On Rich Points and Incidences with Restricted Sets of Lines in 3-Space

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Abstract

Let L be a set of n lines in $\mathbb{R}^3$ that is contained, when represented as points in the four-dimensional Plücker space of lines in $\mathbb{R}^3$, in an irreducible variety T of constant degree which is *non-degenerate* with respect to L (see below). We show:

(1) If T is two-dimensional, the number of r-rich points (points incident to at least r lines of L) is $O(n^{4/3+\varepsilon}/r^2)$, for $r \geqslant 3$ and for any $\varepsilon > 0$, and, if at most $n^{1/3}$ lines of L lie on any common regulus, there are at most $O(n^{4/3+\varepsilon})$ 2-rich points. For r larger than some sufficiently large constant, the number of r-rich points is also $O(n/r)$.
As an application, we deduce (with an ε-loss in the exponent) the bound obtained by Pach and de Zeeuw [16] on the number of distinct distances determined by n points on an irreducible algebraic curve of constant degree in the plane that is not a line nor a circle.

(2) If T is two-dimensional, the number of incidences between L and a set of m points in $\mathbb{R}^3$ is $O(m+n)$.

(3) If T is three-dimensional and nonlinear, the number of incidences between L and a set of m points in $\mathbb{R}^3$ is $O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right)$, provided that no plane contains more than s of the points. When $s = O(\min\{n^{3/5}/m^{2/5}, m^{1/2}\})$, the bound becomes $O(m^{3/5}n^{3/5} + m + n)$.

As an application, we prove that the number of incidences between m points and n lines in $\mathbb{R}^4$ contained in a quadratic hypersurface (which does not contain a hyperplane) is $O(m^{3/5}n^{3/5}+m+n)$.

The proofs use, in addition to various tools from algebraic geometry, recent bounds on the number of incidences between points and algebraic curves in the plane.

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1 Introduction

The setup: Incidences between a set of points and a restricted set of lines in $\mathbb{R}^3$. Let P be a set of m points and L a set of n lines in $\mathbb{R}^3$. We consider the problem of obtaining sharp incidence bounds between the points of P and the lines of L, when the lines of L, considered as points in the four-dimensional Plücker space of lines in $\mathbb{R}^3$, are restricted to lie on a two- or three-dimensional constant-degree algebraic variety T. The topic of incidences between points and lines is a fundamental topic in incidence geometry, significantly boosted since Guth and Katz's seminal work [11] on point-line incidences in $\mathbb{R}^3$. Instead of asking for a bound on the number of incidences between points and lines, we can ask, for each $r \geqslant 3$, for a bound on the number of *r-rich points* in a set of lines, which are the points that are incident to at least r of the lines. As it turns out, the two questions are equivalent. The related, and finer problem of bounding the number of 2-rich points, determined by a set of n lines in $\mathbb{R}^3$, studied in [11], is also discussed in this paper, under the restricted setup considered here. Building on recent works of Sharir and Zahl [25] and Zahl [29], we are able to improve Guth and Katz's point-line incidence bounds when the lines in L are restricted to lie on a two- or three-dimensional variety T in the Plücker space.

Background: Points and curves, the planar case. The study of incidences between points and curves has a rich history, starting with the simplest instance of points and lines in the plane, where we have (see also [4, 27]):

▶ **Theorem 1** (Szemerédi and Trotter [28]). *The maximum number of incidences between m points and n lines in the plane is $\Theta(m^{2/3}n^{2/3} + m + n)$.*

In fact, an equivalent formulation of Szemerédi-Trotter theorem asserts that, given n lines in the plane, the number of points that are incident to at least r of the lines, for any parameter $2 \leqslant r \leqslant n$, which we call *$r$-rich points* and denote the set of these points by $P_{\geqslant r}(L)$, satisfies

$$|P_{\geqslant r}(L)| = O\left(\frac{n^2}{r^3} + \frac{n}{r}\right). \tag{1}$$

Still in the plane, Pach and Sharir [17] extended this bound to incidence bounds between points and curves with k *degrees of freedom*, namely, for each set of k distinct points, there are only $\mu = O(1)$ curves that pass through all of them, and each pair of curves intersect in at most μ points; μ is called the *multiplicity* (of the degrees of freedom). Here is their result, tailored to the case of algebraic curves.

▶ **Theorem 2** (Pach and Sharir [17]). *Let P be a set of m points in $\mathbb{R}^2$ and let $\mathcal{C}$ be a set of n bounded-degree algebraic curves in $\mathbb{R}^2$ with k degrees of freedom and with multiplicity μ. Then (where the constant of proportionality depends on k and μ)*

$$I(P,\mathcal{C}) = O\left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + n\right).$$

Except for the case $k = 2$ (lines have two degrees of freedom), the bound is not known, and is strongly suspected not to be tight in the worst case (see [1, 2, 15] for an improvement for the case of circles and similar curves).

Recently, Sharir and Zahl [25] have considered general families of constant-degree algebraic curves in the plane that belong to an *s-dimensional family of curves*. This means that each curve in such a family can be represented by a constant number of real parameters, so that, in this parametric space, the points representing the curves lie in an s-dimensional algebraic variety $\mathcal{F}$ of some constant degree (the so-called "complexity" of $\mathcal{F}$). See [25] for details.

▶ **Theorem 3** (Sharir and Zahl [25]). *Let $\mathcal{C}$ be a set of n algebraic plane curves that belong to an s-dimensional family $\mathcal{F}$ of curves of maximum constant degree E, no two of which share a common irreducible component, and let P be a set of m points in the plane. Then, for any $\varepsilon > 0$, the number $I(P,\mathcal{C})$ of incidences between the points of P and the curves of $\mathcal{C}$ satisfies*

$$I(P,\mathcal{C}) = O\Big(m^{\frac{2s}{5s-4}} n^{\frac{5s-6}{5s-4}+\varepsilon} + m^{2/3}n^{2/3} + m + n\Big),$$

where the constant of proportionality depends on ε, s, E, and the complexity of the family $\mathcal{F}$.

Except for the factor $O(n^{\varepsilon})$, this is a significant improvement over the bound in Theorem 2 (for $s \geqslant 3$) when $\mathcal{C}$ has $k = s$ degrees of freedom (as it often does).

Incidences with lines in three dimensions. The groundbreaking work of Guth and Katz [11] implies[1] the sharper bound $O(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + m + n)$ on the number of incidences between m points and n lines in $\mathbb{R}^3$, provided that no plane contains more than q of the given lines. We use the following variant (see the full version [21] for the proof).

▶ **Theorem 4.** *Let P be a set of m points in $\mathbb{R}^3$, and let L be a set of n lines in $\mathbb{R}^3$, so that no 2-flat contains more than s points of P. Then*

$$I(P, L) = O(m^{1/2}n^{3/4} + m^{1/3}n^{2/3}s^{1/3} + m + n).$$

Plugging the bound of Theorem 4 into the proof of [23, Theorem 1.3(a)], we get

▶ **Theorem 5.** *Let P be a set of m points and L a set of n lines in $\mathbb{R}^d$, for $d \geqslant 3$, so that all the points and lines lie in a two-dimensional algebraic variety V of degree D that does not contain any 2-flat, and so that no 2-flat contains more than s points of P. Then*

$$I(P, L) = O(m^{1/2}n^{1/2}D^{1/2} + m^{1/3}D^{4/3}s^{1/3} + m + n).$$

Guth and Katz's work has lead to many recent works on incidences between points and lines or other curves in three and higher dimensions; see [3, 12, 19, 22, 23, 20] for a sample.

Of particular significance is the recent work of Guth and Zahl [12] on the number of 2-rich points in a collection of algebraic curves of constant degree, namely, points incident to at least two of the given curves, which extends Guth and Katz's bound of $O(n^{3/2})$, obtained for the case of lines, when no plane or regulus contains more than $O(n^{1/2})$ lines [11]. The extension requires analogous (but stricter) restrictive assumptions (concerning surfaces that are doubly ruled by the given family of curves).

Our new bounds require the extension to three dimensions of the notions of having k degrees of freedom and of being an s-dimensional family of curves. The definitions of these concepts, as given above for the planar case, extend, basically verbatim, to three (or higher) dimensions, but, even in typical situations, these two concepts do not coincide anymore.

Our results. We obtain improved incidence bounds when the lines of L, as points in Plücker space, lie on a two- or three-dimensional variety T. When T is two-dimensional and non-planar, the number of r-rich points is $O(n^{4/3+\varepsilon}/r^2)$, for $r \geqslant 3$ and for any $\varepsilon > 0$, and, if at most $n^{1/3}$ lines of L lie on any common regulus, there are at most $O(n^{4/3+\varepsilon})$ 2-rich points. For r larger than some sufficiently large constant, the number of r-rich points is also $O(n/r)$, which is a better bound when $r = O(n^{1/3})$. These bounds improve significantly, for the restricted context at hand, the bound $O(n^{3/2}/r^2)$ due to Guth and Katz [11] (which holds when no plane or regulus contains more than $O(n^{1/2})$ lines). Moreover, the number of incidences between L and a set of m points in $\mathbb{R}^3$ is $O(m+n)$, again a significant improvement, in our context, over the previous bound in [11].

As an application, we show that the number of distinct distances determined by n points on an irreducible algebraic curve of constant degree in the plane that is not a line nor a circle, is $\Omega(n^{4/3-\varepsilon})$, for any $\varepsilon > 0$, which is (with an ε-loss in the exponent) the bound obtained by Pach and de Zeeuw [16].

[1] This bound is not explicitly stated in [11], but it readily follows from the analysis given there, and by now it is generally attributed to that work.

If T is three-dimensional and nonlinear, the number of incidences between L and a set of m points in $\mathbb{R}^3$ is $O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right)$, provided that no plane contains more than s of the points. When $s = O(\min\{n^{3/5}/m^{2/5}, m^{1/2}\})$, the bound becomes $O(m^{3/5}n^{3/5} + m + n)$.

An interesting novel feature of our results is that, like Theorem 4, it is obtained under an assumption that restricts the number of *points* that can lie on a common plane (instead of restricting the number of coplanar lines in the previous studies). Very few earlier works have used this kind of restriction; see Elekes et al. [5] for one of the few exceptions.

Similar bounds have recently been obtained by the authors for other special cases of the incidence problem [24, 25], using related but different approaches.

As an application, we prove that the number of incidences between m points and n lines in $\mathbb{R}^4$ contained in a quadratic hypersurface (which does not contain a hyperplane) is $O(m^{3/5}n^{3/5} + m + n)$.

All our bounds are significant improvements, under the restricted scenarios assumed in this work, over the standard incidence bounds in three dimensions, and shed, as we believe, new light on the structure of point-line incidences in three dimensions.

As is standard in the "modern" study of incidence geometry, the analysis is based on the *polynomial partitioning* technique (see [10, 11] for details), combined with a variety of tools from algebraic geometry. Due to lack of space, some details are missing in this version; they can be found in the full version [21].

2 Rich points determined by a two-dimensional family of lines

We first remark that, wherever needed in the analysis, we switch to the (projective 3-space over) the complex field, which simplifies it and lets us use numerous tools from algebraic geometry, available in this setting. The passage from the complex projective setup back to the real affine one is easy – the former is a generalization of the latter. The real affine setup is needed only for constructing a polynomial partitioning, which is meaningless over $\mathbb{C}$. Once we are, say, within the zero set $Z(f)$ of the partitioning polynomial f, we can switch to the complex projective setup, and reap the benefits just noted.

As already said, we parameterize lines in three dimensions by their *Plücker coordinates*, as follows (see, e.g., Griffiths and Harris [9, Section 1.5]). For a pair of distinct points $x, y \in \mathbb{P}^3$, given in projective coordinates as $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, let $\ell_{x,y}$ denote the (unique) line in $\mathbb{P}^3$ incident to both x and y. The Plücker coordinates of $\ell_{x,y}$ are given in projective coordinates in $\mathbb{P}^5$ as $(\pi_{0,1}, \pi_{0,2}, \pi_{0,3}, \pi_{2,3}, \pi_{3,1}, \pi_{1,2})$, where $\pi_{i,j} = x_i y_j - x_j y_i$. Under this parameterization, the set of lines in $\mathbb{P}^3$ corresponds bijectively to the set of points in $\mathbb{P}^5$ lying on the *Klein quadric* Q given by the quadratic equation

$$\pi_{0,1}\pi_{2,3} + \pi_{0,2}\pi_{3,1} + \pi_{0,3}\pi_{1,2} = 0 \tag{2}$$

(which is indeed always satisfied by the Plücker coordinates of a line).

Given a surface V in $\mathbb{P}^3$, the set of lines fully contained in V, represented by their Plücker coordinates in $\mathbb{P}^5$, is a subvariety of the Klein quadric Q, which is denoted by $F(V)$, and is called the *Fano variety* of V; see Harris [13, Lecture 6, page 63] and [13, Example 6.19].

Let H denote a plane in $\mathbb{R}^3$, and let H^* denote the 2-flat in the Plücker coordinates consisting of the points that represent the lines fully contained in H (see Rudnev [18] for why H^* is indeed a 2-flat and for more details).

For a set L of lines, we put $\mathcal{H}(L) = \{H^*_{\ell,\ell'} \mid \ell, \ell' \in L \text{ and } \ell,\, \ell' \text{ are coplanar}\}$, where for coplanar lines ℓ, ℓ', $H_{\ell,\ell'}$ is the (unique) 2-flat containing ℓ and ℓ'.

In this paper, we study incidences between a set of points $P \subset \mathbb{R}^3$, and a set of lines L in $\mathbb{R}^3$, whose Plücker images lie on some irreducible algebraic subvariety T of the Klein quadric Q, which is of constant degree, and which has dimension either 2 or 3.

In this section we restrict ourselves to the case where $\dim(T) = 2$. For a set L of lines, we say that the (two-dimensional) variety T is *non-degenerate* with respect to L if

(i) T is irreducible of constant degree,

(ii) T is not a 2-flat, and

(iii) the intersection of T with each 2-flat $H^* \in \mathcal{H}(L)$ consists of $O(1)$ Plücker points.

Note that condition (iii) is what one would expect to hold in a generic situation in a four-dimensional space. The simpler case where T is a 2-flat can be handled via the Szemerédi–Trotter theorem (Theorem 1) [28], but we will not consider this case in this work.

► **Theorem 6.**

(a) *Let L be a set of n lines in $\mathbb{R}^3$, such that, in Plücker space, L is a subset of some two-dimensional variety T that is non-degenerate with respect to L. Then the number of r-rich points determined by L is $|P_{\geqslant r}(L)| = O(n^{4/3+\varepsilon}/r^2)$, for any $\varepsilon > 0$ and $r \geqslant 3$.*

(b) *If the number of lines of L contained in any common regulus[2] is at most $n^{1/3}$ then the number of 2-rich points determined by L is $|P_{\geqslant 2}(L)| = O(n^{4/3+\varepsilon})$, for any $\varepsilon > 0$.*

Proof. First here is a high-level overview of the proof. After a pruning step, we may assume that the set γ_ℓ, for a line $\ell \in L$, of the lines coplanar with ℓ and lying in T, is a one-dimensional curve in T. An r-rich point generates $\Omega(r^2)$ incidences between the Plücker points of the lines of L and the curves γ_ℓ, so it suffices to bound the number of such incidences. There are two kinds of curves, those that represent the lines in one ruling of some regulus, and those that do not. For r-rich points, with $r \geqslant 3$, only the latter kind of curves matter, and a suitable application of Theorem 3 allows us to obtain an upper bound for the number of such incidences. For 2-rich points (part (b) of the theorem), the regulus-curves also play a part, and the analysis is complicated because these curves do not have to be distinct. Still, the assumptions in (b) allow us to handle this case and get the desired bound.

To simplify the presentation, we use the same notation for a line in 3-space and for its Plücker point in Q (we will deviate from this convention in Section 5). The following notation will also be useful later on in the paper. For each line $\ell \in Q$, define the variety S_ℓ to be

$$S_\ell = \{\ell' \in Q \mid \text{ and } \ell, \ell' \text{ are coplanar}\}.$$

If the Plücker coordinates of ℓ are $(\pi_{0,1}, \pi_{0,2}, \pi_{0,3}, \pi_{2,3}, \pi_{3,1}, \pi_{1,2})$, then

$$S_\ell = \{(\pi'_{0,1}, \pi'_{0,2}, \pi'_{0,3}, \pi'_{2,3}, \pi'_{3,1}, \pi'_{1,2}) \in Q \mid \\ \pi_{0,1}\pi'_{2,3} + \pi_{0,2}\pi'_{3,1} + \pi_{0,3}\pi'_{1,2} + \pi'_{0,1}\pi_{2,3} + \pi'_{0,2}\pi_{3,1} + \pi'_{0,3}\pi_{1,2} = 0\}.$$

In particular, Equation (2) implies that $\ell \in S_\ell$. We see that, for every line ℓ, the variety S_ℓ is the intersection of Q with a hyperplane, so it is a three-dimensional quadratic surface contained in Q, and we clearly have $\ell \in S_\ell$. We say that a line ℓ is *exceptional* with respect to T if $T \subset S_\ell$. We say that a point $p \in \mathbb{R}^3$ is *exceptional* with respect to T if the set of lines incident to p in 3-space, which we denote by S_p and which is known to be a 2-plane (see the proof below), is equal to T. Clearly, since T is non-degenerate, there are no exceptional points with respect to T (see the full version [21] for an additional discussion).

[2] This assumption is needed only for bounding the number of 2-rich points.

▶ **Lemma 7.** *There are at most two exceptional lines with respect to T.*

Proof. Assume to the contrary that there are three exceptional lines. Assume first that two of these lines are coplanar, and denote them by ℓ_1 and ℓ_2. Then $T \subseteq S_{\ell_1} \cap S_{\ell_2}$, i.e., T is contained in the set of lines intersecting both ℓ_1 and ℓ_2. If ℓ_1 and ℓ_2 do not intersect one another, then $S_{\ell_1} \cap S_{\ell_2} = H^*_{\ell_1,\ell_2}$. Otherwise, letting $p = \ell_1 \cap \ell_2$, we have $S_{\ell_1} \cap S_{\ell_2} = H^*_{\ell_1,\ell_2} \cup S_p$, i.e., it is a union of two 2-flats. In both cases, T is a 2-flat, contradicting our assumption.

We may thus assume there are (at least) three lines ℓ_1, ℓ_2 and ℓ_3 that are pairwise skew, such that $T \subseteq S_{\ell_1} \cap S_{\ell_2} \cap S_{\ell_3}$. As is well known (see, e.g., [7, Theorem 16.4] and [23, Lemma 2.2]), the Plücker points of lines that intersect $r \geqslant 3$ pairwise-skew lines $\ell_1, \ldots, \ell_r$ belong to one ruling of a regulus, and the Plücker points of $\ell_1, \ldots, \ell_r$ belong to the other ruling of this regulus. That is, $S_{\ell_1} \cap S_{\ell_2} \cap S_{\ell_3}$ is one ruling of the regulus generated by the lines intersecting ℓ_1, ℓ_2 and ℓ_3, which is a quadratic curve in the Plücker space, contradicting the fact that T is two-dimensional. ◀

We prune away, as we may, the (at most) two exceptional lines, thereby losing at most $2(n-1) < 2n$ 2-rich points.

For each of the (remaining) lines $\ell \in L$, the intersection $S_\ell \cap T$ is a curve contained in T (possibly also containing a discrete finite subset), which we denote by γ_ℓ. Define

$$\mathcal{C} = \{\gamma_\ell \mid \ell \text{ is not exceptional}\}. \tag{3}$$

We have the following simple observation, whose trivial proof is omitted.

▶ **Lemma 8.** *Let $p \in P$ be an r-rich point, with $r \geqslant 2$, and denote the lines of L incident to p as $\ell_1, \ldots, \ell_s$, for some $s \geqslant r$. Then, for each pair of indices $1 \leqslant i \neq j \leqslant s$, ℓ_i, viewed as a point in Q, is incident to γ_{ℓ_j}, and for every such incidence there is at most one point $p \in P$ that induces it, in the sense stated above.*

The lemma asserts that each r-rich point contributes at least $r(r-1)$ incidences between the lines of L (as points in Q) and the curves γ_ℓ of $\mathcal{C}$ (as curves in Q). Hence, to bound the number of r-rich points, it suffices to bound the number of incidences between the lines in L and the curves of $\mathcal{C}$ (and then divide the bound by $r(r-1)$). For any curve γ_ℓ, its corresponding discrete subset of $O(1)$ points contributes only $O(1)$ incidences, for a total of $O(n)$ incidences. We may therefore ignore all these discrete subsets.

The notion of *dimensionality* for families of curves (see the definition preceding Theorem 3) easily extends in a natural way to collections $\mathcal{C}$ of higher-dimensional algebraic varieties.

▶ **Lemma 9.** *The family $\mathcal{C}$ defined in (3) is two-dimensional.*

Proof. Each curve γ_ℓ of $\mathcal{C}$ can be parameterized by the parameters of the corresponding line ℓ, and these lines lie in the two-dimensional variety T in the Plücker parametric space. ◀

Denote by L^* the set of points in the Klein quadric Q that represent the lines of L. When analyzing incidences between the points of L^* and the curves γ_j, as in Lemma 8, some care has to be exercised, to handle situations in which many of the curves γ_ℓ share a common irreducible component (or even coincide).

($\geqslant$ 3)-rich points. Assume that $\ell_1, \ldots, \ell_\xi \in L$ are such that $\gamma_{\ell_1}, \ldots, \gamma_{\ell_\xi}$ all share a common curve, for some $\xi \geqslant 3$. If some pair of lines ℓ_i, ℓ_j are coplanar, we write $H_{i,j}$ for the (unique) plane H_{ℓ_i,ℓ_j} containing both ℓ_i and ℓ_j. As in the proof of Lemma 7, (i) if ℓ_i and ℓ_j are parallel then $S_{\ell_i} \cap S_{\ell_j} = H^*_{i,j}$, and (ii) if ℓ_i and ℓ_j intersect in a point p then $S_{\ell_i} \cap S_{\ell_j} = H^*_{i,j} \cup S_p$, so $S_{\ell_i} \cap S_{\ell_j}$ is either a 2-flat or the union of two 2-flats. Therefore,

$$\gamma_{\ell_i} \cap \gamma_{\ell_j} = S_{\ell_i} \cap S_{\ell_j} \cap T = H^*_{i,j} \cap T \quad \text{or} \quad (H^*_{i,j} \cup S_p) \cap T,$$

and the right hand sides of these equations are the intersection of one or two 2-flats with T. Since T is assumed to be non-degenerate, it follows that $\gamma_{\ell_i} \cap \gamma_{\ell_j}$ is a finite set of points, and thus γ_{ℓ_i} and γ_{ℓ_j} cannot intersect in a common curve. We can thus assume that $\ell_1, \ldots, \ell_\xi$ are pairwise skew (and that $\gamma_{\ell_1}, \ldots, \gamma_{\ell_\xi}$ intersect in a common curve).

▶ **Lemma 10.** *Assume that the arc (in Plücker space) $\gamma := \bigcap_{i=1}^{\xi} \gamma_{\ell_i}$ is nonempty (and is not a finite set), where $\ell_1, \ldots, \ell_\xi$ are ξ pairwise-skew lines, $\xi \geqslant 3$. Then γ parameterizes one ruling of a regulus, and, for each line $\ell \in T$ whose Plücker point is in γ, ℓ intersects $\ell_1, \ldots, \ell_\xi$, so its Plücker point lies in the curve that represents the other ruling of the same regulus.*

Proof. The proof is similar to that of Lemma 7. The intersection $\bigcap_{i=1}^{\xi} S_{\ell_i}$ consists of the Plücker points of the lines that intersect the $\xi \geqslant 3$ pairwise-skew lines $\ell_1, \ldots, \ell_\xi$. Thus, as already noted (see [7]), all these lines belong to one ruling of a regulus, and the Plücker points of $\ell_1, \ldots, \ell_\xi$ belong to the other ruling of this regulus. Therefore, γ parameterizes one ruling of a regulus, and $\ell_1, \ldots, \ell_\xi$ belong to the other ruling of this regulus, as asserted. ◀

Partition the set of irreducible components of the curves γ_ℓ, over all lines $\ell \in L$ that are not exceptional, into two subsets $\mathcal{C}_0$ and $\mathcal{C}_1$, where $\mathcal{C}_0$ (resp., $\mathcal{C}_1$) contains all the components that do not (resp., do) parameterize one ruling of some regulus. Since $\deg(\gamma_\ell) \leqslant \deg(T) = O(1)$, for each $\ell \in L$, it follows that $|\mathcal{C}_0| = |\mathcal{C}_1| = O(n)$. We partition the set of incidences into incidences between the Plücker points of the lines in L and the curves in $\mathcal{C}_0$, and incidences with the curves in $\mathcal{C}_1$. We remind the reader that at this stage we are only concerned with incidences induced by a concurrence of $r \geqslant 3$ lines of L at some (r-rich) point p.

By Lemma 8, any r-rich point p, for $r \geqslant 3$, corresponds to incidences between the points in Plücker space that represent lines ℓ of L that are incident to p and the (at least three) curves $\gamma_{\ell'}$ that are associated with these lines, and any such incidence can arise for at most one point p. One possibility is that the Plücker point of a line ℓ (incident to p), is incident to a common component of at least three of these curves, call them $\gamma_{\ell_1^p}, \gamma_{\ell_2^p}, \gamma_{\ell_3^p}$. However, the analysis preceding Lemma 10 implies that ℓ_1^p, ℓ_2^p, ℓ_3^p are pairwise skew, which is impossible as they are all incident to p. Hence an incidence between a line and a common component of at least three curves γ_{ℓ_i} does not generate any r-rich points, for $r \geqslant 3$, and, by construction, curves in $\mathcal{C}_1$ also do not generate any r-rich points, for $r \geqslant 3$. We may therefore assume that every curve in $\mathcal{C}_0$ is an irreducible component of at most two curves in $\mathcal{C}$.

Summarizing, the number of r-rich points, with $r \geqslant 3$, is proportional to the number of incidences between the Plücker points of the lines in L and the distinct curves in $\mathcal{C}_0$, divided by $\binom{r}{2}$, and there is no contribution by the curves in $\mathcal{C}_1$.

2-rich points. The situation is different for 2-rich points, which may arise also as incidences between the Plücker points of lines in L and curves in $\mathcal{C}_1$. Handling them requires more care, and is done as follows. A *proper* 2-rich point p, namely a point that is incident to precisely two lines ℓ_p and ℓ'_p of L, corresponds to an incidence between the Plücker point of ℓ_p and the curve $\gamma_{\ell'_p}$ (and also between the Plücker point of ℓ'_p and the curve γ_{ℓ_p}). We count this incidence at most $\deg(\gamma_{\ell'}) = O(1)$ times, once for each irreducible component of the curve

$\gamma_{\ell'_p}$. It therefore suffices to count incidences between the Plücker points of the lines in L and the curves in $\mathcal{C}_0$ (as we have just argued, this is relevant only for curves of multiplicity at most 2) and in $\mathcal{C}_1$ (which may have an arbitrary multiplicity).

We now combine the arguments in the two subcases in the final phase of the analysis. Consider first incidences with curves of $\mathcal{C}_0$. By projecting T onto some generic plane, the number of incidences between the n points representing the lines of L and the curves of $\mathcal{C}_0$ is the same as the number of incidences between the projected points and the projected curves. Since $\mathcal{C}$ is a two-dimensional family of curves (Lemma 9), so is $\mathcal{C}_0$. It therefore follows, by Theorem 3, that the number of these incidences is $O(n^{4/3+\varepsilon})$, for any $\varepsilon > 0$. As argued above, this gives us the bound $O(n^{4/3+\varepsilon}/r^2)$ on the number of r-rich points, for $r \geqslant 3$, thereby establishing part (a) of Theorem 6.

This also gives us the bound $O(n^{4/3+\varepsilon})$ for the number of 2-rich points that correspond to incidences formed with the curves of $\mathcal{C}_0$. For the remaining 2-rich points, which correspond to incidences between lines in L (points of L^*) with curves of $\mathcal{C}_1$ (which may appear with arbitrarily large multiplicity), we recall that each of the curves in $\mathcal{C}_1$ represents one ruling of some regulus, and that we have assumed that no regulus contains more than $n^{1/3}$ lines of L. Hence the each curve in $\mathcal{C}_1$ is incident to at most $n^{1/3}$ lines in L (points in L^*), which implies that the number of incidences with these curves, counted with multiplicity, is at most $O(n^{4/3})$. Hence part (b) of the theorem also follows, and the proof is thus completed. ◀

3 Application: Distinct distances between points on an algebraic curve in the plane

Let P be a set of n points on an irreducible algebraic curve γ of constant degree in the plane, which is not a line or a circle. We derive a lower bound on the number of distinct distances between the points of P; we only sketch our approach, with details in the full version [21]. To derive the bound, we apply the Elekes-Sharir-Guth-Katz framework [6, 11], and define a set L of $n(n-1)$ lines in the parametric 3-space of rotations (rigid motions) in the plane, as $L = \{h_{a,b} \mid a \neq b \in P\}$, where $h_{a,b}$ is the locus of all rotations that map a to b. Then L is contained in the two-dimensional family of lines $\mathcal{C} = \{h_{x,y} \mid x, y \in \gamma\}$. We show that $\mathcal{C}$ is irreducible and is not a 2-flat, and, after pruning away some lines of L, the number of remaining lines in L that are contained in a common plane or regulus in 3-space is $O(1)$, and thus $\mathcal{C}$ is non-degenerate with respect to L. We can then apply the machinery of the previous section to L and $\mathcal{C}$, and derive our bound.

In more detail, let Δ denote the number of distinct distances determined by P. We count the number of quadruples $\{(a, b, a', b') \in P^4 \mid |ab| = |a'b'|\}$ in two different ways. First, let N_k (resp., $N_{\geqslant k}$) denote the number of rotations of multiplicity exactly (resp., at least) k; that is, rotations that map exactly (resp., at least) k points of P to k other points of P. By construction, a rotation of multiplicity at least k is mapped to a k-rich point with respect to the lines of L. Then Theorem 6 implies

$$N_{\geqslant k} = O((n^2)^{4/3+\varepsilon/2}/k^2) = O(n^{8/3+\varepsilon}/k^2), \tag{4}$$

provided that the number of lines in L contained in a common plane or regulus is $O(|L|^{1/3})$, a property that we establish. In fact, we show that no plane or regulus contains more than a constant number of lines of L, except for at most $O(1)$ special planes, whose effect on the asserted bound is negligible, and which we ignore by removing all the lines contained in these planes. Hence, arguing as in [6, 11] and using (4), the number of quadruples is at most

$$\sum_{k=2}^{n}\binom{k}{2}N_k \leqslant \sum_{k=2}^{n} kN_{\geqslant k} = O\left(\sum_{k=2}^{n}\frac{n^{8/3+\varepsilon}}{k}\right) = O(n^{8/3+\varepsilon}\log n) = O(n^{8/3+2\varepsilon}). \quad (5)$$

On the other hand, by Elekes's analysis, which is based on the Cauchy-Schwarz inequality (see, e.g., Guth and Katz [6, 11]), the number of quadruples is also $\Omega(n^4/\Delta)$, implying that the number of distinct distances satisfies $\Delta = \Omega(n^{4/3-2\varepsilon})$. This result was obtained earlier in [16], without the ε-loss in the exponent, but the proof here is much simpler, and we hope that it will find similar applications of this kind.

4 Incidences between points and lines in a two-dimensional family of lines

The main result of this section is the following theorem.

▶ **Theorem 11.** *Let P be a set of m points in $\mathbb{R}^3$, and let L be a set of n lines in $\mathbb{R}^3$, such that, in Plücker coordinates, L is contained in some two-dimensional, non-planar, irreducible variety T of constant degree, as in Section 2. Then $I(P, L) = O(m + n)$.*

Proof. We only provide a sketch here; full details are given in the full version [21]. As observed above, for a point $p \in \mathbb{C}^3$, the set of lines S_p that are incident to p form a 2-flat in the parametric Plücker space, which is contained in Q. We assume that for every $p \in \mathbb{C}^3$, $T \neq S_p$; otherwise, all the lines in T would be incident to p, so the number of incidences would be $O(m + n)$, as asserted.

As T is two-dimensional, Bézout's theorem [8] implies that for every $p \in \mathbb{C}^3$, the intersection $S_p \cap T$ is a union of a constant number of curves of constant degree and a discrete set of a constant number of points.

Put $V := \{p \in \mathbb{C}^3 \mid S_p \cap T \text{ is a curve}\}$. Similar to [22, Theorem 2.16], one can define a polynomial of constant degree, via multivariate resultants, whose vanishing at a point p is equivalent to $S_p \cap T$ being one-dimensional. Hence, V is a complex algebraic variety, and, as T is of constant degree, so is V. We may assume that V is irreducible.

We argue that if $V = \mathbb{C}^3$ then T has to be three-dimensional, so we have $V \neq \mathbb{C}^3$. We then argue that V cannot be two-dimensional, using a somewhat involved argument, whose details are given in [21]. Hence, V must be one-dimensional. By definition of V, every point $p \in P \setminus V$ is incident to at most $O(1)$ lines of L, for a total of $O(m)$ incidences. Thus, we may assume that all the points of P are contained in the curve V. For each line $\ell \in L$, if ℓ is not contained in V it contributes at most $O(\deg V) = O(1)$ incidences with P. Thus, we get a total of $O(n)$ incidences, except for at most $O(\deg V) = O(1)$ lines that are contained in V, for a total of $O(m)$ additional incidences. This completes the proof of the theorem. ◀

The following corollary is an immediate consequence of the theorem.

▶ **Corollary 12.** *Let T be a two-dimensional, non-planar, irreducible subvariety, of constant degree, of the Klein quadric Q. Then, there exists a constant $r_0 = r_0(\deg(T))$ so that, if L is a set of n lines in $\mathbb{R}^3$ whose Plücker images are points in T then, for $r \geqslant r_0$, the set $P_{\geqslant r}(L)$ of r-rich points determined by L satisfies $|P_{\geqslant r}(L)| = O(n/r)$.*

5 Incidences between points and lines in a three-dimensional family of lines

In this section we prove the following main result.

▶ **Theorem 13.** *The number of incidences between m points in $\mathbb{R}^3$ and n lines in $\mathbb{R}^3$ whose Plücker images are contained in an irreducible nonlinear constant-degree three-dimensional variety T is*

$$O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right),$$

provided that no plane contains more than s of the points. If $s = O(\min\{n^{3/5}/m^{2/5}, m^{1/2}\})$, the bound becomes $O(m^{3/5}n^{3/5} + m + n)$.

Proof. Recal that S_p is the 2-flat in Plücker space that consists of all lines passing through a point $p \in \mathbb{R}^3$, and define

$$W := \{p \in \mathbb{C}^3 \mid S_p \cap T \text{ is two-dimensional}\}.$$

▶ **Lemma 14.** *W is an algebraic variety of dimension at most 2 and of constant degree.*

Proof. Since S_p is a 2-flat, $S_p \cap T$ is two-dimensional if and only if $S_p \subset T$. Similarly to the Fano variety of lines, the Grassmannian manifold of 2-flats contained in a constant-degree variety is an algebraic variety [9] of constant degree, so W is algebraic of constant degree. To bound the dimension of W, we repeat the proof of [23, Theorem 2.3(a)], which proceeds by counting the dimensions of the fibers that arise in the problem. Here we omit the details and give the high-level idea. Assume to the contrary that W is three-dimensional, i.e., $W = \mathbb{C}^3$, so for every point $p \in \mathbb{C}^3$, the 2-flat S_p is contained in T. Omitting details, we note that each $p \in \mathbb{C}^3$ contributes a two-dimensional set ($\dim(S_p) = 2$), but then every line is counted by the infinitely many points incident to it. A standard dimension counting argument then implies that $\dim(F(T)) \geqslant 4$, where $F(T)$ is the Fano variety of lines contained in T. By [22, Theorem 3.11], this implies that T has to be a 3-flat, contradicting our assumption. ◀

We first treat incidences with points $p \in P \cap W$. We decompose W into its $O(1)$ irreducible components, and treat each component separately. If a component W_0 of W is not a 2-flat then, by [23, Corollary 1.4], the number of incidences between points contained in W_0 and lines in L is $O(m+n)$. If W_0 is a 2-flat, we invoke the Szemerédi-Trotter bound in Theorem 1, and get the bound $O(s^{2/3}n^{2/3} + s + n)$, using our assumption that no 2-flat contains more than s points of P. This in turn can be upper bounded by $O(s^{1/3}m^{1/3}n^{2/3} + m + n)$, which is subsumed by the bound asserted by the theorem.

Next, we treat incidences involving points in $\mathbb{R}^3 \setminus W$, i.e., points that are incident to a one-dimensional family of lines in T. In this case we use duality, replacing each point p in $\mathbb{R}^3$ with the one-dimensional curve γ_p of lines incident to p and contained in T, in Plücker space. This yields a family of m constant-degree curves that is a family of pseudo-lines. (Two such curves γ_p and γ_q intersect in at most one point, corresponding to the (unique) line connecting p and q, if it lies in T.) We replace each of the n lines in L by its Plücker image, and obtain an incidence problem between n points and m pseudo-lines within the variety T, a three-dimensional subset of the four-dimensional Klein quadric Q. Using a generic projection of T onto $\mathbb{R}^3$ (in which all projected points are distinct and no pair of projected curves overlap), the analysis then proceeds by invoking Zahl [29, Lemma 4.1], which extends the Guth–Katz incidence bound from incidences with lines to incidences with pseudo-lines. Specifically, Zahl shows that the number of incidences between n points and m pseudo-lines in $\mathbb{R}^3$, assuming that these pseudo-lines are constant-degree algebraic curves, is $O(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}\xi^{1/3} + m + n)$, where ξ is an upper bound on the number of pseudo-lines that are contained in any common two-dimensional surface contained in T that is infinitely ruled by curves from the infinite family from which our pseudo-lines are taken.

As argued in Guth and Zahl [12], any such surface must be of degree at most $100E^2$, where E is the degree of the pseudo-lines γ_p, so it is sometimes convenient, especially when no simple characterization of such infinitely ruled surfaces is known, to impose the stronger assumption that no surface of degree at most $100E^2$ contains more than ξ pseudo-lines. In general, this assumption is too restrictive, and difficult to verify. One of the main technical contribution of the analysis in this section is to exploit the dual nature of the present setup, and replace this assumption by the simpler and more natural assumption that, in the original "primal" 3-space, *no plane contains more than s points of P*, allowing us to replace ξ by s, and obtain the incidence bound

$$I(P,L) = O(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}s^{1/3} + m + n).$$

Since $n^{1/2}m^{3/4} \leqslant m^{3/2}$ when $n \leqslant m^{3/2}$, and $n^{1/2}m^{3/4} \leqslant n$ otherwise, we get the following bootstrapping bound

$$I(P,L) = O(m^{3/2} + n^{2/3}m^{1/3}s^{1/3} + n). \tag{6}$$

The analysis then proceeds by "starting over" in primal space, i.e., by constructing a partitioning polynomial g of degree $O(D)$, for a suitable value of D, to be fixed shortly, using the techniques in [10, 11], so that each connected component (cell) τ of $R^3 \setminus Z(g)$ contains at most m/D^3 points of P and is crossed by at most n/D^2 lines of L (but any number of points and lines can be contained in the zero set $Z(g)$).

Incidences within the cells. We first bound the number of incidences within the partition cells. We apply the bootstrapping bound in (6) to each cell τ and sum the bound over all components, to obtain the bound

$$O\left(D^3((m/D^3)^{3/2} + (n/D^2)^{2/3}(m/D^3)^{1/3}s^{1/3} + n/D^2)\right) = O\left(\frac{m^{3/2}}{D^{3/2}} + n^{2/3}m^{1/3}D^{2/3}s^{1/3} + nD\right).$$

To balance the first and last terms, we choose $D = m^{3/5}/n^{2/5}$. For this to make sense, we require that $1 \leqslant D \leqslant \min\{m^{1/3}, n^{1/2}\}$, or, equivalently, that $n \leqslant m^{3/2}$ and $m \leqslant n^{3/2}$. When the first inequality does not hold, we do not use any partitioning and just apply (6) to obtain the bound $I(P,L) = O(n^{2/3}m^{1/3}s^{1/3} + n)$. When the second inequality does not hold, we choose $D = an^{1/2}$, for a suitable constant a, which satisfies the inequalities. In fact, we can construct a polynomial g of this degree so that all the lines of L are fully contained in $Z(g)$ (see, e.g., [14]), and we may therefore assume that all the points of P are also contained in $Z(g)$, as the other points contribute no incidences. That is, in this case there are no incidences within the cells.

In the middle range, our choice of D yields the bound $O(m^{3/5}n^{3/5} + m^{11/15}n^{2/5}s^{1/3})$. Combining all the bounds, the number of incidences within the partition cells is

$$O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right). \tag{7}$$

Incidences on the zero set. Consider next incidences involving points that lie on $Z(g)$. A line ℓ that is not fully contained in $Z(g)$ crosses it in at most $O(D)$ points, for an overall $O(nD)$ bound, which is subsumed by the bound (7) for incidences within the cells. It therefore remains to bound the number of incidences between the points of P on $Z(g)$ and the lines that are fully contained in $Z(g)$.

We handle each irreducible component of $Z(g)$ separately. For non-planar components, Theorem 5, combined with Hölder's inequality (for summing up the bounds over the irreducible components) implies that the number of incidences between points and lines contained in $Z(g)$, but not in any planar component of $Z(g)$, is

$$O(m^{1/2}n^{1/2}D^{1/2} + m^{1/3}D^{4/3}s^{1/3} + m + n).$$

In the middle range $n^{2/3} \leqslant m \leqslant n^{3/2}$, the choice of $D = m^{3/5}/n^{2/5}$ is easily seen to yield the desired bound $O(m^{3/5}n^{3/5} + m + n)$. The case $m < n^{2/3}$ has already been handled, by a single application of (6), which yields the bound $O(n^{2/3}m^{1/3}s^{1/3} + n)$. When $m > n^{3/2}$, the choice of $D = an^{1/2}$, as made above, yields the bound $O(n^{2/3}m^{1/3}s^{1/3} + m)$.

For the planar components, we use the standard technique of assigning each point and line to the first planar component that contains it (according to some arbitrary enumeration of the components). The number of incidences between points and lines assigned to different components is $O(nD) = O(m^{3/5}n^{3/5} + m + n)$ (the right-hand side does indeed bound the left-hand side for each of the sub-ranges). For incidences between points and lines assigned to the same planar component, we apply the Szemerédi-Trotter bound (Theorem 1) to each component and sum the resulting bounds over the components. The assumption that each plane contains at most s points, combined with Hölder's inequality, yields the bound $O(m^{1/3}n^{2/3}s^{1/3} + m + n)$.

That is, the number of incidences with points on $Z(g)$ is bounded by

$$O\left(m^{3/5}n^{3/5} + m^{1/3}n^{2/3}s^{1/3} + m + n\right). \tag{8}$$

Combining with the bound (7) for incidences within the cells, we get the overall bound

$$I(P, L) = O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right),$$

thereby completing the proof of the theorem. ◀

▶ Remark. An interesting challenge in incidence geometry is to sharpen the Guth-Katz bound [11] when the number of lines in any common plane is at most some constant. When the lines in L are contained, as points in Plücker space, in an irreducible nonlinear constant-degree three-dimensional variety T then, while we cannot deduce that the number of lines contained in a common plane is constant, we can nevertheless show: For any plane $\Pi \subset \mathbb{C}^3$, $T \cap \Pi^*$ (recall that Π^* is the 2-flat dual to Π, consisting of all the points dual to lines that are contained in Π) is a constant-degree curve, and thus, except for $O(1)$ points, every point in Π is incident to $O(1)$ lines in T, implying that the number of incidences in a common plane is $O(m_\Pi + n_\Pi)$, where m_Π (n_Π) is the number of points (lines) contained in Π. Such a linear bound on the number of incidences within a plane is a key property for deriving improved incidence bounds, as demonstrated in this work. For Theorem 13, we also added the condition that $m_\Pi \leqslant s$, for every plane Π, to further improve the bound.

6 Application: Incidences between points and lines on a quadric in four dimensions

Solomon and Zhang [26] give a configuration of m points and n lines in a quadratic hypersurface in $\mathbb{R}^4$, having $\Omega(m^{2/3}n^{1/2} + m + n)$ incidences. The following theorem follows as a corollary from the previous section.

▶ **Theorem 15.** *Let P be a set of m points and L a set of n lines contained in a quadratic hypersurface $S \subset \mathcal{C}^4$ such that no 2-flat contains more than $s = O(n^{3/5}/m^{2/5})$ of the points of P. Then $I(P, L) = O(m^{3/5}n^{3/5} + m + n)$.*

▶ Remark. When $m = O(n^{3/2})$, the lower bound $\Omega(m^{2/3}n^{1/2} + m + n)$ obtained in [26] is (asymptotically) smaller than the upper bound $O(m^{3/5}n^{3/5} + m + n)$ asserted in Theorem 15. Closing this gap remains a challenging open problem.

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