# Von Neumann Regularity, Split Epicness and Elementary Cellular Automata

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**Abstract**

We show that a cellular automaton on a mixing subshift of finite type is a von Neumann regular element in the semigroup of cellular automata if and only if it is split epic onto its image in the category of sofic shifts and block maps. It follows from [S.-Törmä, 2015] that von Neumann regularity is decidable condition, and we decide it for all elementary CA.

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## 1 Introduction

The von Neumann regular elements – elements $a$ having a weak inverse $b$ such that $aba = a$ – of cellular automaton (CA) semigroups are studied in [1]. We show that in the context of cellular automata on mixing subshifts of finite type, von Neumann regularity coincides with the notion of split epicness onto the image, another generalized invertibility notion from category theory.

Question 1 of [1] asks which of the so-called elementary cellular automata (ECA) are von Neumann regular. They determine this for all ECA except ones equivalent to those with numbers 6, 7, 9, 23, 27, 28, 33, 41, 57, 58 and 77, see the next section for the definition of the numbering scheme.

What makes this question interesting is that von Neumann regularity of one-dimensional cellular automata is not obviously[1] decidable – clearly checking if $g$ is a weak inverse is semidecidable, but it is not immediately clear how to semidecide the nonexistence of a weak inverse. However, split epicness has been studied previously in [9], and in particular it was shown there that split epicness of a morphism between two sofic shifts is a decidable condition. This means Question 1 of [1] can in theory be decided algorithmically.

As the actual bound stated in [9] is beyond astronomical, it is an interesting question whether the method succeeds in actually deciding each case. With a combination of this method, computer and manual searches, and some ad hoc tricks, we prove that ECA 6, 7, 23, 33, 57 and 77 are von Neumann regular, while 9, 27, 28, 41 and 58 are not, answering the remaining cases of Question 1 of [1].

[1] Specifically, many things about "one-step behavior" of cellular automata (like surjectivity and injectivity) are decidable using automata theory, or the decidability of the MSO of the natural numbers under successor. No decision algorithm for split epicness using these methods is known.

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The von Neumann regular CA on this list have weak inverses of radius at most five. Non-regularity is proved in each case by looking at eventually periodic points of eventual period one. The non-regularity of all but ECA 9 and ECA 28 can be proved by simply observing that their images are proper sofic, though we also explain why they are not regular using the method of [9].

## 2 Preliminaries

The *full shift* is $\Sigma^{\mathbb{Z}}$ where $\Sigma$ is a finite alphabet, carrying the product topology, It is a dynamical system under the *shift* $\sigma(x)_i = x_{i+1}$. Its subsystems (closed shift-invariant subsets) are called *subshifts*. A *cellular automaton (CA)* is a shift-commuting continuous function $f : X \to X$ on a subshift $X$. The cellular automata on a subshift $X$ form a monoid $\mathrm{End}(X)$. A CA $f$ is *reversible* if $\exists g : f \circ g = g \circ f = \mathrm{id}$.

A cellular automaton has a local rule, that is, there exists a *radius* $r \in \mathbb{N}$ such that $f(x)_i$ is determined by $x|_{[i-r,i+r]}$ for all $x \in X$ (and does not depend on $i$). The *elementary cellular automata* (ECA) are the CA on the binary full shift $\{0,1\}^{\mathbb{Z}}$ which can be defined with radius 1. There is a numbering scheme for such CA: If $n \in [0, 255]$ has base 2 representation $b_7 b_6 ... b_1 b_0$, then ECA number $n$ is the one mapping

$$f(x)_i = 1 \iff b_{(x_{[i-1,i+1]})_2} = 1$$

where $(x_{[i-1,i+1]})_2$ is the number represented by $x_{[i-1,i+1]}$ in base 2. This numbering scheme is from [11].

We recall [1, Definition 3]: define maps $R, S : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ by the formulas $R(x)_i = x_{-i}$ and $S(x)_i = 1 - x_i$. Two cellular automata $f, g \in \mathrm{End}(\{0,1\}^{Z})$ are *equivalent* if $f \in \langle S \rangle \circ g \circ \langle S \rangle \cup \langle S \rangle \circ R \circ g \circ R \circ \langle S \rangle$, where $\circ$ denotes function composition and $\langle S \rangle = \{\mathrm{id}, S\}$.

The usage of base 10 in this notation is standard, and many CA researchers remember ECA by these numbers. However, for clarity we switch to hexadecimal notation from radius 2 upward.

A subshift can be defined by forbidding a set of finite words from appearing as subwords of its points (which themselves are infinite words), and this is in fact a characterization of subshifts. A subshift is *of finite type* or *SFT* if it can be defined by a finite set of forbidden patterns, and *sofic* if it can be defined by a forbidden regular language in the sense of automata and formal languages.

The *language* $\mathcal{L}(X)$ of a subshift $X$ is the set of finite words that appear in its points. An SFT $X$ is *mixing* if $L = \mathcal{L}(X)$ satisfies $\exists m : \forall u, v \in L : \exists w \in L : |w| = m \wedge uwv \in L$.

If $u \in A^*$ is a finite word, we write ${}^\infty u^\infty$ for the $|u|$-periodic point (i.e. fixed point of $\sigma^{|u|}$) in $A^{\mathbb{Z}}$ whose subword at $\{0, 1, ..., |u| - 1\}$ is equal to $u$.

See standard references for more information on symbolic dynamics [6] or automata theory [4].

## 3 Split epicness and von Neumann regularity

In this section, we show split epicness and von Neumann regularity are equivalent concepts on mixing SFTs. On the full shift, this is simply a matter of defining these terms.

If $S$ is a semigroup, then $a \in S$ is *(von Neumann) regular* if $\exists b \in S : aba = a \wedge bab = b$. We say $b$ is a *generalized inverse* of $a$. If $aba = a$, then $b$ is a *weak generalized inverse* of $a$.

▶ **Lemma 1.** *If $a$ has a weak generalized inverse, then it has a generalized inverse and thus is regular.*

**Proof.** If $aba = a$, then letting $c = bab$, we have $aca = ababa = aba = a$ and $cac = bababab = babab = bab = c$. ◀

If $\mathcal{C}$ is a category, a morphism $f : X \to Y$ is *split epic* if there is a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. Such a $g$ is called a *right inverse* or a *section*.

Note that in general category-theoretic concepts depend on the particular category at hand, but if $\mathcal{C}$ is a full subcategory of $\mathcal{D}$ (meaning a subcategory induced by a subclass of the objects, by taking all the morphisms between them), then split epicness for a morphism $f : X \to Y$ where $X, Y$ are objects of $\mathcal{C}$ means the same in both.

We are in particular interested in the category $K3$ (in the naming scheme of [9]) with sofic shifts as objects, and *block maps*, i.e. shift-commuting continuous functions $f : X \to Y$ as morphisms. More generally, morphisms between general subshifts have the same definition.

The following theorem is essentially only a matter of translating terminology, and works in many concrete categories.

▶ **Theorem 2.** *Let $X$ be a subshift, and $f : X \to X$ a cellular automaton. Then the following are equivalent:*

- *$f : X \to f(X)$ has a right inverse $g : f(X) \to X$ which can be extended to a morphism $h : X \to X$ such that $h|_{f(X)} = g$,*
- *$f$ is regular as an element of $\mathrm{End}(X)$.*

**Proof.** Suppose first that $f$ is regular, and $h \in \mathrm{End}(X)$ satisfies $fhf = f$ and $hfh = h$. Then the restriction $g = h|_{f(X)} : f(X) \to X$ is still shift-commuting and continuous, and $\forall x : fg(f(x)) = f(x)$ implies that for all $y \in f(X)$, $fg(y) = y$, i.e. $g$ is a right inverse for the codomain restriction $f : X \to f(X)$ and it extends to the map $h : X \to X$ by definition.

Suppose then that $fg = \mathrm{id}_{f(X)}$ for some $g : f(X) \to X$, as a right inverse of the codomain restriction $f : X \to f(X)$. Let $h : X \to X$ be such that $h|_{f(X)} = g$, which exists by assumption. Then $fh(f(x)) = fg(f(x)) = f(x)$. Thus $f$ is regular, and $hfh$ is a generalized inverse for it. ◀

Note that when $X$ is a full shift, extending morphisms is trivial: simply fill in the local rule arbitrarily. The Extension Lemma generalizes this idea to mixing SFTs:

▶ **Theorem 3.** *Let $X$ be a mixing SFT, and $f : X \to X$ a cellular automaton. Then the following are equivalent:*

- *$f : X \to f(X)$ is split epic in $K3$.*
- *$f$ is regular as an element of $\mathrm{End}(X)$.*

**Proof.** It is enough to show that any right inverse $g : f(X) \to X$ can be extended to $h : X \to X$ such that $h|_{f(X)} = g$. By the Extension Lemma [6], it is enough to show the "$X \searrow X$ condition" [6], which means that for every point $x \in X$ with minimal period $p$, there is a point $y \in X$ with minimal period dividing $p$. This holds trivially. ◀

Theorem 2 clearly implies that regularity respects equivalence (this is not difficult to obtain directly from the definition either).

▶ **Corollary 4.** *if $f, g \in \mathrm{End}(\{0,1\}^{\mathbb{Z}})$ are equivalent, then $f$ is regular if and only if $g$ is regular.*

## 4 Deciding split epicness

We recall the characterization of split epicness [9, Theorem 1]. This is Theorem 7 below.

▶ **Definition 5.** *Let $X, Y$ be subshifts and let $f : X \to Y$ be a morphism. Define*

$$\mathcal{P}_p(Y) = \{u \in \mathcal{L}(Y) \mid {}^{\infty}u^{\infty} \in Y, |u| \leq p\}.$$

*We say $f$ satisfies the* strong $p$-periodic point condition *if there exists a length-preserving function $G : \mathcal{P}_p(Y) \to \mathcal{L}(X)$ such that for all $u, v \in \mathcal{P}_p(Y)$ and $w \in \mathcal{L}(Y)$ with ${}^{\infty}u.wv^{\infty} \in Y$, there exists an $f$-preimage for ${}^{\infty}u.wv^{\infty}$ of the form ${}^{\infty}G(u)w'.w''w'''G(v)^{\infty} \in X$ where $|u|$ divides $|w'|$, $|v|$ divides $|w'''|$ and $|w| = |w''|$. The* strong periodic point condition *is that the strong $p$-periodic point condition holds for all $p \in \mathbb{N}$.*

Note that $G$ is simply a notation for a choice of periodic preimage for each periodic point, and the condition simply states that periodic tails of eventually periodic points eventually map according to $G$.

The strong periodic point condition is an obvious necessary condition for having a right inverse, as the right inverse must consistently pick preimages for periodic points, and they must satisfy these properties. Let us show the XOR CA with neighborhood $\{0, 1\}$ is not regular using this method – this is clear from the fact it is surjective, and from the fact there are 1-periodic points with no inverse of period 1, but it also neatly illustrates the strong periodic point method.

▶ **Example 6.** The CA $f : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ defined by

$$f(x)_i = 1 \iff x_i + x_{i+1} \equiv 1 \bmod 2$$

is not regular. To see this, consider the strong $p$-periodic point condition for $p = 1$. Since $f(0^{\mathbb{Z}}) = f(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$, the point $0^{\mathbb{Z}}$ has two preimages, and we must have either $G(0) = 0$ or $G(0) = 1$. It is enough to show that neither choice of $a = G(0)$ is consistent, i.e. there is a point $y$ which is in the image of $f$ such that $y$ has no preimage that is left and right asymptotic to $a^{\mathbb{Z}}$. This is shown by considering the point

$$y = ...0000001000000...$$

(which is in the image of $f$ since $f$ is surjective). It has two preimages, and the one left-asymptotic to $a^{\mathbb{Z}}$ is right-asymptotic to $(1 - a)^{\mathbb{Z}}$. ○

In [9, Theorem 1], it is shown that the strong periodic point condition actually characterizes split epicness, in the case when $X$ is an SFT and $Y$ is a sofic shift.

▶ **Theorem 7.** *Given two objects $X \subset S^{\mathbb{Z}}$ and $Y \subset R^{\mathbb{Z}}$ and a morphism $f : X \to Y$ in K3, it is decidable whether $f$ is split epic. If $X$ is an SFT, split epicness is equivalent to the strong periodic point condition.*

We note that Definition 5 is equivalent to a variant of it where $G$ is only defined on Lyndon words [7], i.e. lexicographically minimal representative words of periodic orbits: if $G$ is defined on those, it can be extended to all of $\mathcal{P}_p$ in an obvious way, and the condition being satisfied by minimal representatives implies it for all eventually periodic points.

▶ Remark 8. It is observed in [1, Theorem 1] that if $f : X \to Y$ is split epic, then every periodic point in $Y$ must have a preimage of the same period in $X$ – this is a special case of the above, and could thus be called the *weak periodic point condition*. In [9, Example 5], an example is given of morphism between mixing SFTs which satisfies the weak periodic point condition but not the strong one. We have not attempted to construct an example of a CA on a full shift which has this property onto its image, and we did not check whether any non-regular ECA satisfies it. In [1, Theorem 4], for full shifts on finite groups, the weak periodic point condition is shown to be equivalent to split epicness (when CA are considered to be morphisms onto their image). In the context of CA on $\mathbb{Z}^2$, there is no useful strong periodic point condition in the sense that split epicness is undecidable, see Corollary 13.

In the proof of Theorem 7 in [9], decidability is obtained from giving a bound on the radius of a minimal inverse, and a very large one is given, as we were only interested in the theoretical decidability result. The method is, however, quite reasonable in practise:

- To semidecide non-(split epicness), look at periodic points one by one, and try out different possible choices for their preimages. Check by automata-theoretic methods (or "by inspection") which of these are consistent in the sense of Definition 5.
- To semidecide split epicness, invent a right inverse – note that here we can use the other semialgorithm (running in parallel) as a tool, as it tells us more and more information about how the right inverse must behave on periodic points, which tells us more and more values of the local rule.

One of these is guaranteed to finish eventually by [9].

Proposition 10 below is a slight generalization of [9, Proposition 1]. We give a proof here, as the proof in [9] unnecessarily applies a more difficult result of S. Taati (and thus needs the additional assumption of "mixing"). This Proposition allows us to obtain non-regularity of all of the non-regular ECA considered here apart from ECA 9 and 28, though we do also provide a strong periodic point condition argument for all the non-regular ECA.

▶ **Lemma 9.** *If $X$ is an SFT and $f : X \to X$ is idempotent, i.e. $f^2 = f$, then $f(X)$ is an SFT.*

**Proof.** Clearly $x \in f(X) \iff f(x) = x$, which is an SFT condition. ◀

▶ **Proposition 10.** *If $X$ is an SFT and $f : X \to X$ is regular, then $f(X)$ is of finite type.*

**Proof.** Let $g : X \to X$ be a weak inverse. Then $g \circ f : X \to X$ is idempotent, so $g(f(X))$ is an SFT. Note that the domain-codomain restriction $g|_{f(X),g(f(X))} : f(X) \to g(f(X))$ is a conjugacy between $f(X)$ and $g(f(X))$: its two-sided inverse is $f|_{g(f(X))} \to f(X)$ by a direct computation. Thus $f(X)$ is also an SFT. ◀

We also mention another condition, although it is not applicable in the proofs.

▶ **Lemma 11.** *Let $X$ be a subshift with dense periodic points and $f : X \to X$ a cellular automaton. If $f$ is injective, it is surjective.*

**Proof.** The set $X_p = \{x \in X \mid \sigma^p(x) = x\}$ satisfies $f(X_p) \subset X_p$. Since $f$ is injective and $X_p$ is finite, we must have $f(X_p) = X_p$. Thus $f(X)$ is a closed set containing the periodic points. If periodic points are dense, $f(X) = X$. ◀

We are interested mainly in mixing SFTs, where periodic points are easily seen to be dense. We remark in passing that in the case of mixing SFTs, the previous lemma can also be proved with an entropy argument: An injective CA cannot have a *diamond*[2] when seen as

[2] This means a pair of distinct words whose long prefixes and suffixes agree, and which the local rule maps the same way, see [6].

a block map, so [6, Theorem 8.1.16] shows that the entropy of the image $f(X)$ of an injective CA is equal to the entropy of $X$. By [6, Corollary 4.4.9], $X$ is *entropy minimal*, that is, has no proper subshifts of the same entropy, and it follows that $f(X) = X$.

▶ **Proposition 12.** *Let $X$ be a mixing SFT and $f : X \to X$ a surjective cellular automaton. Then $f$ is injective if and only if it is regular.*

**Proof.** Suppose $f$ is a surjective CA on a mixing SFT. If it is also injective, it is thus bijective, thus reversible, thus regular. Conversely, let $f$ be surjective and regular, and let $g : X \to X$ be a weak generalized inverse. Then $g$ is injective, so it is surjective by the previous lemma. Thus $f$ must be bijective as well. ◀

More generally, the previous proposition works on *surjunctive subshifts* in the sense of [2, Exercise 3.29], i.e., subshifts where injective cellular automata are surjective. In particular this is the case for full shifts on surjunctive groups [3, 10] such as abelian ones. Since injectivity is undecidable for surjective CA on $\mathbb{Z}^d$, $d \geq 2$ by [5], we obtain the following corollary.

▶ **Corollary 13.** *Given a surjective CA $f : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$, it is undecidable whether $f$ is split epic.*

## 5 Von Neumann regularity of elementary CA

▶ **Theorem 14.** *The elementary CA with numbers 6, 7, 23, 33, 57 and 77 are regular.*

**Proof.** It is a finite case analysis to verify that the CA defined in Figure 1, Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6 in Appendix A are generalized inverses of the respective CA. Code for verifying this and discussion on how such rules were found is included in the arXiv version [8]. ◀

▶ **Theorem 15.** *The elementary CA with numbers 9, 27, 28, 41 and 58 are not regular.*

**Proof.** See the lemmas below. ◀

▶ **Lemma 16.** *The elementary CA 9 is not regular.*

**Proof.** Let $f$ be the ECA 9, i.e. $f(x)_i = 1 \iff x_{[i-1,i+1]} \in \{000, 011\}$. The image $X$ of $f$ is the SFT with forbidden patterns 1011, 10101, 11001, 11000011 and 110000101. One can verify[3] this with standard automata-theoretic methods.

We have $f(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$ and $f(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$, so if $g : X \to \{0,1\}^{\mathbb{Z}}$ is a right inverse for $f$, then $g(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$. Consider now the configuration

$$x = ...0000011.00000... \in X$$

where coordinate 0 is to the left of the decimal point (i.e. the rightmost 1 or the word 11). Let $g(x) = y$. Then $y_i = 1$ for all large enough $i$ and $y_i = 0$ for some $i$. Let $n$ be maximal such that $y_n = 0$. Then $y_{[n,n+2]} = 011$ so $f(y)_{n+1} = 1$ and $f(y)_{n+1+i} = 0$ for all $i \geq 1$. Since $f(y) = x$, we must have $n = -1$ and since $\{000, 011\}$ does not contain a word of the form $a01$, it follows that $f(y)_{-1} = 0 \neq x_{-1}$, a contradiction. ◀

[3] For verifying only the proof of this lemma, i.e. the non-regularity of ECA 9, it is enough to show that the point $x$ below is in $X$, that is, it has some preimage (...0100100001001001... is one). Knowing the SFT is, however, essential for finding such an argument, so we argue in this way, again to illustrate the method.

The proof shows that the CA does not have the strong periodic point property for $p = 1$. In general, for fixed $p$ one can use automata-theory to decide whether it holds up to that $p$, though here (and in all other proofs) we found the contradictions by hand before we had to worry about actually implementing this.

▶ **Lemma 17.** *The ECA* 27 *is not regular.*

**Proof.** Let $f$ be the ECA 27, i.e. $f(x)_i = 1 \iff x_{[i-1,i+1]} \in \{000, 001, 011, 100\}$. The image $X$ of $f$ is proper sofic, we omit the automaton and argue directly in terms of configurations. Proposition 10 directly shows that the CA can not be regular in the case when the image is proper sofic, but we give a direct proof to illustrate the method (and so that we do not have to provide a proof that the image is sofic, which is straightforward but lengthy).

Again, we will see that this CA does not satisfy the strong periodic point condition for $p = 1$. Observe that $f(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$ and $f(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$ so if $g$ is a right inverse from the image to $\{0,1\}^{\mathbb{Z}}$, then $g(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$ and $g(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$. Let $y = ...000001100.10101010...$ and observe that

$$f(y) = f(...000001100.10101010...) = \\ ...111111011.00000000... = x \in X.$$

We now reason similarly as in Lemma 16. We have $g(x)_i = 1$ for all large enough $i$, and if $n$ is maximal such that $g(x)_n = 0$, then $f(g(x))_{n+1} = 1$ and $f(g(x))_{n+1+i} = 0$ for all $i \geq 1$, so again necessarily $n = -1$. A short combinatorial analysis shows that no continuation to the left from $n$ produces $f(g(x))_n = 1$ and $f(g(x))_{n-1} = 0$, that is, the image of $g$ has no possible continuation up to coordinate $-1$. ◀

▶ **Lemma 18.** *The ECA* 28 *is not regular.*

**Proof.** Let $f$ be the ECA 28, i.e. $f(x)_i = 1 \iff x_{[i-1,i+1]} \in \{010, 011, 100\}$. The image $X$ of $f$ is the SFT with the single forbidden pattern 111.

We have $f(0^{\mathbb{Z}}) = f(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$. The point

$$...0000.10000... \in X$$

contradicts the choice $g(0^{\mathbb{Z}}) = 0^{\mathbb{Z}}$ by a similar analysis as in previous theorems; similarly as in Example 6, computing the preimage from right to left, the asymptotic type necessarily changes to 1s. Thus we must have $g(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$.

On the other hand, if $g(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$, then going from right to left, we cannot find a preimage for

$$...0001.10000... \in X.$$

(Alternatively, going from left to right, the asymptotic type necessarily changes to 0s or never becomes 1-periodic.)

It follows that $g(0^{\mathbb{Z}})$ has no consistent possible choice, a contradiction. ◀

▶ **Lemma 19.** *The ECA* 41 *is not regular.*

**Proof.** Let $f$ be the ECA 41, i.e. $f(x)_i = 1 \iff x_{[i-1,i+1]} \in \{000, 011, 101\}$. The image $X$ of $f$ is proper sofic, we omit the automaton and argue directly in terms of configurations. Again Proposition 10 would also yield the result.

Again, we will see that this CA does not satisfy the strong periodic point condition for $p = 1$. Observe that $f(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$ and $f(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$ so if $g$ is a right inverse from the image to $\{0,1\}^{\mathbb{Z}}$, then $g(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$ and $g(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$. Let $y = ...00000001.00100100...$ so

$$f(y) = f(...00000001.00100100...) = \\ ...11111100.00000000... = x \in X.$$

In the usual way (right to left), we verify that $x$ has no preimage that is right asymptotic to $1^{\mathbb{Z}}$, obtaining a contradiction. ◀

▶ **Lemma 20.** *The ECA* 58 *is not regular.*

**Proof.** Let $f$ be the ECA 58, i.e. $f(x)_i = 1 \iff x_{[i-1,i+1]} \in \{001, 011, 100, 101\}$. The image $X$ of $f$ is proper sofic, we omit the automaton. Again Proposition 10 would also yield the result.

The point $0^{\mathbb{Z}}$ has two 1-periodic preimages. We show neither choice satisfies the strong periodic point condition: if $g(0^{\mathbb{Z}}) = 1^{\mathbb{Z}}$, then $g$ cannot give a preimage for

$...0000000.10000000...$

If $g(0^{\mathbb{Z}}) = 0^{\mathbb{Z}}$, then it cannot give a preimage for

$...0000000.11000000...$

It is easy to find preimages for these two configurations, however, so ECA 58 is not regular. ◀

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## A Weak generalized inverses

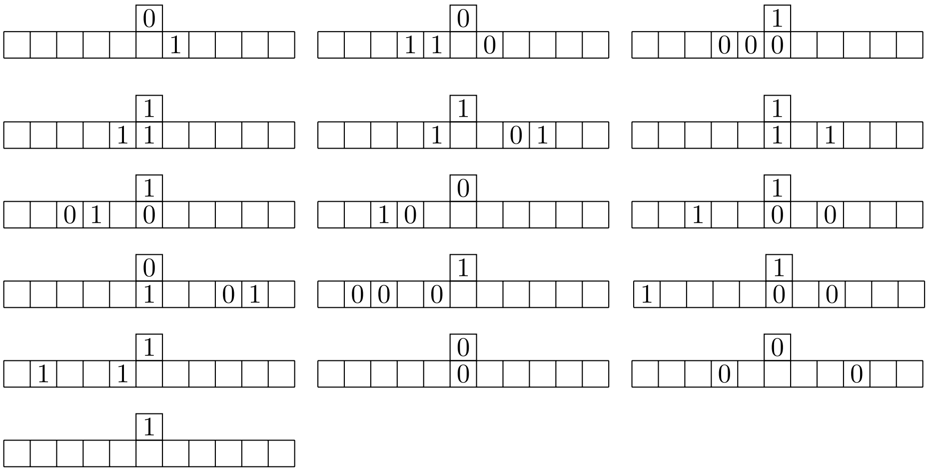

**Figure 1** A weak generalized inverse of ECA 6. The rules are applied row by row, and on each row from left to right. An empty box denotes a wildcard symbol, and the first rule to apply is used. The rightmost coordinate is not actually read by any rule. This is the radius 5 binary CA with the hex number

```
0000000000000000000FFF3000000FF0000FFFF000000F00000FF000000FFFF
```

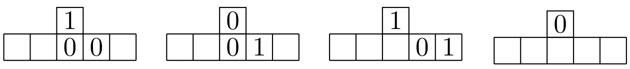

**Figure 2** A weak generalized inverse of ECA 7. This is the radius 2 binary CA with the hex number `23232323`, equal to ECA 35 composed with $\sigma$.

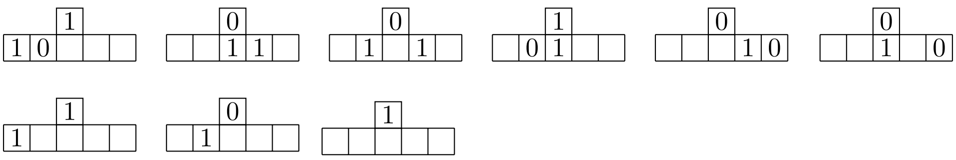

**Figure 3** A weak generalized inverse of ECA 23. This is the radius 2 binary CA with the hex number `23FF003B`.

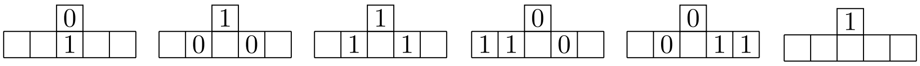

**Figure 4** A weak generalized inverse of ECA 33. This is the radius 2 binary CA with the hex number `0C070F07`.

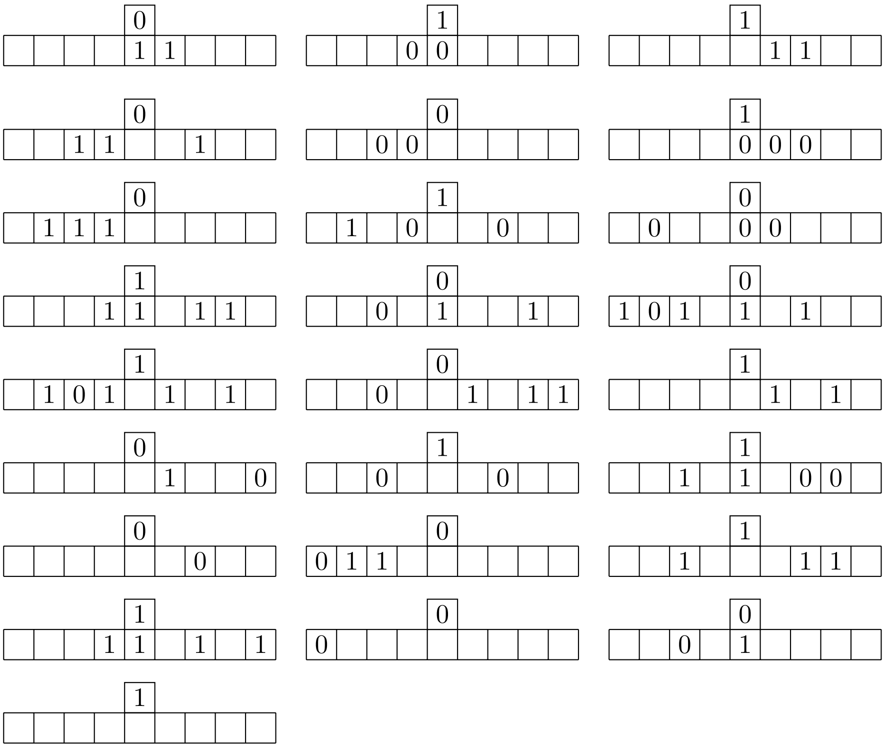

**Figure 5** A weak generalized inverse of ECA 57. This is the radius 4 binary cellular automaton with the hex number

`0000F00F00FFFFFF00E3FEFF0000FFFF0003FC0F0003FFFF00E3F60F0000FFFF`
`0000F00F000FFFFF00E3FE0F0000FFFF0003FC0F00C3FFFF00E3F60F0000FFFF`

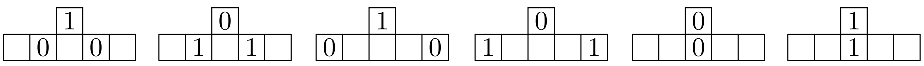

**Figure 6** A weak generalized inverse of ECA 77. This is the radius 2 binary CA with the hex number `107331F7`.