The planar Hanani-Tutte theorem can be turned into a linear system of equations over \mathbb{Z}_2 , which is solvable in polynomial time (see [32, Section 1.4.2]). Unfortunately, modeling the

handle of the torus requires quadratic equations, which loses us polynomial-time solvability. This is similar to the situation for the projective plane, where the projective handle also leads to quadratic equations.

► **Question 21.** *Can the Hanani-Tutte criteria for the projective plane and the torus be turned into a polynomial-time test?*

Such tests would not be competitive with existing algorithms – the running time in the plane is $O(n^6)$, but they have the potential to be significantly simpler (as is true for the plane).

8.2 Arf and Approximating the Hanani-Tutte Theorem

The fact that the Hanani-Tutte theorem cannot be extended to all orientable surfaces has some positive practical consequences. Some results about graphs embedded on a surface remain true if we only require that the graph has an independently even drawing on the surface, and this is a strictly larger class of graphs in general (starting at genus 4).

One notable example is the Arf invariant formula for the number of perfect matchings in a graph embedded on an orientable surface [19, Remark 1.4, Theorem 1]⁷, see also [3, 22]. The proof of the similar result by Tesler [34], which applies to all closed surfaces, still works for independently even drawings instead of embeddings. In both cases, the complexity of the formula depends exponentially on the genus of the underlying surface, so for graphs which have an independently even drawing in a surface in which they cannot be embedded, the complexity of the formula is improved; its length does not depend on the genus of the graph, but rather its \mathbb{Z}_2 -genus, the smallest genus of a surface on which the graph has an independently even drawing.

It is therefore interesting to study how large the gap between the genus and the \mathbb{Z}_2 -genus of a graph may be. It is known that the gap can be at least linear [8, Corollary 4.1]. There also is an upper bound on this gap, but it is not effective [11].

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⁷ The authors attribute the extension of the result to independently even drawings to Norine. The extension is even more general, since instead of independently even drawings Norine considers so-called (perfect) matching even drawings, in which every perfect matching induces an even number of crossings. It is easy to see that independently even drawings form a proper sub-class of matching even drawings on every surface.

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