A Bicategorical Model for Finite Nondeterminism

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Abstract

Finiteness spaces were introduced by Ehrhard as a refinement of the relational model of linear logic. A finiteness space is a set equipped with a class of finitary subsets which can be thought of being subsets that behave like finite sets. A morphism between finiteness spaces is a relation that preserves the finitary structure. This model provided a semantics for finite non-determism and it gave a semantical motivation for differential linear logic and the syntactic notion of Taylor expansion. In this paper, we present a bicategorical extension of this construction where the relational model is replaced with the model of generalized species of structures introduced by Fiore et al. and the finiteness property now relies on finite presentability.

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1 Introduction

1.1 Quantitative semantics

In quantitative semantics, the interpretation of a program provides information on the number of times the program uses its input to compute a given output whereas qualitative semantics only allows us to recover which inputs were used. Quantitative semantics originates from Girard's normal functor semantics of system F [16]. His original intuition was to interpret types as vector spaces such that linear maps between them correspond to programs using their arguments exactly once and analytic functions correspond to general programs.

This approach led to the birth of linear logic but it does not directly provide a model for it. Indeed, the exponential modality of linear logic leads to infinite dimensional vector spaces which are no longer isomorphic to their double dual, a property required to model classical negation. Topological vector spaces were therefore considered to circumvent this issue [17, 6, 8]. In this setting, the series interpreting a program usually has infinite support describing all its possible behaviors for all possible inputs which allows for the study of non-deterministic languages.

1.2 Controlling non-deterministic computation

Finiteness spaces are a model of linear logic introduced by Ehrhard as a way to enforce finite interactions between programs and reject infinite computations [9]. Finiteness spaces do not provide a model of PCF since the fixpoint operator is not a morphism in the model. Vaux showed however that it allows for primitive recursion and is hence a model of Gödel's system T [26].

The construction of the finiteness spaces model is done in two steps: the first step is a double glueing construction (in the sense of Hyland and Schalk [20]) on the relational model **Rel**. A finiteness space $A = (|A|, \mathscr{F}A)$ is a countable set $|A|$ together with a set of

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finitary subsets $\mathscr{F}A$ such that the intersection of a finitary subset in $\mathscr{F}A$ together with a finitary subset in the dual type $\mathscr{F}A^{\perp}$ is always finite. Morphisms between finiteness spaces are relations that preserve the finitary structure backward and forward.

The second step is parameterized by a fixed field (or commutative semi-ring) $\mathcal{R}$: for every finiteness space, one can define a vector space (or semi-module) whose vectors are linear combinations with finitary support, and this space is endowed with a topology induced by the duality. In this setting, morphisms in the linear category correspond to linear continuous maps between these vector spaces and non-linear maps correspond to analytic maps for which there is a natural notion of differentiation. This construction provided the semantical motivation for differential linear logic and the syntactic notion of Taylor expansion which associates a formal sum of resource terms to a given term [11, 10]. Finiteness spaces were also used to characterize strongly normalizing terms in non-deterministic λ-calculus [25]. More recently, finiteness spaces were used in the theory of generalized power series rings and topological groupoids [5, 1].

This finiteness space construction yields a model of controlled non-determinism: the objects can be infinite dimensional vector spaces and the morphisms are series with possibly infinite support but whenever an explicit computation is made, the result is always finite. It corresponds to the operational property that a program always has a finite number of reduction paths for a given input and output.

1.3 Generalized species of structures

In this paper, we use species of structures to extend the finiteness construction on the relational model to a bicategorical setting. Species of structures were originally introduced by Joyal as a unified framework for the theory of generating series in enumerative combinatorics [21]. Fiore et al. then presented a generalized definition that both encompasses Joyal's species and constitutes a model of differential linear logic [13]. This model of generalized species is based on the bicategory of profunctors **Prof** and it can be considered as a generalization of the differential relational model **Rel**. It follows the line of research of categorifying λ-calculus models by replacing sets or preorders by richer categorical structures [7, 19]. Generalized species are also connected to the Girard's normal functor model [16] which was later extended by Hasegawa [18].

The exponential modality in the model of generalized species is based on the free symmetric monoidal completion for small categories which generalizes the finite multiset construction for the relational model. Morphisms in the co-Kleisli bicategory correspond to the notion of analytic functors which provide the series counterpart of generalized species [12].

1.4 Finiteness spaces with profunctors

In the original model of relational finiteness spaces, types are interpreted as pairs $A = (|A|, \mathscr{F}A)$ of countable set $|A|$ with a set of so-called finitary subsets $\mathscr{F}A \subseteq \mathcal{P}(|A|)$ satisfying $\mathscr{F}A = (\mathscr{F}A)^{\perp\perp}$. In our setting, the types will correspond to pairs $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ of a locally finite category $|\mathbf{A}|$ equipped with a full subcategory of finite presheaves $\mathscr{F}\mathbf{A} \hookrightarrow [|\mathbf{A}|^{\mathrm{op}}, \mathbf{FinSet}]$ such that $\mathscr{F}\mathbf{A} \cong (\mathscr{F}\mathbf{A})^{\perp\perp}$.

The categorification of the orthogonality relation allows us to work in a better behaved setting of *focused orthogonalities* where forward preservation is equivalent to backward preservation for morphisms preserving the finiteness structure [20]. In our case, a morphism from $(|\mathbf{A}|, \mathscr{F}\mathbf{A})$ to $(|\mathbf{B}|, \mathscr{F}\mathbf{B})$ will be a finite profunctor $P : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|$ such that $(P)\mathscr{F}\mathbf{A} \hookrightarrow \mathscr{F}\mathbf{B}$ which will imply that $(P^{\perp})\mathscr{F}\mathbf{B}^{\perp} \hookrightarrow \mathscr{F}\mathbf{A}^{\perp}$. We follow the same pattern of the double-glueing construction for 1-categories to obtain a bicategory of finiteness spaces and profunctors between them where computations are enforced to be finite and show that all the differential linear logic constructions in **Prof** can be refined to our bicategory.

Notation

- For an integer $n \in \mathbb{N}$, we write $\underline{n}$ for the set $\{1, \ldots, n\}$.
- For a small category $\mathcal{A}$, we denote by $\widehat{\mathcal{A}}$ the presheaf category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and write $\mathbf{y}_{\mathcal{A}} : \mathcal{A} \to \widehat{\mathcal{A}}$ for the Yoneda embedding.
- We denote by $\mathbf{1}$ the category with one object and one morphism and by $\mathbf{0}$ the empty category.
- We use $\cong$ for natural isomorphisms between functors or category isomorphisms and $\simeq$ for equivalences.

2 Relational Finiteness Spaces

The model of relational finiteness spaces is obtained from **Rel** via a glueing construction along hom-functors using the following orthogonality relation:

▶ **Definition 1.** *For a countable set S, subsets $x \in \mathbf{Rel}(1, S) \cong \mathcal{P}(S)$ and $x' \in \mathbf{Rel}(S, 1) \cong \mathcal{P}(S)$, we say that x and x' are* orthogonal *if $x \cap x'$ is a finite set and we denote it by $x \perp_S x'$.*

The idea is that morphisms in $\mathbf{Rel}(1, S)$ are thought of as closed programs of type S and morphisms in $\mathbf{Rel}(S, 1)$ correspond to counter-programs or environments. The orthogonality relation allows for more control on interactions between programs and environments as we require their interaction to always be finite even if the type S is infinite. For a subset $\mathscr{F} \subseteq \mathcal{P}(S)$, we define its *orthogonal* as $\mathscr{F}^{\perp} := \{x \in \mathcal{P}(S) \mid \forall x' \in \mathscr{F}, x \perp x'\} \subseteq \mathcal{P}(S)$. This orthogonality relation induces a Galois connection on $\mathcal{PP}(S)$

$$\mathcal{PP}(S) \underset{(-)^{\perp}}{\overset{(-)^{\perp}}{\rightleftarrows}} \mathcal{PP}(S) \qquad (\perp)$$

where finiteness spaces, introduced below, are its fixpoints $\mathscr{F} = \mathscr{F}^{\perp\perp}$.

▶ **Definition 2.** *A* relational finiteness space *is a pair $A = (|A|, \mathscr{F}(A))$ where $|A|$ is a countable set and $\mathscr{F}(A)$ is a subset of $\mathcal{P}(|A|)$ satisfying $\mathscr{F}(A) = \mathscr{F}(A)^{\perp\perp}$.*

For any countable set S, the smallest finiteness structure is given by the set of finite subsets of S, $\mathcal{P}_{\mathrm{fin}}(S)$ whose orthogonal is given by the whole powerset $\mathcal{P}(S)$. For a relational finiteness space A, while elements of $\mathscr{F}(A)$ may be infinite subsets of $|A|$, they are called *finitary subsets* as they "behave" like finite sets in that $\mathscr{F}(A)$ is closed under inclusion (for $x \in \mathscr{F}(A)$, if $x' \subseteq x$, then $x' \in \mathscr{F}(A)$) and finite unions.

▶ **Definition 3.** *The category* **FinRel** *has objects finiteness spaces and morphisms are relations that preserve the finitary structure forward and backward. Explicitly, for finiteness spaces $A = (|A|, \mathscr{F}(A))$ and $B = (|B|, \mathscr{F}(B))$, a relation $R \subseteq |A| \times |B|$ induces two functions $R^{\star}$ and $R_{\star}$ given by $R_{\star} : x \mapsto \{b \in |B| \mid \exists a \in |A|, (a, b) \in R\}$ and $R^{\star} : y \mapsto \{a \in |A| \mid \exists b \in |B|, (a, b) \in R\}$. The relation R is said to be a* morphism of finiteness spaces *from A to B if for all $x \in \mathscr{F}(A)$, $R_{\star} \cdot x \in \mathscr{F}(B)$ and for all $y \in \mathscr{F}(B)^{\perp}$, $R^{\star} \cdot y \in \mathscr{F}(A)^{\perp}$.*

$$
\begin{array}{ccc}
\mathcal{P}(|A|) & \xrightarrow{R_\star} & \mathcal{P}(|B|) \\
\uparrow & & \uparrow \\
\mathscr{F}(A) & \dashrightarrow & \mathscr{F}(B)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(|B|) & \xrightarrow{R^\star} & \mathcal{P}(|A|) \\
\uparrow & & \uparrow \\
\mathscr{F}(B)^\perp & \dashrightarrow & \mathscr{F}(A)^\perp
\end{array}
$$

Formally, the category **FinRel** is the tight orthogonality category in the sense of Hyland and Schalk obtained from the orthogonality relation defined above [20]. Ehrhard showed that the linear logic structure from **Rel** can be lifted to **FinRel** which constitutes a model of differential linear logic [10]. The morphisms in the co-Kleisli category of **FinRel** play the role of supports for power series for the second part of the construction: for a fixed field (or semi-ring) $\mathcal{R}$, we can define for every relational finiteness space $A = (|A|, \mathscr{F}(A))$, the following vector space (or semi-module): $\mathcal{R}\langle A\rangle := \{X \in \mathcal{R}^{|A|} \mid \text{support}(X) \in \mathscr{F}(A)\}$. Ehrhard showed that $\mathcal{R}\langle A\rangle$ can be endowed with a topology $\mathcal{T}_A$ such that a matrix $M \in \mathcal{R}\langle A \multimap B\rangle$ corresponds to a linear continuous map $\mathcal{R}\langle A\rangle \to \mathcal{R}\langle B\rangle$ and a matrix $M \in \mathcal{R}\langle !A \multimap B\rangle$ corresponds to an analytic map $\mathcal{R}\langle A\rangle \to \mathcal{R}\langle B\rangle$ [9].

3 Profunctorial Finiteness Spaces

3.1 Orthogonality on bicategories

We work with a fragment of **Prof** where the objects are locally finite categories, it has the important consequence that finitely presentable presheaves are always finite presheaves as we will see below.

▶ **Definition 4.** *A small category $\mathcal{A}$ is said to be* locally finite *if it is enriched over finite sets i.e. for any objects $a, a' \in \mathcal{A}$, the homset $\mathcal{A}(a, a')$ is finite.*

▶ **Definition 5.** *For a category $\mathcal{A}$, a presheaf $X : \mathcal{A}^{op} \to$ **Set** is said to be* finite *if for all $a \in \mathcal{A}$, $X(a)$ is a finite set. We denote by $\widehat{\mathcal{A}}_{\text{fin}} \hookrightarrow \widehat{\mathcal{A}}$ the full subcategory of finite presheaves. Note that the Yoneda embedding $\mathbf{y}_{\mathcal{A}}$ for a locally finite category $\mathcal{A}$ factors through the inclusion $\widehat{\mathcal{A}}_{\text{fin}} \hookrightarrow \widehat{\mathcal{A}}$ by an embedding $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}_{\text{fin}}$.*

For presheaf categories, finitely presentable objects can be characterized as presheaves that are isomorphic to a finite colimit of representables. For a locally finite category $\mathcal{A}$, since a finite colimit of finite presheaves is also a finite presheaf, there is an embedding from the subcategory of finitely presentable objects $\widehat{\mathcal{A}}_{\text{fp}}$ to $\widehat{\mathcal{A}}_{\text{fin}}$.

▶ **Definition 6.** *A profunctor $F : \mathcal{A} \nrightarrow \mathcal{B}$ between two small categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \times \mathcal{B}^{\text{op}} \to$ **Set** or equivalently a functor $F : \mathcal{A} \to \widehat{\mathcal{B}}$. F is said to be a* finite profunctor *if it can be factored as a functor $F : \mathcal{A} \to \widehat{\mathcal{B}}_{\text{fin}}$ through the embedding $\widehat{\mathcal{B}}_{\text{fin}} \hookrightarrow \widehat{\mathcal{B}}$. In other words, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $F(a, b)$ is a finite set. A finite profunctor will be denoted by $F : \mathcal{A} \nrightarrow_{\mathsf{f}} \mathcal{B}$.*

The composite of two profunctors $F : \mathcal{A} \nrightarrow \mathcal{B}$ and $G : \mathcal{B} \nrightarrow \mathcal{C}$ is the profunctor $G \circ F : \mathcal{A} \nrightarrow \mathcal{C}$ given by the coend formula:

$$(a, c) \quad \mapsto \quad \int^{b \in \mathcal{B}} F(a, b) \times G(b, c) \cong \left(\sum_{b \in \mathcal{B}} F(a, b) \times G(b, c) \right) /\!\sim$$

where $\sim$ is the least equivalence relation such that $(b, F(a, f)(s), t) \sim (b', s, G(f, c)(t))$ for $s \in F(a, b')$, $t \in G(b, c)$ and $f : b \to b' \in \mathcal{B}$. Composition of profunctors is associative only up to natural isomorphisms which puts us in the setting of a bicategory [3]. Note that the composite of two finite profunctors between locally finite categories need not to be finite (since the sum above can be infinite if $\mathcal{B}$ has an infinite object set for example) but we will see how finiteness structures will enable us to make this notion compositional.

▶ **Definition 7.** *Let $\mathcal{A}$ be a locally finite category, $X : \mathcal{A}^{op} \to \mathbf{Set}$ a presheaf and $X' : \mathcal{A} \to \mathbf{Set}$ a copresheaf, we say that X and X' are* orthogonal *and write $X \perp_{\mathcal{A}} X'$ if the set $\langle X, X' \rangle := \int^{a\in\mathcal{A}} X(a) \times X'(a)$ is finite.*

In the bicategorical case, presheaves in $\widehat{\mathcal{A}}$ or equivalently profunctors $\mathbf{1} \nrightarrow \mathcal{A}$ (where $\mathbf{1}$ is the terminal category) are thought of as closed programs of type $\mathcal{A}$ and co-presheaves in $\widehat{\mathcal{A}^{\mathrm{op}}}$ or profunctors $\mathcal{A} \nrightarrow \mathbf{1}$ correspond to environments. In our setting, the interaction between a program $X : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ and an environment $X' : \mathcal{A} \to \mathbf{Set}$ corresponds to their composition in $\mathbf{Prof}$: $X' \circ X = \int^{a\in\mathcal{A}} X(a) \times X'(a)$.

Adding the orthogonality structure on categories allows us to work in a setting where we enforce this composite to always be finite. Note that the condition in Definition 7 becomes $X' \circ X \in \mathbf{FinSet} \hookrightarrow \mathbf{Set} \cong \mathbf{Prof}(\mathbf{1}, \mathbf{1})$. In the case of 1-categories, for $\mathcal{C}$ a model of linear logic with monoidal units $\mathbf{1}$ and $\perp$, and for $\perp\!\!\!\perp \subseteq \mathcal{C}(\mathbf{1}, \perp)$ a distinguished *pole*, if the orthogonality relation $\perp_c \hookrightarrow \mathcal{C}(\mathbf{1}, c) \times \mathcal{C}(c, \perp)$ is given by:

$$\perp_c \quad = \quad \{(x, x') \in \mathcal{C}(\mathbf{1}, c) \times \mathcal{C}(c, \perp) \mid x' \circ x \in \perp\!\!\!\perp\}$$

we say that the orthogonality is *focused* and it is one of the better behaved cases [20]. It implies in particular that for all $x \in \mathcal{C}(\mathbf{1}, c)$, $f \in \mathcal{C}(c, d)$ and $y \in \mathcal{C}(d, \perp)$, $f \circ x \perp_d y$ if and only if $x \perp_c y \circ f$. In the general case, a morphism preserving the orthogonality needs to preserve it forward and backward whereas in the focused case, forward preservation becomes equivalent to backward preservation which simplifies the proofs significantly since we do not have to prove both directions every time. Unlike the relational case, the orthogonality in the categorified setting becomes focused so that the two preservation conditions for relations of Definition 3 reduce to a single preservation condition for profunctors as we will see in Definition 14.

▶ **Lemma 8.** *For all $X : \mathbf{1} \nrightarrow_{\mathsf{f}} \mathcal{A}$, $Y : \mathcal{B} \nrightarrow_{\mathsf{f}} \mathbf{1}$ and $F : \mathcal{A} \nrightarrow_{\mathsf{f}} \mathcal{B}$, we have:*

$$F \circ X \perp_{\mathcal{B}} Y \qquad \Leftrightarrow \qquad X \perp_{\mathcal{A}} Y \circ F.$$

Proof. It follows from the fact that the sets $\langle F \circ X, Y \rangle$ and $\langle X, Y \circ F \rangle$ are both isomorphic to $\int^{a\in\mathcal{A}} \int^{b\in\mathcal{B}} F(a, b) \times X(a) \times Y(b)$. ◀

For a set A considered as a discrete category, a subset $x \subseteq A$ can be viewed as a presheaf $x : A^{\mathrm{op}} \to \mathbf{Set}$ (or a copresheaf $x : A \to \mathbf{Set}$) that maps $a \in A$ to the singleton $\{\star\}$ if $a \in x$ and to the empty set otherwise. Hence, for $x \subseteq A$ viewed as a presheaf and $x' \subseteq A$ viewed as a copresheaf, $x \cap x'$ is finite is equivalent to the set $\int^{a\in A} x(a) \times x'(a)$ being finite. This analogy provides the connection between the bicategorical case and the relational case.

▶ **Definition 9.** *For a subcategory $\mathcal{C} \hookrightarrow \widehat{\mathcal{A}}_{\mathrm{fin}}$, we denote by $\mathcal{C}^{\perp}$, the full subcategory of $\widehat{\mathcal{A}^{op}}_{\mathrm{fin}}$ of finite copresheaves X' such that for all $X \in \mathcal{C}$, $X' \perp_{\mathcal{A}} X$.*

Let $\mathbf{Sub}(\widehat{\mathcal{A}})$ be the poset of full subcategories of $\widehat{\mathcal{A}}$, the orthogonality relation induces a Galois connection:

$$\mathbf{Sub}(\widehat{\mathcal{A}}) \underset{(-)^{\perp}}{\overset{(-)^{\perp}}{\rightleftarrows}} \mathbf{Sub}(\widehat{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} \quad (\perp)$$

whose fixed points are full subcategories $\mathcal{C}$ verifiying $\mathcal{C}^{\perp\perp} \cong \mathcal{C}$.

▶ **Definition 10.** *A* finiteness structure *is a pair $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ of a locally finite category $|\mathbf{A}|$ and a full subcategory $\mathscr{F}\mathbf{A} \hookrightarrow \widehat{|\mathbf{A}|}_{\mathrm{fin}}$ verifying $\mathscr{F}\mathbf{A} \cong \mathscr{F}\mathbf{A}^{\perp\perp}$.*

▶ **Lemma 11.** *For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, the subcategory of finitely presentable objects $\widehat{|\mathbf{A}|}_{\mathrm{fp}} \hookrightarrow \widehat{|\mathbf{A}|}_{\mathrm{fin}}$ is always a full subcategory of $\mathscr{F}\mathbf{A}$.*

Proof. If X is finitely presentable, then X is isomorphic to a finite colimit of representables $X \cong \varinjlim_{i\in I} |\mathbf{A}|(-, a_i) : |\mathbf{A}|^{\mathrm{op}} \to \mathbf{Set}$. For any $X' \in (\mathscr{F}\mathbf{A})^{\perp}$,

$$\langle X, X' \rangle = \int^{a\in|\mathbf{A}|} X(a) \times X'(a) \cong \varinjlim_{i\in I} \int^{a\in|\mathbf{A}|} |\mathbf{A}|(a, a_i) \times X'(a) \cong \varinjlim_{i\in I} X'(a_i).$$

Since a finite colimit of finite sets is finite, we obtain that $X \perp_{\mathbf{A}} X'$ as desired. ◀

The minimal finiteness structure is $(|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\mathrm{fp}})$ and its orthogonal is the maximal finiteness structure $(|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\mathrm{fin}})$ so for any finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, we have

$$(|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\mathrm{fp}}) \hookrightarrow \mathbf{A} \hookrightarrow (|\mathbf{A}|, \widehat{|\mathbf{A}|}_{\mathrm{fin}}).$$

▶ **Lemma 12.** *If $\mathcal{A}$ is a finite category (both the object and morphism sets are finite), then there is a unique finiteness structure given by $\widehat{\mathcal{A}}_{\mathrm{fin}}$.*

Proof. By Lemma 11, it suffices to show that if $\mathcal{A}$ is finite, then any finite presheaf $X : \mathcal{A}^{op} \to \mathbf{FinSet}$ is finitely presentable. If $\mathcal{A}$ is finite, then the category of elements $\int X$ of X is finite as well and since $X \cong \varinjlim(\int X \to \mathcal{A} \to \widehat{\mathcal{A}})$, X is a finite colimit of representables and hence is finitely presentable. ◀

In the relational case, for a finiteness structure $A = (|A|, \mathscr{F}A)$, $\mathscr{F}A$ can be larger than $\mathcal{P}_{\mathrm{fin}}(|A|)$ but its elements “behave” like finite sets in the sense that $x \subseteq y \in \mathscr{F}(A)$ implies $x \in \mathscr{F}(A)$ and a finite union of finitary elements is finitary. In the categorical case, $\mathscr{F}(\mathbf{A})$ can be thought of as a category larger than $\widehat{|\mathbf{A}|}_{\mathrm{fp}}$ but its elements “behave” like finitely presentable elements as $\mathscr{F}(\mathbf{A})$ is closed under retractions and finite colimits.

▶ **Lemma 13.** *Let $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}(\mathbf{A}))$ be a finiteness structure, then the following two properties hold:*
1. *if X' is a retract of an element $X \in \mathscr{F}(\mathbf{A})$, then $X' \in \mathscr{F}(\mathbf{A})$;*
2. *$\mathscr{F}(\mathbf{A})$ is closed under finite colimits.*

Proof. Let $\alpha : X \Rightarrow X'$ be a retraction in $\widehat{|\mathbf{A}|}$. Since a retraction is an epimorphism and colimits in $\widehat{|\mathbf{A}|}$ are computed pointwise, for every $a \in |\mathbf{A}|$, $\alpha_a : X(a) \to X'(a)$ is a surjection. Hence, for every $Y \in \mathscr{F}(\mathbf{A})^{\perp}$,

$$\langle Y, X \rangle = \int^{a\in|\mathbf{A}|} Y(a) \times X(a) \quad \twoheadrightarrow \quad \int^{a\in|\mathbf{A}|} Y(a) \times X'(a) = \langle Y, X' \rangle$$

which implies that $\langle Y, X' \rangle$ is a finite set as well so that $X' \in \mathscr{F}(\mathbf{A})$. The second property follows from the fact that a finite colimit of finite sets is finite. ◀

▶ **Definition 14.** *Given two finiteness structures* $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$*, a finite profunctor* $F : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|$ *is called a* finiteness profunctor *if* $\widehat{F} := \mathbf{Lan}_{\mathbf{y}_{|\mathbf{A}|}} F : \widehat{|\mathbf{A}|} \to \widehat{|\mathbf{B}|}$ *verifies* $\widehat{F}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathscr{F}\mathbf{B}$ *i.e if there exists a functor* $\mathscr{F}\mathbf{A} \to \mathscr{F}\mathbf{B}$ *making the diagram below commute:*

$$\begin{array}{ccc} \widehat{|\mathbf{A}|} & \xrightarrow{\widehat{F}} & \widehat{|\mathbf{B}|} \\ \uparrow & & \uparrow \\ \mathscr{F}\mathbf{A} & \dashrightarrow & \mathscr{F}\mathbf{B} \end{array}$$

▶ **Lemma 15.** *Given two finiteness structures* $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ *and* $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$*, a profunctor* $F : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|$ *is a finiteness profunctor* $\mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ *if and only if* $F^{\perp} : (|\mathbf{B}|^{op}, \mathscr{F}\mathbf{B}^{\perp}) \nrightarrow_{\mathsf{f}} (|\mathbf{A}|^{op}, \mathscr{F}\mathbf{A}^{\perp})$ *is also a finiteness profunctor.*

Proof. Direct consequence of Lemma 8. ◀

Since the categories $\mathbf{Prof}(\mathbf{1}, |\mathbf{A}|)$ and $\widehat{|\mathbf{A}|}$ are isomorphic, we will abuse notation and identify presheaves $X \in \widehat{|\mathbf{A}|}$ with profunctors $\mathbf{1} \nrightarrow |\mathbf{A}|$ and write $F \circ X$ instead of $\widehat{F}X$. Under this isomorphism, we can reformulate the condition of Definition 14 as follows: F is a finiteness profunctor if for all presheaves X in $\mathscr{F}\mathbf{A}$, $F \circ X$ is in $\mathscr{F}\mathbf{B}$. Likewise, using the isomorphism $\mathbf{Prof}(|\mathbf{B}|, \mathbf{1}) \cong \widehat{|\mathbf{B}|^{\mathrm{op}}}$, $F^{\perp}$ is a finiteness profunctor if for all copresheaves Y in $\mathscr{F}\mathbf{B}^{\perp}$, $Y \circ F$ is in $\mathscr{F}\mathbf{A}^{\perp}$.

▶ **Definition 16.** *Define* $\mathbf{FinProf}$ *to be the bicategory whose* 0*-cells are finiteness structures,* 1*-cells are finiteness profunctors as in Definition 14 and* 2*-cells are natural transformations between such profunctors.*

Proof. We show below that $\mathbf{FinProf}$ is indeed a bicategory.

Identity For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $\mathrm{id}_{|\mathbf{A}|} : |\mathbf{A}| \nrightarrow |\mathbf{A}|$ is a finite profunctor as $|\mathbf{A}|$ is a locally finite category. Since $\mathrm{id}_{|\mathbf{A}|}$ verifies $\widehat{id_{|\mathbf{A}|}} \cong id_{\widehat{|\mathbf{A}|}}$, it is a finiteness profunctor $\mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{A}$.

Composition Let $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A}), \mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$ and $\mathbf{C} = (|\mathbf{C}|, \mathscr{F}\mathbf{C})$ be finiteness structures and $F : \mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ and $G : \mathbf{B} \nrightarrow_{\mathsf{f}} \mathbf{C}$ be finiteness profunctors. It is clear that if $\widehat{F}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathscr{F}\mathbf{B}$ and $\widehat{G}(\mathscr{F}\mathbf{B}) \hookrightarrow \mathscr{F}\mathbf{C}$, then $\widehat{G \circ F}(\mathscr{F}\mathbf{A}) \cong \widehat{G} \circ \widehat{F}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathscr{F}\mathbf{C}$. It remains to show that $G \circ F$ is a finite profunctor. For all $a \in |\mathbf{A}|$ and $c \in |\mathbf{C}|$, we have

$$(G \circ F)(a, c) = \int^{b \in |\mathbf{B}|} F(a, b) \times G(b, c) \cong \widehat{G}(\widehat{F}(\mathbf{y}(a)))(c).$$

Since $\mathbf{y}(a) \in \mathscr{F}\mathbf{A}$, $\widehat{G}(\widehat{F}(\mathbf{y}(a))$ is an element of $\mathscr{F}\mathbf{C}$ so it is a finite presheaf, which implies that $\widehat{G}(\widehat{F}(\mathbf{y}(a)))(c)$ is finite as desired. ◀

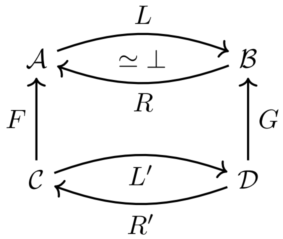
We obtain as a corollary of Lemma 15 that the mapping $\mathbf{A} \mapsto \mathbf{A}^{\perp} := (|\mathbf{A}|^{\mathrm{op}}, \mathscr{F}\mathbf{A}^{\perp})$ can be extended to a full and faithful functor $\mathbf{FinProf}^{\mathrm{op}} \to \mathbf{FinProf}$.

▶ **Lemma 17.** *The forgetful functor* $\mathcal{U} : \mathbf{FinProf} \to \mathbf{Prof}$ *is locally fully faithful and injective on* 1*-cells. Explicitely, for finiteness structures* $\mathbf{A}$ *and* $\mathbf{B}$*, the induced functor* $\mathbf{FinProf}(\mathbf{A}, \mathbf{B}) \to \mathbf{Prof}(|\mathbf{A}|, |\mathbf{B}|)$ *is injective on objects and fully faithful.*

4 Linear Logic Structure

In this section, we prove that the differential linear logic structure in **Prof** can be lifted to **FinProf**. While the full definition of a bicategorical model of linear logic has yet to be spelled out, the standard 1-categorical constructions have canonical bicategorical analogues which we use. The proofs will make use of the lemma below that shows how certain families of adjoint equivalences needed for the linear logic structure can be lifted from **Prof** to **FinProf** using the fact that the forgetful functor is locally fully faithful.

▶ **Lemma 18.** *Let* $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ *be categories and* $(L : \mathcal{A} \to \mathcal{B}, R : \mathcal{B} \to \mathcal{A}, \eta, \varepsilon)$ *be an adjoint equivalence. Let* $L' : \mathcal{C} \to \mathcal{D}$*,* $R' : \mathcal{D} \to \mathcal{C}$*,* $F : \mathcal{C} \to \mathcal{A}$ *and* $G : \mathcal{D} \to \mathcal{B}$ *be functors such that* F *and* G *are fully faithful,* $GL' = LF$ *and* $FR' = RG$*. Then* L' *and* R' *are adjoint equivalent* $L' \dashv R'$*.*



Proof. For objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$, using the hypotheses above, we have:

$$\mathcal{C}(c, R'd) \cong \mathcal{A}(Fc, FR'(d)) = \mathcal{A}(Fc, RGd) \cong \mathcal{B}(LFc, Gd) = \mathcal{B}(GL'c, Gd) \cong \mathcal{D}(L'c, d)$$

which implies that $L' \dashv R'$.

For $c \in \mathcal{C}$, the component of the unit η' of the adjunction $L' \dashv R'$ is the morphism η'_c determined by $F(\eta'_c) = \eta_{F(c)}$. It is an isomorphism since F is fully faithful and hence conservative. We can show that the counit of the adjunction $L' \dashv R'$ is an isomorphism in a similar fashion. ◀

4.1 Additive structure

Similarly to the 1-categorical case, **FinProf** is endowed with a finite biproduct structure. For a family of categories $(\mathcal{A}_i)_{i\in I}$, we denote by $\&_i\mathcal{A}_i$ their coproduct in **Cat**. There is an isomorphism $\widehat{\&_i\mathcal{A}_i} \cong \prod_i \widehat{\mathcal{A}_i}$, so we will often identify a presheaf $Z \in \widehat{\&_i\mathcal{A}_i}$ with a tuple of presheaves $(Z_i)_{i\in I} \in \prod_i \widehat{\mathcal{A}_i}$.

▶ **Lemma 19.** *For a finite family of finiteness structures* $(\mathbf{A}_i)_{i\in I}$*,* $\&_i\mathbf{A}_i := (\&_i |\mathbf{A}_i|, \prod_i \mathscr{F}\mathbf{A}_i)$ *is a finiteness structure.*

Proof. It suffices to show that $(\prod_i \mathscr{F}\mathbf{A}_i)^\perp \cong \prod_i(\mathscr{F}\mathbf{A}_i)^\perp$. ◀

▶ **Definition 20.** *For a family of finiteness structures* $(\mathbf{A}_i)_{i\in I}$*, we define the finiteness structure* $\oplus_i\mathbf{A}_i$ *by* $(\&_i |\mathbf{A}_i|, (\mathscr{F}(\&_i\mathbf{A}_i^\perp))^\perp)$*.*

▶ **Lemma 21.** *The empty category* $\mathbf{0}$ *with its presheaf category* $(\mathbf{0}, \widehat{\mathbf{0}})$ *forms a finiteness structure that is the neutral for* $\&$ *and* $\oplus$*.*

▶ **Lemma 22.** *For a finite family of finiteness structures* $(\mathbf{A}_i)_{i\in I}$*, the profunctors* $\pi_i : \&_i |\mathbf{A}_i| \nrightarrow |\mathbf{A}_i|$ *and* $\mathrm{inj}_i : |\mathbf{A}_i| \nrightarrow \&_i |\mathbf{A}_i|$ *are finiteness profunctors* $\&_i\mathbf{A}_i \nrightarrow_{\mathsf{f}} \mathbf{A}_i$ *and* $\mathbf{A}_i \nrightarrow_{\mathsf{f}} \oplus_i\mathbf{A}_i$ *respectively. They induce adjoint equivalences:*

$$\mathbf{FinProf}(\mathbf{X}, \&_i\mathbf{A}_i) \simeq \prod_i \mathbf{FinProf}(\mathbf{X}, \mathbf{A}_i) \text{ and } \mathbf{FinProf}(\oplus_i\mathbf{A}_i, \mathbf{X}) \simeq \prod_i \mathbf{FinProf}(\mathbf{A}_i, \mathbf{X}).$$

Proof. The profunctors π_i and inj_i are given by $\pi_i : ((i, a_i), a) \mapsto |\mathbf{A}_i|(a, a_i)$ and $\mathrm{inj}_i : (a, (i, a_i)) \mapsto |\mathbf{A}_i|(a_i, a)$ so they are finite profunctors since the category $|\mathbf{A}_i|$ is locally finite. For $Z \in \mathscr{F}(\&_i \mathbf{A}_i)$ and $X \in \mathscr{F}\mathbf{A}_i^{\perp}$, $\langle \pi_i Z, X\rangle \cong \langle Z_i, X\rangle \in \mathbf{FinSet}$ which implies that $\pi_i \in \mathbf{FinProf}(\&_i \mathbf{A}_i, \mathbf{A}_i)$. Likewise, for X in $\mathscr{F}\mathbf{A}_i$ and $Z \in \mathscr{F}(\oplus_i \mathbf{A}_i)^{\perp}$, $\langle \mathrm{inj}_i X, Z\rangle \cong \langle X, Z_i\rangle \in \mathbf{FinSet}$ so that $\mathrm{inj}_i \in \mathbf{FinProf}(\mathbf{A}_i, \oplus_i \mathbf{A}_i)$.

Using Lemma 18, the adjoint equivalences above follow from the biproduct structure in **Prof** where we have adjoint equivalences $\mathbf{Prof}(|\mathbf{X}|, \&_i |\mathbf{A}_i|) \simeq \prod_i \mathbf{Prof}(|\mathbf{X}|, |\mathbf{A}_i|)$ and $\mathbf{Prof}(\&_i |\mathbf{A}_i|, |\mathbf{X}|) \simeq \prod_i \mathbf{Prof}(|\mathbf{A}_i|, |\mathbf{X}|)$. ◀

4.2 Star-Autonomous Structure

The bicategory **Prof** is symmetric monoidal with tensor product given by the cartesian product of categories $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \times \mathcal{B}$ and monoidal unit $\mathbf{1}$. The duality functor $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$ provides **Prof** with a compact closed structure. Adding the orthogonality structure allows for less degenerate model as the bicategory **FinProf** is now $*$-autonomous with dualizing object $\mathbf{1}$. To show that the symmetric monoidal structure in **Prof** lifts to **FinProf**, it suffices to prove that the tensor product lifts to a pseudo-functor $\mathbf{FinProf} \times \mathbf{FinProf} \to \mathbf{FinProf}$ and that the symmetry, associator and left and right unitors pseudo-natural transformations have components in **FinProf**.

For relational finiteness spaces, the tensor product of $A = (|A|, \mathscr{F}(A))$ and $B = (|B|, \mathscr{F}(B))$ is the smallest structure that contains all products $x \times y$ of subsets $x \in \mathscr{F}(A)$ and $y \in \mathscr{F}(B)$. Since the set $\{x \times y \mid x \in \mathscr{F}(A), y \in \mathscr{F}(B)\}$ is not necessarily closed under double orthogonality $A \otimes B$ is defined as $(|A| \times |B|, \{x \times y \mid x \in \mathscr{F}(A), y \in \mathscr{F}(B)\}^{\perp\perp})$. In the categorified case, the construction is similar, for finiteness structures $\mathbf{A}$ and $\mathbf{B}$, $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ is the smallest finiteness structure containing all products $X \times Y$ for $X \in \mathscr{F}(\mathbf{A})$ and $Y \in \mathscr{F}(\mathbf{B})$ where $X \times Y : (|\mathbf{A}| \times |\mathbf{B}|)^{\mathrm{op}} \to \mathbf{Set}$ is the presheaf given by the pointwise product $(a, b) \mapsto X(a) \times Y(b)$.

▶ **Definition 23.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, their tensor product is defined as $\mathbf{A} \otimes \mathbf{B} := (|\mathbf{A}| \times |\mathbf{B}|, \mathscr{F}(\mathbf{A} \otimes \mathbf{B}))$ where $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ is the full subcategory of $\widehat{|\mathbf{A}| \times |\mathbf{B}|}_{\mathrm{fin}}$ whose object set is given by $\{X \times Y \mid X \in \mathscr{F}\mathbf{A}$ and $Y \in \mathscr{F}\mathbf{B}\}^{\perp\perp}$.*

▶ **Lemma 24.** *For finiteness profunctors $F_1 : \mathbf{A}_1 \nrightarrow_{\mathsf{f}} \mathbf{B}_1$ and $F_2 : \mathbf{A}_2 \nrightarrow_{\mathsf{f}} \mathbf{B}_2$, the profunctor $F_1 \otimes F_2 : |\mathbf{A}_1| \times |\mathbf{A}_2| \nrightarrow |\mathbf{B}_1| \times |\mathbf{B}_2|$ given by $(F_1 \otimes F_2)((a_1, a_2), (b_1, b_2)) := F_1(a_1, b_1) \times F_2(a_2, b_2)$ is in $\mathbf{FinProf}(\mathbf{A}_1 \otimes \mathbf{A}_2, \mathbf{B}_1 \otimes \mathbf{B}_2)$.*

Proof. Using Lemma 15, we show that $(F_1 \otimes F_2)^{\perp} \mathscr{F}(\mathbf{B}_1 \otimes \mathbf{B}_2)^{\perp} \hookrightarrow \mathscr{F}(\mathbf{A}_1 \otimes \mathbf{A}_2)^{\perp}$. Let Z be in $\mathscr{F}(\mathbf{B}_1 \otimes \mathbf{B}_2)^{\perp}$ i.e. for all $Y_1 \in \mathscr{F}\mathbf{B}_1$ and $Y_2 \in \mathscr{F}\mathbf{B}_2$, $\langle Z, Y_1 \times Y_2\rangle \in \mathbf{FinSet}$. $(F_1 \otimes F_2)^{\perp}(Z) \in \mathscr{F}(\mathbf{A}_1 \otimes \mathbf{A}_2)^{\perp}$ is equivalent to:

$$\begin{aligned} &\forall X_1 \in \mathscr{F}\mathbf{A}_1, \forall X_2 \in \mathscr{F}\mathbf{A}_2, \langle (F_1 \otimes F_2)^{\perp}(Z), X_1 \times X_2\rangle \in \mathbf{FinSet} \\ \Leftrightarrow &\forall X_1 \in \mathscr{F}\mathbf{A}_1, \forall X_2 \in \mathscr{F}\mathbf{A}_2, \langle Z, (F_1 X_1) \times (F_2 X_2)\rangle \in \mathbf{FinSet} \end{aligned}$$

Since $F_1 X_1$ is in $\mathscr{F}\mathbf{B}_1$ and $F_2 X_2$ is in $\mathscr{F}\mathbf{B}_2$, we obtain the desired result. ◀

▶ **Lemma 25.** $(\mathbf{1}, \mathbf{FinSet})$ *is the tensor unit.*

Proof. Let $\mathbf{A}$ be a finiteness structure, we show that $\mathscr{F}(\mathbf{A})^{\perp} \cong \mathscr{F}(\mathbf{A} \otimes \mathbf{1})^{\perp} \cong \mathscr{F}(\mathbf{1} \otimes \mathbf{A})^{\perp}$ so that the components of the left unitor $l_{|\mathbf{A}|} : |\mathbf{A}| \times |\mathbf{1}| \nrightarrow |\mathbf{A}|$ and right unitor $r_{|\mathbf{A}|} : |\mathbf{1}| \times |\mathbf{A}| \nrightarrow |\mathbf{A}|$ are in **FinProf**. Let $Y \in \mathscr{F}(\mathbf{A})^{\perp}$, $X \in \mathscr{F}(\mathbf{A})$ and $S \in \mathbf{FinSet}$. We have $\langle Y, X \times S\rangle \in \mathbf{FinSet} \Leftrightarrow \langle Y \times S, X\rangle \in \mathbf{FinSet}$. Since $\mathscr{F}(\mathbf{A})^{\perp}$ is closed under finite colimits, $Y \times S$ is in $\mathscr{F}(\mathbf{A})^{\perp}$ which implies the desired result. Now, for $Y \in \mathscr{F}(\mathbf{A} \otimes \mathbf{1})^{\perp}$ and $X \in \mathscr{F}(\mathbf{A})$, $\langle Y, X \rangle \cong \langle Y, X \times \{*\} \rangle \in \mathbf{FinSet}$ so that $Y \in \mathscr{F}(\mathbf{A})^{\perp}$ as desired. The proof for $\mathscr{F}(\mathbf{A})^{\perp} \cong \mathscr{F}(\mathbf{1} \otimes \mathbf{A})^{\perp}$ is similar. ◀

▶ **Lemma 26.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, the categories $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ and $\mathscr{F}(\mathbf{B} \otimes \mathbf{A})$ are isomorphic which implies that the component of the symmetry $\sigma_{|\mathbf{A}|,|\mathbf{B}|} : |\mathbf{A}| \times |\mathbf{B}| \nrightarrow |\mathbf{B}| \times |\mathbf{A}|$ is in $\mathbf{FinProf}(\mathbf{A} \otimes \mathbf{B}, \mathbf{B} \otimes \mathbf{A})$.*

Proof. Immediate. ◀

Showing that the associator has components in **FinProf** is difficult to prove directly so we make use of the duality between the tensor and the internal hom to do it.

▶ **Lemma 27.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, define the finiteness structure $\mathbf{A} \multimap \mathbf{B}$ as $(|\mathbf{A}|^{\mathrm{op}} \times |\mathbf{B}|, \mathscr{F}(\mathbf{A} \multimap \mathbf{B}))$ where $\mathscr{F}(\mathbf{A} \multimap \mathbf{B})$ is the full subcategory of finite presheaves $\widehat{|\mathbf{A}|^{op} \times |\mathbf{B}|}_{\mathrm{fin}}$ that verify Definition 14.*

Proof. We prove that $\mathbf{A} \multimap \mathbf{B}$ is indeed a finiteness structure. We first show that for $X \in \mathscr{F}\mathbf{A}$ and $Y' \in \mathscr{F}\mathbf{B}^{\perp}$, $X \times Y' \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})^{\perp}$. Indeed, for $F \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})$, we have:

$$\langle X \times Y', F \rangle = \int^{a \in |\mathbf{A}|, b \in |\mathbf{B}|} X(a) \times Y'(b) \times F(a,b) \cong \langle Y', FX \rangle \in \mathbf{FinSet}.$$

Now, let $W \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})^{\perp\perp}$, we want to show that $W \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B})$, i.e. that for all $X \in \mathscr{F}\mathbf{A}$, $WX \in \mathscr{F}\mathbf{B}$. Let $Y' \in \mathscr{F}\mathbf{B}^{\perp}$, $\langle Y', WX \rangle \cong \langle X \times Y', W \rangle \in \mathbf{FinSet}$ by the previous remark. ◀

▶ **Lemma 28.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, the categories $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ and $\mathscr{F}(\mathbf{A} \multimap \mathbf{B}^{\perp})^{\perp}$ are isomorphic.*

Proof. We prove that $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})^{\perp} \cong \mathscr{F}(\mathbf{A} \multimap \mathbf{B}^{\perp})$. Let $F : \mathbf{A} \nrightarrow \mathbf{B}^{op}$, we have:

$$\begin{aligned} F \in \mathscr{F}(\mathbf{A} \otimes \mathbf{B})^{\perp} &\Leftrightarrow \forall X \in \mathscr{F}(\mathbf{A}), \forall Y \in \mathscr{F}(\mathbf{B}) \langle F, X \times Y \rangle \in \mathbf{FinSet} \\ &\Leftrightarrow \forall X \in \mathscr{F}(\mathbf{A}), \forall Y \in \mathscr{F}(\mathbf{B}) \langle FX, Y \rangle \in \mathbf{FinSet} \\ &\Leftrightarrow \forall X \in \mathscr{F}(\mathbf{A}), FX \in \mathscr{F}(\mathbf{B})^{\perp} \Leftrightarrow F \in \mathscr{F}(\mathbf{A} \multimap \mathbf{B}^{\perp}) \end{aligned}$$ ◀

▶ **Lemma 29.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$ and $\mathbf{C} = (|\mathbf{C}|, \mathscr{F}\mathbf{C})$, the categories $\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C})$ and $\mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}))$ are isomorphic.*

Proof. Let $F : |\mathbf{A}| \times |\mathbf{B}| \nrightarrow_{\mathsf{f}} |\mathbf{C}|$ be in $\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C})$ and denote by $\overline{F} : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|^{\mathrm{op}} \times |\mathbf{C}|$ the corresponding profunctor obtained from the isomorphism $\mathbf{Prof}(|\mathbf{A}| \times |\mathbf{B}|, |\mathbf{C}|) \cong \mathbf{Prof}(|\mathbf{A}|, |\mathbf{B}|^{\mathrm{op}} \times |\mathbf{C}|)$. Let $X \in \mathscr{F}(\mathbf{A})$, we want to show that $\overline{F}X$ is in $\mathscr{F}(\mathbf{B} \multimap \mathbf{C})$, i.e. for all $Y \in \mathscr{F}(\mathbf{B})$, $\overline{F}(X)(Y) \in \mathscr{F}(\mathbf{C})$. We have that $X \times Y$ is in $\mathscr{F}(\mathbf{A} \otimes \mathbf{B})$ so that $F \circ (X \times Y) \cong \overline{F}(X)(Y)$ is in $\mathscr{F}(\mathbf{C})$.

For the other direction, let $G : |\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|^{\mathrm{op}} \times |\mathbf{C}|$ be in $\mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}))$ and denote by $\overline{G}$ the corresponding profunctor in $\mathbf{Prof}(|\mathbf{A}| \times |\mathbf{B}|, |\mathbf{C}|)$. We show that $\overline{G}^{\perp} \in \mathscr{F}(\mathbf{C}^{\perp} \multimap (\mathbf{A} \otimes \mathbf{B})^{\perp})$. Let $Z \in \mathscr{F}(\mathbf{C})^{\perp}$, we want $\overline{G}^{\perp} Z \in \mathscr{F}(\mathbf{A} \otimes \mathbf{B})^{\perp}$ i.e. for all $X \in \mathscr{F}\mathbf{A}$ and $Y \in \mathscr{F}\mathbf{B}$, $\langle \overline{G}^{\perp} Z, X \times Y \rangle \in \mathbf{FinSet}$. Since $\langle \overline{G}^{\perp} Z, X \times Y \rangle \cong \langle G(X)(Y), Z \rangle$, we obtain the desired result. ◀

▶ **Corollary 30.** *For finiteness structures* $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$*,* $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$ *and* $\mathbf{C} = (|\mathbf{C}|, \mathscr{F}\mathbf{C})$*, the component of the associator* $\alpha_{|\mathbf{A}|,|\mathbf{B}|,|\mathbf{C}|} : (|\mathbf{A}| \times |\mathbf{B}|) \times |\mathbf{C}| \nrightarrow |\mathbf{A}| \times (|\mathbf{B}| \times |\mathbf{C}|)$ *given by:*

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto |\mathbf{A}|(a_2, a_1) \times |\mathbf{B}|(b_2, b_1) \times |\mathbf{C}|(c_2, c_1)$$

is a finiteness profunctor in $\mathbf{FinProf}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}, \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))$*.*

Proof. It suffices to show that the categories $\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C})$ and $\mathscr{F}(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))$ are isomorphic. By Lemmas 28 and 29, we have

$$\begin{aligned}\mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}) &\cong \mathscr{F}((\mathbf{A} \otimes \mathbf{B}) \multimap \mathbf{C}^\perp)^\perp \cong \mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}^\perp))^\perp \\ &\cong \mathscr{F}(\mathbf{A} \multimap (\mathbf{B} \multimap \mathbf{C}^\perp)^{\perp\perp})^\perp \cong \mathscr{F}(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})).\end{aligned}$$ ◀

A symmetric monoidal bicategory $\mathscr{B}$ is $*$-autonomous if there exists a full and faithful functor $(-)^* : \mathscr{B}^{\mathrm{op}} \to \mathscr{B}$ verifying $\mathbf{A} \simeq \mathbf{A}^{**}$ and for every objects $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, a pseudo-natural family of adjoint equivalences $\mathscr{B}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}^*) \simeq \mathscr{B}(\mathbf{A}, (\mathbf{B} \otimes \mathbf{C})^*)$.

▶ **Proposition 31.** $\mathbf{FinProf}$ *a* $\star$*-autonomous bicategory.*

Proof. The duality $(-)^\perp : \mathbf{A} \mapsto \mathbf{A}^\perp = (|\mathbf{A}|^{\mathrm{op}}, \mathscr{F}\mathbf{A}^\perp)$ induces a full and faithful functor by Lemma 15. For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$ and $\mathbf{C} = (|\mathbf{C}|, \mathscr{F}\mathbf{C})$, by Lemma 18, there is a pseudo-natural family of adjoint equivalences $\mathbf{FinProf}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}^\perp) \simeq \mathbf{FinProf}(\mathbf{A}, (\mathbf{B} \otimes \mathbf{C})^\perp)$. ◀

The interpretation of the $\parr$ connective is defined by dualizing the tensor $\mathbf{A} \parr \mathbf{B} = (\mathbf{A}^\perp \otimes \mathbf{B}^\perp)^\perp$. In the compact closed bicategory $\mathbf{Prof}$, the two connectives have the same interpretation whereas in $\mathbf{FinProf}$, adding the orthogonality eliminates this degeneracy. The inclusion $\mathscr{F}(\mathbf{A} \otimes \mathbf{B}) \hookrightarrow \mathscr{F}(\mathbf{A} \parr \mathbf{B})$ always hold which implies that we can interpret the mix rule in $\mathbf{FinProf}$. It can be derived from the set inclusion

$$\{X \times Y \mid X \in \mathscr{F}\mathbf{A}^\perp \text{ and } Y \in \mathscr{F}\mathbf{B}^\perp\} \hookrightarrow \{X \times Y \mid X \in \mathscr{F}\mathbf{A} \text{ and } Y \in \mathscr{F}\mathbf{B}\}^\perp$$

and the fact that $\mathscr{F}(\mathbf{A} \parr \mathbf{B})$ has object set $\{X \times Y \mid X \in \mathscr{F}(\mathbf{A})^\perp \text{ and } Y \in \mathscr{F}(\mathbf{B})^\perp\}^\perp$.

The other inclusion does not hold in general: consider the presheaf $P : ((!\mathbf{1})^{\mathrm{op}} \times !\mathbf{1})^{\mathrm{op}} \to \mathbf{Set}$ given by $(n, m) \mapsto !\mathbf{1}(m, n)$ corresponding to the identity profunctor $!\mathbf{1} \nrightarrow_{\mathsf{f}} !\mathbf{1}$. P is in $\mathscr{F}(!\mathbf{1} \multimap !\mathbf{1}) \cong \mathscr{F}((!\mathbf{1})^\perp \parr !\mathbf{1})$ but it is not in $\mathscr{F}((!\mathbf{1})^\perp \otimes !\mathbf{1})$. Indeed, let $Q : (!\mathbf{1})^{\mathrm{op}} \times !\mathbf{1}) \to \mathbf{Set}$ be dually given by $(n, m) \mapsto !\mathbf{1}(n, m)$, it verifies that for all $X \in \mathscr{F}(!\mathbf{1})^\perp$ and $Y \in \mathscr{F}(!\mathbf{1})$,

$$\langle X \times Y, Q \rangle = \int^{n,m} X(n) \times Y(m) \times !\mathbf{1}(n, m) \cong \langle X, Y \rangle \in \mathbf{FinSet}$$

which implies that Q is in $\mathscr{F}((!\mathbf{1})^\perp \otimes !\mathbf{1})^\perp$. However, $\langle P, Q \rangle = \int^{n,m} !\mathbf{1}(m, n) \times !\mathbf{1}(n, m) \cong \int^n !\mathbf{1}(n, n) \notin \mathbf{FinSet}$.

4.3 Exponential structure

The exponential modality in the setting of generalized species presented by Fiore et al. relies on the free symmetric strict monoidal completion construction for a small category.

▶ **Definition 32.** *For a small category* $\mathcal{A}$*, define* $!\mathcal{A}$ *as the category whose objects are finite sequences* $\langle a_1, \ldots, a_n \rangle$ *of objects of* $\mathcal{A}$ *and a morphism* f *between two sequences* $\langle a_1, \ldots, a_n \rangle$ *and* $\langle b_1, \ldots b_n \rangle$ *consists of a pair* $(\sigma, (f_i)_{i \in \underline{n}})$ *where* σ *is a permutation in the symmetric group* $\mathfrak{S}_n$ *and* $(f_i : a_i \to b_{\sigma(i)})_{i \in \underline{n}}$ *is a sequence of morphisms in* $\mathcal{A}$*.*

The category $!\mathcal{A}$ described above is symmetric monoidal with tensor product $\otimes : (u,v) \mapsto u \otimes v$ given by the concatenation of sequences and unit the empty sequence. This construction induces a 2-monad on **Cat** which lifts to a pseudo-monad on **Prof** [14]. By dualization, one obtains a pseudo-comonad on **Prof** where the counit der and the comultiplication dig have the following components:

$$\begin{aligned} \mathrm{der}_{\mathcal{A}} &: !\mathcal{A} \nrightarrow \mathcal{A} \\ (u,a) &\mapsto !\mathcal{A}(\langle a \rangle, u) \end{aligned} \qquad \begin{aligned} \mathrm{dig}_{\mathcal{A}} &: !\mathcal{A} \nrightarrow !!\mathcal{A} \\ (u, \langle u_1, \dots, u_n \rangle) &\mapsto !\mathcal{A}(u_1 \otimes \dots \otimes u_n, u) \end{aligned}$$

For a profunctor $P : \mathcal{A} \nrightarrow \mathcal{B}$, the action of the pseudo-comonad is given by

$$!P : (u,v) \mapsto \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} P(u_i, v_{\sigma(i)})$$

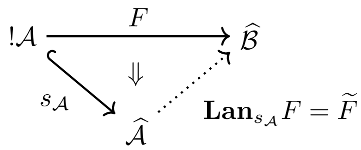
if $u \in !\mathcal{A}$ and $v \in !\mathcal{B}$ are sequences of length n and $!P : (u,v) \mapsto \emptyset$ if u and v have different lengths. Generalized species correspond to the 1-cells in the co-Kleisli bicategory $\mathbf{Prof}_!$. For a species $F : !\mathcal{A} \nrightarrow \mathcal{B}$, its comonadic lifting $F^! : !\mathcal{A} \nrightarrow !\mathcal{B}$ is given by $(!F) \circ \mathrm{dig}_{\mathcal{A}}$. The composite of two species $G : !\mathcal{B} \nrightarrow \mathcal{C}$ and $F : !\mathcal{A} \nrightarrow \mathcal{B}$ in $\mathbf{Prof}_!$ is then given by $G \circ F^! : !\mathcal{A} \nrightarrow \mathcal{C}$.

We show in this section that the comonadic structure described above can be refined to the setting of finiteness structures.

▶ **Definition 33.** *For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, we define $!(\mathbf{A}, \mathscr{F}\mathbf{A}) := (!\,|\mathbf{A}|, \mathscr{F}!\mathbf{A})$ where $\mathscr{F}!\mathbf{A}$ is the full subcategory of $\widehat{!\,|\mathbf{A}|}_{\mathrm{fin}}$ with object set $\{X^! \mid X \in \mathscr{F}\mathbf{A}\}^{\perp\perp}$.*

Note that for a presheaf $X : |\mathbf{A}|^{\mathrm{op}} \to \mathbf{Set}$ (seen as a species $!\,\mathbf{0} \nrightarrow |\mathbf{A}|$), its lifting $X^! : (!\mathbf{A})^{\mathrm{op}} \to \mathbf{Set}$ is given by $\langle a_1, \dots, a_n \rangle \mapsto !X \circ \mathrm{dig}_{\mathbf{0}}(\langle a_1, \dots, a_n \rangle) \cong \prod_{i \in \underline{n}} X(a_i)$. In particular, if X is a finite presheaf, then so is $X^!$.

Joyal presented the notion of analytic functor as the Taylor series counterpart of combinatorial species [22]. Fiore extended Joyal's results in the setting of generalized species and showed that there is a biequivalence between the bicategory of generalized species (restricted to groupoids) and the 2-category of analytic functors [12]. For small categories $\mathcal{A}$ and $\mathcal{B}$, a functor $P : \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ is said to be *analytic* if there exists a generalized species $F : !\mathcal{A} \nrightarrow \mathcal{B}$ such that P is isomorphic to $\mathbf{Lan}_{s_{\mathcal{A}}} F$ (denoted by $\widetilde{F}$):



where $s_{\mathcal{A}} : !\mathcal{A} \to \widehat{\mathcal{A}}$ is the functor that maps a sequence $\langle a_1, \dots a_n \rangle$ in $!\mathcal{A}$ to the presheaf $\sum_{i=1}^{n} \mathbf{y}_{\mathcal{A}}(a_i)$ in $\widehat{\mathcal{A}}$ so that $\widetilde{F}$ is given by $X \mapsto \int^{u \in !\mathcal{A}} F(u) \times \widehat{\mathcal{A}}(s_{\mathcal{A}}(u), X) \cong \int^{u \in !\mathcal{A}} F(u) \times X^!(u)$.

▶ **Lemma 34.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, a species $F : |!\mathbf{A}| \nrightarrow_{\mathsf{f}} |\mathbf{B}|$ is in $\mathscr{F}(!\mathbf{A} \multimap \mathbf{B})$ (viewed as a finite presheaf $(|!\mathbf{A}|^{\mathrm{op}} \times |\mathbf{B}|)^{\mathrm{op}} \to \mathbf{Set}$) if and only if for all $X \in \mathscr{F}(\mathbf{A})$, $\widetilde{F}X$ is in $\mathscr{F}(\mathbf{B})$.*

Proof. Assume that F is in $\mathscr{F}(!\mathbf{A} \multimap \mathbf{B})$ and let X be in $\mathscr{F}(\mathbf{A})$. Since $X^!$ is in $\mathscr{F}(!\mathbf{A})$, we have that $FX^! \cong \widetilde{F}X \in \mathscr{F}(\mathbf{B})$.

For the other direction, it suffices to show that if for all $X \in \mathscr{F}(\mathbf{A})$, $FX^!$ is in $\mathscr{F}(\mathbf{B})$, then $F^\perp(\mathscr{F}(\mathbf{B})^\perp) \hookrightarrow (\mathscr{F}(!\mathbf{A})^\perp)$. Let Y be in $(\mathscr{F}\mathbf{B})^\perp$ and $X \in \mathscr{F}\mathbf{A}$, since $\langle F^\perp Y, X^! \rangle \cong \langle FX^!, Y \rangle \in \mathbf{FinSet}$, we obtain the desired result. ◀

We obtain as a corollary that for a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, $(\mathscr{F}!\mathbf{A})^\perp$ is isomorphic to the full subcategory of finite copresheaves $P : |!\mathbf{A}| \to \mathbf{Set}$ (or equivalently finite profunctors $|!\mathbf{A}| \nrightarrow_{\mathsf{f}} \mathbf{1}$) such that $\widetilde{P}(\mathscr{F}\mathbf{A}) \hookrightarrow \mathbf{FinSet}$.

▶ **Example 35.** In particular, $\mathscr{F}(!\,\mathbf{1})^\perp$ is isomorphic to species whose analytic functor maps finite sets to finite sets. In other words, $F : !\,\mathbf{1} \to \mathbf{Set}$ must verify that for all $S \in \mathbf{FinSet}$, $\sum\limits_{n\in\mathbb{N}} F(n) \times_{\mathfrak{S}_n} S^n$ is finite.

Similarly to relational finiteness spaces, we can see here that the fixpoint operator cannot be interpreted in **FinProf**. Indeed, consider the species of binary trees $B : !\,\mathbf{1} \nrightarrow \mathbf{1}$, it is a solution of the fixpoint equation $B = 1 + X \cdot B^2$ where $1 : !\,\mathbf{1} \nrightarrow \mathbf{1}$ is the species $(u, \star) \mapsto !\,\mathbf{1}(\langle\rangle, u)$ whose analytic functor $\mathbf{Set} \to \mathbf{Set}$ is the constant $S \mapsto \{\star\}$ and $X : !\,\mathbf{1} \nrightarrow \mathbf{1}$ is the species $(u, \star) \mapsto !\,\mathbf{1}(\langle\star\rangle, u)$ whose analytic functor $\mathbf{Set} \to \mathbf{Set}$ is the identity $S \mapsto S$ (see [4] for more details). Both 1 and X are finiteness species since their analytic functors restrict to $\mathbf{FinSet} \to \mathbf{FinSet}$. The species of binary trees however has analytic functor $\mathbf{Set} \to \mathbf{Set}$ given by $S \mapsto \sum_{n\in\mathbb{N}} C_n \times S^n$ where C_n is the nth Catalan number so this functor can not be restricted as a functor $\mathbf{FinSet} \to \mathbf{FinSet}$.

▶ **Lemma 36.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, if $F : \mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ is a finiteness profunctor, then $!F : |!\mathbf{A}| \nrightarrow |!\mathbf{B}|$ is in $\mathbf{FinProf}(!\mathbf{A}, !\mathbf{B})$.*

Proof. We show that $(!F)(\mathscr{F}!\mathbf{B}^\perp) \hookrightarrow \mathscr{F}!\mathbf{A}^\perp$. Let P be in $\mathscr{F}!\mathbf{B}^\perp$, i.e. for all Y in $\mathscr{F}\mathbf{B}$, $\widetilde{P}Y$ is in **FinSet**.

$$\begin{aligned}(!F)(P) \in \mathscr{F}!\mathbf{A}^\perp &\Leftrightarrow \forall X \in \mathscr{F}\mathbf{A}, \int^{u\in|!\mathbf{A}|, v\in|!\mathbf{B}|} !F(u,v) \times P(v) \times X^!(u) \in \mathbf{FinSet}\\ &\Leftrightarrow \forall X \in \mathscr{F}\mathbf{A}, \int^{v\in|!\mathbf{B}|} P(v) \times (!F \circ X^!)(v) \in \mathbf{FinSet}\\ &\Leftrightarrow \forall X \in \mathscr{F}\mathbf{A}, \int^{v\in|!\mathbf{B}|} P(v) \times (F \circ X)^!(v) \in \mathbf{FinSet}\end{aligned}$$

Since FX is in $\mathscr{F}\mathbf{B}$, $(FX)^! \in \mathscr{F}!\mathbf{B}$ which implies the desired result. ◀

We now show that the pseudo-comonad structure in **Prof** can be restricted to **FinProf**.

▶ **Lemma 37.** *For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, the component of the counit pseudo-natural transformation $\mathrm{der}_{|\mathbf{A}|} : |!\mathbf{A}| \nrightarrow |\mathbf{A}|$ is in $\mathbf{FinProf}(!\mathbf{A}, \mathbf{A})$.*

Proof. Since $!\,|\mathbf{A}|$ is locally finite, $\mathrm{der}_{|\mathbf{A}|}$ is a finite profunctor. By Lemma 15, it remains to show that $\mathrm{der}^\perp_{|\mathbf{A}|}((\mathscr{F}\mathbf{A})^\perp) \hookrightarrow (\mathscr{F}!\mathbf{A})^\perp$ i.e. that for all $X' \in (\mathscr{F}\mathbf{A})^\perp$ and $X \in \mathscr{F}\mathbf{A}$, $(\mathrm{der}^\perp_{|\mathbf{A}|})X' \perp X^!$.

$$\begin{aligned}&\langle(\mathrm{der}^\perp_{|\mathbf{A}|})X', X^!\rangle = \int^{u\in!|\mathbf{A}|} X^!(u) \times \int^{a\in|\mathbf{A}|} !\,|\mathbf{A}|\,(\langle a\rangle, u) \times X'(a)\\ &\cong \int^{a,a'\in|\mathbf{A}|} X(a') \times |\mathbf{A}|\,(a,a') \times X'(a) \cong \int^{a\in|\mathbf{A}|} X(a) \times X'(a) \in \mathbf{FinSet}\end{aligned}$$ ◀

▶ **Lemma 38.** *For a finiteness structure $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$, the component of the comultiplication pseudo-natural transformation $\mathrm{dig}_{|\mathbf{A}|} : |!\mathbf{A}| \nrightarrow |!!\mathbf{A}|$ is in $\mathbf{FinProf}(!\mathbf{A}, !!\mathbf{A})$.*

Proof. Since $!\,|\mathbf{A}|$ is locally finite, $\mathrm{dig}_{|\mathbf{A}|}$ is a finite profunctor. We show that $(\mathrm{dig}^\perp_{|\mathbf{A}|})(\mathscr{F}!!\mathbf{A})^\perp \hookrightarrow (\mathscr{F}!\mathbf{A})^\perp$.

For a presheaf X in $\mathscr{F}\mathbf{A}$ considered as a species $!\mathbf{0} \nrightarrow |\mathbf{A}|$, we have $\mathrm{dig}_{|\mathbf{A}|} \circ X^! = \mathrm{dig}_{|\mathbf{A}|} \circ !X \circ \mathrm{dig}_{\mathbf{0}} \cong !!X \circ \mathrm{dig}_{!\mathbf{0}} \circ \mathrm{dig}_{\mathbf{0}} \cong !!X \circ !\mathrm{dig}_{\mathbf{0}} \circ \mathrm{dig}_{\mathbf{0}} \cong X^{!!}$, the first isomorphism follows from the pseudo-naturality of dig and the last from the pseudo-comonad axioms. Hence, for W in $\mathscr{F}!!\mathbf{A}^{\perp}$ and X in $\mathscr{F}\mathbf{A}$, we have $\langle(\mathrm{dig}^{\perp}_{|\mathbf{A}|})W, X^!\rangle \cong \langle W, \mathrm{dig}_{|\mathbf{A}|}X^!\rangle \cong \langle W, X^{!!}\rangle$. Since $X^{!!}$ is in $\mathscr{F}!!\mathbf{A}$, we obtain the desired result. ◀

4.4 Cartesian closed structure

We show in this section that the cartesian closed structure of $\mathbf{Prof}_!$ exhibited by Fiore et al. [13] can be extended to $\mathbf{FinProf}$.

▶ **Definition 39.** *A cartesian bicategory $\mathscr{B}$ is* closed *if for every pair of objects $A, B \in \mathscr{B}$, we have:*

1. *an exponential object $A \Rightarrow B$ together with an evaluation map $\mathrm{Ev}_{A,B} \in \mathscr{B}((A \Rightarrow B)\&A, B)$ and*
2. *for every $X \in \mathscr{B}$, an adjoint equivalence pseudo-natural in A, B and X:*

$$\mathscr{B}(X, B^A) \underset{\Lambda}{\overset{\mathrm{Ev}_{A,B} \circ ((-) \,\&\, A)}{\rightleftarrows}} \mathscr{B}(X \,\&\, A, B) \quad (\perp)$$

For finiteness structures $\mathbf{A}$ and $\mathbf{B}$, the exponential object $\mathbf{A} \Rightarrow \mathbf{B}$ is given by $!\mathbf{A} \multimap \mathbf{B}$. We first show that the Seely adjoint equivalence in $\mathbf{Prof}$ lifts to $\mathbf{FinProf}$.

▶ **Lemma 40.** *For finiteness structures $\mathbf{A} = (|\mathbf{A}|, \mathscr{F}\mathbf{A})$ and $\mathbf{B} = (|\mathbf{B}|, \mathscr{F}\mathbf{B})$, the Seely profunctors $S_{|\mathbf{A}|,|\mathbf{B}|} : !(|\mathbf{A}| \,\&\, |\mathbf{B}|) \nrightarrow !\,|\mathbf{A}| \otimes !\,|\mathbf{B}|$ and $I_{|\mathbf{A}|,|\mathbf{B}|} : !\,|\mathbf{A}| \otimes !\,|\mathbf{B}| \nrightarrow !(|\mathbf{A}| \,\&\, |\mathbf{B}|)$ induce an adjoint equivalence $!(\mathbf{A} \,\&\, \mathbf{B}) \simeq !\mathbf{A} \otimes !\mathbf{B}$ in* $\mathbf{FinProf}$.

Proof.

- We first show that $S_{|\mathbf{A}|,|\mathbf{B}|} : !(|\mathbf{A}| \,\&\, |\mathbf{B}|) \nrightarrow !\,|\mathbf{A}| \otimes !\,|\mathbf{B}|$ given by $(w, (u, v)) \mapsto !\,|\mathbf{A}|\,(u, \pi_1 w) \times !\,|\mathbf{B}|\,(v, \pi_2 w)$ is in $\mathbf{FinProf}(!(\mathbf{A}\&\mathbf{B}), !\mathbf{A}\otimes!\mathbf{B})$ i.e. $(S^{\perp}_{|\mathbf{A}|,|\mathbf{B}|})\mathscr{F}(!\mathbf{A}\otimes!\mathbf{B})^{\perp} \hookrightarrow (\mathscr{F}!(\mathbf{A} \,\&\, \mathbf{B}))^{\perp}$.
  Let T be in $\mathscr{F}(!\mathbf{A} \otimes !\mathbf{B})^{\perp}$, we want to show that for all $W = (W_1, W_2) \in \mathscr{F}(\mathbf{A} \,\&\, \mathbf{B})$, $\langle S^{\perp}_{|\mathbf{A}|,|\mathbf{B}|}(T), W^!\rangle \in \mathbf{FinSet}$. The set $\langle S^{\perp}_{|\mathbf{A}|,|\mathbf{B}|}(T), W^!\rangle$ is isomorphic to:

$$\int^{w \in !(|\mathbf{A}|\&|\mathbf{B}|)} W^!(w) \times \int^{u \in !|\mathbf{A}|, v \in !|\mathbf{B}|} !\,|\mathbf{A}|\,(u, \pi_1 w) \times !\,|\mathbf{B}|\,(v, \pi_2 w) \times T(u, v)$$
$$\cong \int^{u \in !|\mathbf{A}|, v \in !|\mathbf{B}|} W_1^!(u) \times W_2^!(v) \times T(u, v)$$

  Since W is in $\mathscr{F}(\mathbf{A} \,\&\, \mathbf{B})$, W_1 and W_2 are in $\mathscr{F}(\mathbf{A})$ and $\mathscr{F}(\mathbf{B})$ respectively, so that $W_1^!$ and $W_2^!$ are in $\mathscr{F}(!\mathbf{A})$ and $\mathscr{F}(!\mathbf{B})$ respectively. Hence, $T \perp W_1^! \times W_2^!$ as desired.
- We show that $I_{|\mathbf{A}|,|\mathbf{B}|} : !\,|\mathbf{A}| \otimes !\,|\mathbf{B}| \nrightarrow !(|\mathbf{A}| \,\&\, |\mathbf{B}|)$ given by $((u, v), w) \mapsto !\,|\mathbf{A}|\,(\pi_1 w, u) \times !\,|\mathbf{B}|\,(\pi_2 w, v)$ is in $\mathscr{F}((!\mathbf{A}\otimes!\mathbf{B}) \multimap !(\mathbf{A}\&\mathbf{B}))$. By Lemma 29, $\mathscr{F}((!\mathbf{A}\otimes!\mathbf{B}) \multimap !(\mathbf{A}\&\mathbf{B})) \cong \mathscr{F}(!\mathbf{A} \multimap (!\mathbf{B} \multimap !(\mathbf{A} \,\&\, \mathbf{B}))$ and using Lemma 27 twice, it suffices to show that for all $X \in \mathscr{F}\mathbf{A}$ and $Y \in \mathscr{F}\mathbf{B}$, $(I_{|\mathbf{A}|,|\mathbf{B}|}X^!)Y^!$ is in $\mathscr{F}!(\mathbf{A} \,\&\, \mathbf{B})$. Let Z be $\mathscr{F}(!(\mathbf{A} \,\&\, \mathbf{B}))^{\perp}$, the set $\langle(I_{\mathbf{A},\mathbf{B}}X^!)Y^!, Z\rangle$ is isomorphic to:

$$\int^{w\in !(\mathbf{A}\&\mathbf{B}),u\in !\mathbf{A},v\in !\mathbf{B}} Z(w)\times !\mathbf{A}(\pi_1 w,u)\times !\mathbf{B}(\pi_2 w,v)\times X^!(u)\times Y^!(v)$$
$$\cong \int^{w\in !(\mathbf{A}\&\mathbf{B})} Z(w)\times (X,Y)^!(w)$$

Since $(X,Y)^!$ is in $\mathscr{F}(!(\mathbf{A}\,\&\,\mathbf{B}))$, we obtain the desired result. ◀

It remains to show that the non-linear evaluation and currying preserve the finiteness structure. The non-linear evaluation $\mathrm{Ev}_{|\mathbf{A}|,|\mathbf{B}|} : !((|\mathbf{A}|\Rightarrow|\mathbf{B}|)\,\&\,|\mathbf{A}|) \nrightarrow |\mathbf{B}|$ is given by the composite $\mathrm{ev}_{!|\mathbf{A}|,|\mathbf{B}|}\circ(\mathrm{der}_{|\mathbf{A}|\Rightarrow|\mathbf{B}|}\otimes\mathrm{id})\circ S_{|\mathbf{A}|\Rightarrow|\mathbf{B}|,|\mathbf{A}|}$ where $\mathrm{ev}_{|\mathbf{A}|,|\mathbf{B}|} : \mathbf{A}\otimes(\mathbf{A}\multimap\mathbf{B})\nrightarrow\mathbf{B}$ is the linear evaluation coming from the monoidal closed structure in the linear bicategory **FinProf**. As a composite of finiteness profunctors, $\mathrm{Ev}_{|\mathbf{A}|,|\mathbf{B}|}$ is in $\mathbf{FinProf}_!((\mathbf{A}\Rightarrow\mathbf{B})\,\&\,\mathbf{A},\mathbf{B})$. For a finiteness species P in $\mathbf{FinProf}_!(\mathbf{A}\,\&\,\mathbf{B},\mathbf{C})$, its *currying* $\Lambda(P)\in\mathbf{FinProf}_!(\mathbf{A},\mathbf{B}\Rightarrow\mathbf{C})$ is given by $\lambda(P\circ I_{|\mathbf{A}|,|\mathbf{B}|})$ where $\lambda : \mathbf{FinProf}(!\mathbf{A}\otimes!\mathbf{B},\mathbf{C})\to\mathbf{FinProf}(!\mathbf{A},!\mathbf{B}\multimap\mathbf{C})$ is provided by the monoidal closed structure on **FinProf**.

▶ **Theorem 41.** $\mathbf{FinProf}_!$ *is cartesian closed.*

Proof. Direct consequence of the remarks above and Lemma 18. ◀

4.5 Differential structure

The bicategory of generalized species $\mathbf{Prof}_!$ is a model of differential linear logic where differentiation on analytic functors generalises the standard differential operation on formal power series [13]. We show in this section that the differential structure extends to **FinProf**. It suffices to show that the codereliction, coweakening and cocontraction pseudo-natural transformations have components in **FinProf** and all the coherence axioms will be immediately verified.

▶ **Lemma 42.** *For a finiteness structure* $\mathbf{A}=(|\mathbf{A}|,\mathscr{F}\mathbf{A})$*, the component of codereliction pseudo-natural transformation* $\overline{\mathrm{der}}_{|\mathbf{A}|} : |\mathbf{A}|\nrightarrow\,!\,|\mathbf{A}|$ *given by* $(a,u)\mapsto\,!\,|\mathbf{A}|\,(u,\langle a\rangle)$ *is a finiteness profunctor* $\mathbf{A}\nrightarrow_{\mathsf{f}} !\mathbf{A}$.

Proof. Since $|\mathbf{A}|$ is locally finite, $\overline{\mathrm{der}}_{|\mathbf{A}|}$ is a finite profunctor. By Lemma 15, it remains to show that $\overline{\mathrm{der}}^{\perp}_{|\mathbf{A}|}((\mathscr{F}!\mathbf{A})^{\perp})\hookrightarrow(\mathscr{F}\mathbf{A})^{\perp}$ i.e. that for all $Z\in(\mathscr{F}!\mathbf{A})^{\perp}$ and $X\in\mathscr{F}\mathbf{A}$, $(\overline{\mathrm{der}}^{\perp}_{|\mathbf{A}|})Z\perp X$.

$$\langle(\overline{\mathrm{der}}^{\perp}_{|\mathbf{A}|})Z,X\rangle = \int^{u\in !|\mathbf{A}|,a\in|\mathbf{A}|} Z(u)\times\,!\,|\mathbf{A}|\,(u,\langle a\rangle)\times X(a)$$
$$\cong \int^{a\in|\mathbf{A}|} Z(\langle a\rangle)\times X(a)\hookrightarrow\int^{u\in !|\mathbf{A}|} Z(u)\times X^!(u)\in\mathbf{FinSet}$$

The last inclusion follows from the isomorphism $X^!(\langle a\rangle)\cong X(a)$. ◀

Since the components of the coweakening $\overline{\mathrm{w}}_{|\mathbf{A}|} : \mathbf{1}\nrightarrow\,!\,|\mathbf{A}|$ and cocontraction $\overline{\mathrm{c}}_{|\mathbf{A}|} : !\,|\mathbf{A}|\times\,!\,|\mathbf{A}|\nrightarrow\,!\,|\mathbf{A}|$ pseudo-natural transformations are obtained from the Seely equivalences and the biproduct structure, it is immediate that they can be extended to **FinProf**. It implies that the deriving pseudo-natural transformation $\delta_{|\mathbf{A}|} : |\mathbf{A}|\nrightarrow\,!\,|\mathbf{A}|\times|\mathbf{A}|$ given by

$$!\,|\mathbf{A}|\times|\mathbf{A}| \xrightarrow{\mathrm{id}\times\overline{\mathrm{der}}_{|\mathbf{A}|}} !\,|\mathbf{A}|\times\,!\,|\mathbf{A}| \xrightarrow{\overline{\mathrm{c}}_{|\mathbf{A}|}} !\,|\mathbf{A}|$$

is therefore a finiteness profunctor $!\mathbf{A} \otimes \mathbf{A} \nrightarrow_{\mathsf{f}} !\mathbf{A}$ so that for a finiteness species $F : !\mathbf{A} \nrightarrow \mathbf{B}$ its differential $F \circ \delta_{|\mathbf{A}|} : !\mathbf{A} \otimes \mathbf{A} \nrightarrow_{\mathsf{f}} \mathbf{B}$ given by $((u, a), b) \mapsto F(u \otimes \langle a \rangle, b)$ is also a finiteness species.

Conclusion and perspectives

We have constructed a new bicategorical model of differential linear logic categorifying the finiteness model first introduced by Ehrhard [9]. The resulting cartesian closed bicategory refines the model of generalized species by Fiore et al. [13]. The objects are endowed with an additional structure which enables to enforce finite computations as morphisms are species that preserve the finiteness structure.

In future work, we aim to prove that our construction can be generalized to the setting of enriched species studied by Gambino and Joyal [15]. In the 1-categorical model of finiteness spaces, we can express various forms of non-determinism depending on the semi-ring of scalars chosen for the series coefficients. In our case, the analogous variation would come from changing the enrichment basis. In particular, for species enriched over vector spaces, our construction will guarantee that computations are always finite dimensional even if we work in an infinite dimensional setting which could lead to interesting applications for the semantics of quantum λ-calculus [24] and stochastic rewriting systems [2].

In this paper, we have worked on a focused orthogonality on the subclass of finitely presented objects. Our construction opens the way for a lot of variation in terms of the chosen class of objects: for example, restricting the interactions to absolutely presentable objects could yield to a model of totality in the spirit of the one studied by Loader [23].

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