A Residual Service Curve of Rate-Latency Server Used by Sporadic Flows Computable in Quadratic Time for Network Calculus

Marc Boyer
ONERA / DTIS – Université de Toulouse, F-31055 Toulouse, France

Pierre Roux
ONERA / DTIS – Université de Toulouse, F-31055 Toulouse, France

Hugo Daigmorte
RealTime-at-Work, F-54600 Villers-lès-Nancy, France

David Puechmaille
RealTime-at-Work, F-54600 Villers-lès-Nancy, France

Abstract

Computing response times for resources shared by periodic workloads (tasks or data flows) can be very time consuming as it depends on the least common multiple of the periods. In a previous study, a quadratic algorithm was provided to upper bound the response time of a set of periodic tasks with a fixed-priority scheduling. This paper generalises this result by considering a rate-latency server and sporadic workloads and gives a response time and residual curve that can be used in other contexts. It also provides a formal proof in the Coq language.

2012 ACM Subject Classification
Networks → Formal specifications; Networks → Network performance evaluation; Networks → Network reliability; Software and its engineering → Formal methods; General and reference → Verification

Keywords and phrases
Network Calculus, response time, residual curve, rate-latency server, sporadic workload, formal proof, Coq

Digital Object Identifier 10.4230/LIPIcs.ECRTS.2021.14

Supplementary Material The code of the Coq proof is provided.
Software: http://doi.org/10.5281/zenodo.4518843
Software (ECRTS 2021 Artifact Evaluation approved artifact): https://doi.org/10.4230/DARTS.7.1.2

1 Introduction

Network calculus is a theory designed to compute upper bounds on delays and memory usage in distributed real-time systems. Given such a system, network calculus offers different ways to model it and different algorithms, producing different bounds at different computation costs.

Even if network calculus is able to analyse realistic industrial configurations in a few seconds [10], some operations have an exponential worst case complexity, related to the least common multiple (lcm) of the periods of the involved flows.

Currently, when modelling periodic or sporadic flows, one often use either an affine (i.e. fluid) model, with linear complexity, or a staircase model, with exponential complexity. This paper presents a quadratic solution for a very common operation, involved in the computation of a residual service for common scheduling policies.

This paper is inspired by [2], that gave a quadratic algorithm for the response time of a set of periodic real-time tasks on a CPU with fixed-priority scheduling. Since network calculus also offers methods to compute upper bounds on the response time of such systems,
we had a look on the proof itself, and we found that it relies on the computation of the CPU capacity that is left to some task by the higher priority flows. This notion also exists in network calculus, where it is called “residual service” or “left-over capacity”. This paper adapts the result in [2] to the network calculus framework and generalises it.

Since the proof is quite long, a formal proof, checked by the Coq proof assistant [13], is also provided.

After a presentation of a relevant subset of network calculus in Section 2, and an overview of related work in Section 3, the result itself is presented in Section 4, and evaluated on benchmarks in Section 5.

2 Network calculus

This section provides a recall of network calculus formalism in Section 2.1, with a focus on sporadic workload and rate-latency servers in Section 2.2.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the subset of non-negative real numbers, $\mathbb{Z}$ the set of integers, for any $i, j \in \mathbb{Z}$, $[i, j] = \{i, i+1, \ldots, j\}$, $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ the ceiling function ($\lceil 1.2 \rceil = 2$, $\lceil 4 \rceil = 4$, $\lceil -1.2 \rceil = -1$). For any set $X$, $|X|$ denotes its cardinal. For any number $x \in \mathbb{R}$, $\lfloor x \rfloor^+$ is defined to be the non-decreasing non-negative closure (illustrated in Figure 1) is defined by $[f]_{t}^+(t) = \max_{0 \leq s \leq t} [f(s)]^+$.

2.1 Generic results

Network calculus is a theory for deriving deterministic upper bounds in networks. Network calculus mainly manipulates non decreasing functions to model flows, workload and server capacity. This section provides a short introduction. A more thorough treatment can be found in [12, 20, 5].

In network calculus, input and output flows of data are modelled by cumulative functions which represent the amount of data observed at some point the flow up to time $t$. Servers are just relations between input and output flows: a server $S$ receives an arrival/input flow, $A(t)$, and delivers the data after some delay, as a departure/output flow, $D(t)$. We always have the relation $D \leq A$, meaning that data can only go out after its arrival.

If the order of data within the flow is preserved, the delay at time $t$ is defined as $hDev(A, D, t)$, and the worst delay is $hDev(A, D)$, with

$$hDev(A, D, t) \overset{\text{def}}{=} \inf \left\{ d \in \mathbb{R}^+ \mid A(t) \leq D(t + d) \right\} , \quad hDev(A, D) \overset{\text{def}}{=} \sup_{t \in \mathbb{R}^+} hDev(A, D, t)$$

(see Figure 3 for an illustration).

However, the exact input/output data flows are in general unknown at design time, or too complex, and the calculus of these cumulative functions cannot be obtained. Nevertheless, the evolution of input/output data flows can be bounded considering contracts on the traffic and the services in the network. For this purpose, network calculus provides the concepts of arrival curve (illustrated in Figure 2) and service curve.

▶ Definition 1 (Arrival curve). Let $A$ be a flow, and $\alpha$ a function. Then, $\alpha$ is said to be an arrival curve for flow $A$, iff $\forall (t, d) \in \mathbb{R}^+ \times \mathbb{R}^+$, $A(t + d) - A(t) \leq \alpha(d)$.

The expression $\alpha(d)$ is an upper bound on the amount of data that can be generated on any interval of duration $d$. For a given flow $A$, one may consider several arrival curves.
Figure 1 Non-negative non-decreasing closure.

Figure 2 Arrival curve.

Figure 3 Delay of the flow A.

Figure 4 Common curves.

Definition 2 (Minimal service). A server S offers a strict minimal service curve β iff for all input/output A,D and for any backlogged period (s,t] (i.e. such that ∀x ∈ (s,t] : A(x) > D(x))

\[ D(t) - D(s) \geq \beta(t - s). \]  

Let us now present the main network calculus result which allows, considering contracts, to compute bounds on delay.

Theorem 3 (Delay bound). Let S be a server transforming an arrival A into a departure D. If A has arrival curve α and S offers a strict minimal service of curve β then

\[ hDev(A, D) \leq hDev(\alpha, \beta). \]  

A key point in network calculus is that arrival and service curves do not have to be tight. Mathematically they only have to be, respectively, upper and lower bounds (cf. eq. (1), eq. (1)). From a modelling point of view, they are not the exact behaviour, but only contracts. It has two complementary consequences. On one hand, if the contract is too far away from the real behaviour, the computed bounds will be large w.r.t. the real worst case. On the other hand, a complex contract can be approximated by a simpler one and all results still hold.

2.2 Sporadic workload, rate-latency servers and NP-SP policy

This paper focuses on periodic or sporadic flows and rate-latency servers.

Given a flow sending frames of maximal size or cost \( C \in \mathbb{R}^+ \) with a period or minimal inter-arrival time \( T \in \mathbb{R}^+ \) and a jitter \( J \in \mathbb{R}^+ \), it admits the arrival curve \( \nu_{T,C,J} : \mathbb{R}^+ \to \mathbb{R}^+ \), \( t \mapsto C \left\lfloor \frac{t+J}{T} \right\rfloor \) but also \( \gamma_{r,b} : \mathbb{R}^+ \to \mathbb{R}^+ \), \( t \mapsto rt + b \) \(^1\), with \( r = \frac{C}{T} \) and \( b = r(J + T) \), as

---

\(^1\) Readers with some background in network calculus may notice that in our definition, \( \gamma_{r,b}(0) = b \) whereas the common practice is to set \( \gamma_{r,b}(0) = 0 \). But the results are simpler to prove with this definition, and can be easily extended to the case where the function is null at origin.
illustrated in Figure 4. Using $\nu_{T,C,J}$ is called “staircase modelling” while using $\gamma_{r,b}$ is called “fluid modelling”.

Servers often offer a rate-latency service, i.e. a constant rate $R$ (a data link bandwidth for example) after some latency $L$ (some switching delay for example), modelled by a function $\beta_{R,L} : t \mapsto R[t - L]^+$. When several flows share a server, its capacity $\beta$ is shared between the flows, and to compute an upper bound on the delay for a flow of interest, network calculus offers to compute a residual service (aka. left-over service). The expression depends on the scheduling policy.

**Theorem 4 (NP-SP residual service).** Let $S$ be a server shared by $n$ flows using a non-preemptive static priority scheduling policy. If $S$ offers a strict minimal service curve $\beta_{R,0}$, and each flow $i$ has arrival curve $\alpha_i$ and a maximal frame size $S_i$, then each flow $j$ receives a residual service

$$\beta_j = \left[\beta_{R,S_j^{\max} / R} - \sum_{k \in hp(j)} \alpha_k\right]^+$$

with $S_j^{\max} = \max_{k \in lp(j)} S_k$, and $hp(j)$ (resp. $lp(j)$) the set of flows with higher (resp. lower) priority than $j$.

The same kind of result holds, with some variations, with other type of service curves, or preemptive static priority server, FIFO or even EDF [5].

**2.3 Illustrative example**

Consider a bus with a bandwidth of 125kb/s, a non-preemptive static priority arbitration rule, and a latency of 0.75ms. Three periodic flows, with period and packet sizes given in Table 1 are sharing this bus. Flow 3 has the lowest priority, then $hp(3) = \{1, 2\}$.

To compute the delay of this flow 3, one may choose to apply eq. (3). One may then either set $\alpha_i = \nu_{T,C,J_i}$ (staircase modelling) or $\alpha_i = \gamma_{r,b_i}$ (fluid modelling). In the top plot of Figure 5 are plotted the two staircase arrival curves, $\nu_1, \nu_2$ and the corresponding fluid arrival curves $\gamma_1, \gamma_2$. Eq. (3) involves the sum $\sum_{k \in hp(j)} \alpha_k$ in equation 3. With a fluid model (when it exists real values $r_k, b_k$ such that $\alpha_k = \gamma_{r_k,b_k}$) there exists a closed-form formula whose cost is linear w.r.t. the number of curves (it holds $\sum_{k \in hp(j)} \gamma_{r_k,b_k} = \gamma_{r,b}$ with

---

2 Readers with a background in network calculus may have noticed only strict minimal service is presented, whereas applications of these results also involve min-plus minimal service. Since the contribution of this paper is independent of the service type, only one notion has been presented.
Table 1 Flow parameters of illustrative example.

<table>
<thead>
<tr>
<th>i</th>
<th>$T_i$</th>
<th>$C_i$</th>
<th>$r_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5 ms</td>
<td>125 b</td>
<td>50 kb/s</td>
<td>125 b</td>
</tr>
<tr>
<td>2</td>
<td>3.5 ms</td>
<td>125 b</td>
<td>35.72 kb/s</td>
<td>125 b</td>
</tr>
<tr>
<td>3</td>
<td>3 ms</td>
<td>100 b</td>
<td>33.33 kb/s</td>
<td>100 b</td>
</tr>
</tbody>
</table>

$r = \sum_{k \in \mathcal{E}(i)} r_k$ and $b = \sum_{k \in \mathcal{E}(i)} b_k$. On the contrary, the addition with a staircase model is hard: there exists no closed-form formula, only algorithms [6], and the computation requires to unroll the function up to the least common multiple of the periods\(^3\), leading to exponential complexity.

Another problem is the absence of a closed-form formula. Closed formulae, and especially those involving linear terms, allow to perform explicit and efficient optimisations.

A last problem is related to the implementation: not all tools are able to handle staircase functions, and several only consider linear arrival curves, as presented in the next section.

The contribution of this paper is to give a rate-latency residual service that lies between the staircase and the fluid residual service curves, denoted $\beta_{R',C/R'}$ in Figure 5.

### 3 Related work

#### 3.1 Implementation of algebraic operators for network calculus

Practical application of network calculus requires an implementation of algebraic operations on functions.

For years, work has concentrated exclusively on linear functions, using closed-form formulae [20], and some tools were even only using affine arrival curves and rate-latency service curves [3].

The subclass of concave or convex piecewise linear functions has also received some attention [25, 7] and is the class currently used in the DISCO tool [26, 4].

A big step was the development of the (min,plus) library for the RTC toolbox [29], representing piecewise linear functions (called VCCs) as a collection of segments [28, Sec. 7].

A major breakthrough has been achieved with the definition of the class of ultimately pseudo periodic functions, generalising VCCs, and the development of the algorithms allowing effective computation [6].

The problem of computation time has not yet received a lot of attention in academia.

In [9], the idea is to maintain a staircase arrival curve per flow, but to approximate it by a concave piecewise linear function of two segments before summing, to keep linear complexity.

The notion of a “container” is developed in [21], with $O(n \log n)$ complexity on operations.

Another line of work is based on the fact that the computation of the bounds (the horizontal deviation, $hDev$) is based only on the prefix of the involved functions, and that one can maintain only a prefix and approximate the remainder of the function by an affine segment [17, 18, 27].

Lastly, another way to reduce the computation time (at the price of getting larger upper bounds) is to replace some periods $T_i$ by a smaller value $T'_i$ but such that the lcm of the $T'_i$ is smaller than the lcm of the $T_i$ [23, 24].

\(^3\) In practice, periods are often integers or rational numbers that can be mapped to integers once a common denominator is found.
Coq is a proof assistant [13], i.e., a tool offering a language to state theorems and describe their proofs as well as a software verifying the proofs. It can also be used to develop software whose execution is proved to be conform to their (formal) specification such as the CompCert C compiler [22] or the CertiKOS operating system [16]. When used as a proof checker, Coq will complain when attempting to use a lemma without providing a proof for one of its hypotheses or if the proved hypotheses do not match the expected ones.

Proving that a systems guarantees some real-time property is often a complex task, requiring long and complex proofs. One way to build correct analyses is to use a proof assistant, like Coq [11] or Isabelle/HOL.

4 Contribution

This section details the main contribution of the paper: given a rate-latency curve $\beta_{R,T}$ and a set of staircase functions $\nu_{T_i,C_i,J_i}$, there exists a rate-latency function $\beta'_{R,C/R}$ which is a lower bound, as shown in eq. (4), that can be used to compute residual service. The main results are presented in Section 4.1. Since the proof of the main theorem is quite long, it is presented in Appendix A. The statement of the theorem in Coq is presented in Section 4.2.

---

4 Think of it as a compiler (in practice it is indeed a compiler for a very strongly typed language).
The problem is illustrated in Figure 5. It shows that each function $\gamma_i$ is a good fluid approximation of the function $\nu_i$ (e.g. $\gamma_1(t) = \nu_1(t)$ for $t = 2.5 \times k$, $k \in \mathbb{N}$) and even if there are less equality point between the two sums $((\gamma_1 + \gamma_2)(t) = (\nu_1 + \nu_2)(t)$ for $t = 17.5 \times k$, $k \in \mathbb{N}$), $\gamma_1 + \gamma_2$ is still the best possible affine upper approximation of $\nu_1 + \nu_2$. And the distance between $(\beta - \gamma_1 - \gamma_2)$ and $(\beta - \nu_1 - \nu_2)$ is exactly the same as the one between $\gamma_1 + \gamma_2$ and $\nu_1 + \nu_2$. But the non-decreasing closure has a major impact on the expression based on staircase, but none on the fluid one, creating a larger distance between both functions (e.g. at $t = 11$, $(\beta - \nu_1 - \nu_2)(11)$ is close to $(\beta - \gamma_1 - \gamma_2)(11)$ but far away from $[\beta - \nu_1 - \nu_2]_t^+$ (11)).

### 4.1 A quadratic rate-latency bound

#### Theorem 5 (Quadratic rate-latency bound). Let $R, L, C_1, \ldots, C_n, T_1, \ldots, T_n$ (resp. $J_1, \ldots, J_n$) be a set of positive real (resp. non negative real) values such that $\sum_{i=1}^{n} \frac{C_i}{T_i} < R$. Then

$$\left[\beta_{R,L} - \sum_{i=1}^{n} \frac{\nu_{T_i,C_i,J_i}}{J_i}\right]^+ \geq \beta_{R',C/R'}$$

(4)

with

$$R' = R - \sum_{i=1}^{n} \frac{C_i}{T_i}, \quad C = RL + W - \max \left\{ \frac{G^i, G^q}{R} \right\}, \quad W = \sum_{i=1}^{n} \left( T_i + J_i - \frac{C_i}{R} \right) \frac{C_i}{T_i},$$

$$G^i = \min_{k \in [1,n]} C_k \left( \sum_{i=1}^{n} \frac{C_i}{T_i} - \max_{i \in [1,n]} \frac{C_i}{T_i} \right), \quad G^q = \sum_{i=1}^{n} \sum_{j=1}^{n} \min \left\{ T_i, T_j \right\} \frac{C_i C_j}{T_i T_j}.$$  

In the context of a rate-latency server shared by several sporadic flows with a static priority policy, the term $R'$ represents the residual rate, made of the initial rate minus the utilisation of higher priority flow $\frac{C_i}{T_i}$, and the term $C$ represents an upper bound on the backlog of higher priority flows.

The expression of the function $\beta_{R',C/R'}$ involves only simple sums (sub-terms $R'$, $W$ and $G^i$) and one double sum (sub-term $G^q$) leading to quadratic complexity $O(n^2)$. To obtain a linear complexity, one may omit the term $G^q$, leading to a smaller curve (i.e. a worst service) but in a shorter time.

Two proofs are given. In Appendix A is given a “pen and paper” proof. This proof being non trivial, we chose to get a high level of confidence in its soundness by formalizing and verifying it with Coq. A feedback of this use is given at the end of the current section and an overview of the Coq proof is given in Section 4.2.

The next theorem states that the previous result is an enhancement w.r.t. a fluid modelling.

#### Theorem 6. Let $R, L, C_1, \ldots, C_n, T_1, \ldots, T_n, J_1, \ldots, J_n, R', C$ be as in Theorem 5. Then

$$\left[\beta_{R,L} - \sum_{i=1}^{n} \frac{\gamma_{r_i, b_i}}{b_i}\right]^+ = \left[\beta_{R',C/R'} - G\right]^+$$

with $G = \left( \sum_{i=1}^{n} \frac{C_i^2}{T_i} + \max \left\{ G^i, G^q \right\} \right)$

(5)

$r_i = \frac{C_i}{T_i}$, $b_i = r_i(J_i + T_i)$, and $R', C, G^i, G^q$ defined as in Theorem 5.

The term $G$ is the global gain obtained with the new result from Theorem 5 w.r.t. a fluid modelling.
**Proof.** The first step consists in an expression of linear residual service. First, $\sum_{i=1}^{n} r_i, b_i = \gamma \sum_{i=1}^{n} r_i, b_i$, then for any $t \in \mathbb{R}^+$:

$$\beta_{R,L}(t) - \frac{\sum_{i=1}^{n} r_i, b_i(t)}{\gamma} = \left[ (R(t - L))^{+} - \left( \sum_{i=1}^{n} r_i \right) t - \sum_{i=1}^{n} b_i \right]^{+}$$

$$= \left[ (R - \sum_{i=1}^{n} r_i) t - (RL + \sum_{i=1}^{n} b_i) \right]^{+}$$

$$= \beta_{R,C'/R'}$$

(6)

with $C' = RL + (\sum_{i=1}^{n} b_i)$. Let now compare $C$ and $C'$

$$C' = RL + \sum_{i=1}^{n} (J_i + T_i) \frac{C_i}{T_i} = C + \frac{1}{R} \left( \sum_{i=1}^{n} \frac{C_i^2}{T_i} + \max \{ G_i, G_q \} \right).$$

(9)

Figure 5 illustrates the differences between the functions and highlights the influence of the non-decreasing closure. As expected, since fluid modelling gives a larger arrival curve than staircase modelling ($\gamma_i \geq \nu_i$), then the fluid residual curve is less than or equal to the staircase one: $\beta - \sum_{i=1}^{n} \gamma_i \leq \beta - \sum_{i=1}^{n} \nu_i$. As stated by the Theorem, $\beta_{R,C'/R'} \leq \left[ \beta - \sum_{i=1}^{n} \gamma_i \right]^{+}$, but $\beta_{R,C'/R'}$ is not smaller than $\beta - \sum_{i=1}^{n} \gamma_i$.

**Comparison with [2]**

This result is of course closely related to the one in [2], and once the equation is given, the amount of generalisation can be detailed. Using network calculus, the response time of a task of execution time $C_0$ and period $T_0$ on a CPU with speed one ($R = 1$) and no latency ($L = 0$) can be bounded by $hDev(\gamma_{C_0/T_0}, C_0, \beta_{R,C'/R'}) = C_0 + \frac{W_{\max} \{ L, Q \}}{R}$, whereas the expression in [2, Thm. 1] is $\frac{C_0 + W - Q}{R}$. The contribution of this paper is then: the modelling of the speed $R$ and the latency $L$ of a server, the introduction of the linear term $G'$ and the extraction of the residual curve, that can be used in more contexts than the fixed priority scheduling.

**Feedback on the use of Coq**

The use of Coq gave us the opportunity to fix a few small mistakes in a preliminary version of the proof of Theorem 5 and one of its hypotheses.

One of the last steps of the proof consists is showing that a value $s$ is non-negative (step 11 in Appendix A). It was claimed as an evidence, even with negative values $J_i$ of the jitters. While trying to encode this “evidence” in Coq, we realised that the current proof holds only for non-negative $J_i$ values, and the hypotheses have been updated. We do not know currently whether the property holds with negative $J_i$ values.

One step of the proof (an index permutation, step 9.c in Appendix A) was using a wrong argument, doing a confusion between values and indexes. The proof has been corrected.

Regarding the cost of the development, it can be considered reasonable as only 1400 lines of Coq code were needed, requiring about two person \times weeks of development, (including above

---

5 214 lines for statements, 989 lines for proofs and 49 lines of comment (the remaining being blank lines).

6 For a developer with a few years of experience with the tool.
mentioned proof fixes). This was made possible thanks to the availability of a formalization
of the real numbers in Coq’s standard library as well as the nice Mathematical Components
library [15] and particularly its big operators [1] to manipulate the \( \Sigma \) notation for sums.

### 4.2 Coq statement of Theorem 5

While the Coq compiler checks that a theorem is well formed and that its proof is correct, it
can not check that the theorem conforms to the author or reader intuition. We will then
describe the formal statement of Theorem 5 in Coq’s language.

The full proof is available, along with instructions to automatically recheck it with Coq,

First comes the loading of the libraries,

```coq
Require Import mathcomp (*...*).
Require Import Reals (*...*).
```

and Coq is instructed to interpret all standard notations, such as \( +, -, \leq \), as real number
ones

```coq
Local Open Scope R_scope.
```

We then give the hypotheses of the theorem

```coq
Section Theorem3.

Variable n' : nat.
Notation n := n'.+1. (* Be sure that n is non zero *)
Variable R T : R+×.*
Variable tC tT : R+×^n.
Variable tJ : R+^n.

For convenience, the \( i \)-th element \( (tC\_i) \) of the \( n \)-tuple \( tC \) will then be denoted \( C\_i' \)

Notation "'C'_' i" := (tC\_i).
Notation "'T'_' i" := (tT\_i).
Notation "'J'_' i" := (tJ\_i).

Hypothesis R_large_enough : \( \sum_i C\_i' / T\_i < R \).
```

And we define the various constants and functions

```coq
Definition R' := R - \( \sum_i C\_i' / T\_i \).
Definition W := \( \sum_i (T\_i + J\_i - C\_i' / R) \times (C\_i' / T\_i) \).
Definition L := (\( \min_k C\_k \)) * ((\( \sum_i C\_i' / T\_i \)) - \( \max_i (C\_i' / T\_i) \)).
Definition Q :=
\( \sum (i < n) \sum (j < n \mid j < i) R_{\min} T\_i T\_j * (C\_i' * C\_j) / (T\_i * T\_j) \).
Definition V t := \( \sum_i C\_i' * IZR (Zceil ((t + J\_i) / T\_i)) \).
Definition beta R T := fun t \in R+ \Rightarrow R * [t - T]+.
```

Before finally stating the theorem itself

```coq
Theorem theorem3 : \forall t, \( (beta R' (C / R')) t \leq [\forall t \Rightarrow beta R T t - V t]^+ t)^%Rbar.
```

where \( %Rbar \) tells Coq that \( \leq \) is the one on \( R = R \cup \{-\infty, +\infty\} \) since the non decreasing
closure contains a least upper bound that could be infinite.
A Quadratic Residual Service Curve of Rate-Latency Server Used by Sporadic Flows

Table 2 Periods of flows (in ms).

<table>
<thead>
<tr>
<th>Set name</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period values</td>
<td>2,5,10,20,25,40,50</td>
<td>2,3,4,5,6,7,8,9,10</td>
<td>2,3,5,7,11,13</td>
</tr>
<tr>
<td>lcm</td>
<td>200</td>
<td>2520</td>
<td>30030</td>
</tr>
</tbody>
</table>

Table 3 Mean computed bounds and computing time.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Periods</th>
<th>Jitters</th>
<th>Fluid</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Staircase</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1 Null</td>
<td>12.3</td>
<td>12.0</td>
<td>10.3</td>
<td>6.1</td>
<td>6.1 (-50%)</td>
<td></td>
</tr>
<tr>
<td>S1 Rand.</td>
<td>17.6</td>
<td>17.2</td>
<td>15.5</td>
<td>6.1</td>
<td>6.1 (-65%)</td>
<td></td>
</tr>
<tr>
<td>S2 Null</td>
<td>7.7</td>
<td>7.4</td>
<td>5.7</td>
<td>3.4</td>
<td>3.4 (-56%)</td>
<td></td>
</tr>
<tr>
<td>S2 Rand.</td>
<td>10.6</td>
<td>10.2</td>
<td>8.6</td>
<td>3.4</td>
<td>3.4 (-68%)</td>
<td></td>
</tr>
<tr>
<td>S3 Null</td>
<td>7.2</td>
<td>6.9</td>
<td>5.6</td>
<td>3.3</td>
<td>3.3 (-54%)</td>
<td></td>
</tr>
<tr>
<td>S3 Rand.</td>
<td>9.9</td>
<td>9.5</td>
<td>8.2</td>
<td>3.3</td>
<td>3.3 (-66%)</td>
<td></td>
</tr>
</tbody>
</table>

Mean computing time, per configuration, in ms (and ratio w.r.t. lcm for staircase)

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Fluid</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Staircase</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1 Null</td>
<td>15</td>
<td>9</td>
<td>567</td>
<td>567 (2.8)</td>
</tr>
<tr>
<td>S1 Rand.</td>
<td>18</td>
<td>10</td>
<td>597</td>
<td>597 (3.0)</td>
</tr>
<tr>
<td>S2 Null</td>
<td>7</td>
<td>6</td>
<td>26</td>
<td>26</td>
</tr>
<tr>
<td>S2 Rand.</td>
<td>7</td>
<td>6</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>S3 Null</td>
<td>6</td>
<td>6</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>S3 Rand.</td>
<td>6</td>
<td>6</td>
<td>21</td>
<td>21</td>
</tr>
</tbody>
</table>

5 Evaluation

This section evaluates the quality of the approximation provided in this paper, in terms of accuracy of the result and computational cost.

To do so, we test the expression on a large set of configurations. Each configuration represents a non-static priority server, with a constant rate of 1Mb/s, no latency, and a set of randomly generated sporadic flows. Let $c_i$ be a configuration, each flow $f_{i,j}$ has priority $j$, a fixed packet size $C_{i,j}$ chosen uniformly between 8 and 16 bytes, a period $T_{i,j}$ also randomly chosen in a subset of values, and a jitter $J_{i,j}$ also randomly chosen. New flows are added up to reaching a global load of 90%, and $n_i$ denotes the number of flows.

One hundred configurations are generated picking periods values from S1 of Table 2 and with no jitter, another hundred using set S2 and also with no jitter, and another hundred using set S3 of the same table and also no jitter. Three others sets are generated in a similar way, but with a jitter uniformly distributed between 0 and the flow period (excluded).

For each configuration $c_i$, let $f_{i,1}, \ldots, f_{i,n_i}$ be the set of flows. For each flow $f_{i,j}$, four bounds on the delay are computed using different methods. The two first have been used in the illustrative example in Section 2.3.

1. $d_{i,j}^{\text{fluid}} = hDev(\alpha_i, \beta_{i,j}^{\text{fluid}})$, where $\beta_{i,j}^{\text{fluid}}$ is computed using eq. (3) with $\alpha_k = \gamma_{C_k/T_k,C_k(1+J_k/T_k)}$. It is called the fluid modelling.
2. $d_{i,j}^{\text{stc}} = hDev(\alpha_i, \beta_{i,j}^{\text{stc}})$, where $\beta_{i,j}^{\text{stc}}$ is computed using eq. (3) with $\alpha_k = \nu_{T_k,C_k,J_k}$. It is called the staircase modelling.
3. $d_{i,j}^{\text{lin}} = hDev(\alpha_i, \beta_{i,j}^{\text{lin}})$ where $\beta_{i,j}^{\text{lin}}$ is computed using Theorem 5 but only with the linear term $G^l$ (i.e. setting $G^q = 0$). It is called the linear modelling.
4. $d_{i,j}^{\text{quad}} = hDev(\alpha_i, \beta_{i,j}^{\text{quad}})$ where $\beta_{i,j}^{\text{quad}}$ is computed using Theorem 5 but only with the quadratic term $G^q$ (i.e. setting $G^l = 0$). It is called the quadratic modelling.

Experiments have run on a laptop with 4GB of memory and a 2.7GHz Intel Core i5.
Figure 6 Plots related to configurations with periods in S1 set, null w.r.t. random jitters.
A Quadratic Residual Service Curve of Rate-Latency Server Used by Sporadic Flows

Figure 6a plots, for a given configuration $c_k$ with periods chosen in $S_1$ (harmonic periods) and no jitter, the bounds $d_{k,j}^{\text{fluid}}$, $d_{k,j}^{\text{stc}}$, $d_{k,j}^{\text{quad}}$, $d_{k,j}^{\text{lin}}$ computed by the four methods for each flow. Since flows are sorted by priority, the plots are non decreasing. As expected, the fluid modelling gives the larger, i.e. worse, bounds, whereas the linear approximation is smaller, the quadratic approximation even smaller, and staircase modelling leads to the smallest bounds. Only one configuration is plotted, but they all have the same shape.

Now considering the fluid modelling as the reference value, Figure 6c plots, for each configuration with harmonic periods, the sum of all bounds computed by a method divided by the sum of all bounds computed by fluid modelling: $\frac{\sum_{i,j} d_{k,j}^X}{\sum_{i,j} d_{k,j}^{\text{fluid}}}$ with $X \in \{\text{fluid}, \text{stc}, \text{lin}, \text{quad}\}$. Figure 6e plots the computation time required to analyse each configuration, depending on the modelling, with a log-scale on time axis. Last, Figure 7 plots the computation time as a function of the mean delay, for each configuration and each method.

In the same figure group are also plotted the same graphs but considering the jitter of each flow picked up between 0 and the flow period. As expected, the jitter increases the delay of the affine models, but has no influence on the staircase one (cf. Figure 6b). Then, the gain obtained by the staircase model w.r.t. the fluid model increases, whereas the gain of the quadratic model is less (12% instead of 16%).

The Table 3 summarises, for each set of hundred configurations, the mean bound on delays for all flows, and the mean computing time for a single configuration. For the staircase modelling is added this computation time divided by the lcm of the periods, showing that this computation time is almost linear w.r.t. this lcm.

The same kind of information is plotted in the group of Figures 8 when the periods are taken from the set $S_3$. The relations between the methods in terms of accuracy of results are in the same order of magnitude (from 16% to 22% without jitter, from 11% to 17% with jitter), but the computation time of the staircase methods is three orders of magnitude larger (50s vs. 21ms).

The results for the set $S_2$ are not plotted but are summarised in Table 3.
(a) Per flow delay bound, for one configuration, null jitter.

(b) Per flow delay bound, for one configuration, random jitter.

(c) Per configuration mean delay w.r.t. to fluid modelling, null jitter.

(d) Per configuration mean delay w.r.t. to fluid modelling, random jitter.

(e) Per configuration computing time, null jitter.

(f) Per configuration computing time, random jitter.

Figure 8 Plots related to configurations with periods in S3 set, null w.r.t. random jitters.

6 Conclusion

In network calculus, the computation of residual services with staircase arrival curves has exponential complexity, whereas fluid arrival curves offer a linear complexity but give larger, i.e. worse, upper bounds.
This paper generalises a result from [2], and develops a residual service curve with either linear or quadratic computational complexity. The correctness of the result is enforced by providing a formal Coq proof. The different approaches are evaluated on 600 systems with sporadic workload and non-preemptive static priority scheduling.

Whereas the staircase model computes bounds that are half of those of the fluid model\(^7\), at the expense of a computation time from \(10^2\) to \(10^4\) times larger, the quadratic approach already enhances the results by about 20\% while being only 10 times slower. The linear model offers a limited enhancement (2\%-5\%). Having accurate results in short computation times helps real-time system designers when exploring several configurations (in design space exploration). A comparison with prefix-based approach [17, 18, 27] is left to further studies.

Moreover, the analytic formula of the residual service curves opens some opportunities. First, having a residual curve allows to use the Pay Burst Only Once principle, to compute an end-to-end network delay smaller than the sum of per switch delays. Second, a closed form formula gives opportunities for optimisation. Third, getting rid of least common multiple allows the use of directed rounding floating-point arithmetic that could lower the computation cost by one or two additional orders of magnitude.

The formalization of the main result of the paper using Coq enabled to fix some glitches and reach a very high level of confidence in this result. This was done at a moderate extra cost and follows the direction impulsed by the call for action\(^8\) “Real Proofs for Real Time: Let’s do better than “almost right” at ECRTS 2016 [14].

---

### References


\(^7\) It must be mentioned that a previous study on a realistic avionic configuration, based on Ethernet and 2 priority levels, has shown a gain related to staircase of only 6\% [8] and another on a more loaded configuration gave a gain of 18\% [10]. But these realistic configurations had lower load.

\(^8\) A similar impulsion was given a decade ago in the programming language community and a number of mechanized formalisations (using either Coq or other tools) now appear each year at their main conference POPL.


A Proof of Theorem 5

Regarding only correctness issues, we may have omitted this section since the Coq proof already provides a formal correctness insurance. Nevertheless, this section can be considered as the documentation of the Coq proof. But the main justification of this section relies in the opportunity to adapt or generalise the results. The same way as we have converted and extended the result on response time presented in [2] by a study of its proof, we provide a human-oriented proof, as a complement of the formal Coq proof.

The proof presentation is inspired by [19]. Each sub-step of the proof will start with some ordering value, followed by the statement of the sub-step, using bold font. Thereafter will come the proof of the sub-step itself.

For the proof, let
\[ V(t) = \sum_{i=1}^{n} \nu_{T_i, C_i, J_i}, \]
i.e.
\[ V(t) = \sum_{i=1}^{n} C_i \left\lceil \frac{t + J_i}{T_i} \right\rceil, \]
and \( \rho = \sum_{i=1}^{n} \frac{C_i}{T_i} \) the long term rate of \( V \), and recall that \( \rho < R \).

1. Definitions of \( s^M \) and first properties: For any \( M \in \mathbb{R} \), let
\[ s^M \overset{\text{def}}{=} \min \{ t \in \mathbb{R} \mid V(t) + M = R(t - L) \}. \tag{10} \]

This \( s^M \) is the minimal solution to \( V(t) + M = R(t - L) \). The first step consists in showing that \( s^M \) exists (there are solutions, and there exists a minimal one), and the second on their relative positions (cf. Figure 9).

a. The minimum exists: By definition of the ceiling function, \( x \leq [x] < x + 1 \). Then, for any \( i, t \frac{C_i}{T_i} + J_i \frac{C_i}{T_i} \leq C_i \left\lceil \frac{t + J_i}{T_i} \right\rceil < t \frac{C_i}{T_i} + J_i \frac{C_i}{T_i} + C_i \). Making the sum for all \( i \in [1, n] \)

![Figure 9](https://example.com/figure9.png)  
**Figure 9** Illustration of \( s^M \) definition, with \( n = 2, C_1 = C_2 = \frac{1}{2}, T_1 = 2, T_2 = 3, J_1 = J_2 = 0. \)
leads to
\[ \forall t \in \mathbb{R} : \rho t + b \leq V(t) < \rho t + b', \] (11)
with \( b = \sum_{i=1}^{n} J_i C_i, \) \( b' = b + \sum_{i=1}^{n} C_i. \)
These are affine functions, and since \( R > \rho, \) there exists \( x < x' \) such that \( \rho x + b = R(x - L) - M \) and \( \rho x' + b' = R(x' - L) - M \) (cf. Figure 9). Set \( y = \rho x + b, y' = \rho x' + b'. \) From eq. 11, for any \( t \in [x, x'] : y \leq V(t) \leq y'. \) Set \( Y = \{ V(t) | t \in [x, x'] \} \) the set of values of \( V \) on \([x, x']\). This set is non-empty and finite. If \((v_i)_{i \in \mathbb{N}} \) is the ordered set of values of the function \( V \), there exists \( k \leq k' \) such that \( Y = \{ v_k, v_{k+1}, \ldots, v_{k'} \} \) \( Y = \{ v_3, v_4 \} \) on the example in Figure 9), and to each one corresponds one \( s_k^M \) such that \( v_k = R(s_k^M - L) - M. \) Then the set \( \{ t \in \mathbb{R} | V(t) + M = R(t - T) \} = \{ s_k^M, \ldots, s_k'^M \} \) is non-empty and finite, and its minimum, \( s^M \) exists.

b. **Before \( s^M, \) \( R(t - L) - M \) is below \( V(t) \) i.e.**
\[ \forall t < s^M : R(t - L) - M < V(t); \] (12)
By contradiction, assume there exists \( t < s^M \) such that \( R(t - L) - M \geq V(t). \) The case \( R(t - L) - M = V(t) \) leads to \( t \geq s^M \) by definition of \( s^M. \) In case of \( R(t - L) - M > V(t), \) then \( R(t - L) - M > \rho t, \) so \( t > x. \)

c. **A lower bound on \( s^M: ** In step 1a, \( x \) has been defined such that \( s^M \in [x, x'], \) with \( \rho x + b = R(x - L) - M, \) then
\[ s^M \geq x = \frac{M + RL + \sum_{i=1}^{n} J_i C_i}{R'} \] (13)
This relation will be used in one of the last step of the proof.

2. **Definitions of \( q_i^M, r_i^M \) and first properties:** Let introduce for any \( i \in [1, n], \)
\[ q_i^M \overset{\text{def}}{=} \left[ \frac{s_i^M + J_i}{T_i} \right], \quad r_i^M \overset{\text{def}}{=} T_i q_i^M - (s_i^M + J_i) \] (14)
keep in mind that \( q_i^M = \frac{s_i^M + J_i + r_i^M}{T_i} \) and that \( T_i > r_i^M \geq 0 \) (from \( \frac{\rho}{T} \leq \left[ \frac{\rho}{T} \right] < \frac{\rho}{T} + 1 \) comes \( 0 \leq L \left[ \frac{\rho}{T} \right] - x < L, \) and setting \( x = s_i^M + J_i). \)

3. **Expression of \( s^M \) in terms of \( r_i^M: ** Since \( s^M \) is a minimum, it satisfies \( R(s^M - L) = V(s^M) + M \) i.e.
\[ R(s^M - L) = \sum_{i=1}^{n} C_i \left[ \frac{s_i^M + J_i}{T_i} \right] + M = \sum_{i=1}^{n} C_i q_i^M + M \] (15)
\[ = \sum_{i=1}^{n} C_i s_i^M + J_i + r_i^M + M \] (16)
\[ \iff s^M(R - \sum_{i=1}^{n} \frac{C_i}{T_i}) = M + RL + \sum_{i=1}^{n} \frac{C_i}{T_i}(J_i + r_i^M) \] (17)
\[ \iff R' s^M = M + RL + \sum_{i=1}^{n} C_i J_i + \sum_{i=1}^{n} C_i r_i^M \] (18)

4. **Two definitions for a reordering \( l_i^M \) and \( \sigma: ")**
\[ \forall i \in [1, n]: l_i^M \overset{\text{def}}{=} (q_i^M - 1) T_i - J_i. \] (19)
Remark that \( s^M > l^M_i \) (from \( 0 \leq r^M_i < T_i \) comes \( 0 \leq T_i q^M_i - (s^M + J_i) < T_i \) and \( T_i(q^M_i - 1) - J_i < s^M \).

Now, let \( \sigma : [1, n] \to [1, n] \) be a permutation such that the sequence \( l^M_{\sigma(i)} \) is non-increasing, i.e. \( s^M > l^M_{\sigma(1)} \geq l^M_{\sigma(2)} \geq \cdots \geq l^M_{\sigma(n)} \).

5. For all \( k \in [1, n] \) it holds

\[
C_k \left[ \frac{s^M + J_k}{T_k} \right] = C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] + C_k \tag{20}
\]

By definition \( q^M_k \) is an integer, so \( q^M_k - 1 \) also is and \( q^M_k = [q^M_k - 1] + 1 \). By definition of \( l^M_k \), we then get \( q^M_k = \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] + 1 \) hence the result by definition of \( q^M_k \).

6. For all \( i \in [1, n] : s^M \geq l^M_{\sigma(i)} + \sum_{k=1}^i \frac{C_{\sigma(k)}}{R} \):

Let \( i \in [1, n] \), \( S_i = \{ \sigma(1), \ldots, \sigma(i) \} \) and \( S_i = [1, n] \setminus S_i \).

From previous relation, for any \( k \in S_i \),

\[
C_k \left[ \frac{s^M + J_k}{T_k} \right] \geq C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] + C_k.
\]

But by definition, \( (l^M_{\sigma(m)})_{m \in [1, n]} \) is non-increasing sequence, so for all \( k \in S_i \), \( l^M_k \geq l^M_{\sigma(i)} \), which yields

\[
C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] + C_k \geq C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] + C_k.
\]

To conclude

\[
\forall k \in S_i : C_k \left[ \frac{s^M + J_k}{T_k} \right] \geq C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] \tag{21}
\]

Now, consider \( k \in S_i \). By the definition of \( l^M_j \) (cf. proof step 4), \( \forall j \in [1, n] : s^M > l^M_j \), and in particular, for \( j = \sigma(i) \). Then, it holds

\[
\forall k \in S_i : C_k \left[ \frac{s^M + J_k}{T_k} \right] \geq C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] \tag{22}
\]

Summing over eq. (21) and eq. (22), it comes

\[
\sum_{k \in S_i \cup S_i} C_k \left[ \frac{s^M + J_k}{T_k} \right] \geq \sum_{k \in S_i \cup S_i} C_k \left[ \frac{l^M_{\sigma(i)} + J_k}{T_k} \right] + \sum_{k \in S_i} C_k
\]

i.e. \( V(s^M) \geq V(l^M_{\sigma(i)}) + \sum_{j=1}^i C_{\sigma(j)} \)

By the definition of \( s^M \), one has \( V(s^M) = R(s^M - L) - M \). Conversely, since \( s^M \geq l^M_{\sigma(i)} \), from eq. 12, it comes \( V(l^M_{\sigma(i)}) \geq R(l^M_{\sigma(i)} - L) - M \), so

\[
R(s^M - L) - M \geq R(l^M_{\sigma(i)} - L) - M + \sum_{j=1}^i C_{\sigma(j)} \tag{23}
\]

\[
\iff s^M \geq l^M_{\sigma(i)} + \sum_{j=1}^i \frac{C_{\sigma(j)}}{R} \tag{24}
\]

7. For all \( i \in [1, n] : r^M_{\sigma(i)} \leq T_{\sigma(i)} - \sum_{j=1}^i \frac{C_{\sigma(j)}}{R} \) : This is a direct consequence of definition
of $t_i^M$, $q_i^M$ and previous relation.

$$s^M \geq t_{\sigma(i)}^M + \sum_{j=1}^{i} \frac{C_{\sigma(j)}}{R}$$

$$\iff s^M \geq (q_{\sigma(i)}^M - 1)T_{\sigma(i)} - J_{\sigma(i)} + \sum_{j=1}^{i} \frac{C_{\sigma(j)}}{R}$$

$$\iff s^M + J_{\sigma(i)} - T_{\sigma(i)}q_{\sigma(i)}^M \geq -T_{\sigma(i)} + \sum_{j=1}^{i} \frac{C_{\sigma(j)}}{R}$$

$$\iff r_{\sigma(i)}^M \leq T_{\sigma(i)} - \sum_{j=1}^{i} \frac{C_{\sigma(j)}}{R}$$

8. $\sum_{i=1}^{n} \frac{r_i^M C_i}{T_i} \leq \sum_{i=1}^{n} \left( T_i - \frac{C_i}{R} \right) C_i - \frac{1}{R} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}}$;

$$\sum_{i=1}^{n} \frac{r_i^M C_i}{T_i} = \sum_{i=1}^{n} \frac{r_{\sigma(i)}^M C_{\sigma(i)}}{T_{\sigma(i)}} \quad \text{since } \sigma \text{ is a permutation}$$

$$\leq \sum_{i=1}^{n} \left( T_{\sigma(i)} - \sum_{j=1}^{i} \frac{C_{\sigma(j)}}{R} \right) \frac{C_{\sigma(i)}}{T_{\sigma(i)}}$$

$$= \sum_{i=1}^{n} \left( T_{\sigma(i)} - \frac{C_{\sigma(i)}}{R} \right) \frac{C_{\sigma(i)}}{T_{\sigma(i)}} - \frac{1}{R} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(j)}C_{\sigma(i)}}{T_{\sigma(i)}}$$

The next step consists in having a lower bound on $\sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(j)}C_{\sigma(i)}}{T_{\sigma(i)}}$.

9. $\sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(j)}C_{\sigma(i)}}{T_{\sigma(i)}} \geq \max \left\{ G^3, G^l \right\}$ The goal in this step is to get rid of the $\sigma$ permutation, since it depends on $M$. :

a. $\sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(j)}C_{\sigma(i)}}{T_{\sigma(i)}} \geq G^l$ :

$$\sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(j)}C_{\sigma(i)}}{T_{\sigma(i)}} \geq \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)} \min_{k \in [1,n]} C_k}{T_{\sigma(i)}}$$

$$= \min_{k \in [1,n]} \sum_{i=1}^{n} \frac{C_{\sigma(i)}}{T_{\sigma(i)}} \times (i - 1)$$

$$\geq \min_{k \in [1,n]} \sum_{i=1}^{n} \frac{C_{\sigma(i)}}{T_{\sigma(i)}} \times \min(i - 1, 1)$$

and since one does not know the value of $\sigma(1)$ (i.e. when $i - 1 = 0$)

$$\geq \min_{k \in [1,n]} \sum_{i=1}^{n} \frac{C_i}{T_i} - \max_{i \in [1,n]} \frac{C_i}{T_i} = G^l$$
b. \[ \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}} \geq \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}T_{\sigma(j)}} \min \{T_{\sigma(i)}, T_{\sigma(j)}\} \] 

\[ \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}} = \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}T_{\sigma(j)}} T_{\sigma(j)} \geq \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}T_{\sigma(j)}} \min \{T_{\sigma(i)}, T_{\sigma(j)}\} \] (38)

\[ \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}T_{\sigma(j)}} \min \{T_{\sigma(i)}, T_{\sigma(j)}\} = \sum_{p=1}^{n} \sum_{q=1}^{n-1} \frac{C_pC_q}{T_pT_q} \min \{T_p, T_q\} = G^2 \] (39)

(40)

c. \[ \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}T_{\sigma(j)}} \min \{T_{\sigma(i)}, T_{\sigma(j)}\} = \sum_{(i,j)\in X} x_{\sigma(i),\sigma(j)} = \sum_{(i,j)\in X} x_{h(i,j)} \] (41)

One can then prove that \( h \) is injective, meaning it is bijective, which enables the following reindexing

\[ \sum_{h(i,j)\in X} x_{h(i,j)} = \sum_{(ij)\in X} x_{i,j} = G^2 \] (42)

10. \( s^M \leq \frac{M+C}{R} \): This is just, going from equations (18), application of steps 8 and 9.

\[ R's^M = M + RL + \sum_{i=1}^{n} \frac{C_i}{T_i} j_i + \sum_{i=1}^{n} \frac{C_i}{T_i} r_i^M \]

\[ \leq M + RL + \sum_{i=1}^{n} \frac{C_i}{T_i} j_i + \sum_{i=1}^{n} \left( T_i - \frac{C_i}{R} \right) \frac{C_i}{T_i} - \frac{1}{R} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{C_{\sigma(i)}C_{\sigma(j)}}{T_{\sigma(i)}} \]

\[ \leq M + RL + \sum_{i=1}^{n} \left( J_i + T_i - \frac{C_i}{R} \right) \frac{C_i}{T_i} - \max \{G^2, G' \} \frac{R}{R} \]

\[ \implies s^M \leq \frac{M+C}{R} \]

11. Here comes the \( M \) elimination: Let \( t \in \mathbb{R}^+ \).

If \( t \leq \frac{C}{R} \), \( \beta_{R', \frac{C}{R}} (t) = 0 \), so \( \beta_{R', \frac{C}{R}} (t) \leq [\beta_{R,L} - V_1] (t) \) trivially holds.

If \( t \geq \frac{C}{R} \). By definition of \( s^M \), for any \( M \in \mathbb{R} \),

\[ M = R(s^M - L) - V(s^M) \] (43)

so, for any interval \( I^M \) such that \( s^M \in I^M \)

\[ M \leq \sup_{u\in I^M} \{ R(u - L) - V(u) \}. \] (44)
Set $M = R't - C$ (this can be done safely since there is no hidden $M$ in $R,R',L,V(\cdot)$).

From step 10, $s^M \leq \frac{M+C}{R} = t$, so

$$R't - C \leq \sup_{s^M \leq u \leq t} \{R(u - L) - V(u)\}. \quad (45)$$

But from $t \geq \frac{C}{R}$ and $M = R't - C$ comes $M \geq 0$, and introducing it in eq. 13 yields $s^M \geq 0$, so

$$R't - C \leq \sup_{0 \leq u \leq t} \{R(u - L) - V(u)\}, \quad (46)$$

and by doing the maximum with 0

$$R' \left[ t - \frac{C}{R} \right]^+ \leq \sup_{0 \leq u \leq t} [R(u - L) - V(u)]^+ \quad (47)$$

$$\iff \beta_{R',\frac{C}{R}}(t) \leq [\beta_{R,L} - V]^+ (t) \quad (48)$$