

Process-As-Formula Interpretation: A Substructural Multimodal View

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Abstract

In this survey, we show how the processes-as-formulas interpretation, where computations and proof-search are strongly connected, can be used to specify different concurrent behaviors as logical theories. The proposed interpretation is parametric and modular, and it faithfully captures behaviors such as: Linear and spatial computations, epistemic state of agents, and preferences in concurrent systems. The key for this modularity is the incorporation of multimodalities in a resource aware logic, together with the ability of quantifying on such modalities. We achieve tight adequacy theorems by relying on a focusing discipline that allows for controlling the proof search process.

2012 ACM Subject Classification Theory of computation → Proof theory; Theory of computation → Process calculi

Keywords and phrases Linear logic, proof theory, process calculi

Digital Object Identifier 10.4230/LIPIcs.FSCD.2021.3

Category Invited Talk

Funding *Elaine Pimentel*: Partially funded by the project MOSAIC (MSCA RISE 101007627).

Carlos Olarte: Partially funded by CNPq.

1 Introduction

Computational logic research has produced deep and fruitful cross-fertilizations between programming languages and proof theory. Arguably, the most well-known one is the *Curry-Howard correspondence* (also known as types-as-formulas) where (functional) programs correspond to formal proofs and their execution to cut-elimination. A second type of correspondence, *processes-as-formulas* (also known as computation-as-proof-search), was initiated by Miller [21] where, instead, (logic) programs correspond to formulas and their execution to proof search. These two foundational correspondences have been exploited to propose new programming language paradigms as well as greatly extend the expressiveness of existing ones.

When processes or programs are specified as formulas, one has to be careful with the level of adequacy obtained. In particular, it is expected that logical steps in derivations correspond to steps of computations in programs. However, different from computational systems, where *one step of computation* is rigidly determined by the operation semantics, *one step of logical reasoning* depends strongly on the logical framework chosen. Also, the logic should capture, in a natural way, the behavior of programs. For instance, intuitionistic logic (IL) is not adequate to specify systems that may consume information (substructural behavior), execute

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processes in different locations (spatial modalities) or time instances (timed reasoning), or when the information shared by processes is subject to quantitative information (such as preferences or costs).

Hence the need for a more expressive logic (such as *multimodal and resource aware logics*) and an appropriate notion of normal proofs as the logical counterpart of the processes-as-formulas correspondence. This paper surveys one of such choices: focused linear logic with subexponentials (SELLF) [28]. We present different mechanisms previously explored by the authors to both: extend SELLF with quantification over subexponentials; and give adequate characterizations of existing concurrent languages. This fruitful collaboration between the two areas has been useful to provide reasoning techniques for process calculi with the motto *reachability as entailment*, and also to propose declarative extensions of concurrent languages with solid logical grounds.

The *focusing discipline* [1] determines an alternating mechanism on proofs (between *focused* and *unfocused* phases), which controls the non-determinism during proof search, producing normal form proofs. Such normalization of proofs leads to a practical approach to identify logical steps: a *focused step* is a block determined by a focused phase followed by an unfocused one, in a (bottom-up) focused proof.

In Section 2 we recall the proof theory of focused intuitionistic linear logic (ILLF), which will be the base logical language for the processes-as-formulas correspondences addressed in this paper. Section 3 then introduces the base computational counterpart of the correspondence, Concurrent Constraint Programming (CCP) [42], a declarative model for concurrency. We show how to adequately capture the behavior of CCP processes in ILLF.

The *level of adequacy* attained in such interpretations will be important in order to justify the choice of the underlying logic: the closer the two systems are, the easier is to prove the correspondence. Also, a strong adequacy allows for the use of the logical system for proving properties of the computational system, or reconstructing counter-examples from failing derivations. Following [29], we classify the level of adequacy into two classes:

- **FCP** (*full completeness of proofs*) claims that processes outputting an observable are in 1-1 correspondence with the corresponding completed proofs.
- **FCD** (*full completeness of derivations*) claims that one step of computation should correspond to one step of logical reasoning.

In the first case, even though the outputs of a program are characterized by proofs in the underlying logic, it may be the case that there are steps in the logical reasoning that do not correspond to computational steps and vice-versa. In the second case, computational and (in our case, focused) logical steps are in one-to-one correspondence. We present a careful discussion about these different levels of adequacy regarding CCP and ILLF in Section 3.2, and indicate throughout the text, in each result, its level of adequacy.

Even though (focused, intuitionistic) linear logic is suitable for the encoding of (vanilla) CCP, the situation changes when *modalities* are added to concurrent systems: For that, linear logic *subexponentials* are needed. In Section 4 we present SELLF, which shares with ILL all its connectives except the exponential: instead of having a single $!$, it may contain as many subexponentials as one needs (written $!^a$). Such labels are organized in a pre-order, and different organizations give rise to different CCP flavors. Section 5 is then devoted to show how to add such structures *parametrically* to SELLF, obtaining strongly adequate specifications. In this way, processes may be executed and add/query constraints in different *locations*, where the meaning of such locations may vary, for example: Spaces of computation, the epistemic state of agents, time units, levels of preferences, etc. *Modularity* is guaranteed by the fact that the underline interpretation is the same: Locations in CCP become labels in SELLF. Finally, Section 6 concludes the paper.

2 Focused intuitionistic linear logic

Linear logic (LL) is a substructural logic proposed by Girard [13] as a refinement of classical and intuitionistic logics, joining the dualities of the former with many of the constructive properties of the latter.

In this paper, we will concentrate in the *intuitionistic* version of linear logic (ILL) [13], with formulas built from the following grammar

$$F, G ::= A \mid 1 \mid 0 \mid \top \mid F \otimes G \mid F \& G \mid F \oplus G \mid F \multimap G \mid !F \mid \forall x.F \mid \exists x.F$$

Here, A denotes an atomic formula; \multimap , \otimes , 1 represent the *multiplicative* implication, conjunction and true, respectively; $\&$, \top , \oplus , 0 are the *additive* conjunction, true, disjunction, and false, respectively; $!$ is the *exponential*; and \exists, \forall represent the existential and universal quantifiers, respectively.²

These connectives can be separated into two classes, the *negative*: $\multimap, \&, \top, \forall$ and the *positive*: $\otimes, \oplus, !, 1, 0, \exists$. The polarity of non-atomic formulas is inherited from its outermost connective (e.g., $F \otimes G$ is a positive formula) and any bias can be assigned to atomic formulas.³ This partition induces an alternating mechanism on proofs, known as *focusing*, which aims at reducing the non-determinism during proof search. In this sense, focused proofs can be interpreted as *normal form* proofs.

The focusing discipline [1] is determined by the alternation of *focused* and *unfocused* phases in the proof construction. In the unfocused phase, inference rules can be applied eagerly and no backtracking is necessary; in the focused phase, on the other hand, either context restrictions apply, or choices within inference rules can lead to failures for which one may need to backtrack. These phases are totally determined by the polarities of formulas: provability is preserved when applying right/left rules for negative/positive formulas respectively, but not necessarily in other cases.

The focused intuitionistic linear logic system (ILLF) is depicted in Figure 1.

There are three contexts on the left side of ILLF sequents: the set Θ denotes the *unbounded* context, containing only formulas with a banged scope; Γ is a *linear* context containing only negative or atomic formulas; and Δ is the general linear context. Observe that formulas in the context Θ behave as in classical logic: they can be weakened (erased) or contracted (duplicated). Formulas in the other contexts are linear, and are consumed when used.

The phase distinction is reflected in the design of sequents in ILLF: the presence of “ \uparrow ” indicates unfocused sequents, while “ \Downarrow ” marks the formula under focus in focused sequents. Sequents in ILLF have one of the following shapes:

- i. $\Theta; \Gamma \uparrow \Delta \vdash F \uparrow$ is an unfocused sequent.
- ii. $\Theta; \Gamma \uparrow \cdot \vdash \cdot \uparrow F$ is an unfocused sequent representing the end of an unfocused phase.
- iii. $\Theta; \Gamma \vdash F \Downarrow$ is a sequent focused on the right.
- iv. $\Theta; \Gamma \Downarrow F \vdash R$ is a sequent focused on the left.

The swing between focused and unfocused phases is described below.

- At the beginning of an unfocused phase, sequents have the shape (i) and: non-atomic negative formulas appearing in the right context, and positive non-atomic formulas appearing in Δ are eagerly introduced; atomic/negative left formulas are stored in Γ using the store rule S_l ; atomic/positive right formulas are stored in the outermost right context using the store rule S_r .

When this phase ends, sequents have the form (ii).

² Observe that the multiplicative false \perp could be added to ILL’s syntax. However, this would break the nice feature of having *exactly* one formula on succedent of sequents.

³ Although the bias assigned to atoms does not interfere with provability, it changes *considerably* the shape of proofs (see, e.g., [19]).

UNFOCUSED INTRODUCTION RULES

$$\begin{array}{c}
\frac{\Theta; \Gamma \uparrow F, \Delta \vdash G \uparrow}{\Theta; \Gamma \uparrow \Delta \vdash F \multimap G \uparrow} \multimap_r \quad \frac{\Theta; \Gamma \uparrow F, G, \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow F \otimes G, \Delta \vdash \mathcal{R}} \otimes_l \quad \frac{F, \Theta; \Gamma \uparrow \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow !F, \Delta \vdash \mathcal{R}} !_l \\
\\
\frac{\Theta; \Gamma \uparrow \Delta \vdash F \uparrow \quad \Theta; \Gamma \uparrow \Delta \vdash G \uparrow}{\Theta; \Gamma \uparrow \Delta \vdash F \& G \uparrow} \&_r \quad \frac{\Theta; \Gamma \uparrow F, \Delta \vdash \mathcal{R} \quad \Theta; \Gamma \uparrow G, \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow F \oplus G, \Delta \vdash \mathcal{R}} \oplus_l \\
\\
\frac{\Theta; \Gamma \uparrow \Delta \vdash F[y/x] \uparrow}{\Theta; \Gamma \uparrow \Delta \vdash \forall x. F \uparrow} \forall_r \quad \frac{\Theta; \Gamma \uparrow F[y/x], \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow \exists x. F, \Delta \vdash \mathcal{R}} \exists_l \\
\\
\frac{}{\Theta; \Gamma \uparrow \Delta \vdash \top \uparrow} \top_r \quad \frac{\Theta; \Gamma \uparrow \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow 1, \Delta \vdash \mathcal{R}} 1_l \quad \frac{}{\Theta; \Gamma \uparrow 0, \Delta \vdash \mathcal{R}} 0_l
\end{array}$$

FOCUSED INTRODUCTION RULES

$$\begin{array}{c}
\frac{\Theta; \Gamma_1 \vdash F \Downarrow \quad \Theta; \Gamma_2 \Downarrow G \vdash R}{\Theta; \Gamma_1, \Gamma_2 \Downarrow F \multimap G \vdash R} \multimap_l \quad \frac{\Theta; \Gamma \vdash F_i \Downarrow}{\Theta; \Gamma \vdash F_1 \oplus F_2 \Downarrow} \oplus_{r_i} \quad \frac{\Theta; \Gamma \Downarrow F_i \vdash R}{\Theta; \Gamma \Downarrow F_1 \& F_2 \vdash R} \&_{l_i} \\
\\
\frac{\Theta; \Gamma_1 \vdash F \Downarrow \quad \Theta; \Gamma_2 \vdash G \Downarrow}{\Theta; \Gamma_1, \Gamma_2 \vdash F \otimes G \Downarrow} \otimes_r \quad \frac{\Theta; \cdot \uparrow \cdot \vdash F \uparrow}{\Theta; \cdot \vdash !F \Downarrow} !_r \\
\\
\frac{\Theta; \Gamma \Downarrow F[t/x] \vdash R}{\Theta; \Gamma \Downarrow \forall x. F \vdash R} \forall_l \quad \frac{\Theta; \Gamma \vdash F[t/x] \Downarrow}{\Theta; \Gamma \vdash \exists x. F \Downarrow} \exists_r \quad \frac{}{\Theta; \cdot \vdash 1 \Downarrow} 1_r
\end{array}$$

STRUCTURAL AND IDENTITY RULES

$$\begin{array}{c}
\frac{\Theta; \Gamma \Downarrow N \vdash R}{\Theta; \Gamma, N \uparrow \cdot \vdash \cdot \uparrow R} D_l \quad \frac{\Theta, F; \Gamma \Downarrow F \vdash R}{\Theta, F; \Gamma \uparrow \cdot \vdash \cdot \uparrow R} Du \quad \frac{\Theta; \Gamma \vdash P \Downarrow}{\Theta; \Gamma \uparrow \cdot \vdash \cdot \uparrow P} D_r \\
\\
\frac{\Theta; \Gamma \uparrow P \vdash \cdot \uparrow R}{\Theta; \Gamma \Downarrow P \vdash R} R_l \quad \frac{\Theta; \Gamma \uparrow \cdot \vdash N \uparrow}{\Theta; \Gamma \vdash N \Downarrow} R_r \quad \frac{\Theta; C, \Gamma \uparrow \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow C, \Delta \vdash \mathcal{R}} S_l \quad \frac{\Theta; \Gamma \uparrow \cdot \vdash \cdot \uparrow D}{\Theta; \Gamma \uparrow \cdot \vdash D \uparrow} S_r \\
\\
\frac{}{\Theta; A \vdash A \Downarrow} ! \quad \frac{}{\Theta, A; \cdot \vdash A \Downarrow} !_c
\end{array}$$

Here, P is positive, N is negative, C is a negative formula or positive atom, D a positive formula or negative atom, and A is a positive atom. Other formulas are arbitrary. \mathcal{R} denotes $\Delta_1 \uparrow \Delta_2$ where the union of Δ_1 and Δ_2 contains exactly one formula. In the rules \forall_r and \exists_l the eigenvariable y does not occur free in any formula of the conclusion.

■ **Figure 1** The focused intuitionistic linear sequent calculus ILLF.

- The focused phase begins by choosing, via one of the decide rules D_l , D_u or D_r , a formula to be focused on, enabling sequents of the forms (iii) or (iv). Rules are then applied on the focused formula until either: an axiom is reached (in which case the proof ends); the right promotion rule $!_r$ is applied; or a negative formula on the right or a positive formula on the left is derived. At this point, focusing will be lost, and the proof switches to the unfocused phase again.

We will call a *focused step* a focused phase followed by an unfocused one, in a (bottom-up) focused proof.

Observe that the design of the axioms I and I_c in ILLF induces a *positive* polarity to atoms. As it will become clear in Section 3.2, this is necessary for guaranteeing the higher level of adequacy on encodings.

Sequents in ILL will be denoted by $\Gamma \vdash A$. Rules for ILL are the same as in ILLF, only not considering focusing, and the structural rules being substituted by the usual bang left rules: dereliction (D), weakening (W) and contraction (C):

$$\frac{\Gamma, F \vdash G}{\Gamma, !F \vdash G} \text{ D} \quad \frac{\Gamma \vdash G}{\Gamma, !F \vdash G} \text{ W} \quad \frac{\Gamma, !F, !F \vdash G}{\Gamma, !F \vdash G} \text{ C}$$

Note that, in ILLF, dereliction is embedded into the bang left ($!_l$) and unbounded decide (Du) rules.

3 Concurrent Constraint Processes as LL Formulas

In this section we shall see how the process-as-formula interpretation can be used for both, providing verification techniques for a process calculus and characterizing different semantics for it in a uniform way. We start by describing the model of computation of *Concurrent Constraint Programming* (CCP) to later show that ILLF provides a suitable framework for interpreting CCP processes.

Concurrent Constraint Programming (CCP) [41, 42, 43, 37] is a model for concurrency based upon the shared-variables communication model. CCP traces its origins back to the ideas of *computing with constraints* [25], *Concurrent Logic Programming* [45] and *Constraint Logic Programming* (CLP) [15]. Different from other models for concurrency, based on point-to-point communication as in CCS [23], the π -calculus [24], CSP [14] among several others, the CCP model focuses on the concept of *partial information*, traditionally referred to as *constraints*. Under this paradigm, the conception of *store as valuation* in the von Neumann model is replaced by the notion of *store as constraint*, and processes are seen as *information transducers*.

The model of concurrency in CCP is quite simple: concurrent agents (or processes) interact with each other and their environment by posting and asking information (i.e., constraints) in a medium, a so-called *store*. As we shall see, CCP processes can be seen as both computing processes (behavioral style) and as formulas in logic (logical declarative style). In particular, we shall see a strong connection between ILL and CCP originally developed in [11] and later refined in [34].

3.1 Constraint system and processes

We start by defining the language of processes and constraints. The type of constraints processes may act on is not fixed but parametric in a constraint system. Such systems can be formalized as Scott information systems [44] as in [40], or they can be built upon a suitable fragment of logic e.g. as in [46, 11, 26]. Here we shall follow the second approach. More precisely, a constraint system is a tuple $\mathbf{C} = (\mathcal{C}, \models_{\Delta})$ where the set of constraints \mathcal{C} is built from a first-order signature and the grammar

$$F ::= \text{true} \mid A \mid F \wedge F \mid \exists \bar{x}. F$$

where A is an atomic formula. We shall use c, c', d, d' , etc, to denote elements in \mathcal{C} . The entailment relation \models_{Δ} is parametric on a set of non-logical axioms Δ of the form $\forall \bar{x}. [c \supset c']$ where all free variables in c and c' are in \bar{x} . We say that d *entails* c , written as $d \models_{\Delta} c$, iff

the sequent $\Delta, d \vdash c$ is provable in intuitionistic logic (IL). Intuitively, the entailment relation specifies inter-dependencies between constraints: $c \models_{\Delta} d$ means that the information d can be deduced from the information represented by c , e.g. $x > 42 \models_{\Delta} x > 0$.

The constraint store, shared by processes, is a conjunction of constraints and **true** denotes the empty store. The existential quantifier is used to specify variable hiding.

Processes are built from constraint as follows:

$$P, Q ::= \mathbf{tell}(c) \mid \sum_{i \in I} \mathbf{ask} c_i \mathbf{then} P_i \mid P \parallel Q \mid (\mathbf{local} x) P \mid p(\bar{x})$$

A process $\mathbf{tell}(c)$ adds the constraint c to the store, thus incrementing the information in it. The guarded choice $\sum_{i \in I} \mathbf{ask} c_i \mathbf{then} P_i$, where I is a finite set of indexes, chooses non-deterministically one of the processes P_j whose guard c_j can be deduced from the current store. If none of the guards can be deduced, this process remains blocked until more information is added. Hence, ask agents implement a synchronization mechanism based on entailment of constraints. The interleaved parallel composition of P and Q is denoted as $P \parallel Q$. The agent $(\mathbf{local} x) P$ behaves as P and binds the variable x to be local to it. Finally, given a possibly recursive process definition $p(\bar{y}) \triangleq P$, where all free variables of P are in the set of pairwise distinct variables \bar{y} , the process $p(\bar{x})$ evolves into $P[\bar{x}/\bar{y}]$.

The operational semantics of CCP is given by the transition relation $\gamma \longrightarrow \gamma'$ satisfying the rules in Figure 2. Here we follow the semantics in [11] and a *configuration* γ is a triple of the form $(X; \Gamma; c)$, where c is a constraint specifying the store, Γ is a multiset of processes, and X is the set of hidden (local) variables of c and Γ . The multiset $\Gamma = P_1, P_2, \dots, P_n$ represents the process $P_1 \parallel P_2 \dots \parallel P_n$. We shall indistinguishably use both notations to denote parallel composition of processes.

Processes are quotiented by a structural congruence relation \cong satisfying: (1) $P \cong Q$ if they differ only by a renaming of bound variables (alpha-conversion); (2) $P \parallel Q \cong Q \parallel P$; and (3) $P \parallel (Q \parallel R) \cong (P \parallel Q) \parallel R$. Furthermore, $\Gamma = \{P_1, \dots, P_n\} \cong \{P'_1, \dots, P'_n\} = \Gamma'$ iff $P_i \cong P'_i$ for all $1 \leq i \leq n$. Finally, $(X; \Gamma; c) \cong (X'; \Gamma'; c')$ iff $X = X'$, $\Gamma \cong \Gamma'$ and $c \equiv_{\Delta} c'$ (i.e., $c \models_{\Delta} c'$ and $c' \models_{\Delta} c$).

Rules R_T and R_C are self-explanatory. Rule R_{EQUIV} says that structurally congruent processes have the same transitions. Rule R_L adds the variable x to the set of variables X when it is fresh (otherwise, Rule R_{EQUIV} can be used to apply alpha conversion). The rule R_A says that the process $\sum_{i \in I} \mathbf{ask} c_i \mathbf{then} P_i$ evolves into P_j if the current store d entails c_j .

► **Definition 1 (Observables).** Let \longrightarrow^* be the reflexive and transitive closure of \longrightarrow . If $(X; \Gamma; d) \longrightarrow^* (X'; \Gamma'; d')$ and $\exists X'. d' \models_{\Delta} c$ we write $(X; \Gamma; d) \Downarrow_c$. If $X = \emptyset$ and $d = \mathbf{true}$ we simply write $\Gamma \Downarrow_c$.

Intuitively, if P is a process then $P \Downarrow_c$ says that P outputs c under input **true**.

3.2 Interpretation and adequacy

We shall present different encodings for processes ($\mathcal{P}[\cdot]$) and constraints ($\mathcal{C}[\cdot]$) as formulas in ILL. Our goal is to show that the outputs of a process P can be characterized by proofs in ILLF. More precisely, we shall show that P outputs c iff a sequent of the form $\mathcal{P}[\Psi], \mathcal{C}[\Delta] : \cdot \uparrow \mathcal{P}[P] \vdash \mathcal{C}[c] \otimes \top \uparrow$ is provable in ILLF, where Ψ is a set of process definitions and Δ is the set of non-logical axioms in the constraint system. Note the use of \top : we shall erase the formulas corresponding to processes that were not executed. Below, we will see how to tune the process interpretation to get the highest level of adequacy possible.

$$\begin{array}{c}
\frac{(X; \Gamma; c) \cong (X'; \Gamma'; c') \longrightarrow (Y'; \Delta'; d') \cong (Y; \Delta; d)}{(X; \Gamma; c) \longrightarrow (Y; \Delta; d)} \text{R}_{\text{EQUIV}} \\
\\
\frac{}{(X; \text{tell}(c), \Gamma; d) \longrightarrow (X; \Gamma; c \wedge d)} \text{R}_T \quad \frac{d \models_{\Delta} c_j}{(X; \sum_{i \in I} \text{ask } c_i \text{ then } P_i, \Gamma, d) \longrightarrow (X; P_j, \Gamma, d)} \text{R}_A \\
\\
\frac{}{(X; (\text{local } x) P, \Gamma; d) \longrightarrow (X \cup \{x\}; P, \Gamma; d)} \text{R}_L \quad \frac{p(\bar{x}) \triangleq P}{(X; p(\bar{y}), \Gamma; d) \longrightarrow (X; P[\bar{y}/\bar{x}], \Gamma; d)} \text{R}_C
\end{array}$$

■ **Figure 2** Operational semantics of CCP. In R_L , $x \notin X$ and it does not occur free in Γ nor in d .

► **Definition 2.** *Constraints and axioms in CCP are encoded in ILL as follows:*

$$\begin{array}{lll}
\mathcal{C}[\text{true}] = 1 & \mathcal{C}[A] = !A & \mathcal{C}[F_1 \wedge F_2] = \mathcal{C}[F_1] \otimes \mathcal{C}[F_2] \\
\mathcal{C}[\exists x.F] = \exists x.\mathcal{C}[F] & \mathcal{C}[\forall \bar{x}.(c \supset d)] = \forall \bar{x}.(\mathcal{C}[c] \multimap \mathcal{C}[d]) &
\end{array}$$

For the processes and process definition, the interpretation is the following:

$$\begin{array}{lll}
\mathcal{P}[\text{tell}(c)] & = \mathcal{C}[c] & \mathcal{P}[P \parallel Q] = \mathcal{P}[P] \otimes \mathcal{P}[Q] \\
\mathcal{P}[\sum_{i \in I} \text{ask } c_i \text{ then } P_i] & = \&_{i \in I} (\mathcal{C}[c_i] \multimap \mathcal{P}[P_i]) & \mathcal{P}[(\text{local } x) P] = \exists x.\mathcal{P}[P] \\
\mathcal{P}[p(\bar{y})] & = p(\bar{y}) & \mathcal{P}[p(\bar{x}) \triangleq P] = \forall \bar{x}.(p(\bar{x}) \multimap \mathcal{P}[P])
\end{array}$$

Since the store in CCP is monotonic, i.e., constraints cannot be removed, we mark atomic formulas with a bang (to be stored in the unbounded context). Parallel composition is identified with multiplicative conjunction and the act of choosing one of the branches in a non-deterministic choice is specified with additive conjunction. The action of querying the store in ask agents is specified with a linear implication. Similarly, the unfolding of a process definition is guarded by the atomic proposition $p(\bar{y})$ (denoting the call).

If Γ is a set of constraints, or axioms of the form $\forall \bar{x}.[c \supset c']$, we write $\mathcal{C}[\Gamma]$ to denote the set $\{\mathcal{C}[d] \mid d \in \Gamma\}$. A similar convention applies for $\mathcal{P}[\cdot]$. Moreover, $!\Gamma = \{!F \mid F \in \Gamma\}$.

► **Theorem 3** (Adequacy – ILL [11]). *Let $(\mathcal{C}, \models_{\Delta})$ be a constraint system, P be a process and Ψ be a set of process definitions. Then, for any constraint c , $P \Downarrow_c$ iff there is a proof of the sequent $!\mathcal{P}[\Psi], !\mathcal{C}[\Delta], \mathcal{P}[P] \vdash \mathcal{C}[c] \otimes \top$ in ILL. The level of adequacy is **FCP**.*

Without focusing (as originally done in [11]), the proof of this theorem is not straightforward and a low level of adequacy is obtained: there may be logical steps not corresponding to any operational step and vice-versa. Let us focus first in the case where logical steps do not correspond to the operational ones. We will come back to the other direction later.

Consider the two derivations bellow.

$$\frac{\Gamma, c_1 \multimap F_1 \vdash d}{\Gamma, (c_1 \multimap F_1) \& (c_2 \multimap F_2) \vdash d} \&_l \quad \frac{\Gamma_1, F_1 \vdash d \quad \Gamma_2 \vdash c_1}{\Gamma_1, \Gamma_2, c_1 \multimap F_1 \vdash d} \multimap_l \quad (1)$$

In the first, one of the branches is chosen but, in π_1 , it could be the case that c_1 is never proved (and F_1 is never added to the context). This is not the intended meaning in Rule R_A , that first checks the entailment of c_j to *immediately* add the corresponding process P_j to the context. In the second example, π_3 could contain sub-derivations that have nothing to do with the proof of the guard c_1 . For instance, process definitions could be unfolded or other processes could be executed. This would correspond, operationally, to the act of triggering an ask process **ask** c **then** P with no guarantee that its guard c will be derivable only from the set of non-logical axioms Δ and the current store. For instance, it may be the case, in π_3 , that c_1 will be later produced by a process Q such that $\mathcal{P}[Q] \in \Gamma_2$. This is clearly not allowed by the operational semantics.

Let's now put focusing into play. An inspection in the encoding reveals that the fragment of ILL used is restricted to the following grammar:

$$\begin{array}{ll}
G & := 1 \mid !A \mid G \otimes G \mid \exists x.G & \text{Guards and Goals} \\
P & := G \mid P \otimes P \mid P \& P \mid G \multimap P \mid \exists x.P \mid p(\bar{t}) & \text{Processes} \\
PD & := \forall \bar{x}.p(\bar{x}) \multimap P. & \text{Process Definitions}
\end{array}$$

where A is an atomic formula (constraint) in \mathcal{C} and p (a process identifier) is also atomic but $p \notin \mathcal{C}$. In any derivation, the only formulas that can appear on the right are guards/goals G and heads p . The other formulas, including processes, process definitions and axioms, appear on the left. Hence, only instances of the unfocused rules $1_l, \otimes_l, \exists_l, !_l, \top_r$ and the focused rules $\otimes_r, \multimap_l, \exists_r, !_r, \&_l, \forall_l$ are used.

Observe that formulas G, p are *strictly positive*. Thus, focusing on such a formula on the right either forces finishing the proof, or the formula will be *entirely* decomposed into formulas of the shape 1 or $!A$. This means that a proof of A can use only the theory Δ , the encoding of constraints and process definitions (since all of them are unbounded). In fact, we can show that the encoding of process definitions can be weakened (since calls of the form $p(\bar{y})$ are necessarily stored in the linear context). Hence, when a goal is focused on, it must be completely decomposed, and the atomic constraints must be proved only from the current store and the non-logical axioms.

Formulas occurring on the left of sequents can be positive or negative. *Positive formulas* on the left (that cannot be focused on) come from the interpretation of *tell*, *parallel composition* and *locality* that do not need any interaction with the context. Note, for instance, that the formula $\exists x. !G_1 \otimes !G_2$, resulting from the encoding of $\mathbf{tell}(\exists x.G_1 \wedge G_2)$, can be entirely decomposed in an unfocused phase using the rules \otimes_l, \exists_l and $!_l$. On the other hand, *negative formulas* on the left (that can be chosen for focusing) come from the encodings of *guarded choices* and *process definitions*. They *do need* to interact with the environment, either for choosing a path to follow (in non-deterministic choices), or waiting for a guard to be available (in asks or procedure calls).

Due to completeness of focusing [1], Theorem 3 trivially holds if we replace in it ILL with ILLF. But using directly the focused system, the proof of the theorem becomes simpler. For instance, it is a routine exercise to show that non-logical axioms permute up, and it is always possible to apply them at the top of proofs. Moreover, situations as the ones described after the derivations in Equation (1) are not longer valid in the focused system: focusing over $c_1 \multimap F_1$ implies immediately proving c_1 (from the logical axioms and accumulated constraints), thus reflecting *exactly* the operational semantics of CCP.

► **Example 4.** Consider a *community coffee machine*, which is triggered by the insertion of a coin, always available at the side of the machine. When the user inserts the coin, the machine delivers a coffee and returns the coin, which will be available for the next user. This machine can be specified as the CCP process

$$P = \mathbf{tell}(coin) \parallel \mathbf{m}() \text{ where } \mathbf{m}() \triangleq \mathbf{ask} \text{ coin then } (\mathbf{tell}(coffee) \parallel \mathbf{m}())$$

Hence, $P \Downarrow_c$, where $c = coin \wedge coffee$:

$$\langle \emptyset, P, \mathbf{true} \rangle \longrightarrow \langle \emptyset, \mathbf{m}(), coin \rangle \longrightarrow \langle \emptyset, \mathbf{tell}(coffee) \parallel \mathbf{m}(), coin \rangle \longrightarrow \langle \emptyset, \mathbf{m}(), coin \wedge coffee \rangle$$

On the other hand, the sequent $\mathcal{P}[[P]] \vdash \mathcal{C}[[c]] \otimes \top$ has the following focused proof

We define the encoding $\mathcal{P}[\cdot]_+$ as $\mathcal{P}[\cdot]$ but replacing the following cases:

$$\begin{aligned} \mathcal{P}[\sum_{i \in I} \mathbf{ask} \ c_i \ \mathbf{then} \ P_i]_+ &= \&_{i \in I} (\mathcal{C}[c_i] \multimap \delta^+(\mathcal{P}[P_i]_+)) \\ \mathcal{P}[p(\bar{x}) \triangleq P]_+ &= \forall \bar{x}. p(\bar{x}) \multimap \delta^+(\mathcal{P}[P]_+) \end{aligned}$$

The use of delays forces the focused phase to end, e.g., once the guard of the ask agent is entailed. In this encoding, we can prove a stronger adequacy theorem.

► **Theorem 6** (Strong adequacy [34]). *Let $(\mathcal{C}, \models_{\Delta})$ be a constraint system, P be a process and Ψ be a set of process definitions. Then, for any constraint c ,*

$$P \Downarrow_c \text{ iff there is a proof of the sequent } \mathcal{P}[\Psi]_+, \mathcal{C}[\Delta]; \cdot \uparrow \mathcal{P}[P]_+ \vdash \cdot \uparrow \mathcal{C}[c] \otimes \top$$

in ILLF. The adequacy level is **FCD**.

Now derivations in logic have a one-to-one correspondence with traces of a computation in a CCP program.

It is possible to modify the encoding to introduce negative actions (*tell*, *parallel* and *local*) during a focused phase (thus counting them as a focused step). For that, it suffices to introduce, in the encoding, negative delays $\delta^-(F)$. By using a multi-focusing systems [38], maximal parallelism semantics [9] - where all the enabled agents must all proceed in one step - can be also captured. Finally, if recursive definitions are interpreted as fixed points, more interesting properties of infinite computations can be specified and proved. See [34] for further details.

4 LL with multi-modalities

A careful analysis of the rules for the exponential $!$ in Figure 1 reveals that this connective has a differentiated behavior w.r.t. the other ones. In fact, $!$ is the only operator having a positive/negative behavior: the application of the right rule ($!_r$) immediately breaks focusing. Also, this is the only rule in ILLF that is *context dependent*, in the sense that it demands the linear context Γ to be empty in order to be applied.

This distinguished character of the exponential in linear logic is akin to the behavior found in *modal connectives*. In particular, the connective $!$ is not *canonical*, in the sense that, if we label $!$ with different colors, say b (for blue - $!^b$) and r (for red - $!^r$), but with the same introduction rules, then it is not possible to prove, in the resulting proof system, the equivalence $!^r A \equiv !^b A$ for an arbitrary formula A , where $H \equiv G$ denotes the formula $(H \multimap G) \& (G \multimap H)$. Not surprisingly, this exercise would have a different outcome for any other linear logic connective. For instance, if we construct a proof system with two labeled connectives, e.g., \otimes^r and \otimes^b , together with their introduction rules, then it would be possible to prove $A \otimes^b B \equiv A \otimes^r B$ for any A and B . This opens the possibility of defining new connectives: the colored exponentials, known as *subexponentials* [8].

4.1 Linear logic with subexponentials

Linear logic with subexponentials (SELL)⁴ shares with intuitionistic linear logic all its connectives except the exponential: instead of having a single $!$, SELL may contain as many subexponentials, written $!^a$ for a label (or color) a , as one needs.

⁴ Although in this paper we are mostly interested in the intuitionistic version of SELL, it was proven in [3] that classical and intuitionistic subexponential logics are equally expressive. Hence we will abuse the notation and use SELL for intuitionistic linear logic system with subexponentials.

Such labels are organized in a pre-order, giving rise to a *subexponential signature* $\Sigma = \langle I, \preceq, U \rangle$, where I is a set of labels, $U \subseteq I$ is a set specifying which subexponentials behave classically (i.e., those labels that allow for weakening and contraction), and \preceq is a pre-order among the elements of I . We shall use a, b, \dots to range over elements in I , and we will assume that \preceq is upwardly closed with respect to U , i.e., if $a \in U$ and $a \preceq b$, then $b \in U$.

The division of *unbounded* ($a \in U$) and linear or *bounded* ($a \notin U$) subexponentials induces also a partition of the subexponential context Θ , which is split into two: a set Θ^u and a multiset Θ^b of labeled formulas, having the form

$$\Theta^u = \{a_1 : \Theta_1^u, \dots, a_n : \Theta_n^u\} \quad \Theta^b = \{b_1 : \Theta_1^b, \dots, b_m : \Theta_m^b\}$$

The formulas in Θ_i^u are under the scope of the unbounded subexponential $!^{a_i}$, and formulas in Θ_j^b are under the scope of the bounded subexponential $!^{b_j}$. The linear context Γ continues containing only negative or atomic formulas, as in ILLF.

The focused proof system SELLF [28] is constructed by adding all the rules for the intuitionistic linear logic connectives as shown in Figure 1,⁵ except for the exponentials. The rules for subexponentials are the following:

- A formula F under the scope of $!^a$ is stored in the exponential context Θ accordingly: if a is unbounded/bounded, then F is added to the set/multiset Θ_a , which is created if it does not exist. This action is represented by $\Theta \uplus \{a : F\}$.

$$\frac{\Theta \uplus \{a : F\}; \Gamma \uparrow \Delta \vdash \mathcal{R}}{\Theta; \Gamma \uparrow !^a F, \Delta \vdash \mathcal{R}} !^a_l$$

- The unbounded decide rule in ILLF is split into bounded and unbounded versions, depending of the nature of the subexponential.

$$\frac{\Theta^u, \Theta^b; \Gamma \downarrow F \vdash R}{\Theta^u, \Theta^b \uplus \{a : F\}; \Gamma \uparrow \cdot \vdash \cdot \uparrow R} \text{Db} \quad \frac{\Theta^u \uplus \{a : F\}, \Theta^b; \Gamma \downarrow F \vdash R}{\Theta^u \uplus \{a : F\}, \Theta^b; \Gamma \uparrow \cdot \vdash \cdot \uparrow R} \text{Du}$$

- The promotion rule has the form

$$\frac{\Theta_{\geq a}^u, \Theta^b; \cdot \uparrow \cdot \vdash F \uparrow}{\Theta^u, \Theta^b; \cdot \vdash !^a F \downarrow} !^a_r$$

with the proviso that, for all $b_j : \Theta_j^b$ in Θ^b , it must be the case that $a \preceq b_j$. In the premise of the rule, $\Theta_{\geq a}^u \subseteq \Theta^u$ contains only elements of the form $a_i : \Theta_i^u$ where $a \preceq a_i$ (the other contexts are weakened). That is, $!^a F$ is provable only if F can be proved in the presence of subexponentials greater than a .

It is known that subexponentials greatly increase the expressiveness of the system when compared to linear logic. For instance, subexponentials can be used to represent contexts of proof systems [32], to mark the epistemic state of agents [27], or to specify locations in sequential computations [28]. The key difference is that, while linear logic has only seven logically distinct prefixes of bangs and question-marks ($?$ is the dual of $!$), SELL allows for an unbounded number of such prefixes, e.g., $!^i$, or $!^{i?j}$. As we show later, by using different prefixes, we can interpret subexponentials in more creative ways, such as linear constraints, epistemic modalities or preferences. The interested reader can also check in [30, 35, 31] the interpretation of subexponentials as temporal units, and the study of dynamical subexponentials in distributed systems.

⁵ Taking the extra-care of splitting the bounded context Θ^b for the multiplicative rules \multimap_l and \otimes_r .

The organization of subexponentials in pre-orders brings at least two interesting aspects that can be further investigated: what kind of refinements of the proof system can be obtained by adopting richer algebraic structures for subexponentials (Section 4.2 below); and what is the proof-theoretic notion of quantification over modalities (Section 4.3 below).

Being able to quantify over subexponentials is important, e.g., for specifying properties that are valid in an unbounded number of locations or agents. It is also crucial for establishing a certain notion of *mobility*, or *permissibility* of resources, that can be available, e.g., iff they are marked with a label of some specific sort. But one has to be careful here: the pre-order structure is a minimal requirement in subexponential signatures in order to guarantee the *cut-elimination* property [8]. Since, in the presence of quantifiers, proving cut-elimination requires *substitution lemmas*, a naive approach of exchanging labels could invalidate such results (see [31] for an extensive discussion on the topic).

On the other hand, if we move above the pre-order minimality and consider, e.g., \wedge -semi-lattices as subexponential structures, then the side condition in the promotion rule, $a \preceq a_i$ for all $a_i \in \Theta_{\geq a}$, is equivalent to $a \preceq \bigwedge_i a_i$. And this reflects certain kinds of preferences, as explained next.

4.2 Richer subexponential signatures

We now explore a refinement of SELLF, where richer structures are considered as subexponential signatures. For that, we shall use an algebraic structure that defines a mean to compare (\preceq) and accumulate (\bullet) values.

More precisely, a complete lattice monoid [12] is a tuple $CLM = \langle \mathcal{D}, \preceq, \bullet \rangle$ such that $\langle \mathcal{D}, \preceq \rangle$ is a complete lattice, \perp and \top are, respectively, the least and the greatest elements of \mathcal{D} and $\langle \mathcal{D}, \bullet, \top \rangle$ is an abelian monoid. Moreover, \bullet distributes over lubs, i.e., for all $v \in \mathcal{D}$ and $X \subseteq \mathcal{D}$, $v \bullet \sqcup X = \sqcup \{v \bullet x \mid x \in X\}$. Due to distributivity, \bullet is monotone and decreasing: $a \bullet b \preceq a$.

Observe that, if the SELL signature structure is a lattice, then $a \preceq \{b, c\}$ is equivalent to $a \preceq \text{glb}(b, c)$. Moreover, in the presence of \bullet , promotion can be refined so to consider the combination of values as follows.

Given a SELL signature $\Sigma = \langle \mathcal{D}, \preceq, U \rangle$ with $\langle \mathcal{D}, \preceq, \bullet \rangle$ a CLM , the promotion rule $!^a_{r, \bullet}$ is defined as:

$$\frac{\Theta_{\geq a}^u, \Theta^b; \cdot \uparrow \cdot \vdash F \uparrow}{\Theta^u, \Theta^b; \cdot \vdash !^a F \downarrow} !^a_{r, \bullet}, \text{ provided } a \preceq \bullet \{a_i, b_j\}$$

Note that, if the CLM is \bullet -idempotent (i.e. $a \bullet a = a$), then $\text{glb}(a, b) = a \bullet b$, and the above rule coincides with SELLF's promotion rule.

► **Example 7.** Consider the signature $\Sigma = \langle \mathcal{D}, \preceq, \mathcal{D} \rangle$, with the following instances of CLM .

- $\langle \{\text{pub}, \text{sec}\}, \preceq, \wedge \rangle$, where **pub** and **sec** represent public and private information, respectively. The ordering is $\text{pub} \prec \text{sec}$ and $a \wedge b = \text{sec}$ iff $a = b = \text{sec}$. Hence, any proof of $\Theta; \cdot \vdash !^{\text{sec}} F \downarrow$ does not make use of any *public* information.
- $\langle [0, 1], \leq_{\mathbb{R}}, \min \rangle$ (fuzzy), where $[0, 1] \subset \mathbb{R}$, and $\leq_{\mathbb{R}}$ is the usual order in \mathbb{R} . In this case, we can interpret $!^{0.2}c$ as “ c is believed with preference 0.2”. Note that the sequent $!^{0.2}c \otimes !^{0.7}d \vdash !^a(c \otimes d)$ is provable only if $a \leq_{\mathbb{R}} 0.2$.
- $\langle [0, 1], \leq_{\mathbb{R}}, \times \rangle$ (probabilistic), where \times is the multiplication operator in \mathbb{R} . This is a non-idempotent CLM , and the sequent $!^{0.2}c \otimes !^{0.7}d \vdash !^a(c \otimes d)$ is provable only if $a \leq_{\mathbb{R}} 0.14$.

In [39] we have showed that this new version of the promotion rule is not at all ad-hoc. The resulting system, SELLS, is a smooth extension of ILLF and it is a closed subsystem of SELLF, which is strict when non-idempotent *CLMs* are considered. Hence SELLS inherits all SELLF good properties such as cut-elimination.

The SELLS system has inspired the development of new CCP-based calculi where processes can tell and ask soft constraints, understood as formulas of the form $!^a c$ where a is an element of a given *CLM* [39]. Also, since the underlying logic is the same, it is possible to obtain adequate interpretations of processes as formulas as the ones in Section 3.2. More interestingly, it is also possible to combine, in a uniform way, different modalities [35], all of them grounded on linear logic principles. Some of these modalities will be explored in Section 5.

4.3 Subexponential Quantifiers

This section introduces the focused system $\text{SELLF}^{\mathfrak{m}}$, containing two novel connectives \mathfrak{m} and \mathfrak{w} , representing, respectively, a universal and existential quantifiers over *subexponentials*.⁶

As mentioned in Section 4.1, in order to guarantee cut-elimination of the resulting system, the substitution of subexponentials in the rules for quantification should be done carefully. As showed in [31], it is enough to require that labels are substituted, bottom-up, for *smaller* ones. Also, the possibility of creating new labels dynamically implies that there should be two sorts of labels: constants and variables. This justifies the next definition.

► **Definition 8.** *Given a pre-order (I, \preceq) and $a \in I$, the ideal generated by a is the set $\downarrow a = \{b \in I \mid b \preceq a\}$.*

The subexponential signature of $\text{SELL}^{\mathfrak{m}}$ is the triple $\Sigma = \langle I, \preceq, U \rangle$, where I is a set of subexponential constants, \preceq is a pre-order over I and $U \subseteq I$ is the upwardly closed set of unbounded constants.

The sets of typed subexponential constants and typed subexponential variables are denoted respectively by

$$\mathcal{T}_{\Sigma} = \{b : a \mid b \in \downarrow a\} \quad \mathcal{T}_x = \{l_{x_1} : a_1, \dots, l_{x_n} : a_n\}$$

where $\{l_{x_1}, \dots, l_{x_n}\}$ is a disjoint set of subexponential variables, and $\{a_1, \dots, a_n\} \subseteq I$ are subexponential constants.

Formally, only these subexponential constants and variables may appear free in an index of subexponential bangs and question marks.

Sequents in $\text{SELLF}^{\mathfrak{m}}$ have the same form as in SELLF, with the difference that there is an extra context $\mathcal{T} = \mathcal{T}_{\Sigma} \cup \mathcal{T}_x$.

The rules for \mathfrak{m} and \mathfrak{w} are the novelty with respect to the focused proof system for SELLF. They behave exactly as the first-order quantifiers: the \mathfrak{m}_r and \mathfrak{w}_l belong to the negative phase because they are invertible, while \mathfrak{m}_l and \mathfrak{w}_r are positive since they are not invertible.

$$\frac{\mathcal{T} \cup \{l_e : a\}; \Theta; \Gamma \uparrow \Delta \vdash F[l_e/l_x] \uparrow}{\mathcal{T}; \Theta; \Gamma \uparrow \Delta \vdash \mathfrak{m}l_x : a.F \uparrow} \mathfrak{m}_r \quad \frac{\mathcal{T} \cup \{l_e : a\}; \Theta; \Gamma \uparrow \Delta, F[l_e/l_x] \vdash \mathcal{R}}{\mathcal{T}; \Theta; \Gamma \uparrow \Delta, \mathfrak{w}l_x : a.F \vdash \mathcal{R}} \mathfrak{w}_l$$

$$\frac{\mathcal{T}; \Theta; \Gamma \downarrow F[l/l_x] \vdash R}{\mathcal{T}; \Theta; \Gamma \downarrow \mathfrak{m}l_x : a.F \vdash R} \mathfrak{m}_l \quad \frac{\mathcal{T}; \Theta; \Gamma \vdash F[l/l_x] \downarrow}{\mathcal{T}; \Theta; \Gamma \vdash \mathfrak{w}l_x : a.F \downarrow} \mathfrak{w}_r$$

⁶ Some motivation for the symbols \mathfrak{m} and \mathfrak{w} . The former resembles the symbol for intersection, which is the usual semantics assigned to for all quantifiers, namely, the intersection of all models, while the latter is same for exists and union.

In the left rule of \mathfrak{M} and the right rule of \mathfrak{U} , l_x is substituted with a subexponential of the right type: $l : b \in \mathcal{T}$, $b \in \downarrow a$. In the rules \mathfrak{M}_r and \mathfrak{U}_l , a fresh variable l_e of type a is created and added to the context \mathcal{T} .

Next, we shall see that the quantifiers allows for encoding, in a modular way, systems dealing with an unbounded number of modalities.

5 Parametric interpretations

This section illustrates how focusing, subexponentials and quantifiers in $\text{SELLF}^{\mathfrak{M}}$ can be used to give adequate interpretations to CCP calculi featuring different modalities. The interpretation is *modular*: there is only one base logic – $\text{SELL}^{\mathfrak{M}}$; and *parametric*: each modal flavor of CCP is specified by a signature in SELL having a particular algebraic structure. In this way, processes may be executed and add/query constraints in different *locations*, where the meaning of such locations may vary, for example: spaces of computation, the epistemic state of agents, time units, levels of preferences, etc. But the underline interpretation is the same: locations in CCP become labels in SELL .

Another modular aspect of our process-as-formula interpretation is the organization of the encodings of constraints, processes and process definitions, into non-comparable *families* of subexponentials, so that focusing on an element of a family forces all elements of the other families to be erased during proof search. This ensures the discipline necessary for guaranteeing the highest level of adequacy (**FCD**).

Formally, let M be an underlying set of labels, with least and greatest elements represented by nil and ∞ respectively, ordered with a pre-order \preceq_M . The families of subexponentials are built with marked copies of elements of M : $\mathfrak{c}(\cdot)$ for constraints, $\mathfrak{p}(\cdot)$ for processes, and $\mathfrak{d}(\cdot)$ for process definitions. The subexponential signature $\Sigma = \langle I, \preceq, U \rangle$ is built from M in the following way:

- The set of labels is: $I = \{l, \mathfrak{c}(l), \mathfrak{p}(l), \mathfrak{d}(l) \mid l \in M\}$; that is, besides the elements in M , we consider three additional distinct copies of the labels, each of them marked with the appropriate family.
- The subexponential pre-order is: $l \preceq l'$ iff $l \preceq_M l'$ and $\mathfrak{f}(l) \preceq \mathfrak{f}(l')$ iff $l \preceq_M l'$ where $\mathfrak{f} \in \{\mathfrak{c}, \mathfrak{p}, \mathfrak{d}\}$; note that subexponentials pertaining to different families are not related.
- The set U of unbounded subexponentials will vary depending on the encoded system.

Constraints and CCP processes are encoded into $\text{SELLF}^{\mathfrak{M}}$ by using the functions $\mathcal{C}[\cdot]_l$ and $\mathcal{P}[\cdot]_l$ as in Definition 2, now parametric w.r.t. subexponentials $l \in M$ as follows.⁷

► **Definition 9 (General Encoding).** *Constraints and axioms of the constraint system are encoded in $\text{SELL}^{\mathfrak{M}}$ as:*

$$\begin{aligned} \mathcal{C}[\text{true}]_l &= 1 & \mathcal{C}[A]_l &= !^{\mathfrak{c}(l)} A & \mathcal{C}[c_1 \wedge c_2]_l &= \mathcal{C}[c_1]_l \otimes \mathcal{C}[c_2]_l \\ \mathcal{C}[\exists \bar{x}.c]_l &= \exists \bar{x}. \mathcal{C}[c]_l & \mathcal{C}[\forall \bar{x}.(d \supset c)] &= \mathfrak{M}l_x : \infty. \forall \bar{x}. (\mathcal{C}[d]_{l_x} \multimap \mathcal{C}[c]_{l_x}) \end{aligned}$$

⁷ We observe that, technically, the encoding functions should also consider subexponential variables. However, the encoded processes/axioms are stored on left contexts, and the left introduction rule for universal quantifiers does not create fresh variables.

The encoding of processes and process definitions is:

$$\begin{aligned}
\mathcal{P}[\mathbf{tell}(c)]_l &= !^{\mathfrak{p}(l)}[\mathfrak{m}l_x : l.(\mathcal{C}[c]_{l_x})] \\
\mathcal{P}[\sum_{i \in I} \mathbf{ask} c \text{ then } P]_l &= !^{\mathfrak{p}(l)}[\mathfrak{m}l_x : l.(\&\mathcal{Z}[\mathcal{C}[c_i]_{l_x} \multimap \mathcal{P}[P_i]_{l_x}])] \\
\mathcal{P}[(\mathbf{local} \bar{x}) P]_l &= !^{\mathfrak{p}(l)}[\mathfrak{m}l_x : l.\exists \bar{x}.(\mathcal{P}[P]_{l_x})] \\
\mathcal{P}[P \parallel Q]_l &= \mathcal{P}[P]_l \otimes \mathcal{P}[Q]_l \\
\mathcal{P}[p(\bar{x})]_l &= !^{\mathfrak{d}(l)}p(\bar{x}) \\
\mathcal{P}[p(\bar{x}) \triangleq P] &= \mathfrak{m}l_x : \infty.\forall \bar{x}.(!^{\mathfrak{d}(l_x)}p(\bar{x}) \multimap \mathcal{P}[P]_{l_x})
\end{aligned}$$

The main difference between the encodings in $\text{SELL}^{\mathfrak{m}}$ and ILL is the presence of *mobility* of processes, given by the universal quantifier \mathfrak{m} over subexponentials. This enables the specification of systems to govern an unbounded number of modalities.

Intuitively, when (left) focusing over a quantified clause of the form $\mathfrak{m}l_x : l.!\mathfrak{f}(l_x)F$, a location $a \in \downarrow l$ is chosen, and F becomes available in the location a , inside a family \mathfrak{f} , which is totally determined by the nature of the encoded object: \mathfrak{c} for constraints, \mathfrak{p} for processes, \mathfrak{d} for process definitions. In the special case of $l = \infty$, F can be allocated *anywhere* inside the family. This is the case for example, of axioms and process definitions.

Let us now illustrate how the use of subexponentials and quantifiers allow for attaining the highest level of adequacy. The first thing to note is that, due to the shape of the encoding, the subexponential context can be divided into 3 zones: \mathcal{C} , \mathcal{D} and \mathcal{P} , containing the formulas marked, respectively, with subexponentials of the form $\mathfrak{c}(\cdot)$, $\mathfrak{d}(\cdot)$ and $\mathfrak{p}(\cdot)$.

Using simple logical equivalences, we can rewrite the encoding of a constraint $\mathcal{C}[c]_l$ so that it has the following shape $\exists \bar{x}. (!^{\mathfrak{c}(l_1)}A_1 \otimes \dots \otimes !^{\mathfrak{c}(l_n)}A_n)$, where A_1, \dots, A_n are atomic (positive) formulas. Whenever such a formula appears in the left-hand side, it is completely decomposed and stored in the \mathcal{C} context:

$$\frac{\frac{\mathcal{C} \uplus \{\mathfrak{c}(l_1) : A_1, \dots, \mathfrak{c}(l_n) : A_n\}, \mathcal{D}, \mathcal{P}; \cdot \uparrow \Delta \vdash \mathcal{R}}{\mathcal{C}, \mathcal{D}, \mathcal{P}; \cdot \uparrow !^{\mathfrak{c}(l_1)}A_1, \dots, !^{\mathfrak{c}(l_n)}A_n, \Delta \vdash \mathcal{R}} !^a_l}{\mathcal{C}, \mathcal{D}, \mathcal{P}; \cdot \uparrow !^{\mathfrak{c}(l_1)}A_1 \otimes \dots \otimes !^{\mathfrak{c}(l_n)}A_n, \Delta \vdash \mathcal{R}} \exists_l, \otimes_l$$

That is, in the negative phase, the atomic formulas A_1, \dots, A_n appearing in the premise of this derivation are moved to the contexts \mathcal{C} .

Consider now a derivation that focuses on the encoding of a process. For instance, let $Q = \mathbf{ask} c \text{ then } P$, and $\mathcal{P}[Q]_l = !^{\mathfrak{p}(l)}F$, with $F = \mathfrak{m}l_x : a.(\mathcal{C}[c]_{l_x} \multimap \mathcal{P}[P]_{l_x})$. Focusing on F results necessarily in a focused derivation of the following shape:

$$\frac{\frac{\frac{\frac{\mathcal{C}''; \cdot \vdash \mathcal{C}[c]_{l'} \downarrow \quad \frac{\mathcal{C}'', \mathcal{D}, \mathcal{P}' \uplus \{\mathfrak{p}(l') : F_P\}; \cdot \uparrow \cdot \vdash G \uparrow}{\mathcal{C}'', \mathcal{D}, \mathcal{P}'; \cdot \downarrow \mathcal{P}[P]_{l'} \vdash G} \text{R}_l, !^a_l}{\mathcal{C}, \mathcal{D}, \mathcal{P}'; \cdot \downarrow \mathfrak{m}l_x : a.(\mathcal{C}[c]_{l_x} \multimap \mathcal{P}[P]_{l_x}) \vdash G} \mathfrak{m}_l, \multimap_l}{\mathcal{C}, \mathcal{D}, \mathcal{P} \uplus \{\mathfrak{p}(l) : F\}; \cdot \uparrow \cdot \vdash \cdot \uparrow G} \text{Du/Db}$$

If $\mathfrak{p}(l) \in U$ (resp. $\mathfrak{p}(l) \notin U$) the rule Du (resp. Db) is applied and $\mathcal{P}' = \mathcal{P} \uplus \{\mathfrak{p}(l) : F\}$ (resp. $\mathcal{P}' = \mathcal{P}$). Since $\mathcal{C}[c]_{l'}$ contains only positive formulas, it will be totally decomposed, and every exponential context in π will be a \mathcal{C} context. That is, only constraints and axioms from the constraint system can be used in the proof π .

A similar analysis can be done when a process definition is selected: only the context \mathcal{D} , storing all the calls, can be used to entail the needed guard.

In the following, we instantiate the general definition of the encoding for different flavors of CCP. The adequacy we obtain, in each case is at the **FCD** level.

Classical and linear CCP

For encoding the language in Section 3, the set of modalities is the simplest one: $M = \{\text{nil}, \infty\}$. All the subexponentials but $\mathfrak{p}(\text{nil})$ and $\mathfrak{d}(\cdot)$ are unbounded.

► **Theorem 10.** *Let $(\mathcal{C}, \models_{\Delta})$ be a constraint system, P be a CCP process and Ψ be a set of process definitions. Then, for any constraint c ,*

$$P \downarrow_c \text{ iff } \cdot \uparrow!^{c(\infty)} \mathcal{C}[\Delta], !^{c(\infty)} [\Psi], \mathcal{P}[P]_{\text{nil}} \vdash \mathcal{C}[c]_{\text{nil}} \otimes \top \uparrow$$

It is worth noticing that all the processes remain in the location *nil* (denoting “without modality”) and then, the universal quantification in the encoding is always forced to instantiate l_x with *nil*.

Linear CCP. As we already know, the store in CCP increases monotonically: once a constraint is added, it cannot be removed from the store. This can be problematic for the specification of systems where resources can be consumed. In linear CCP (**lcc**) [11], constraints are built from formulas in the following fragment of **ILL**:

$$F ::= A \mid 1 \mid F \otimes F \mid \exists x.F \mid !F$$

In this setting, the empty store is 1 and constraints are accumulated using \otimes . The extra case $!F$, as expected, is used to denote persistent constraints.

► **Example 11.** The *vending coffee machine* has the same CCP specification as the community coffee machine presented in Example 4. However, as expected, linear asks consume constraints when querying the store and the coin does not come back after delivering the coffee:

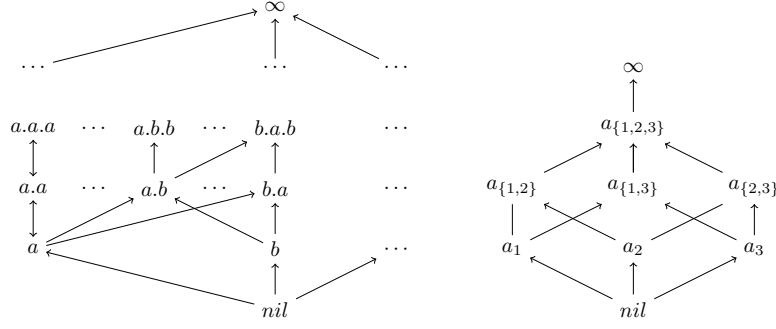
$$\langle \emptyset, P, 1 \rangle \longrightarrow \langle \emptyset, \mathfrak{m}(), \text{coin} \rangle \longrightarrow \langle \emptyset, \mathbf{tell}(\text{coffee}) \parallel \mathfrak{m}(), 1 \rangle \longrightarrow \langle \emptyset, \mathfrak{m}(), \text{coffee} \rangle$$

In order to characterize the semantics of **lcc**, we configure the encoding in Definition 9 as follows. We declare $c(\text{nil}) \notin U$ (i.e., constraints can be consumed) and $c(\infty) \in U$. Moreover, the encoding is extended for the case of unbounded constraints: $\mathcal{C}[!c]_l = \mathcal{C}[c]_{\infty}$. In this way, we obtain an adequacy theorem as the one in Theorem 10, also at the **FCD** level, in contrast to the weakest level of adequacy (**FCD**) obtained originally in [11] (for linear logic and without focusing).

It is important to note that the characterization in Theorem 6, that uses (vanilla) linear logic, does not work for **lcc** at the **FCD** level. Take for instance the process $Q = \mathbf{ask} \ c \otimes d \ \mathbf{then} \ P$ being executed in the store $!(c \otimes d)$. Clearly, Q reduces to P and the store remains unchanged. If we were to use the encoding in Theorem 6, before focusing on $\mathcal{P}[Q]$, we have to do an intermediary step without an operational counterpart: focus on $c \otimes d$, stored in the classical context, to produce a copy of c and d in the linear context. Only after that, the implication in $\mathcal{P}[Q]$ is able to entail the guard $c \otimes d$. In the encoding of the present section, proving the query of Q results in focusing on $!^{c(\text{nil})} c \otimes !^{c(\text{nil})} d$. After decomposing the tensor, focusing is lost and only linear $c(\text{nil})$ and replicated ($c(\infty)$) constraints and the axioms of the constraint systems can be used to deduce the atoms c and d . This adequately reflects the semantics of linear asks.

Epistemic CCP

Now let us consider a richer system where different modalities will play a fundamental role. Epistemic CCP (**eccp**) [16] is a CCP-based language where systems of agents are considered for distributed and epistemic reasoning. In **eccp**, the constraint system is extended to consider space of agents, denoted as $s_a(c)$, and meaning “ c holds in the space –store– of agent a .” The function $s_a(\cdot)$ satisfies certain conditions to reflect epistemic behaviors:



■ **Figure 3** Subexponential signature for *eccp*.

$$\frac{(X; P, \Gamma; c) \longrightarrow (X'; P', \Gamma; d)}{(X; [P]_a, \Gamma; c) \longrightarrow (X'; [P]_a, P', \Gamma; d)} \text{R}_E \quad \frac{(X; P, \Gamma; d^a) \longrightarrow (X'; P', \Gamma; d')}{(X; [P]_a, \Gamma; d) \longrightarrow (X'; [P]_a, \Gamma; d \wedge s_a(d'))} \text{R}_S$$

■ **Figure 4** Operational rules for *eccp* and *sccp*.

1. $s_a(1) = 1$ (bottom preserving)
2. $s_a(c \wedge d) = s_a(c) \wedge s_a(d)$ (lub preserving)
3. If $d \vdash_{\Delta_e} c$ then $s_a(d) \vdash_{\Delta_e} s_a(c)$ (monotonicity)
4. $s_a(c) \vdash_{\Delta_e} c$ (believes are facts –extensiveness–)
5. $s_a(s_a(c)) = s_a(c)$ (idempotence)

In *eccp*, the language of processes is extended with the constructor $[P]_a$ that represents P running in the space of the agent a . The operational rules for $[P]_a$ are specified in Figure 4. In epistemic systems, agents are trustful, i.e., if an agent a knows some information c , then c is necessarily true. Furthermore, if b knows that a knows c , then b also knows c . For example, given a hierarchy of agents as in $[[[P]_a]_b]$, it should be possible to propagate the information produced by P in the space a to the outermost space b . This is captured exactly by the rule R_E , which allows a process P in $[P]_a$ to run also outside the space of agent a . Notice that the process P is contracted in this rule. The rule R_S , on the other hand, allows us to observe the evolution of processes inside the space of an agent. There, the constraint d^a represents the information the agent a may see or have of d , i.e., $d^a = \bigwedge \{c \mid d \vdash_{\Delta_e} s_a(c)\}$. For instance, a sees c from the store $s_a(c) \wedge s_b(c')$ but it does not see c' .

We now configure the encoding in Definition 9 so to capture the behavior of *eccp* processes. We consider a possibly infinite set of agents $\mathcal{A} = \{a_1, a_2, \dots\}$ and the set of locations/modalities M , besides nil and ∞ , contains the set \mathcal{A}^+ of non-empty strings of elements in \mathcal{A} ; for example, if $a, b \in \mathcal{A}$, then $a, b, a.a, b.a, a.b.a, \dots \in \mathcal{A}^+$. We use \bar{a}, \bar{b} , etc to denote elements in \mathcal{A}^+ and nil will denote the empty string. The only linear subexponentials are $\mathfrak{d}(nil)$ and $\mathfrak{p}(nil)$. This reflects the fact that both constraints and processes in the space of an agent are unbounded, as specified by rule R_E . Intuitively, $!^{\mathfrak{p}(1.2.3)}$ specifies a process in the structure $[[[\cdot]_3]_2]_1$, denoting “agent 1 knows that agent 2 knows that agent 3 knows” expressions. The connective $!^{c(1.2.3)}$, on the other hand, specifies a constraint of the form $s_1(s_2(s_3(\cdot)))$. We thus extend the encoding accordingly: $\mathcal{C}[[s_i(c)]]_{\bar{l}} = \mathcal{C}[[c]]_{\bar{l}.i}$ and $\mathcal{P}[[[P]_i]]_{\bar{l}} = \mathcal{P}[[P]]_{\bar{l}.i}$.

The pre-order \preceq is as depicted in Figure 3 on the left. Note that for every two different agent names a and b in \mathcal{A} , the subexponentials a and b are unrelated. Moreover, $a \approx a.\bar{a}$ and $b_1.b_2.\dots.b_n \preceq \bar{a}_1.b_1.\bar{a}_2.b_2.\dots.\bar{a}_n.b_n.\bar{a}_{n+1}$ where each \bar{a}_i is a possible empty string of elements in \mathcal{A} . The shape of the pre-order is key for our encoding. For instance, the formula

$\mathbb{M}l_x : a.b.b.\mathcal{P}[[P]]_{l_x}$ on the left, allows us to place P on the (outer) location $a.b$ and b as required by R_E . In fact, we can show that the sequent $\mathcal{P}[[P]]_{\bar{l}.i} \vdash \mathcal{P}[[P]]_{\bar{l}}$ is provable in $\text{SELL}^{\mathbb{M}}$ for any process P and subexponentials \bar{l} and i . We can also show that the encoding of constraints satisfy the axioms of an epistemic constraint system. For instance, the sequent $\mathcal{C}[[s_i(c)]]_{\bar{l}} \vdash \mathcal{C}[[c]]_{nil}$ is provable, showing that believes are facts. Hence, a tailored version of Theorem 10 applies for this language, with the same level of adequacy.

As an interesting example of epistemic behavior, it is possible to specify common knowledge by extending the subexponential signature as in Figure 3 on the right, where for all $\mathcal{S} \subseteq \mathcal{A}$, $\bar{a} \preceq a_{\mathcal{S}}$ for any string $\bar{a} \in \mathcal{S}^+$. Then, the announcement of c on the group of agents \mathcal{S} can be represented by $!^{c(a_{\mathcal{S}})}c$. Notice that the sequent $!^{c(a_{\mathcal{S}})}c \vdash !^{c(\bar{a})}c \otimes \top$ can be proved for any $\bar{a} \in \mathcal{S}^+$. For instance, if $\mathcal{S} = \{a_i, a_j\}$, from $!^{c(a_{\mathcal{S}})}c$ one can prove that a_i knows that a_j knows that a_i knows that a_i knows ... c , i.e., c is common knowledge between a_i and a_j .

Spatial CCP

Inconsistent information in CCP arises when considering theories containing axioms such as $c \wedge d \vdash_{\Delta} 0$. Unlike epistemic scenarios, in spatial computations, a space can be locally inconsistent and it does not imply the inconsistency of the other spaces (i.e., $s_a(0)$ does not imply $s_b(0)$). Moreover, the information produced by a process in a space is not propagated to the outermost spaces (i.e., $s_a(s_b(c))$ does not imply $s_a(c)$).

In [16], spatial computations are specified in spatial CCP (sccp) by considering processes of the form $[P]_a$ as in the epistemic case, but excluding the rule R_E in the system shown in Figure 4. Furthermore, some additional requirements are imposed on the representation of agents' spaces $s_a(\cdot)$. In particular, $s_a(\cdot)$ must satisfy false containment, i.e., if $c \wedge d \vdash_{\Delta} 0$, it does not necessarily imply that $s_a(c) \wedge s_b(d) \vdash_{\Delta} 0$ if $a \neq b$.

We build the subexponential signature as we did in the epistemic case but the pre-order is much simpler: for any $\bar{a} \in \mathcal{A}^+$, $\bar{a} \preceq \infty$. That is, two different elements of \mathcal{A}^+ are unrelated. Moreover, since sccp does not contain the R_E rule, processes in spaces are again treated linearly. Thus: $U = \{c(a) \mid a \in I\} \cup \{p(\infty)\}$.

By modifying the pre-order we partially capture the behavior of spatial systems. However, it is not enough to confine inconsistencies. In particular, note that $!^a 0 \vdash G$ for any a and G . The solution for information confinement, as shown in [31], is to consider combinations of bangs and question marks (the dual of bang). In this case, $!^a ?^a 0 \vdash !^a ?^a G$ but $!^a ?^a 0 \not\vdash !^b ?^b G$ for a, b not related. Hence, the encoding remains the same, but for the base cases: atomic propositions are encoded as $!^{c(l)} ?^{c(l)} A$, and procedure calls as $!^{d(l)} ?^{d(l)} p(\vec{x})$.

6 Conclusion and future work

We have shown that the process-as-formula interpretation can provide useful reasoning techniques for process calculi, by faithfully capturing the behavior of processes. The interpretations we have achieved are *modular* and *parametric*, and they can capture different modal behaviors as Table 1 summarizes.

Other examples of processes-as-formulas interpretations, relating computation and proof search, include linear logic-based models for the π -calculus [22], abstract transition systems and operational semantics [20], CCS [10], Bigraphs [5], P-systems [33] and concurrent object oriented programming languages [36]. Also, in [4] we have tailored the notion of fixed points in linear logic [2] to the system $\text{SELL}^{\mathbb{M}}$, and this allowed the encoding of CTL (Computational Tree Logic) formulas as SELL theories, thus opening the possibility of specifying and proving temporal properties inside the same logical framework.

■ **Table 1** Encoding of CCP modalities in SELL^{m} .

General Encoding	
Connective	Meaning
$\nabla_s = !^s$	$!^s P$ is located at s .
$\nabla_s = !^s ?^s$	$!^s ?^s P$ is confined to s .
$\text{m}l : a P$	P can move to locations below (outside) a
Epistemic Modalities	
Pre-order	Meaning
$a.a \sim a$	Modalities are idempotent : $[[P]_a]_a \sim [P]_a$
$a \preceq a.b$	Processes can move outside $[[P]_b]_a \rightarrow [P \parallel [P]_b]_a$
Spatial Modalities	
Pre-order	Meaning
$a \not\leq b$	P does not communicate with Q in $[P]_a \parallel [Q]_b$
$a.a \not\sim a$	Modalities are not necessarily idempotent.
$a \not\preceq a.b$	Processes are confined: $[[P]_b]_a \not\sim [P \parallel [P]_b]_a$

Regarding future work, in [17] we have shown how to incorporate other modal behaviors (besides the structural ones of weakening and contraction) in linear logic, thus extending the multiplicative and additive fragment of LL with *simply dependent* multi-modalities. The interpretations we have presented here have inspired new CCP-based calculi [35]. We foresee that the finer control of modalities given in [17], as well as the extensions with *non-normal modalities* [6, 18, 7], may contribute with other declarative models of concurrency with strong logical foundations.

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