Secure Merge with $O(n \log \log n)$ Secure Operations

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Abstract

Data-oblivious algorithms are a key component of many secure computation protocols.

In this work, we show that advances in secure multiparty shuffling algorithms can be used to increase the efficiency of several key cryptographic tools.

The key observation is that many secure computation protocols rely heavily on secure shuffles. The best data-oblivious shuffling algorithms require $O(n \log n)$, operations, but in the two-party or multiparty setting, secure shuffling can be achieved with only $O(n)$ communication.

Leveraging the efficiency of secure multiparty shuffling, we give novel, information-theoretic algorithms that improve the efficiency of securely sorting sparse lists, secure stable compaction, and securely merging two sorted lists.

Securely sorting private lists is a key component of many larger secure computation protocols. The best data-oblivious sorting algorithms for sorting a list of n elements require $O(n \log n)$ comparisons. Using black-box access to a linear-communication secure shuffle, we give a secure algorithm for sorting a list of length n with $t \ll n$ nonzero elements with communication $O(t \log^2 n + n)$, which beats the best oblivious algorithms when the number of nonzero elements, t, satisfies $t < n/\log^2 n$.

Secure compaction is the problem of removing dummy elements from a list, and is essentially equivalent to sorting on 1-bit keys. The best oblivious compaction algorithms run in $O(n)$-time, but they are unstable, i.e., the order of the remaining elements is not preserved. Using black-box access to a linear-communication secure shuffle, we give an information-theoretic stable compaction algorithm with only $O(n)$ communication.

Our main result is a novel secure merge protocol. The best previous algorithms for securely merging two sorted lists into a sorted whole required $O(n \log n)$ secure operations. Using black-box access to an $O(n)$-communication secure shuffle, we give the first multi-party secure merge algorithm that requires only $O(n \log \log n)$ communication. Our algorithm takes as input n secret-shared values, and outputs a secret-sharing of the sorted list.

All our algorithms are generic, i.e., they can be implemented using generic secure computations techniques and make black-box access to a secure shuffle. Our techniques extend naturally to the multiparty situation (with a constant number of parties) as well as to handle malicious adversaries without changing the asymptotic efficiency.

These algorithm have applications to securely computing database joins and order statistics on private data as well as multiparty Oblivious RAM protocols.

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1 Introduction

Secure sorting protocols allow two (or more) participants to privately sort a list of n encrypted or secret-shared [41] values without revealing any data about the underlying values to any of the participants. Secure sorting is an important building block for many more complex secure multiparty computations (MPCs), including Private Set Intersection (PSI) [32], secure database joins [33, 10, 43], secure de-duplication and securely computing order statistics as well as Oblivious RAMs [38, 24].

Secure sorting algorithms, and secure computations in general, must have control flows that are input-independent, and most secure sorting algorithms are built by instantiating a data-oblivious sorting algorithm using a generic secure computation framework (e.g. garbled circuits [47, 48], GMW [23], BGW [13]). This method is particularly appealing because it is composable – the sorted list can be computed as secret shares, and used in a further (secure) computations.

Most existing secure sorting algorithms make use of *sorting networks*. Sorting networks are inherently data oblivious because the sequence of comparisons in a sorting network is fixed and thus independent of the input values. The AKS sorting network [2] requires $O(n \log n)$ comparators to sort n elements. The AKS network matches the lower bound on the number of comparisons needed for any (not necessarily data independent) comparison-based sorting algorithm. Unfortunately, the constants hidden by the big-O notation are extremely large, and the AKS sorting network is never efficient enough for practical applications [3]. In practice, some variant of Batcher's sort [9] is often used[1]. The MPC compilers Obliv-c [49], ABY [19] and EMP-toolkit [45] provide Batcher's bitonic sort. Batcher's sorting network requires $O(n \log^2 n)$ comparisons, but the hidden constant is approximately $1/2$, and the network itself is simple enough to be easily implementable.

Although Batcher's sorting network is fairly simple and widely used, the most efficient oblivious sorting algorithms make use of the *shuffle-then-sort* paradigm [31, 30] which builds on the observation that many traditional sorting algorithms (e.g. quicksort, mergesort, radixsort) can be made oblivious by obliviously shuffling the inputs before running the sorting algorithm. Since oblivious shuffling and (non-oblivious) sorting can be done in $O(n \log n)$-time these oblivious sorting algorithms run in $O(n \log n)$ (but unlike AKS the hidden constants are small).

Although the shuffle-then-sort paradigm is extremely powerful, improvements in shuffling (below $O(n \log n)$) are unlikely to improve these protocols because of the $O(n \log n)$ lower-bound on comparison-based sorting.

In the context of secure multiparty computation, however, sorting can often be reduced to the simpler problem of *merging* two sorted lists into a single sorted whole. Each participant in the computation, sorts their list locally, before beginning the computation, and the secure computation itself need only implement a data-oblivious merge.

Merging is an easier problem than sorting, and even in the insecure setting it is known that any comparison-based sorting method requires $O(n \log n)$ comparisons, whereas (non-oblivious) linear-time merging algorithms are straightforward. Unfortunately, no data-oblivious merge algorithms are known with complexity better than simply performing a data-oblivious sort, and the best merging networks require $O(n \log n)$ comparisons.

Our main result is a secure multiparty *merge* algorithm, for merging two (or more) sorted lists (into a single, sorted whole) that requires only $O(n \log \log n)$ secure operations. This is the first secure multiparty merge algorithm requiring fewer than $O(n \log n)$ secure operations.

[1] For example, hierarchical ORAM [38, 24] uses Batcher's sort.

The crucial building block of our algorithm is a linear-communication secure multiparty shuffle. Although no single-party, comparison-based shuffle exists using $O(n)$ comparisons, such shuffles exist in the two-party and multiparty setting (see Section 3), and this allows us to avoid the $O(n \log n)$ lower bound for comparison-based merging networks that exists in the single-party setting.

Our secure multiparty merge algorithm makes use of several novel data-oblivious algorithms whose efficiency can be improved through the use of a linear-communication secure multiparty shuffle.

These include

- **Securely sorting with large payloads:** In Section 4 we show how to securely sort t elements (with payloads of size w) using $O(t \log t + tw)$ communication. Previous sorting algorithms required $O(tw \log(tw))$ communication.
- **Securely sorting sparse lists:** In Section 5 we show how to securely sort a list of size n with only t nonzero elements in $O(t \log^2 n + n)$ communication. This beats naïvely sorting the entire list whenever $t < n/\log^2 n$.
- **Secure stable compaction:** In Section 6.1 we show how to securely *compact* a list (*i.e.*, extract nonzero elements) in linear time, while preserving the order of the extracted elements. Previous linear-time oblivious compaction algorithms (e.g. [6]) are *unstable* *i.e.*, they do not preserve the order of the extracted values.
- **Secure merge:** In Section 7 we give our main algorithm for securely merging two lists with $O(n \log \log n)$ communication complexity. Previous works all required $O(n \log n)$ complexity.

All the results above crucially rely on a linear-communication secure multiparty shuffle. Outside of the shuffle, all the algorithms are simple, deterministic and data-oblivious and thus can be implemented using any secure multiparty computation protocol.

In the two-party setting, we give a protocol for a linear-communication secure shuffle using any additively homomorphic public-key encryption algorithm with constant ciphertext expansion (Section 3.3). In the multiparty setting, a linear-communication secure shuffle can be built from any one-way function [34].

By making black-box use of a secure shuffle, our protocols can easily extend to different security models. If the shuffle is secure against malicious adversaries, then the entire protocol can achieve malicious security simply by instantiating the surrounding (data oblivious) algorithm with an MPC protocol that supports malicious security. One benefit of this is that our protocols can be made secure against malicious adversaries without changing the asymptotic communication complexity. Similarly, as two-party and multi-party linear-communication shuffles exist, all our algorithms can run in the two-party or multi-party settings simply by instantiating the surrounding protocol with a two-party or multi-party secure computation protocol (e.g. Garbled Circuits or GMW).

2 Preliminaries

2.1 Secure multiparty computation

Secure multiparty computation (MPC) protocols allow a group of participants to securely compute arbitrary functions of their joint inputs, without revealing their private inputs to each other or any external party. Secure computation has been widely studied in both theory and practice.

Different MPC protocols provide security in different settings, depending on parameters like the number of participants (e.g. two-party or multiparty), the amount of collusion (e.g. honest majority vs. dishonest majority), and whether the participants are semi-honest, covert [15, 7] or malicious.

In this work, we focus on creating data-oblivious algorithms that can be easily implemented using a variety of MPC protocols.

2.2 Oblivious algorithms

Secure and oblivious algorithms have been widely studied, it is instructive to differentiate between three types of data oblivious algorithms [37].

1. **Deterministically data independent:** In these algorithms, the control flow is deterministic and dependent only on public data. Most sorting networks are deterministically data independent.
2. **Data independent:** In these algorithms, the control flow is determined completely by the public data as well as additional (data-independent) randomness.
3. **Data oblivious:** In these algorithms, data can be "declassified" during the computation, and the control flow can depend on public data, as well as on previously declassified data. To ensure privacy, we require that the distribution of all declassified data (and the point at which it was declassified) is independent of the secret (input) data. The sorting algorithms of [31] are data oblivious, as are many ORAM constructions [38, 24].

All three of these types of algorithms can be easily implemented using generic MPC protocols.

2.3 Secure sorting

One common technique for secure sorting is to implement a *sorting network* under a generic MPC protocol. Since the sequence of comparisons in a sorting network is data-independent, if each comparison is done securely, the entire sorting procedure is secure.

In practice, many secure sorting algorithms are built on Batcher's sorting network [9]. Batcher sorting networks require $O(n \log^2 n)$ comparisons to sort n entries, and is straightforward to implement, and is provided by MPC compilers like EMP-toolkit [45] and Obliv-c [49]. In the two-party setting, when each individual's list is pre-sorted, then the final round of the Batcher sort can be omitted, and Batcher's Bitonic sort provides an efficient *merge* algorithm with $O(n \log n)$ complexity. The AKS sorting network [2] and its improvements [39, 40] are asymptotically better than a Batcher's, and requires only $O(n \log n)$ comparisons, but the hidden constants are enormous and the AKS network is not efficient for practical applications [3].

Zig-zag sort [27] is a deterministic data-independent sorting method, requiring $O(n \log n)$ comparisons, but the hidden constants are much smaller than those in AKS. Unfortunately, Zig-zag sort has a *depth* of $O(n \log n)$ (instead of $O\left(\log^2 n\right)$ for Batcher's sorting network), and this high depth makes it less appealing for some applications.

A randomized version of the Shellsort algorithm can be made data-oblivious, and gives an $O(n \log n)$ randomized algorithm that can be made either Monte Carlo or Las Vegas [25].

In a 2-party computation, when both parties hold their data in the clear, each party can locally sort his or her data, and then apply Batcher's bitonic sorting network to merge the two sorted lists. This results in an algorithm that runs in $O(n \log n)$ time (with small constants). This trick was used, for instance, in private set intersection [32]. Unfortunately, this trick does not apply when the two halves of the list cannot be pre-sorted, e.g. when the list is the (secret-shared) output of a prior computation.

Although sorting networks of size $O(n \log n)$ with a small hidden constant are unknown, secure sorting can be achieved in $O(n \log n)$ time (with a small constant) by combining secure shuffles and a generic sorting algorithm [31]. The core idea is that if the underlying data are randomly shuffled, then the sequence of comparisons in any sorting algorithm (e.g. mergesort, quicksort) are independent of the underlying data.

More concretely, to securely sort a list, data owners can first securely shuffle their lists, then apply an $O(n \log n)$ sorting algorithm (e.g. merge sort) to their shuffled list. Each comparison in the sorting algorithm will be computed under MPC, but the result of the comparison is then revealed, and the players can order the (secret) data based on the output of this public comparison. The Waksman permutation network [44] requires $O(n \log n)$ swaps, to implement a shuffle, so the entire shuffle-then-sort procedure only requires $O(n \log n)$ operations (with small constants). This idea has been implemented using the Sharemind platform [14] and to build efficient mix-nets [4]. These protocols are not data-independent (since the exact sequence of comparisons depends on the underlying data), but instead they are *data-oblivious* which is sufficient for security.

Building on this shuffle-then-sort paradigm, oblivious radix sort [30] requires $O(n \log n)$ communication, but only a constant number of rounds, and is efficient in both theory and practice. This was later improved (in the multiparty setting) [17] by incorporating the linear-time multiparty shuffle algorithm of [34] we review this shuffle in Section 3.2.

See [21] for a survey of data-oblivious sorting methods.

Sorting provides a method for computing all the *order statistics* of the joint list. If, however, only a single order statistic (e.g. the kth largest element) is needed, there are more efficient secure protocols that only require $O(\log n)$ secure comparisons to compute the kth order statistic [1]. The protocols of [1] reveal the order statistics *in the clear*, and it is not clear how to modify them to reveal only *secret shares* of the relevant order statistic, Thus they are not applicable in scenarios where computing order statistics is merely the first step in a larger secure computation. Another way of viewing this distinction is that the algorithms presented in [1] are not data-oblivious – the sequence of comparisons depends on the *output* – but since the output is revealed by the protocol the entire sequence of comparisons could be simulated by a simulator who only sees the protocol's output.

Merging two sorted lists is potentially easier than sorting, and when data-obliviousness is not needed merging can be done in linear-time using a single-scan over each list.

In the deterministic data independent setting, Batcher's merging networks are known to be optimal when one list is small [5]. In the (probabilistic) data independent setting, [35] gives a randomized variant of Batcher's odd-even mergesort using $O(n \log n)$ comparisons (with hidden constant less than one).

In the *three-party* setting, there is a linear-communication secure merge protocol [16], but no similar result is known in the two-party setting.

The main contribution of this work is to provide a new, multiparty secure merge algorithm that only requires $O(n \log \log n)$ secure operations (with small constants). Our construction avoids the lower bound of [35] by using an efficient secure shuffle (see Section 3) that is not comparison-based. Our construction immediately yields efficient, secure algorithms for sorting and obliviously computing order statistics in both the two-party and multiparty settings, and these constructions can easily be made secure against malicious adversaries using standard techniques.

3 Shuffling secret shares

3.1 $O(n \log(n))$-oblivious shuffles

Secure shuffles can be done in $O(n \log n)$-time using a Waksman permutation network [44, 12]. Waksman permutation networks are built using "controlled-swap-gates" which take two inputs and a "control bit" that determines whether to swap the two inputs. Although the Waksman network guarantees that every permutation can be realized through a choice of control bits, a *uniformly* random choice of control bits does not result in a uniformly random permutation [12]. On the other hand, given a permutation, the specific control bits required to realize this permutation can be calculated efficiently.

Waksman networks can be used to facilitate a secure m-party shuffle by simply having each player separately input their control bits and performing m (sequential) shuffles. The resulting shuffle will be random as long as one player was honest, and the entire cost of the protocol is $O(mn \log n)$. Alternatively, the control bits can be set *within* the MPC [42], but this requires $O\left(n^2\right)$ secure multiplications, and is thus *less* efficient than simply repeated executing a Waksman permutation with different control bits provided by each party when the number of players, m, is constant.

Asymptotically efficient oblivious shuffles can also be performed using more complex ORAM-based techniques [6, 20], but these are not nearly as efficient as Waksman shuffles in practice.

3.2 Multiparty secure shuffles

In this section, we review the linear-communication secure multiparty shuffle of [34]. A similar, multiparty secure shuffle was used for efficient multiparty ORAM [16]. The protocol is an *information-theoretic* protocol for executing a *pseudo-random* shuffle. An overview of the multiparty shuffle is given in Figure 1.

The group of participants, C, generates a permutation, $\sigma^{(C)}$. Since $\sigma^{(C)}$ is hidden from players outside C, and every coalition of size t is outside some subset, the final permutation (which is the composition of all the permutations $\sigma^{(C)}$) is hidden from all players [34, Section 4.3].

As noted in [34], simply sharing the (public) permutation among members of C requires $O(n \log n)$ communication. If, however, the players share a *pseudorandom* permutation, this communication cost is essentially eliminated and the total communication complexity becomes $O(n)$ as claimed. This can also be made secure against malicious adversaries, while retaining its $O(n)$ communication complexity [34, Section 4.4].

▶ **Lemma 1** (Multiparty secure shuffle [34]). *If there exists a Pseudorandom permutation (PRP) with λ-bit keys, then for any $m \geq 3$ and any $t < m-1$, then then there is an m-party secure shuffling protocol that remains secure against t corrupted players, that can shuffle vectors of length m, where each player's communication is*

$$\binom{m}{m-t} n(m-t) + \binom{m-1}{m-t-1} (nm + \lambda).$$

In particular, if the number of players, m, is constant, the total communication per player is linear in the database size, n.

Although the communication complexity of this re-sharing based protocol is linear in the database size, n, repeating the resharing procedure for every subset of size t makes the overall communication *exponential* in the threshold size, t. Thus if $t = \Theta(m)$, the communication

MultiPartyShuffle

Input: m parties hold secret shares of a vector $\vec{v}$ of length n. Let $t < m - 1$ be the desired corruption threshold.
Output: Secret shares of the shuffled vector $\vec{v}$.

1. For every subset, C, of $m - t$ players, the protocol does the following:
   a. The players re-share the secret shares of $\vec{v}$ to members of C.
   b. The members of C reconstruct shares of $\vec{v}$ from the shares of shares of $\vec{v}$ using the linearity of the secret sharing scheme.
   c. The members of C choose a (pseudorandom) permutation $\sigma^{(C)} : [n] \to [n]$, known to all members of C.
   d. The members of C shuffle the shares of $\vec{v}$ according to this public permutation.
   e. The members of C re-share the shares of $\sigma^{(C)}(\vec{v})$ to the entire group of players.
   f. The players reconstruct shares of $\sigma^{(C)}(\vec{v})$ from the shares of shares of $\sigma^{(C)}(\vec{v})$ using the linearity of the secret sharing scheme.

Figure 1 The secure m-party shuffle of [34]. This shuffle provides security against semi-honest adversaries when the corruption threshold is $t < m - 1$.

will be exponential in the number of players, m. Thus it only retains asymptotic efficiency for small (constant) m. From an asymptotic standpoint, this is not a restriction, because if m is super-constant, simply secret-sharing the input data among all the participants requires $\omega(n)$ communication per party, so we can't hope to get $O(n)$ communication whenever $m = \omega(1)$.

3.3 2-party secure shuffles

In this section, we give a simple two-party shuffle that relies on an additively homomorphic cryptosystem. Such a cryptosystem is not *information-theoretically* secure, and currently there is no known, linear-communication information-theoretic secure shuffle. We note, however, that all our protocols use only *black-box* access to the underlying shuffle, and thus if the underlying shuffle could be made information-theoretically secure, then the entire protocol would inherit this security.

If the cryptosystem has constant ciphertext expansion, then the resulting shuffle requires only $O(n)$ communication. This is essentially the two-party variant of the linear-comumunication multiparty shuffle [34] described in Section 3.2. A similar 2-party shuffle was described in [22].

Using a lattice-based scheme with ciphertext packing, this can be made extremely efficient in practice. To demonstrate the practical efficiency of this scheme, we implemented it using the PALISADE [18] FHE library, to show that it is dramatically more communication efficient than a simple Waksman shuffle (implemented in EMP [45]). We chose to implement our scheme using lattice-based FHE because ciphertext packing makes these schemes extremely efficient (in terms of ciphertext expansion, and the cost of additively homomorphic operations) when used to encrypt *blocks* of data. See Appendix C for details.

▶ **Lemma 2** (2-Party secure shuffle). *If* PKE *is an additively homomorphic, semantically secure cryptosystem with constant ciphertext expansion, then the shuffle* TwoPartyShuffle *outlined in Figure 2 is secure against passive adversaries, and requires $O(n)$ communication.*

TwoPartyShuffle

Input: Alice and Bob hold secret shares of vector $\vec{v}$. Let $\vec{v}^a$ and $\vec{v}^b$ denote the shares. Assume that the secret sharing is additive over a group G, *i.e.*, $\vec{v}^a[i] + \vec{v}^b[i] = \vec{v}[i]$ where arithmetic is done over G.

Output: Additive secret shares of the shuffled vector $\vec{v}$.

1. Alice generates a key for an additively homomorphic PKE. $pk^a \xleftarrow{\$} \mathsf{Gen}$. Suppose PKE is additively homomorphic over a group, G.
2. Bob generates a key for an additively homomorphic PKE. $pk^b \xleftarrow{\$} \mathsf{Gen}$.
3. For each block $i = 1, \ldots, n$, Alice encrypts her share vector $\vec{v}^a[i]$, setting $c^a[i] = \mathsf{Enc}(pk^a, \vec{v}^a[i])$. Alice sends these ciphertexts to Bob.
4. For each block $i = 1, \ldots, n$, Bob encrypts his share vector $\vec{v}^b[i]$. $c^b[i] = \mathsf{Enc}(pk^b, \vec{v}^b[i])$.
5. Bob locally shuffles the $2n$ ciphertexts, keeping both (encrypted) shares of each element together.
6. Bob re-randomizes the ciphertexts and the shares and sends them to Alice.
7. Alice locally shuffles the $2n$ ciphertexts, keeping both (encrypted) shares of each element together.
8. Alice re-randomizes the ciphertexts and the plaintext shares.
9. Alice sends Bob his encrypted ciphertexts.
10. Bob decrypts his shares.

Figure 2 A 2-party shuffle based on additively homomorphic encryption, secure against semi-honest adversaries.

The proof is straightforward, but for completeness we provide it in Appendix B.

Two-party shuffles of this type can be made secure against malicious adversaries, while retaining their asymptotic efficiency [29, 11].

The linear-communication multiparty secure shuffle in Section 3.2 has been used to create extremely efficient sorting algorithms in the multiparty setting [17]. Using our linear-communication secure 2-party shuffle, TwoPartyShuffle described in Figure 2, the shuffle-then-sort construction of [17] can be extended to the 2-party setting.

4 Securely sorting with large payloads

In this section, we give a simple, linear-communication algorithm for sorting keys with large payloads that makes black-box use of a linear-communication secure shuffle. In large-payload sorting, we have a collection of *blocks* data (payloads), and each block is tagged with a *key*. Each payload must be put into the position determined by its key, but the position of elements *within* each payload remains unchanged. Like all our constructions, this algorithm crucially relies on a black-box access to a linear-communication shuffle (Section 3).

Oblivious sorting algorithms [31] and sorting networks [2] can sort n elements using $O(n \log n)$ comparisons. Now, imagine that instead of n elements, we have n/w blocks, each of size w, and the n/w blocks need to be (obliviously) sorted based on n/w (short) keys. In the insecure setting, this requires $O(n/w \log(n/w))$ comparisons. In the secure setting, using an existing oblivious sorting algorithm, requires $O(n/w \log(n/w))$ secure comparisons.

LargePayloadSort

Input: A secret-shared vector $[\![\vec{x}]\!]$ of keys with $|\vec{x}| = n/w$. A secret-shared vector of payloads, $[\![\vec{y}]\!]$ with $\vec{y} \in (\{0,1\}^w)^{n/w}$ of payloads. The vector $\vec{y}$ is viewed as n/w payloads, each of size w.
Output: Sorted lists $[\![\vec{x}]\!]$ and $[\![\vec{y}]\!]$, sorted by $\vec{x}$.

1. For $i = 1, \ldots, n/w$, generate a random "tag," $[\![r_i]\!]$, with $r_i \in \{0,1\}^\lambda$ (r_i is the tag associated to x_i and block y_i).
2. Shuffle the list $(([\![x_1]\!], [\![y_1]\!], [\![r_1]\!]), \ldots, ([\![x_{n/w}]\!], [\![y_{n/w}]\!], [\![r_{n/w}]\!]))$.
$$(([\![\tilde{x}_1]\!], [\![\tilde{y}_1]\!], [\![\tilde{r}_1]\!]), \ldots, ([\![\tilde{x}_{n/w}]\!], [\![\tilde{y}_{n/w}]\!], [\![\tilde{r}_{n/w}]\!])) = \mathsf{SHUFFLE}\left((([\![x_1]\!], [\![y_1]\!], [\![r_1]\!]), \ldots, ([\![x_{n/w}]\!], [\![y_{n/w}]\!], [\![r_{n/w}]\!]))\right)$$
3. Sort the (shuffled) list $((\tilde{x}_1, \tilde{r}_1), \ldots, (\tilde{x}_{n/w}, \tilde{r}_{n/w}))$, based on their keys $\vec{x}$.
$$(([\![\bar{x}_1]\!], [\![\bar{r}_1]\!]), \ldots, ([\![\bar{x}_{n/w}]\!], [\![\bar{r}_{n/w}]\!])) = \mathsf{SORT}\left((([\![\tilde{x}_1]\!], [\![\tilde{r}_1]\!]), \ldots, ([\![\tilde{x}_{n/w}]\!], [\![\tilde{r}_{n/w}]\!]))\right)$$
4. Reveal the tags $(\tilde{r}_1, \ldots, \tilde{r}_{n/w})$, and $(\bar{r}_1, \ldots, \bar{r}_{n/w})$.
5. Move $((\tilde{r}_1, [\![\tilde{y}_1]\!]), \ldots, (\tilde{r}_{n/w}, [\![\tilde{y}_{n/w}]\!]))$ so that the $\tilde{r}_i$ are in the same order as $\bar{r}_i$. Let $([\![\bar{y}_1]\!], \ldots, [\![\bar{y}_{n/w}]\!])$ denote this ordered list.
6. Return $([\![\bar{x}_1]\!], \ldots, [\![\bar{x}_{n/w}]\!])$ and $([\![\bar{y}_1]\!], \ldots, [\![\bar{y}_{n/w}]\!])$.

Figure 3 Securely sorting keys with large payloads.

Unfortunately, obliviously swapping two blocks (based on the result of the secure comparison) requires $O(w)$ controlled swap gates. Thus the entire process requires $O(n \log(n/w))$ secure operations.

Note that since a secure comparison of λ-bit keys requires λ secure AND gates to implement as a circuit, whereas a controlled-swap gate only requires one, sorting n elements (based on λ-bit keys) requires $O(n\lambda \log n)$ secure AND gates, whereas sorting n/w blocks, requires $O\left(n\left(\frac{\lambda}{w}+1\right)\log(n/w)\right)$ secure AND gates, so sorting on blocks is actually somewhat faster (although still not linear).

Given a linear-communication secure shuffle, the problem of sorting with large payloads can be reduced to the problem of sorting with small payloads as follows. Each key and its corresponding payload ("block") are tagged with a random tag. Then the keys are sorted together with their (short) tags, and the (sorted) tags are revealed. The blocks are shuffled together with their tags, and the tags are revealed. Finally, the blocks are moved into the ordering given by the tags. The key obsevrvation is that shuffle ensures that this final data-movement is independent of the underlying data. The full algorithm is given by LargePayloadSort in Figure 3.

▶ **Lemma 3** (Securely sorting with large payloads). *The sorting algorithm, LargePayloadSort, outlined in Figure 3 can be instantiated using $O(n/w \log(n/w) + n)$ communication, and is (t, m)-secure against semi-honest adversaries if $m = 2$, or $t < m - 1$.*

Proof. First, note that the probability that r_i collides with another r_j is at most $\frac{n}{w2^\lambda}$, so a union bound shows that with probability at least $1 - \frac{n^2}{w^2 2^\lambda}$, all the r_i will be distinct. Note that if the r_i are *not* distinct, correctness may fail, but privacy will still be preserved.

If we choose $w = \omega(\log n)$, then $\frac{n^2}{w^2 2^\lambda}$ will be negligible, and for the rest of the argument, we assume we are in the case where all the r_i are distinct.

First, note that the vectors $\vec{x}$ and $\vec{y}$ can be tagged using a single linear pass (requiring $O(n\lambda/w)$ secure operations). Sorting the vector $\vec{x}$ requires $O\left(n/w \log\left(n/w\right)\right)$ operations, using a standard oblivious sorting algorithm (e.g. [31]). The shuffling algorithm requires $O(n)$ secure operations, and the final step of moving the data can be done in linear time, since it does not need to be done obliviously.

To see that this protocol is secure, note that each player's view consists of the $\{r_i\}$ associated with the sorted $\vec{x}$, and the $\{r_i\}$ associated with the shuffled $\vec{y}$. These distributions can be simulated as follows: the simulator chooses n/w r_i uniformly from $\{0,1\}^\lambda$. The simulator reveals $\{r_i\}$ as associated with $\vec{x}$, then the simulator shuffles the $\{r_i\}$ and reveals the shuffled set as associated with $\vec{y}$. Since the protocol chooses the $\{r_i\}$ uniformly, their distribution is unchanged after sorting them based on $\vec{x}$. Since the shuffle is secure, the $\{r_i\}$ associated with the shuffled $\vec{y}$ are simply a random permutation of the $\{r_i\}$ associated with $\vec{x}$. ◀

5 Sorting sparse lists

The algorithm LargePayloadSort provides a method for sorting *sparse* lists with linear communication. The idea is to divide the list into blocks. Then, with a single pass, we can count the number of nonzero elements in each block. Using LargePayloadSort, we can sort the blocks based on the number of nonzero elements. If the list is sparse enough (relative to the blocksize), we can be sure that only a small fraction of blocks have nonzero entries. These blocks will appear first (after sorting blocks based on the number of nonzero entries), thus it only remains to sort these "top" blocks (using on $O(n \log n)$-sorting algorithm). The complete algorithm is outlined in Figure 4.

▶ **Lemma 4.** *If $\vec{V}$ is a list of length n with t nonzero entries, then $\vec{V}$ can be securely sorted using $O\left(t \log^2(n) + n\right)$ secure operations, which is linear in n when $t < n/\log^2(n)$.*

Proof. The algorithm, SparseSort is provided in Figure 4.

First, we note that this algorithm is correct. Since $\vec{V}$ has at most t nonzero elements, at most t blocks of $\vec{B}$ contain nonzero elements. Thus after sorting $\vec{B}$ (Step 3) all the nonzero elements are in the top t blocks, and after sorting the top t blocks (Step 5) the entire list is sorted.

Next, we analyze the running efficiency. Step 2 requires a linear pass over the list, and requires $O(n)$ communication. Step 3 calls LargePayloadSort which requires $O(n/w \log(n/w) + n)$ communication to sort blocks of size w. Step 5 requires sorting a list of length tw which can be done in time $O(tw \log(tw))$. If $w = \log(n)$, then Step 3 requires $O(n)$ secure operations, and Step 5 requires $O(t \log^2(n)) = O(n)$ secure operations. ◀

6 Oblivious Compaction

In this section, we review the notion of oblivious compaction. The goal of compaction is to remove a set of marked element from a list. Given a secret shared list, where each element is tagged with secret share of 0 or 1, an oblivious compaction procedure removes all elements tagged with a 0, and returns the new (secret shared) list containing only those elements tagged with a 1.

SparseSort

Input: A secret-shared list, $[\![\vec{V}]\!]$, of length n. An upper bound, $t < n$, on the number of nonzero elements of $\vec{V}$.
Output: Secret shares of the sorted list $[\![\vec{V}]\!]$.

1. **Break $\vec{V}$ into blocks:** Set the blocksize, $w = O(\text{polylog}\, n)$. Let $[\![\vec{B}]\!]$ denote a secret-shared vector of length n/w, where

$$\vec{B}_i = [V_{iw+1}, \dots, V_{(i+1)w}]$$

is the ith block of $\vec{V}$.
2. **Count entries in blocks:** For $i = 1, \dots, n/w$
   - Set $[\![C_i]\!] = [\![0]\!]$
   - For $j = 1, \dots, w$
     - If $V_{(i-1)w+j} \neq 0$ then $[\![C_i]\!] = [\![C_i + 1]\!]$.
3. **Sort blocks:**

$$\left(([\![\bar{C}_1]\!], [\![\bar{B}_1]\!])\,,\ \dots, ([\![\bar{C}_{n/w}]\!], [\![\bar{B}_{n/w}]\!])\right) = \mathsf{LargePayloadSort}\left(([\![C_1]\!], [\![B_1]\!])\,, \dots, ([\![C_{n/w}]\!], [\![B_{n/w}]\!])\right)$$

4. **Merge blocks:** Merge the blocks $\bar{B}_i$ into a list of n individual elements, let W_i denote the ith element in this list, *i.e.*,

$$([\![W_1]\!], \dots, [\![W_n]\!]) = \left([\![\bar{B}_1]\!], \dots, [\![\bar{B}_{n/w}]\!]\right).$$

5. **Sort top blocks:**

$$\left([\![\bar{W}_1]\!], \dots, [\![\bar{W}_{tw}]\!]\right) = \mathsf{SORT}\left([\![W_1]\!], \dots, [\![W_{tw}]\!]\right)$$

6. **Return:** $\left([\![\bar{W}_1]\!], \dots, [\![\bar{W}_{tw}]\!], [\![W_{tw+1}]\!], \dots, [\![W_n]\!]\right)$

Figure 4 Securely sorting a sparse list of length n with t nonzero entries using $O(t\log^2(n) + n)$ communication.

The first oblivious compaction algorithm was probabilistic and ran in $O(n \log \log \lambda)$ time with failure probability that was negligible as a function of λ [35]. Follow-up works [37, 36] also gave probabilistic algorithms for solving the problem of oblivious compaction with running time $O(n \log \log n)$. The first deterministic, $O(n)$-time compaction algorithm appeared in [6].

The compaction algorithms of [35, 6, 20] use expander graphs, and while they are asymptotically efficient, the hidden constants in the big-O are large,[2] and the algorithms are likely to be inefficient for lists of reasonable size. The compaction algorithms of [37] and [36] are data independent and run in time $O(n \log \log n)$, (with reasonable constants) and thus are suitable for our purposes. In Appendix D, we review the algorithm of [37] and give a tight analysis of its error probability and running time.

When the list is sparse (*i.e.*, it has $O(n/\operatorname{polylog}(n))$ nonzero elements), the problem of compaction is much simpler, and in Appendix E we give a simple algorithm for compacting sparse lists.

A sorting algorithm is called *stable* if the order of elements with equal keys is retained. In general, 0-1 principle [8] for sorting networks tells us that any deterministic, data-independent stable compaction algorithm is in fact a sorting algorithm. Thus the lower bounds on the size of comparison-based sorting algorithms tell us that any deterministic, comparison-based compaction algorithm with $o(n \log n)$ complexity must be *unstable*.

In Section 6.1, we show that, given black-box access to a linear-communication shuffle, stable compaction with complexity $O(n)$ is achievable. This does not violate the sorting lower bounds since the underlying shuffle is a multiparty protocol.

6.1 Stable compaction

Using the a linear-communication secure shuffle (see Section 3), we give a simple, linear-time *stable* compaction algorithm. Our stable compaction algorithm takes three arguments, a public bound, t, a secret-shared vector of "tags," $\vec{s}$, and a secret shared vector of "payloads," $\vec{x}$.

$$[\![\vec{y}]\!] = \mathsf{StableCompaction}\left(t, [\![\vec{s}]\!], [\![\vec{x}]\!]\right).$$

▶ **Lemma 5** (Stable compaction). *Algorithm, StableCompaction, outlined in Figure 5 is stable, secure against passive adversaries, and requires $O(n)$ communication.*

Proof. It is straightforward to see that if the shuffle can be done with linear time and communication, the entire protocol can be done with linear time and communication.

To see that the protocol is secure, we construct a simulator that simulates the players' views. First, note that, essentially, the players' views consist of the revealed vector $\vec{y}$. Consider the following simulator, S. On inputs n, t, the simulator, S, generates a vector $\vec{z}$ such that $z_i = i$ for $i = 1, \ldots, n-t$, and $z_i = 0$ for $i > n-t$. Then S shuffles $\vec{z}$, and outputs the shuffled vector $\vec{y}$. It is straightforward to check that this has the same distribution as in the real protocol. ◀

[2] The smallest constant being $\sim 16,000$ in [20].

StableCompaction

Input: Secret shares of a vector $[\![\vec{s}]\!]$ of length n of "tags". Secret shares of a vector $[\![\vec{x}]\!]$ of length n of "payloads". An integer $t < n$, such that the number of nonzero elements in $\vec{s}$ is t. Note that in *compaction* the value t is publicly known.

Output: Secret shares of a list of length n consisting of the t nonzero elements in $\vec{x}$, followed by the $n - t$ "dummy" elements.

1. Using a linear pass, tag the real and dummy elements using two distinct counters, c and d, so that the tags of the dummy elements all appear before the real elements.
   - Initialize $[\![c]\!] = 1$.
   - Initialize $[\![d]\!] = t + 1$.
   - For $i = 1, \dots, n$,
     - If $[\![s_i]\!] = [\![\perp]\!]$, set $[\![y_i]\!] = [\![d]\!]$, and set $[\![d]\!] = [\![d + 1]\!]$.
     - If $[\![s_i]\!] \neq [\![\perp]\!]$, set $[\![y_i]\!] = [\![c]\!]$, and set $[\![c]\!] = [\![c + 1]\!]$.
2. Shuffle the vector $\vec{x}$ together with the tags $\vec{y}$, using a linear-communication secure shuffle (see Section 3).

$$(([\![\tilde{x}_1]\!], [\![\tilde{y}_1]\!]), \dots ([\![\tilde{x}_n]\!], [\![\tilde{y}_n]\!])) = \mathsf{SHUFFLE}\,(([\![x_1]\!], [\![y_1]\!]), \dots ([\![x_n]\!], [\![y_n]\!]))$$

3. Reveal the tags $\tilde{y}$.
4. For $i = 1, \dots, n$, move $[\![\tilde{x}_i]\!]$ to location, y_i, *i.e.*, set $[\![\bar{x}_{y_i}]\!] = [\![\tilde{x}_i]\!]$.
5. The first t elements of $[\![\bar{x}]\!]$ are the "true" values and the last $n - t$ entries are the "dummy" elements.

Figure 5 Stable compaction.

7 Securely merging private lists

7.1 Construction overview

In this section, we describe our novel data-oblivious merge algorithm. Our algorithm requires a linear-communication algorithm for shuffling secret shares (see Section 3), an oblivious sorting algorithm, SORT (e.g. [2, 27, 31, 30]) that requires $\mathsf{sort}(\cdot)$ secure operations, and an oblivious, stable compaction algorithm (see Section 6) The rest of the operations are standard operations (e.g. equality test, comparison) that can be easily implemented in any secure computation framework.

At a high-level, the merging algorithm proceeds as follows:

- The input is two (locally) sorted lists, which are then concatenated.
- The players divide the list into blocks of size $w = O(\text{polylog}(n))$. We call the first element of each block a "pivot" element. Then the players sort these blocks based on their pivots using LargePayloadSort. (For efficiency, this step requires the linear-communication shuffle).
- At this point, because the initial lists were sorted, most elements are "close" to their true location in the list. In fact, we can concretely bound the number of "strays" (*i.e.*, the number of elements that may be far from their true location).
- After extracting the strays, every wth element is declared to be a pivot for some parameter $w = O(\text{polylog}\, n)$.

- The players obliviously extract these strays, and match them to their "true" pivots. Since the number of strays and pivots is not too large, this can be done using a linear number of secure operations using the sparse sorting algorithm SparseSort (described in Figure 4).
- The players reinsert the strays next to their true pivot. To avoid revealing the number of strays associated with each pivot, the number of strays associated to each pivot must be padded with "dummy" elements.
- The players use a linear-communication *stable* compaction algorithm to remove the dummy elements that were inserted with the strays.
- The players sort using polylog(N)-sized sliding windows again. At this point, all the elements will be in sorted order, but there will be many dummy elements.

The details of this construction are described in Section 7.2.

7.2 Oblivious merge with $O(n \log \log n)$ secure operations

In this section, we provide the details of our oblivious-merge algorithm.

1. **Public parameters:** A length, $n \in \mathbb{Z}$. A blocksize $w \in \mathbb{Z}$, such that $w \mid n$ (we will set $w = O(\text{polylog}(n))$). A parameter δ, with $0 < \delta < 1$.
2. **Inputs:** Sorted, secret-shared lists $([\![a_1]\!], \ldots, [\![a_\ell]\!])$, and $([\![b_1]\!], \ldots, [\![b_{n-\ell}]\!])$. We let $\vec{v}$ denote this list,
$$[\![v_1]\!], \ldots, [\![v_n]\!] \stackrel{\text{def}}{=} [\![a_1]\!], \ldots, [\![a_\ell]\!], [\![b_1]\!], \ldots, [\![b_{n-\ell}]\!].$$
3. **Creating pivot tags:** For every pivot, assign a random identifier r from the set $1, \ldots, n/w$ as follows
$$\left([\![r_1]\!], \ldots, [\![r_{n/w}]\!]\right) = \mathsf{SHUFFLE}\left([\![1]\!], \ldots, [\![n/w]\!]\right).$$
At this point, the "identifier" or "tag" r_i remains hidden (secret-shared), and will be assigned to the ith pivot in the next step.
Secure Operations: $O\left(n/w\right)$
4. **Sorting based on pivots:** This step uses LargePayloadSort to sort blocks of size w based on their leading entry as follows. Define B_i to be the ith block of size w,
$$B_i \stackrel{\text{def}}{=} \left(v_{(i-1)w+1} \ldots, v_{i \cdot w}\right),$$
and define $p_i \stackrel{\text{def}}{=} v_{(i-1)w+1}$ for $i = 1, \ldots, n/w$ to be the leading element of each block.
$$\begin{aligned}&\left([\![\vec{p}]\!], \left(\left([\![\tilde{r}_1]\!], [\![\tilde{B}_1]\!]\right), \ldots, \left([\![\tilde{r}_{n/w}]\!], [\![\tilde{B}_{n/w}]\!]\right)\right)\right)\\ &= \mathsf{LargePayloadSort}\left([\![\vec{p}]\!], \left(\left([\![r_1]\!], [\![B_1]\!]\right), \ldots, \left([\![r_{n/w}]\!], [\![B_{n/w}]\!]\right)\right)\right).\end{aligned}$$
At this point, the blocks of the vector $\vec{v}$ are sorted according to the leading element in each block,
$$(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \tilde{B}_1 \cdots \tilde{B}_{n/w}.$$
Secure Operations: $O(n/w \log(n/w) + n)$
5. **Revealing pivot Tags:** For $i = 1, \ldots, n/w$, reveal $\tilde{r}_i$. Note that since each pivot, p_i, was assigned a random tag r_i (which remains hidden), revealing $\{\tilde{r}_i\}$, which are sorted based on the p_i reveals no information about the set of pivots, $\{p_i\}$.
Secure Operations: $O\left(n/w\right)$

6. **Tagging:** Using a linear pass, tag each element with its initial index, *i.e.*, the ith element in the list is tagged with a (secret-shared) value i. For $i = 1, \ldots, n$ set $[\![e_i]\!] = [\![i]\!]$. Note that since the tags are publicly known, this step can be done without communication.
   Secure Operations: $O(n)$
7. **Sorting sliding windows:** Fix a threshold, $\delta > 0$ (the exact value of δ is calculated in Lemma 11). Sort the list $[\![\vec{v}]\!]$ together with the tags $\vec{e}$, based on windows of size $4\delta^{-1}w$ as follows: For $i = 1, \ldots, n/(2\delta^{-1}w) - 1$,

$$\left(\left([\![v_{2(i-1)\delta^{-1}w+1}]\!], [\![e_{2(i-1)\delta^{-1}w+1}]\!]\right), \ldots, \left([\![v_{2(i+1)\delta^{-1}w}]\!], [\![e_{2(i+1)\delta^{-1}w}]\!]\right)\right)$$
$$= \mathsf{SORT}\left(\left([\![v_{2(i-1)\delta^{-1}w+1}]\!], [\![e_{2(i-1)\delta^{-1}w+1}]\!]\right), \ldots, \left([\![v_{2(i+1)\delta^{-1}w}]\!], [\![e_{2(i+1)\delta^{-1}w}]\!]\right)\right)$$

   Secure Operations: $O\left(\left(n/\left(2\delta^{-1}w\right)\right) \mathsf{sort}\left(4\delta^{-1}w\right)\right)$
8. **Identifying "strays"** For each element in $\vec{v}$, if its initial index (stored in its tag e) differs from its current position by more than $\delta^{-1}w$, then mark the element with a (secret-shared tag "stray").
   For $i = 1, \ldots, n$,

$$[\![s_i]\!] = \begin{cases} [\![1]\!] \text{ if } |e_i - i| > \delta^{-1}w \\ [\![0]\!] \text{ otherwise.} \end{cases}$$

   and

$$[\![v_i]\!] = \begin{cases} [\![\perp]\!] \text{ if } |e_i - i| > \delta^{-1}w \\ [\![v_i]\!] \text{ otherwise.} \end{cases}$$

   Secure Operations: $O(n)$
9. **Extracting strays** At this point, each stray is tagged with the (secret-shared) tag ($[\![s_i]\!] = [\![1]\!]$) and we can extract these strays using a compaction algorithm.

$$([\![z_1]\!], \ldots, [\![z_{\mathfrak{b}}]\!]) = \mathsf{StableCompaction}\left([\![\vec{s}]\!], [\![\vec{v}]\!]\right).$$

   For an appropriate choice of parameters, δ, w, Lemma 6 shows that the number of strays will be less than $\mathfrak{b}$.
   Secure Operations: $O(n)$
10. **Sorting pivots and strays:** Sort pivots together with strays, using SORT. There are n/w pivots, and the list of strays has $\mathfrak{b}$ elements, so this list has $\mathfrak{b} + n/w$ elements. Pivot i, $\tilde{p}_i$ is tagged with its tag, r (from Step 4), and each stray is tagged with 0.

$$\left(([\![z_1]\!], [\![\rho_1]\!]), \ \ldots, ([\![z_{\mathfrak{b}+n/w}]\!], [\![\rho_{\mathfrak{b}+n/w}]\!])\right)$$
$$= \mathsf{SORT}\left(\left(([\![z_1]\!], [\![0]\!]), \ldots, ([\![z_{\mathfrak{b}}]\!], [\![0]\!])\right) \,||\, \left(([\![\tilde{p}_1]\!], [\![\tilde{r}_1]\!]), \ldots, ([\![\tilde{p}_{n/w}]\!], [\![\tilde{r}_{n/w}]\!])\right)\right)$$

    Secure Operations: $\mathsf{sort}(\mathfrak{b} + n/w)$
11. **Adding pivot IDs to strays** After step 10, the players hold a (sorted) list, $[\![\vec{z}]\!]$, of pivots and strays, and a list of "tags" $[\![\vec{p}]\!]$, where pivot p_i is tagged with r_i and each stray is tagged with 0. Both lists are of length $\mathfrak{b} + n/w$. In this step, they will tag each stray in this list with its corresponding pivot ID as follows. Initialize $[\![c]\!] = [\![\tilde{\rho}_1]\!]$. For $i = 1, \ldots, \mathfrak{b} + n/w$.
    a. If $\rho_i \neq 0$ (*i.e.*, z_i is a pivot), then $[\![c]\!] = [\![\rho_i]\!]$, $[\![\rho_i]\!] = [\![0]\!]$.
    b. If $\rho_i = 0$ (*i.e.*, z_i is not a pivot) , then $[\![\rho_i]\!] = [\![c]\!]$.
    At the end of this process, each of the strays is tagged with a (secret-shared) ID of the nearest pivot above it. To make this step oblivious, the conditional can be implemented with a simple mux. **Secure Operations:** $O(\mathfrak{b} + n/w)$

12. **Counting number of strays associated to each pivot:** Initialize $[\![c]\!] = [\![0]\!]$. Define the share vector $[\![\vec{s}]\!]$ as follows. For $i = \mathfrak{b} + n/w, \ldots, 1$
    a. If $\rho_i \neq 0$ (*i.e.*, z_i is *not* a pivot), then set $[\![c]\!] = [\![c+1]\!]$, and $[\![s_i]\!] = [\![0]\!]$.
    b. If $\rho_i = 0$ (*i.e.*, z_i is a pivot) , then set $[\![s_i]\!] = [\![c]\!]$, $[\![c]\!] = [\![0]\!]$.

    At the end of this process, if z_i is a pivot, then s_i stores the number of strays associated with that pivot.
    Secure Operations: $O(n)$
13. **Removing pivots from stray list:** Using the sparse compaction algorithm SparseCompaction (described in Figure 8), extract a list of $\mathfrak{b}$ strays, together with their tags (recall the "tag" ρ_i gives the pivot ID r_j of the nearest pivot preceding the ith stray). Note that this compaction does not need to be stable.

$$(((\llbracket z_1\rrbracket, \llbracket \rho_1\rrbracket), \ldots, (\llbracket z_{\mathfrak{b}}\rrbracket, \llbracket \rho_{\mathfrak{b}}\rrbracket)) \\ = \mathsf{SparseCompaction}\left(\mathfrak{b}, \llbracket\vec{\rho}\rrbracket, ((\llbracket z_1\rrbracket, \llbracket \rho_1\rrbracket), \ldots, (\llbracket z_{\mathfrak{b}+n/w}\rrbracket, \llbracket \rho_{\mathfrak{b}+n/w}\rrbracket)\right)$$

    Secure Operations: $O\left(\mathfrak{b} \log^2(n)\right)$
14. **Extracting pivot counts:** After Step 12 the $\vec{s}$ is a vector of length $\mathfrak{b} + n/w$ containing the number of strays associated with each of the n/w pivots, and 0s in the locations corresponding to strays. Set

$$\llbracket\vec{s}\rrbracket = \mathsf{StableCompaction}\left(n/w, \llbracket\vec{s}\rrbracket, \llbracket\vec{s}\rrbracket\right).$$

    At this point, $\vec{s}$ is a vector of length n/w, and for $i = 1, \ldots, n/w$, s_i is the number of strays associated with pivot i.
    Secure Operations: $O\left(\mathfrak{b} + n/w\right)$
15. **Padding lists of strays:** Although the total number of strays, $\mathfrak{b}$, is known, revealing the number of strays associated with each pivot would leak information. Thus the number of strays associated with each pivot must be padded to a uniform size. Note that every wth element in the *sorted* inputs $\llbracket\vec{a}\rrbracket$ and $\llbracket\vec{b}\rrbracket$ was defined to be a pivot, thus if the list were completely sorted, there could be at most $2(w-1)$ elements between any two adjacent pivots.
    a. For $i = 1, \ldots, n/w$, for $j = 1, \ldots, w$,

$$B_{(i-1)\cdot w+j} = \begin{cases} (1, (\perp, r_i)) & \text{if } j \leq \llbracket s_i\rrbracket \\ (0, (\perp, r_i)) & \text{otherwise.} \end{cases}$$

    The elements tagged with 1 are the "dummy" elements. Note that among all the B_i, there are at most $\mathfrak{b}$ elements tagged with a 0. The elements tagged with a 0 will be removed in the next step.
    b. Using the algorithm SparseSort (described in Figure 4), sort the B_i.

$$\left(\left(\llbracket\tilde{B}_{1,1}\rrbracket, \left(\llbracket\tilde{B}_{1,2}\rrbracket, \llbracket\tilde{B}_{1,3}\rrbracket\right)\right), \ldots, \left(\llbracket\tilde{B}_{2n(w-1)/w,1}\rrbracket, \left(\llbracket\tilde{B}_{2n(w-1)/w,2}\rrbracket, \llbracket\tilde{B}_{2n(w-1)/w,3}\rrbracket\right)\right)\right) \\ = \mathsf{SparseSort}\left((\llbracket B_{1,1}\rrbracket, (\llbracket B_{1,2}\rrbracket, \llbracket B_{1,3}\rrbracket)), \ldots, \left(\llbracket B_{2n(w-1)/w,1}\rrbracket, \left(\llbracket B_{2n(w-1)/w,2}\rrbracket, \llbracket B_{2n(w-1)/w,3}\rrbracket\right)\right)\right)$$

    c. We remove the first components, $\tilde{B}_{i,1}$, and set

$$\left((\llbracket C_{1,1}\rrbracket, \llbracket C_{1,2}\rrbracket), \ldots, \left(\llbracket C_{2(w-1)n/w-\mathfrak{b},1}\rrbracket, \llbracket C_{2(w-1)n/w-\mathfrak{b},2}\rrbracket\right)\right) \\ = \left(\left(\llbracket\tilde{B}_{1,2}\rrbracket, \llbracket\tilde{B}_{1,3}\rrbracket\right), \ldots, \left(\llbracket\tilde{B}_{2n(w-1)/w-\mathfrak{b},2}\rrbracket, \llbracket\tilde{B}_{2n(w-1)/w-\mathfrak{b},3}\rrbracket\right)\right)$$

    Secure Operations: $O\left(n\right)$

16. **Merging strays and pads** Concatenate the list of $\mathfrak{b}$ strays, $(\llbracket\vec{z}\rrbracket, \llbracket\vec{\rho}\rrbracket)$ (from Step 13) along with the $2(w-1)n/w - \mathfrak{b}$ pads $\llbracket\vec{C}\rrbracket$ from the previous step. Shuffle this list, keeping the associated tags, then, reveal the tags and move strays and pads to the positions given by their tags. This is accomplished as follows.
    a.
    $$\left(\left(\llbracket\tilde{C}_{1,1}\rrbracket, \llbracket\tilde{C}_{1,2}\rrbracket\right), \ldots, \left(\llbracket\tilde{C}_{(2w-1)n/w,1}\rrbracket, \llbracket\tilde{C}_{(2w-1)n/w,2}\rrbracket\right)\right) = \mathsf{SHUFFLE}\left((\llbracket\vec{z}\rrbracket, \llbracket\vec{\rho}\rrbracket) \,||\, \llbracket\vec{C}\rrbracket\right)$$
    b. For each element in this shuffled list, reveal the associated tag, $\tilde{C}_{i,2}$. Note that by Step 15 exactly $2(w-1)$ (secret-shared) elements will have each tag.
    c. For each i, move the block $\tilde{C}_{i,1}$ of size $2(w-1)$ to the location where $\tilde{C}_{i,2} = \tilde{r}_j$ (revealed in Step 5). At the end of Step 8, $\llbracket v_1\rrbracket, \ldots, \llbracket v_n\rrbracket$ was the list of elements with the strays set to $\perp$. To accomplish this, define the function $f(\tilde{r}_j) \stackrel{\text{def}}{=} j$ for $j = 1, \ldots, n/w$, for the public $\tilde{r}_j$ (revealed in Step 5).

```
for i = 0, ..., n/w − 1 do
    Define d_i = 1.
    for j = 1, ..., w do
        set [[ṽ_{i(3w−1)+j}]] = [[v_{iw+j}]].
    end for
end for
for i = 1, ..., (2w − 1) do
    Let j = f(C̃_{i,2}).
    Set [[ṽ_{(j−1)(3w−1)+w+d_j}]] = [[C̃_{i,1}]].
    Set d_j = d_j + 1.
end for
```

    Secure Operations: $O(n)$
17. **Compacting:** Now, we need to remove the $2(w-1)n/w$ dummy elements. We cannot use an off-the-shelf compaction algorithm [37, 6, 36] because these algorithms are not *stable*, Instead, we use the stable compaction algorithm $\mathsf{StableCompaction}$ (described in Figure 5).
    For $i = 1, \ldots, (3w-1)n/w$, if $\llbracket\tilde{v}_i\rrbracket = \llbracket\perp\rrbracket$, then set $\llbracket z_i\rrbracket = \llbracket 0\rrbracket$, otherwise set $\llbracket z_i\rrbracket = \llbracket 1\rrbracket$
    $$\llbracket\vec{v}\rrbracket = \mathsf{StableCompaction}\left(n, \llbracket\vec{z}\rrbracket, \llbracket\vec{\tilde{v}}\rrbracket\right).$$
    Secure Operations: $O(n)$
18. **Sorting sliding windows** At this point, the players have a (secret-shared) list, $\llbracket\vec{\tilde{v}}\rrbracket$, consisting of n elements, and all elements are in approximately their correct positions. In this step, sort overlapping blocks of size $4\left(\left(\delta^{-1}+4\right)w+2\right)$ using a secure sorting algorithm SORT.
    For $i = 1, \ldots, \left\lceil \frac{n}{2(\delta^{-1}+4)w+2} \right\rceil$, set
    $$\left(\llbracket\tilde{v}_{(i-1)2\left(\left(\delta^{-1}+4\right)w+2\right)+1}\rrbracket, \ \ldots, \llbracket\tilde{v}_{(i+1)2\left(\left(\delta^{-1}+4\right)w+2\right)+1}\rrbracket\right) = \mathsf{SORT}\left(\llbracket\tilde{v}_{(i-1)2\left(\left(\delta^{-1}+4\right)w+2\right)+1}\rrbracket, \ldots, \llbracket\tilde{v}_{(i+1)2\left(\left(\delta^{-1}+4\right)w+2\right)+1}\rrbracket\right)$$
    At this point all the elements will be sorted.
    Secure Operations: $O\left(\left\lceil \frac{n}{2(\delta^{-1}+4)w+2} \right\rceil \mathsf{sort}\left(4\left(\left(\delta^{-1}+4\right)w+2\right)\right)\right)$
19. **Return:** The sorted, secret shared list, $\llbracket\vec{\tilde{v}}\rrbracket$.

See Appendix A for a concrete calculation of the communication cost. See Appendix B for a proof of security.

8 Correctness

8.1 Bounding the number of strays

In order to analyze the running time of our algorithm, we need to bound the number of "strays" that appear in Step 8.

▶ **Lemma 6** (Bounding the number of strays). *Suppose a list, L, is created as follows*

1. *L is composed of two sorted sublist $L = \vec{a}||\vec{b}$ with $|\vec{a}| = \ell$, and $|\vec{b}| = n - \ell$. We assume $w \mid \ell$ and $w \mid n - \ell$.*
2. *Break the sorted list $\vec{a}$ into blocks of size w. Call the first element (*i.e.*, the smallest element) in each block a "pivot."*
3. *Break the sorted list $\vec{b}$ into blocks of size w. Call the first element (*i.e.*, the smallest element) in each block a "pivot."*
4. *Alice and Bob sort their joint list of blocks based on their pivots.*

We call an element a "stray" if it is more than tw positions above *its "true" position (*i.e.*, its position in the fully sorted list of Alice and Bob's entries). Then there at most $\frac{n}{t}$ strays.*

Proof. Call the elements with indices $[iw + 1, \ldots, (i + 1)w]$ in L a "block." Let B_i denote the ith block for $i = 1, \ldots, n/w$. Notice that

1. The elements within each block are sorted *i.e.*, $L[iw+j] \leq L[iw+k]$ for each $0 \leq j \leq k \leq w$ and all i.
2. The lead elements in each block are sorted *i.e.*, $L[iw] \leq L[jw]$ for $i \leq j$.
3. Each element is less than or equal to all pivots above it *i.e.*, $L[iw + j] \leq L[kw]$ for all $j < w$, $k > i$.
4. All entries provided by a single party are in sorted order.

With these facts, notice that the only way an element's index in L can be greater than its true position is if it was in a block where the preceding block was provided by the other party. Similarly, for an element to be more than tw from its true position, it must be in a block preceded by t consecutive blocks provided by the other party. If we label blocks provided by Alice with an a, and blocks provided by Bob with a b, then in order for w elements to be more than tw out position, we need a sequence of $\underbrace{a, \cdots, a}_{t}, b$ or $\underbrace{b, \ldots, b}_{t}, a$. There can only be $\frac{n}{tw}$ such sequences, so at most $\frac{n}{t}$ elements can be strays. ◀

Note that the sequence of operations described in Lemma 6 exactly corresponds to the process in the merging algorithm. In Step 2, the initial list is created as the concatenation of $\vec{a}$ and $\vec{b}$. In Step 3, every wth element is tagged as a pivot, and in Step 4, the blocks are sorted based on their pivots. Lemma 6 gives a bound on the number of elements that can be more than tw positions away from their "true" location at the end of this process. In Step 7 (Sorting sliding windows), every element that is more than tw from its true location will move at least tw positions, and thus will be tagged as a "stray" in Step 8. Conversely, every element that moves more than tw positions in Step 7 must have been at least tw positions from its true location, and thus the set of "strays" found in Step 8 will exactly correspond to the set of elements that were tw positions from their true location, and this number is exactly what is bounded in Lemma 6.

▶ **Theorem 7** (Correctness of the merge). *If the input lists $[\![\vec{a}]\!]$ and $[\![\vec{b}]\!]$ in Step 2 are locally sorted, then the output list $[\![\vec{v}]\!]$ in Step 19 is globally sorted.*

Proof. ▬ At the end of Step 2, the two parts of the list $\vec{v}$, $(v_1, \ldots, v_\ell)$ and $(v_{\ell+1}, \ldots, v_n)$ are locally sorted.

- At the end of Step 4 blocks of size w are sorted according to their leading (smallest) elements. Note that if these blocks were non-overlapping (*i.e.*, $v_{iw} < v_{iw+1}$ for $i = 1, \ldots, n/w - 1$), the entire list would already be sorted at this point. In general, however, there may be considerable overlap in the blocks provided from the $\vec{a}$ and those from $\vec{b}$.
- At the end of Step 8, Lemma 10 tells us that all elements that are more than $\delta^{-1}w$ from their true (final) location will be tagged as "stray."
- Corollary 9 shows that after Step 8, no pivot will be tagged as "stray," so no strays will be extracted in Step 9, and thus concatenating the lists of pivots and strays in Step 10 will not introduce any duplications.
- At the beginning of Step 18, Lemma 10 shows that every non-stray will be within $\delta^{-1}w$ of its true location. Lemma 8 shows that at the end of Step 4, every pivot is within w of its true location. By Step 16, every stray is within $3w + 2$ of its true pivot (based on the pivot's location after Step 4. Thus at the beginning of 18, every stray is within $4w + 2$ of its true location. Putting this together, every element is within $(\delta^{-1} + 4)w + 2$ of its true location. Since the sorting windows are chosen so that every element is sorted along with all elements within a distance of $(\delta^{-1} + 4)w + 2$ on either side, at the end of Step 18 all the elements are sorted. ◀

▶ **Lemma 8.** *Let $v_{(i-1)w+1}$ denote the ith pivot at the end of Step 4. The true index, j^*, of $v_{(i-1)w+1}$ (in the completely sorted list) satisfies*

$$(i-2)w < j^* < (i-1)w + 1$$

Proof. At the end of Step 4 the pivots are all in sorted order relative to one another, and all the blocks between the pivots are locally sorted.

First, notice that if $(i-1)w + 1 < j < w$, the $v_j \geq v_{(i-1)w+1}$, since the ith block is locally sorted. Next, notice that if $(i-1)w + 1 \leq j$, then

$$v_{(i-1)w+1} \leq v_{(\lceil \frac{j}{w} \rceil - 1)w+1} \leq v_j \tag{1}$$

where the first inequality holds because the pivots are sorted, and the second inequality holds because v_j is in the $\lceil \frac{j}{w} \rceil$th block which is locally sorted. Thus the true index j^* of $v_{(i-1)w+1}$ satisfies $j^* \leq (i-1)w + 1$.

To see the other side of Equation 1, recall that the list $\vec{v}$ was composed of blocks from two sources $\vec{a}$, and $\vec{b}$ which were locally sorted. Without loss of generality, assume block i came from source $\vec{a}$. Now, consider the i'th block for $i' < i$. If the i'th block came from the same source as the ith block ($\vec{a}$), then since the original lists $\vec{a}$ was sorted, all elements of the i'th block are less than or equal to $v_{(i-1)w+1}$. If the i'th block came from the *other source*, $\vec{b}$, then the elements $v_{(i'-1)w+2}, \ldots, v_{i' \cdot w}$ could be out of order relative to $v_{(i-1)w+1}$. On the other hand, if there exists an i'' with $i' < i'' < i$, with i'' also from the source $\vec{b}$, then since $\vec{b}$ was locally sorted, all elements of the i'th block are less than or equal to those of the i''th block, in particular, they are less than or equal to the i''th pivot which is less than or equal to the ith pivot $v_{(i-1)w+1}$. Thus only *one* block from $\vec{b}$ can be out of order relative to $v_{(i-1)w+1}$. Thus at most $w - 1$ elements v_j with $j < (i-1)w + 1$ can satisfy $v_j > v_{(i-1)w+1}$. ◀

▶ **Corollary 9.** *In Step 8, no pivot will be tagged as a "stray."*

Proof. In Step 8, an element will be tagged as a stray if it is more than $\delta^{-1}w$ from its true location. Lemma 8 shows that a pivot is at most w from its true location, and thus can move at most w positions when we sort on sliding windows in Step 7. ◀

▶ **Lemma 10.** *After Step 8 every element that was more than $\delta^{-1}w$ from its true location before Step 7 will be tagged as a "stray."*

Proof. To show this, it suffices to show that at the beginning of Step 7, if an element is more than $\delta^{-1}w$ from its true location (in the globally sorted list) then it will move at least $\delta^{-1}w$ during the sorting procedure of Step 7.

First, note that (as in the proof of Lemma 6) the only way an element can be more than $\delta^{-1}w$ from its true position is if $\lfloor \delta^{-1} \rfloor$ consecutive, adjacent blocks were provided by the other party. By the choice of sliding windows, every element will be sorted within a window containing at least $\delta^{-1}w$ elements on either side of it. Thus any element that is directly preceded or followed by $\delta^{-1}w$ "out-of-order" elements will move at least $\delta^{-1}w$ and thus be tagged as a stray. ◀

9 Extensions

Malicious adversaries: Our secure-merge algorithm outlined in Section 7 is "MPC-friendly," and aside from the $O(n)$-communication shuffle (discussed in Section 3), the entire algorithm can be naturally represented as an $O(n \log \log n)$-sized circuit. For this reason, extending our merge protocol to provide malicious security requires (1) a linear-communication shuffle and (2) a generic MPC protocol that both provide security against malicious adversaries.

The multiparty shuffle of [34] can be modified to provide security against malicious adversaries, and several generic MPC protocols (e.g. [46, 28]) provide security against malicious adversaries. In the two-party setting, the literature on efficient, verifiable shuffles (e.g. [29, 11]) provide methods for making homomorphic encryption-based shuffles (like that of Section 3.3) secure against malicious adversaries without affecting its asymptotic communication complexity.

Merging more than two lists. Our protocol can also be modified in a straightforward manner to support more than two parties, by merging multiple lists recursively.

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Appendix

A Protocol analysis

A.1 Efficiency analysis

In this section, we examine the efficiency of our construction.

Our algorithm requires a data-oblivious sorting routine, SORT. Sorting networks like Batcher's and the AKS network are deterministic data-independent sorting algorithms. Batcher's sorting network requires $O(n \log^2(n))$ comparisons to securely sort n elements, and the AKS sorting network [2] uses only $O(n \log(n))$ comparisons, the hidden constants are so large that it only begins to beat Batcher's sort for $n > 10^{52}$ [3] and is hence impractical.

The work of [31, 30] provide a data-oblivious sorting routines that requires $O(n \log n)$ comparisons (with small constants) by combining a permutation network, and (public) sorting algorithm. To the best of our knowledge, this is the fastest data-oblivious sort in practice, and has optimal asymptotic guarantees.

Throughout the rest of the analysis, we assume that the subroutine SORT is a data-oblivious sorting algorithm that requires $\mathsf{sort}(n) = O(n \log(n))$ secure operations.

▶ **Lemma 11.** *The merging algorithm in described in Section 7.2 requires $O(n \log\log(n))$ secure operations.*

Proof. Lemma 6 tells us that the maximum number of strays, $\mathfrak{b}$, is δn. In order for Step 15 to run in linear time, we set $\delta = O\left(\log^{-2}(n)\right)$. Note, however, that if we use the asymptotically efficient linear-time compaction algorithm from [6], we can choose a larger value for δ, (*i.e.*, $\delta = O(1)$). With this choice of δ, the runtime is dominated by Steps 10 and 18. Step 10 takes time $\mathsf{sort}(\mathfrak{b} + n/w)$. Setting $t = \delta^{-1}$, Lemma 6 gives $\mathfrak{b} = \delta n$, so $\mathfrak{b} + n/w = O(n/w)$. Since $\mathsf{sort}(n) = O(n \log(n))$ operations, setting $w = O\left(\log^2(n)\right)$, step 10 takes $O(n)$ secure operations. Step 18 takes $O\left(\left\lceil \frac{n}{2(\delta^{-1}+4)w+2} \right\rceil \mathsf{sort}\left(4\left(\left(\delta^{-1}+4\right)w+2\right)\right)\right)$. With our choices of $\delta = O\left(\log^{-2}(n)\right)$, and $w = O\left(\log^2(n)\right)$, $\left\lceil \frac{n}{2(\delta^{-1}+4)w+2} \right\rceil = O\left(nlog^{-4}(n)\right)$, and $4\left(\left(\delta^{-1}+4\right)w+2\right) = O\left(\log^4(n)\right)$. Since $\mathsf{sort}(n) = O(n \log(n))$ operations, Step 18 takes time $O(n \log\log(n))$. ◀

B Obliviousness

In this section, we show that the algorithm given in Section 7 is *data oblivious.*

▶ **Lemma 12** (Obliviousness)**.** *The merge algorithm given in Section 7 is data oblivious.*

Proof. Showing data-obliviousness requires showing

1. All values that affect the control flow are independent of the inputs
2. The value, and time of revelation of all revealed values are independent of the inputs

It is straightforward to check that all revealed values are uniformly and independently chosen, and that the time of their revelation is deterministic (and hence input-independent).

Data are revealed at Steps 4 and 16. At Step 4, the pivot-IDs that are revealed are uniformly random and independent of the input. The pivot locations are deterministic (every wth element).

At Step 16, the same number of pivot IDs of each type are revealed $(2w - 1)$ because of the padding, and their locations are data-independent because of the secure shuffle.

Thus the entire algorithm is data-oblivious as long as the secure shuffle is data oblivious. ◀

With the exception of an asymptotically efficient secure shuffling algorithm, all the steps of our sorting algorithm can be implemented with generic secure computation techniques, and hence can easily be made secure against malicious parties or extended to the multiparty setting.

Note that three steps require the asymptotically efficient secure shuffle. These are Steps 4 ("Sorting based on pivots"), 16 ("Merging strays and pads") and 17 ("Compacting").

In Section 3 we give standard algorithms for instantiating a two-party data-oblivious shuffle (using additively homomorphic encryption) and a multi-party data-oblivious shuffle (using one-way functions).

C Shuffling times

Our secure sorting algorithm requires an efficient method for secure *shuffling* with payloads. We give a simple, linear-time algorithm for this in Figure 2 based on additively homomorphic encryption. To demonstrate the *practical* performance of this shuffle, we implemented and benchmarked it using the PALISADE FHE library [18].

We used PALISADE version 1.6, with the "BFVrns" cryptosystem with a security level set to "HEstd_128_classic." This scheme uses the plaintext modulus 536903681, which can encode plaintexts of length 29 bits. In this scheme, each ciphertext can be "packed" with 16384 plaintexts, so each ciphertext holds $29 \cdot 16384 = 475136$ plaintext bits. Each ciphertext required 1053480 bytes to store, so the ciphertext expansion with these parameters is approximately 17.7.

We benchmarked the running time and communication cost of this scheme, and the results are presented in Figure 6.

For comparison, we also implement the Waksman permutation network [44] using the semi-honest 2pc provided by EMP [45]. The Waksman permutation network has complexity $O(n \log n)$, where n is the number of *bits* being shuffled, rather than the number of *blocks*. Because a uniform setting of control bits in the Waksman network does not yield a uniform permutation, in practice, the Waksman network would usually be run twice (where each player inputs control bits for one of the shuffles). These benchmarks only show a single run of Waksman network.

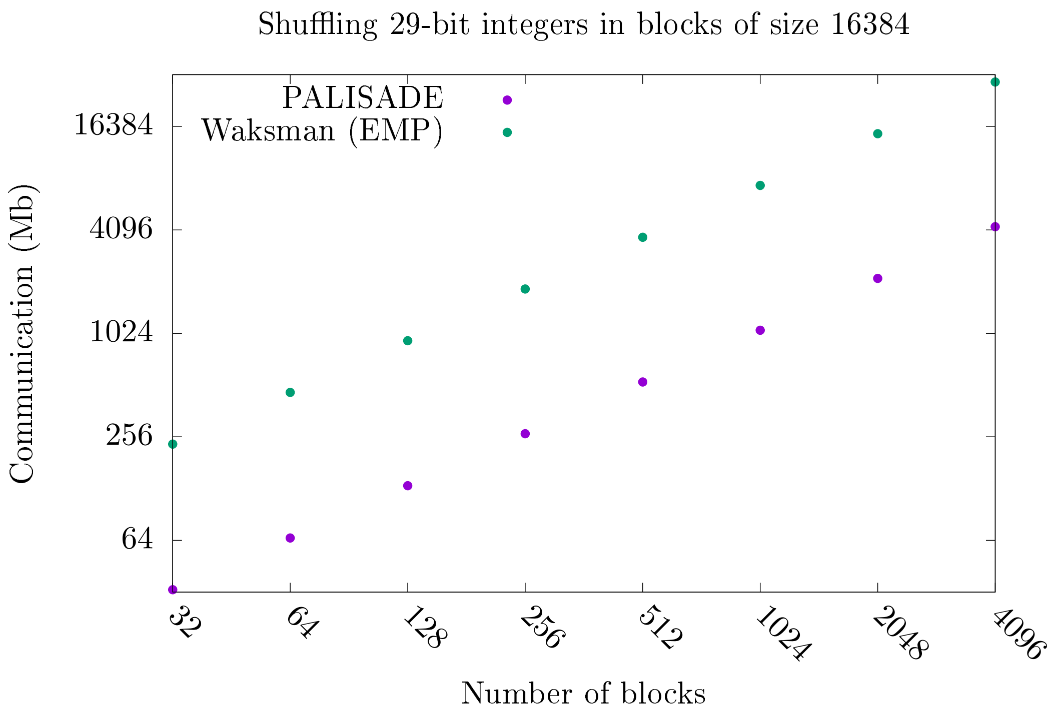

Figure 6 The communication cost of the FHE-based secure shuffling protocol in Section 3. Note that both the x and y axes are on a log scale, and in such a scale, the function $y = x \log x$, will appear as $y = x + \log x$, which is why the $O(n \log n)$ Waksman shuffle appears linear.

D The [37] compaction algorithm

In this section, we review a data independent compaction algorithm described in [37, Theorem 14]. The algorithm runs in $O(n \log \log n)$ time, and works by recursively peeling off 1/6th of the remaining elements (depending on whether the majority of the remaining elements are zero or one).

Algorithm 3 describes a simple randomized procedure that takes an array, $\vec{v}$, of length n with the promise that at least $n/2$ of the elements in $\vec{v}$ are 0. Algorithm 3 reorganizes $\vec{v}$ such that the first $n/6$ elements of $\vec{v}$ are 0 with high probability. The algorithm makes use of a deterministic $O(n \log(n))$ partitioning algorithm, partition, e.g. that of [26] that requires exactly $n \log(n)$ comparisons.

The full algorithm is described in Algorithm 2.

Algorithm 1 The [37] data-independent partitioning algorithm that runs in $O(n \log \log n)$ time.

Private input: A list $\vec{v} \in \{0, 1\}^n$
Initialize $a_0 = 0$
for $i = 0, \ldots, n-1$ **do** ▷ Count the number of 1s in $\vec{v}$
  $a_0 = a_0 + v[i]$
end for
return $\mathsf{MZ}(\vec{v}, a_0)$

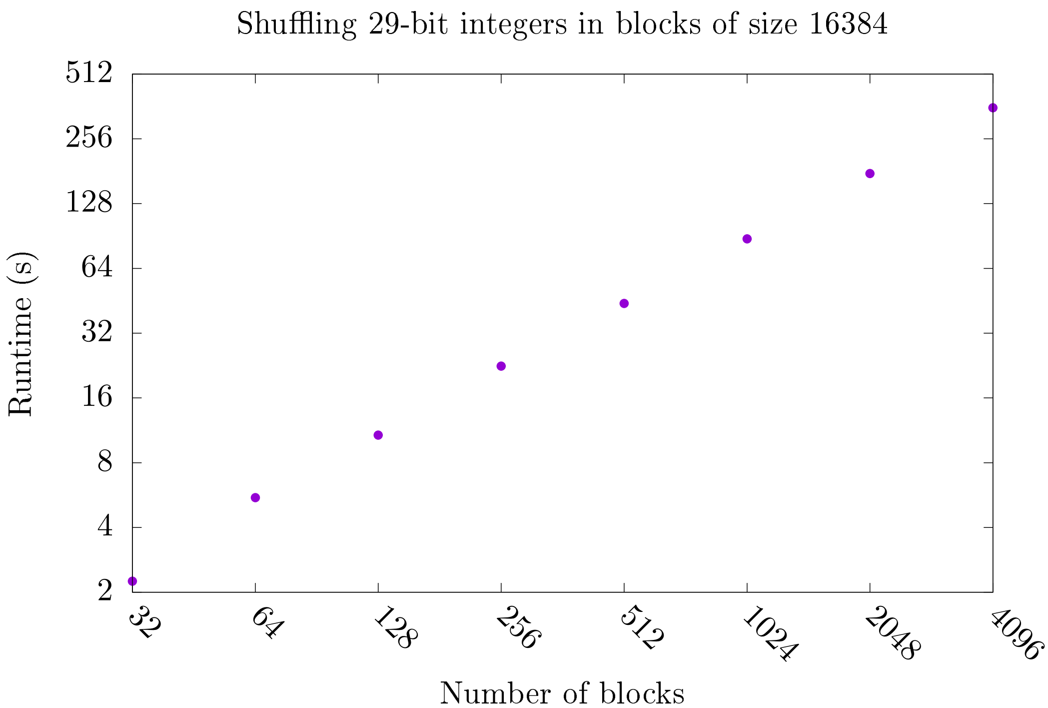

Figure 7 The running time of the FHE-based secure shuffling protocol. Both parties were run on the same machine, so networking costs were minimized.

Algorithm 2 The [37] data-independent partitioning algorithm that runs in $O(n \log \log n)$ time.

Private input: A list $\vec{v} \in \{0,1\}^n$
Private input: a, the number of 1's in $\vec{v}$
if $n < 4s$ **then**
  return partition($\vec{v}$)
end if
Initialize flipped $= 0$
if $a > n/2$ **then** ▷ If majority ones, flip the bits of $\vec{v}$
  flipped $= 1$
end if
$\vec{v} = \text{flipped} \cdot (\vec{v} \oplus 1^n) + (1 - \text{flipped}) \cdot \vec{v}$ ▷ If flipped $= 1$, invert bits of $\vec{v}$
$\vec{v} = \text{MZInner}(\vec{v}, a)$ ▷ First $n/6$ bits are now 0 w.h.p.
Define $\vec{v}_l = (v[0], \ldots, v[\lfloor n/6 \rfloor])$ ▷ $\vec{v}_l$ is sorted portion of list
Define $\vec{v}_r = (v[\lfloor n/6 \rfloor + 1], \ldots, v[n-1])$ ▷ $\vec{v}_r$ is unsorted portion of list
$a = \text{flipped} \cdot (n - a) + (1 - \text{flipped}) \cdot a$ ▷ Number of 1s remaining in $\vec{v}_r$
$\vec{v}_r = \text{MZ}(\vec{v}_r, a)$ ▷ Recurse
$\vec{v}_u = \vec{v}_l || \vec{v}_r$
$\vec{v}_f = \text{reverse}(1^n \oplus \vec{v}_u)$ ▷ Flip bits and reverse list
return $\text{flipped} \cdot \vec{v}_f + (1 - \text{flipped}) \cdot \vec{v}_u$

Algorithm 3 The inner step of the [37] algorithm that moves $n'/6$ elements of type 0 to the beginning of the list.

Input: A list $\vec{v} \in \{0,1\}^{n'}$ with the promise that majority($\vec{v}$) $= 0$
for i from 0 to $n'/3 - 1$ **do** ▷ Boost probability that $\vec{v}[i] = 0$
  for j from 0 to $c - 1$ **do**
    $r \xleftarrow{\$} [n'/3, n' - 1]$
    if $\vec{v}[r] = 0$ **then**
      swap($\vec{v}[i], \vec{v}[r]$)
    end if
  end for ▷ At this point, $\Pr[\vec{v}[i] = 0] > 1 - \left(\frac{1}{2}\right)^{c+1}$
end for
for i from 0 to $n'/(3s) - 1$ **do**
  partition($\vec{v}[i \cdot s, \ldots, (i+1) \cdot s - 1]$) ▷ Sort blocks of size s
  for j from 0 to $s/2$ **do**
    swap($\vec{v}[i \cdot s/2 + j], \vec{v}[(2 \cdot i) \cdot s/2 + j]$) ▷ Move first half of each block to the beginning of the list
  end for
end for

▶ **Lemma 13** (Algorithm 1 correctness). *The probability that Algorithm 1 fails to correctly compact a list is*

$$\frac{6n-20s}{3s}e^{-2\left(\frac{1}{2}-\left(\frac{3}{4}\right)^c\right)^2 s}$$

A straightforward calculation shows that for $c = 6$*, setting* $s = \log(n)^2$*, gives a failure probability of less than* 2^{-40} *for all* $n > 2^{12}$*.*

Proof. At each iteration through the loop, the size of the remaining list drops by a factor of $5/6$. The loop terminates when the list size reaches $4s$. If we let t denote the number of iteration of the algorithm, we have $4s = \left(\frac{5}{6}\right)^t n$, which means

$$t = \frac{\log\left(\frac{n}{4s}\right)}{\log\frac{6}{5}}$$

and $\left(\frac{5}{6}\right)^t = \frac{4s}{n}$.

First, notice that

$$\begin{aligned}\sum_{i=0}^{t}\left(\frac{5}{6}\right)^i &= \left(\frac{1-\left(\frac{5}{6}\right)^{t+1}}{1-\frac{5}{6}}\right)\\ &= 6\left(1-\frac{5}{6}\cdot\left(\frac{5}{6}\right)^t\right)\\ &= 6\left(1-\frac{5}{6}\cdot\frac{4s}{n}\right)\\ &= \frac{6n-20s}{n}.\end{aligned}$$

A given block of size s will fail if it has more than $\frac{s}{2}$ zeros. The right half is guaranteed to have at least $\frac{n'}{2}-\frac{n'}{3}=\frac{n'}{6}$ zeros, so at least $\frac{1}{4}$ of the elements on the right hand side are zero. For a given a_i, after making c attempted swaps, the probability that a_i is zero is at least $1-\left(\frac{3}{4}\right)^c$. Thus by the Hoeffding bound, the probability that a given block fails is at most

$$e^{-2\left(\frac{1}{2}-\left(\frac{3}{4}\right)^c\right)^2 s}$$

Taking a union bound over the $\frac{n'}{3s}$ blocks of size s, and then summing over the n', we have the total failure probability is bounded by

$$\frac{n}{3s}e^{-2\left(\frac{1}{2}-\left(\frac{3}{4}\right)^c\right)^2 s}$$

◀

▶ **Lemma 14** (Algorithm 1 runtime). *Algorithm 1 obliviously compacts a list using*

$$\frac{6n-14s}{3}\log s + \frac{6n-20s}{3}\cdot(c+6)$$

comparisons.

Proof. Algorithm 1 uses n comparisons to compute a before calling Algorithm 2.

Each iteration of the loop (Algorithm 3) requires $\frac{n'}{3s}$ calls to partition (on sets of size s). At iteration i, $n' = \left(\frac{5}{6}\right)^i n$, which gives

$$\frac{n}{3s}\sum_{i=0}^{t}\left(\frac{5}{6}\right)^i = \frac{6n-20s}{3s}.$$

So there are a total of $\frac{6n-20s}{3s}$ calls to partition of size s.

At each iteration of the loop (Algorithm 3) there are also $\frac{n' \cdot c}{3}$ controlled swaps.

Thus the total number of controlled swaps in is

$$\frac{n \cdot c}{3} \sum_{i=0}^{t} \left(\frac{5}{6}\right)^i = \frac{6n-20s}{3} \cdot c$$

At every call to Algorithm 2 there are $2n + \log(n)$ swaps. Finally, there is one call to partition of size $4s$.

Thus total runtime is

$$6n - 20s \cdot \left(\frac{1}{3s} \mathsf{partitionTime}(s) + \frac{c}{3} \frac{6n-20s}{3} + 2 \right) + \mathsf{partitionTime}(4s)$$

Where $\mathsf{partitionTime}(n)$ denotes the number of swaps needed to partition a set of size s. There are many options for the partition algorithm used here. In our implementation, we use the simple, deterministic, data independent partitioning algorithm from [26], which requires $n \log n$ controlled swaps.

Thus the total number of comparisons is

$$\frac{6n-14s}{3} \log s + \frac{6n-20s}{3} \cdot (c+6).$$

As noted above, setting $c = 6$, and $s = \log^2 n$ gives a failure probability below 2^{-40}, for all $n > 2^{12}$, so with these parameters, the overall number of comparisons is

$$4n \log \log n + 14n.$$ ◀

The deterministic data independent partitioning algorithm from [26] runs in time requires exactly $n \log n$, secure comparisons, so the [37] compaction algorithm will start to beat the deterministic [26] solution when $4n \log \log n < n \log n$, *i.e.*, when $n > 2^{16}$.

E Sparse compaction

The sparse sorting algorithm of Figure 4 can also be used for extracting a small number of nonzero values from a list. The resulting *sparse compaction* algorithm takes three arguments, a public bound, t, a secret-shared vector of "tags," $\vec{s}$, and a secret shared vector of "payloads," $\vec{x}$.

$$[\![\vec{y}]\!] = \mathsf{SparseCompaction}\left(t, [\![\vec{s}]\!], [\![\vec{x}]\!]\right).$$

We outline our sparse compaction algorithm in Figure 8.

▶ **Corollary 15** (Sparse compaction). *Given a secret-shared list of length n with at most t nonzero elements, the non-zero elements can be extracted in $O\left(t \log^2(n) + n\right)$ time using the algorithm given in Figure 8.*

F Security proofs

▶ **Lemma 16** (Secure shuffling). *If* $\mathsf{PKE} = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ *is a CPA-secure additively-homomorphic cryptosystem, over the group $\mathbb{G}$, and the scheme is* rerandomizable, *then secure shuffling algorithm in Figure 2 securely implements a 2-party shuffle in the honest-but-curious setting.*

SparseCompaction

Public parameters: An integer, n, and a bound $t < n$.
Input: A secret-shared list, $[\![\vec{x}]\!]$, of length n, each element of $[\![\vec{x}]\!]$ is tagged with a (secret-shared) tag $[\![y_i]\!]$ with $y_i \in \{0, 1\}$. The guarantee is that at most t tags are 1.
Output: A secret shared list, $[\![\vec{w}]\!]$ of length t containing all the nonzero elements of $\vec{x}$.

1. **Sorting:** Use the sparse sorting algorithm, SparseSort to sort the n elements of $\vec{V}$ based on their tags.

$$(([\![\bar{x}_1]\!], [\![\bar{y}_1]\!]), \ldots, ([\![\bar{x}_n]\!], [\![\bar{y}_n]\!])) = \mathsf{SparseSort}\left(([\![x_1]\!], [\![y_1]\!]), \ldots, (\bar{x}_n, \bar{y}_n)\right)$$

2. **Extraction:** Return the top t elements of the sorted list $([\![\bar{x}_1]\!], \ldots, [\![\bar{x}_t]\!])$.

Figure 8 Sparse compaction.

Proof. First, we note that if Alice *or* Bob, generates a random permutation, the resulting permutation will be random. Second, note that since Alice and Bob are honest-but-curious, at every step the ciphertexts they provide correctly encode some ordering of the secret shares.

Consider a series of Bob's views. Let view_0^b denote Bob's view in the real protocol. Let view_1^b be the protocol where, instead of encrypting her shares under pk^a, Alice encrypts the 0 vector. The semantic security of PKE ensures that view_0^b and view_1^b are indistinguishable. Let view_2^b be the protocol where, instead of shuffling the pairs of ciphertexts, Alice simply re-randomizes Bob's shares (through the homomorphic encryption). Since the encryption is re-randomizable, and the plaintext shares are re-randomized over the group $\mathbb{G}$, both the ciphertexts and the decrypted plaintexts are indistinguishable from in view_1^b. Thus, Alice's security is preserved.

The proof of security from Bob's side is essentially identical. ◀