Stackelberg-Pareto Synthesis

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Abstract

In this paper, we study the framework of two-player Stackelberg games played on graphs in which Player 0 announces a strategy and Player 1 responds rationally with a strategy that is an optimal response. While it is usually assumed that Player 1 has a single objective, we consider here the new setting where he has several. In this context, after responding with his strategy, Player 1 gets a payoff in the form of a vector of Booleans corresponding to his satisfied objectives. Rationality of Player 1 is encoded by the fact that his response must produce a Pareto-optimal payoff given the strategy of Player 0. We study the Stackelberg-Pareto Synthesis problem which asks whether Player 0 can announce a strategy which satisfies his objective, whatever the rational response of Player 1. For games in which objectives are either all parity or all reachability objectives, we show that this problem is fixed-parameter tractable and \textit{NEXPTIME}-complete. This problem is already \textit{NP}-complete in the simple case of reachability objectives and graphs that are trees.

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1 Introduction

Two-player zero-sum infinite-duration games played on graphs are a mathematical model used to formalize several important problems in computer science, such as reactive system synthesis. In this context, see e.g. [26], the graph represents the possible interactions between the system and the environment in which it operates. One player models the system to synthesize, and the other player models the (uncontrollable) environment. In this classical setting, the objectives of the two players are opposite, that is, the environment is adversarial. Modelling the environment as fully adversarial is usually a bold abstraction of reality as it can be composed of one or several components, each of them having their own objective.

In this paper, we consider the framework of Stackelberg games [31], a richer non-zero-sum setting, in which Player 0 (the system) called leader announces his strategy and then Player 1 (the environment) called follower plays rationally by using a strategy that is an optimal response to the leader’s strategy. This framework captures the fact that in practical applications, a strategy for interacting with the environment is committed before the interaction actually happens. The goal of the leader is to announce a strategy that
guarantees him a payoff at least equal to some given threshold. In the specific case of Boolean objectives, the leader wants to see his objective being satisfied. The concept of leader and follower is also present in the framework of rational synthesis \cite{17, 24} with the difference that this framework considers several followers, each of them with their own Boolean objective. In that case, rationality of the followers is modeled by assuming that the environment settles to an equilibrium (e.g. a Nash equilibrium) where each component (composing the environment) is considered to be an independent selfish individual, excluding cooperation scenarios between components or the possibility of coordinated rational multiple deviations. Our work proposes a novel and natural alternative in which the single follower, modeling the environment, has several objectives that he wants to satisfy. After responding to the leader with his own strategy, Player 1 receives a vector of Booleans which is his payoff in the corresponding outcome. Rationality of Player 1 is encoded by the fact that he only responds in such a way to receive Pareto-optimal payoffs, given the strategy announced by the leader. This setting encompasses scenarios where, for instance, several components can collaborate and agree on trade-offs. The goal of the leader is therefore to announce a strategy that guarantees him to satisfy his own objective, whatever the response of the follower which ensures him a Pareto-optimal payoff. The problem of deciding whether the leader has such a strategy is called the Stackelberg-Pareto Synthesis problem (SPS problem).

Contributions. In addition to the definition of the new setting, our main contributions are the following ones. We consider the general class of \(\omega\)-regular objectives modelled by parity conditions and also consider the case of reachability objectives for their simplicity\(^1\). We provide a thorough analysis of the complexity of solving the SPS problem for both objectives. Our results are interesting and singular both from a theoretical and practical point of view.

First, we show that the SPS problem is fixed-parameter tractable (FPT) for reachability objectives when the number of objectives of the follower is a parameter and for parity objectives when, in addition, the maximal priority used in each priority function is also a parameter of the complexity analysis (Theorem 3). These are important results as it is expected that, in practice, the number of objectives of the environment is limited to a few. To obtain these results, we develop a reduction from our non-zero-sum games to a zero-sum game in which the protagonist, called Prover, tries to show the existence of a solution to the problem, while the antagonist, called Challenger, tries to disprove it. This zero-sum game is defined in a generic way, independently of the actual objectives used in the initial game, and can then be easily adapted according to the case of reachability or parity objectives.

Second, we prove that the SPS problem is \(\text{NEXPTIME}\)-complete for both reachability and parity objectives (Theorem 6 and Theorem 9), and that it is already \(\text{NP}\)-complete in the simple setting of reachability objectives and graphs that are trees (Theorem 7). To the best of our knowledge, this is the first \(\text{NEXPTIME}\)-completeness result for a natural class of games played on graphs. To obtain the hardness for \(\text{NEXPTIME}\), we present a natural succinct version of the set cover problem that is complete for this class (Theorem 11), a result of potential independent interest. We then show how to reduce this problem to the SPS problem. To obtain the \(\text{NEXPTIME}\)-membership of the SPS problem, we have shown that exponential-size solutions exist for positive instances of the SPS problem and this allows us to design a nondeterministic exponential-time algorithm. Unfortunately, it was not possible to use the FPT algorithm mentioned above to show this membership due to its too high time complexity; conversely, our \(\text{NEXPTIME}\) algorithm is not FPT.

\(^1\) Indeed, in the classical context of two-player zero-sum games, solving reachability games is in \(P\) whereas solving parity games is only known to be in \(\text{NP} \cap \text{co-NP}\), see e.g. \cite{18}.
Related Work. Rational synthesis is introduced in [17] for $\omega$-regular objectives in a setting where the followers are cooperative with the leader, and later in [24] where they are adversarial. Precise complexity results for various $\omega$-regular objectives are established in [13] for both settings. Those complexities differ from the ones of the problem studied in this paper. Indeed, for reachability objectives, adversarial rational synthesis is PSPACE-complete, while for parity objectives, its precise complexity is not settled (the problem is PSPACE-hard and in NEXPTIME). Extension to non-Boolean payoffs, like mean-payoff or discounted sum, is studied in [19, 20] in the cooperative setting and in [1, 16] in the adversarial setting.

When several players (like the followers) play with the aim to satisfy their objectives, several solution concepts exist such as Nash equilibrium [25], subgroup perfect equilibrium [27], secure equilibria [11, 12], or admissibility [2, 4]. The constrained existence problem, close to the cooperative rational synthesis problem, is to decide whether there exists a solution concept such that the payoff obtained by each player is larger than some threshold. Let us mention [13, 29, 30] for results on the constrained existence for Nash equilibria and [5, 6, 28] for such results for subgroup perfect equilibria. Rational verification is studied in [21, 22]. This problem (which is not a synthesis problem) is to decide whether a given LTL formula is satisfied by the outcome of all Nash equilibria (resp. some Nash equilibrium). The interested reader can find more pointers to works on non-zero-sum games for reactive synthesis in [3, 7].

Structure. The paper is structured as follows. In Section 2, we introduce the class of Stackelberg-Pareto games and the SPS problem. We show in Section 3 that the SPS problem is in FPT for reachability and parity objectives. The complexity class of this problem is studied in Section 4 where we prove that it is NEXPTIME-complete and NP-complete in case of reachability objectives and graphs that are trees. In Section 5, we provide a conclusion and discuss future work. Detailed proofs of our results can be found in the full version of this paper.

2 Preliminaries and Stackelberg-Pareto Synthesis Problem

This section introduces the class of two-player Stackelberg-Pareto games in which the first player has a single objective and the second has several. We present a decision problem on those games called the Stackelberg-Pareto Synthesis problem, which we study in this paper.

2.1 Preliminaries

Game Arena. A game arena is a tuple $G = (V, V_0, V_1, E, v_0)$ where $(V, E)$ is a finite directed graph such that: (i) $V$ is the set of vertices and $(V_0, V_1)$ forms a partition of $V$ where $V_0$ (resp. $V_1$) is the set of vertices controlled by Player 0 (resp. Player 1), (ii) $E \subseteq V \times V$ is the set of edges such that each vertex $v$ has at least one successor $v'$, i.e., $(v, v') \in E$, and (iii) $v_0 \in V$ is the initial vertex. We call a game arena a tree arena if it is a tree in which every leaf vertex has itself as its only successor. A sub-arena $G'$ with a set $V' \subseteq V$ of vertices and initial vertex $v'_0 \in V'$ is a game arena defined from $G$ as expected.

Plays. A play in a game arena $G$ is an infinite sequence of vertices $\rho = v_0v_1 \ldots \in V^\omega$ such that it starts with the initial vertex $v_0$ and $(v_j, v_{j+1}) \in E$ for all $j \in \mathbb{N}$. Histories in $G$ are finite sequences $h = v_0 \ldots v_j \in V^+$ defined similarly. A history is elementary if it contains no cycles. We denote by $\text{Plays}_G$ the set of plays in $G$. We write $\text{Hist}_G$ (resp. $\text{Hist}_{G,i}$) the set of histories (resp. histories ending with a vertex in $V_i$). We use the notations $\text{Plays}$, $\text{Hist}$, and $\text{Hist}_G$, when $G$ is clear from the context. We write $\text{Occ}(\rho)$ the set of vertices occurring in $\rho$ and $\text{Inf}(\rho)$ the set of vertices occurring infinitely often in $\rho$. 
Strategies. A strategy $\sigma_i$ for Player $i$ is a function $\sigma_i : \text{Hist} \rightarrow V$ assigning to each history $hv \in \text{Hist}$, a vertex $v' = \sigma_i(hv)$ such that $(v, v') \in E$. It is memoryless if $\sigma_i(hv) = \sigma_i(h'v)$ for all histories $hv, h'v$ ending with the same vertex $v \in V_i$. More generally, it is finite-memory if it can be encoded by a Moore machine $\mathcal{M}$ [18]. The memory size of $\sigma_i$ is the number of memory states of $\mathcal{M}$. In particular, $\sigma_i$ is memoryless when it has a memory size of one.

Given a strategy $\sigma_i$ of Player $i$, a play $\rho = v_0v_1 \ldots$ is consistent with $\sigma_i$ if $v_{j+1} = \sigma_i(v_0 \ldots v_j)$ for all $j \in \mathbb{N}$ such that $v_j \in V_i$. Consistency is naturally extended to histories. We denote by $\text{Plays}_{\sigma_i}$ (resp. $\text{Hist}_{\sigma_i}$) the set of plays (resp. histories) consistent with $\sigma_i$. A strategy profile is a tuple $\sigma = (\sigma_0, \sigma_1)$ of strategies, one for each player. We write $\text{out}(\sigma)$ the unique play consistent with both strategies and we call it the outcome of $\sigma$.

Objectives. An objective for Player $i$ is a set of plays $\Omega \subseteq \text{Plays}$. A play $\rho$ satisfies the objective $\Omega$ if $\rho \in \Omega$. In this paper, we focus on the two following $\omega$-regular objectives. Let $T \subseteq V$ be a subset of vertices called a target set, the reachability objective $\text{Reach}(T) = \{ \rho \in \text{Plays} \mid \text{Occ}(\rho) \cap T \neq \emptyset \}$ asks to visit at least one vertex of $T$. Let $c : V \rightarrow \mathbb{N}$ be a function called a priority function which assigns an integer to each vertex in the arena, the parity objective $\text{Parity}(c) = \{ \rho \in \text{Plays} \mid \min_{v \in \text{Hist}(\rho)}(c(v)) \text{ is even} \}$ asks that the minimum priority visited infinitely often be even.

2.2 Stackelberg-Pareto Synthesis Problem

Stackelberg-Pareto Games. A Stackelberg-Pareto game (SP game) $\mathcal{G} = (G, \Omega_0, \Omega_1, \ldots, \Omega_t)$ is composed of a game arena $G$, an objective $\Omega_0$ for Player 0 and $t \geq 1$ objectives $\Omega_1, \ldots, \Omega_t$ for Player 1. In this paper, we focus on SP games where the objectives are either all reachability or all parity objectives and call such games reachability (resp. parity) SP games.

Payoffs in SP Games. The payoff of a play $\rho \in \text{Plays}$ corresponds to the vector of Booleans $\text{pay}(\rho) \in \{0,1\}^t$ such that for all $i \in \{1, \ldots, t\}$, $\text{pay}_i(\rho) = 1$ if $\rho \in \Omega_i$, and $\text{pay}_i(\rho) = 0$ otherwise. Note that we omit to include Player 0 when discussing the payoff of a play.

Instead we say that a play $\rho$ is won by Player 0 if $\rho \in \Omega_0$ and we write $\text{won}(\rho) = 1$, otherwise it is lost by Player 0 and we write $\text{won}(\rho) = 0$. We write $\text{won}(\rho)$ the extended payoff of $\rho$. Given a strategy profile $\sigma$, we write $\text{won}(\sigma) = \text{won}(\text{out}(\sigma))$ and $\text{pay}(\sigma) = \text{pay}(\text{out}(\sigma))$. For reachability SP games, since reachability objectives are prefix-depandent and given a history $h \in \text{Hist}$, we also define $\text{won}(h)$ and $\text{pay}(h)$ as done for plays.

We introduce the following partial order on payoffs. Given two payoffs $p = (p_1, \ldots, p_t)$ and $p' = (p'_1, \ldots, p'_t)$ such that $p, p' \in \{0,1\}^t$, we say that $p'$ is larger than $p$ and write $p \leq p'$ if $p_i \leq p'_i$ for all $i \in \{1, \ldots, t\}$. Moreover, when it also holds that $p_i < p'_i$ for some $i$, we say that $p'$ is strictly larger than $p$ and we write $p < p'$. A subset of payoffs $P \subseteq \{0,1\}^t$ is an antichain if it is composed of pairwise incomparable payoffs with respect to $\leq$.

Stackelberg-Pareto Synthesis Problem. Given a strategy $\sigma_0$ of Player 0, we consider the set of plays of payoff consistent with $\sigma_0$ which are Pareto-optimal, i.e., maximal with respect to $\leq$. We write this set $P_{\sigma_0} = \max\{\text{pay}(\rho) \mid \rho \in \text{Plays}_{\sigma_0}\}$. Notice that it is an antichain. We say that those payoffs are $\sigma_0$-fixed Pareto-optimal and write $|P_{\sigma_0}|$ the number of such payoffs. A play $\rho \in \text{Plays}_{\sigma_0}$ is called $\sigma_0$-fixed Pareto-optimal if its payoff $\text{pay}(\rho)$ is in $P_{\sigma_0}$.

The problem studied in this paper asks whether there exists a strategy $\sigma_0$ for Player 0 such that every play in $\text{Plays}_{\sigma_0}$ which is $\sigma_0$-fixed Pareto-optimal satisfies the objective of Player 0. This corresponds to the assumption that given a strategy of Player 0, Player 1 will play rationally, that is, with a strategy $\sigma_1$ such that $\text{out}(\sigma_0, \sigma_1))$ is $\sigma_0$-fixed Pareto-optimal. It is therefore sound to ask that Player 0 wins against such rational strategies.
Definition 1. Given an SP game, the Stackelberg-Pareto Synthesis problem (SPS problem) is to decide whether there exists a strategy $\sigma_0$ for Player 0 (called a solution) such that for each strategy profile $\sigma=(\sigma_0,\sigma_1)$ with $\text{pay}(\sigma) \in P_{\sigma_0}$, it holds that $\text{won}(\sigma) = 1$.

Witnesses. Given a strategy $\sigma_0$ that is a solution to the SPS problem and any payoff $p \in P_{\sigma_0}$, for each play $\rho$ consistent with $\sigma_0$ such that $\text{pay}(\rho) = p$ it holds that $\text{won}(\rho) = 1$. For each $p \in P_{\sigma_0}$, we arbitrarily select such a play which we call a witness (of $p$). We denote by $\text{Wit}_{\sigma_0}$ the set of all witnesses, of which there are as many as payoffs in $P_{\sigma_0}$. In the sequel, it is useful to see this set as a tree composed of $|\text{Wit}_{\sigma_0}|$ branches. Additionally for a given history $h \in \text{Hist}$, we write $\text{Wit}_{\sigma_0}(h)$ the set of witnesses for which $h$ is a prefix, i.e., $\text{Wit}_{\sigma_0}(h) = \{ \rho \in \text{Wit}_{\sigma_0} \mid h$ is prefix of $\rho \}$. Notice that $\text{Wit}_{\sigma_0}(h) = \text{Wit}_{\sigma_0}$ when $h = v_0$ and that $\text{Wit}_{\sigma_0}(h)$ decreases as $h$ increases, until it contains a single value or becomes empty.

Example 2. Consider the reachability SP game with arena $G$ depicted in Figure 1 in which Player 1 has $t=3$ objectives. The vertices of Player 0 (resp. Player 1) are depicted as ellipses (resp. rectangles)$^2$. Every objective in the game is a reachability objective defined as follows: $\Omega_0 = \text{Reach}(\{v_0, v_1\})$, $\Omega_1 = \text{Reach}(\{v_4, v_7\})$, $\Omega_2 = \text{Reach}(\{v_3\})$, $\Omega_3 = \text{Reach}(\{v_1, v_5\})$. The extended payoff of plays reaching vertices from which they can only loop is displayed in the arena next to those vertices, and the extended payoff of play $v_0v_2v_1v_3v_5v_6v_7$ is $(0, (0, 1, 0))$.

Consider the memoryless strategy $\sigma_0$ of Player 0 such that he chooses to always move to $v_5$ from $v_3$. The set of payoffs of plays consistent with $\sigma_0$ is $\{(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1)\}$ and the set of those that are Pareto-optimal is $P_{\sigma_0} = \{(1, 0, 0), (0, 1, 1)\}$. Notice that play $\rho = v_0v_2v_1v_3v_5v_6v_7$ consistent with $\sigma_0$ has payoff $(1, 0, 0)$ and is lost by Player 0. Strategy $\sigma_0$ is therefore not a solution to the SPS problem. In this game, there is only one other memoryless strategy for Player 0, where he chooses to always move to $v_7$ from $v_3$. One can verify that it is again not a solution to the SPS problem.

We can however define a finite-memory strategy $\sigma'_0$ such that $\sigma'_0(v_0v_2v_1) = v_5$ and $\sigma'_0(v_0v_2v_3v_5v_6) = v_7$ and show that it is a solution to the problem. Indeed, the set of $\sigma'_0$-fixed Pareto-optimal payoffs is $P_{\sigma'_0} = \{(0, 1, 1), (1, 1, 0)\}$ and Player 0 wins every pay play consistent with $\sigma'_0$ whose payoff is in this set. A set $\text{Wit}_{\sigma'_0}$ of witnesses for these payoffs is $\{v_0v_2v_1v_3v_5v_6, v_0v_2v_3v_5v_6v_7\}$ and in this case the unique set of witnesses. This example shows that Player 0 sometimes needs memory in order to have a solution to the SPS problem.

3 Fixed-Parameter Complexity

In this section, we show that the SPS problem is in $\text{FPT}$ for both cases of reachability and parity SP games. We refer the reader to [15] for the concept of fixed-parameter complexity.

$^2$ This convention is used throughout this paper.
Stackelberg-Pareto Synthesis

![Figure 2](image)

**Theorem 3.** Solving the SPS problem is in FPT for reachability SP games for parameter \( t \) equal to the number of objectives of Player 1 and it is in FPT for parity SP games for parameters \( t \) and the maximal priority according to each parity objective of Player 1.

### 3.1 Challenger-Prover Game

In order to prove Theorem 3, we provide a reduction to a specific two-player zero-sum game, called the Challenger-Prover game (C-P game). This game is a zero-sum game played between Challenger (written \( C \)) and Prover (written \( P \)). We will show that Player 0 has a solution to the SPS problem in an SP game if and only if \( P \) has a winning strategy in the corresponding C-P game. In the latter game, \( P \) tries to show the existence of a strategy \( \sigma_0 \) that is solution to the SPS problem in the original game and \( C \) tries to disprove it. The C-P game is described independently of the objectives used in the SP game and its objective is described as such in a generic way. We later provide the proof of our FPT results by adapting it specifically for reachability and parity SP games.

**Intuition on the C-P Game.** Without loss of generality, the SP games we consider in this section are such that each vertex in their arena has at most two successors. It can be shown that any SP game \( G \) with \( n \) vertices can be transformed into an SP game \( \tilde{G} \) with \( O(n^2) \) vertices such that every vertex has at most two successors and Player 0 has a solution to the SPS problem in \( G \) if and only if he has a solution to the SPS problem in \( \tilde{G} \).

Let \( G \) be an SP game. The C-P game \( G' \) is a zero-sum game associated with \( G \) that intuitively works as follows. First, \( P \) selects a set \( P \) of payoffs which he announces as the set of Pareto-optimal payoffs \( \bar{P} \) to the SP problem in \( G \) he is trying to construct. Then, \( P \) tries to show that there exists a set of witnesses \( \text{Wit}_{\sigma_0} \) in \( G \) for the payoffs in \( P \). After the selection of \( P \) in \( G' \), there is a one-to-one correspondence between plays in the arenas \( G \) and \( G' \) such that the vertices in \( G' \) are augmented with a set \( W \) which is a subset of \( P \). Initially \( W \) is equal to \( P \) and after some history in \( G' \), \( W \) contains payoff \( p \) if the corresponding history in \( G \) is prefix of the witness with payoff \( p \) in the set \( \text{Wit}_{\sigma_0} \) that \( P \) is building. In addition, the objective \( \Omega_P \) of \( P \) is such that he has a winning strategy \( \sigma_0 \) in \( G' \) if and only if the set \( P \) that he selected coincides with the set \( \bar{P} \) for the corresponding strategy \( \sigma_0 \) in \( G \) and the latter strategy is a solution to the SPS problem in \( G \). A part of the arena of the C-P game for Example 2 with a positional winning strategy for \( P \) highlighted in bold is illustrated in Figure 2.

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3 We assume that the reader is familiar with the concept of zero-sum games, see e.g. [18].
**Arena of the C-P Game.** The initial vertex $\bot$ belongs to $\mathcal{P}$. From this vertex, he selects a successor $(v_0, P, W)$ such that $W = \mathcal{P}$ and $P$ is an antichain of payoffs which $\mathcal{P}$ announces as the set $P_{\sigma_0}$ for the strategy $\sigma_0$ in $G$ he is trying to construct. All vertices in plays starting with this vertex will have this same value for their $P$-component. Those vertices are either a triplet $(v, P, W)$ that belongs to $\mathcal{P}$ or $(v, P, (W_I, W_r))$ that belongs to $\mathcal{C}$. Given a play $\rho$ (resp. history $h$) in $G'$, we denote by $\rho_{V}$ (resp. $h_{V}$) the play (resp. history) in $G$ obtained by removing $\bot$ and keeping the $v$-component of every vertex of $\mathcal{P}$ in $\rho$ (resp. $h$), which we call its projection.

- After history $hm$ such that $m = (v, P, W)$ with $v \in V_0$, $\mathcal{P}$ selects a successor $v'$ such that $(v, v') \in E$ and vertex $(v', P, W)$ is added to the play. This corresponds to Player 0 choosing a successor $v'$ after history $h_{V}v$ in $G$.
- After history $hm$ such that $m = (v, P, W)$ with $v \in V_1$, $\mathcal{P}$ selects a successor $(v, P, (W_I, W_r))$ with $(W_I, W_r)$ a partition of $W$. This corresponds to $\mathcal{P}$ splitting the set $W$ into two parts according to the two successors $v_I$ and $v_r$ of $v$. For the strategy $\sigma_0$ that $\mathcal{P}$ tries to construct and its set of witnesses $\text{Wit}_{\sigma_0}$ he is building, he asserts that $W_I$ (resp. $W_r$) is the set of payoffs of the witnesses in $\text{Wit}_{\sigma_0}(h_{V}v)$ (resp. $\text{Wit}_{\sigma_0}(h_{V}v_{r})$).
- From a vertex $(v, P, (W_I, W_r))$, $\mathcal{C}$ can select a successor $(v_I, P, W_I)$ or $(v_r, P, W_r)$ which corresponds to the choice of Player 1.

Formally, the game arena of the C-P game is the tuple $G' = (V', V', V', E', \bot)$ with:
- $V'_0 = \{ \bot \} \cup \{(v, P, W) \mid v \in V, P \subseteq \{0, 1\}^t \text{ is an antichain and } W \subseteq P\}$,
- $V'_1 = \{(v, P, (W_I, W_r)) \mid v \in V_1, P \subseteq \{0, 1\}^t \text{ is an antichain and } W_I, W_r \subseteq P\}$,
- $(\bot, (v, P, W)) \in E' \text{ if } v = v_0 \text{ and } P = W,$
- $((v, P, W), (v', P, W)) \in E'$ if $v \in V_0$ and $(v, v') \in E,$
- $((v, P, W), (v, P, (W_I, W_r))) \in E' \text{ if } v \in V_1 \text{ and } (W_I, W_r) \text{ is a partition of } W,$
- $((v, P, (W_I, W_r)), (v', P, W)) \in E' \text{ if } (v, v') \in E \text{ and } \{v' = v_I \text{ and } W = W_I\} \text{ or } \{v' = v_r \text{ and } W = W_r\}.$

In the definition of $E'$, if $v$ has a single successor $v'$ in $G$, it is assumed to be $v_I$ and $W_r$ is always equal to $\emptyset$. Given the two successors $v_I$ and $v_j$ of $v$, $v_I$ is the left successor if $i < j$.

**Objective of $\mathcal{P}$ in the C-P Game.** Let us now discuss the objective $\Omega_{\mathcal{P}}$ of $\mathcal{P}$. The $W$-component of the vertices controlled by $\mathcal{P}$ has a size that decreases along a play $\rho$ in $G'$. We write $\lim_W(\rho)$ the value of the $W$-component at the limit in $\rho$. Recall that with this $W$-component, $\mathcal{P}$ tries to construct a solution $\sigma_0$ to the SPS problem with associated sets $P_{\sigma_0}$ and $\text{Wit}_{\sigma_0}$. Therefore, for him to win in the C-P game, $\lim_W(\rho)$ must be a singleton or empty in every consistent play such that:

- $\lim_W(\rho)$ must be a singleton $\{p\}$ with $p$ the payoff of $\rho_{V}$ in $G$, showing that $\rho_{V} \in \text{Wit}_{\sigma_0}$ is a correct witness for $p$. In addition, it must hold that $\text{won}(\rho_{V}) = 1$ as $p \in P$ and as $\mathcal{P}$ wants $\sigma_0$ to be a solution.
- $\lim_W(\rho)$ must be the empty set such that either the payoff of $\rho_{V}$ belongs to $P_{\sigma_0}$ and $\text{won}(\rho_{V}) = 1$, or the payoff of $\rho_V$ is strictly smaller than some payoff in $P_{\sigma_0}$.

These conditions verify that the sets $P = P_{\sigma_0}$ and $\text{Wit}_{\sigma_0}$ are correct and that $\sigma_0$ is indeed a solution to the SPS problem in $G$. They are generic as they do not depend on the actual objectives used in the SP game.
Let us give the formal definition of $\Omega_P$. For an antichain $P$ of payoffs, we write $\text{Plays}_{G'}^P$ the set of plays in $G'$ which start with $\perp(v_0, P, P)$ and we define the following set

$$B_P = \{ \rho \in \text{Plays}_{G'}^P \mid (\text{lim}_{W}(\rho) = \{p\} \land \text{pay}(\rho_{\perp}) = p \in P \land \text{won}(\rho_{\perp}) = 1) \lor$$

$$(\text{lim}_{W}(\rho) = \emptyset \land \text{pay}(\rho_{\perp}) \in P \land \text{won}(\rho_{\perp}) = 1) \lor$$

$$(\text{lim}_{W}(\rho) = \emptyset \land \exists p \in P, \text{pay}(\rho_{\perp}) < p) \}.$$  

Objective $\Omega_P$ of $P$ in $G'$ is the union of $B_P$ over all antichains $P$. As the C-P game is zero-sum, objective $\Omega_C$ equals $\text{Plays}_{G'} \setminus \Omega_P$. The following theorem holds.

**Theorem 4.** Player 0 has a strategy $\sigma_0$ that is solution to the SPS problem in $G$ if and only if $P$ has a winning strategy $\sigma_P$ from $\perp$ in the C-P game $G'$.

**Proof.** Let us first assume that Player 0 has a strategy $\sigma_0$ that is solution to the SPS problem in $G$. Let $P_{\sigma_0}$ be its set of $\sigma_0$-fixed Pareto-optimal payoffs and let $\text{Wit}_{\sigma_0}$ be a set of witnesses. We construct the strategy $\sigma_P$ from $\sigma_0$ such that

- $\sigma_P(\perp) = (v_0, P, P)$ such that $P = P_{\sigma_0}$ (this vertex exists as $P_{\sigma_0}$ is an antichain),
- $\sigma_P(hm) = (v', P, W)$ if $m = (v, P, W)$ with $v \in V_0$ and $v' = \sigma_0(h_{V,V})$,
- $\sigma_P(hm) = (v, P, (W_i, W_{r}))$ if $m = (v, P, W)$ with $v \in V_1$ and for $i \in \{l, r\}$, $W_i = \{ \text{pay}(\rho) \mid \rho \in \text{Wit}_{\sigma_0}(h_{V,V})\}$. 

It is clear that given a play $\rho$ in $G'$ consistent with $\sigma_P$, the play $\rho_{\perp}$ in $G$ is consistent with $\sigma_0$. Let us show that $\sigma_P$ is winning for $P$ from $\perp$ in $G'$. Consider a play $\rho$ in $G'$ consistent with $\sigma_P$. There are two possibilities. (i) $\rho_{\perp}$ is a witness of $\text{Wit}_{\sigma_0}$ and by construction $\text{lim}_{W}(\rho) = \{p\}$ with $p = \text{pay}(\rho_{\perp})$; thus $\text{won}(\rho_{\perp}) = 1$ as $\sigma_0$ is a solution and $\rho_{\perp}$ is a witness. (ii) $\rho_{\perp}$ is not a witness and by construction $\text{lim}_{W}(\rho) = \emptyset$; as $\sigma_0$ is a solution, then $p = \text{pay}(\rho_{\perp})$ is bounded by some payoff of $P_{\sigma_0}$ and in case of equality $\text{won}(\rho_{\perp}) = 1$. Therefore $\rho$ satisfies the objective $B_P$ of $\Omega_P$ since it satisfies condition (1) in case (i) and condition (2) or (3) in case (ii).

Let us now assume that $P$ has a winning strategy $\sigma_P$ from $\perp$ in $G'$. Let $P$ be the antichain of payoffs chosen from $\perp$ by this strategy. We construct the strategy $\sigma_0$ from $\sigma_P$ such that $\sigma_0(h_{V,V}) = v'$ given $\sigma_P(hm) = (v', P, W)$ with $m = (v, P, W)$ and $v \in V_0$. Notice that this definition makes sense since there is a unique history $hm$ ending with a vertex of $P$ associated with $h_{V,V}$ showing a one-to-one correspondence between those histories.

Let us show $\sigma_0$ is a solution to the SPS problem with $P_{\sigma_0}$ being the set $P$. First notice that $P$ is not empty. Indeed let $\rho$ be a play consistent with $\sigma_P$. As $\rho$ belongs to $\Omega_P$ and in particular to $B_P$, one can check that $P \neq \emptyset$ by inspecting conditions (1) to (3). Second notice that by definition of $E'$, if $((v, P, W), (v, P, (W_i, W_{r}))) \in E'$ with $W \neq \emptyset$, then either $W_i$ or $W_r$ is not empty. Therefore given any payoff $p \in P$, there is a unique play $\rho$ consistent with $\sigma_P$ such that $\text{lim}_{W}(\rho) = \{p\}$. By construction of $\sigma_0$ and as $\sigma_P$ is winning, the play $\rho_{\perp}$ is consistent with $\sigma_0$, has payoff $p$, and is won by Player 0 (see (1)).

Let $\rho_{\perp}$ be a play consistent with $\sigma_0$ and $\rho$ be the corresponding play consistent with $\sigma_P$. It remains to consider (2) and (3). These conditions indicate that $\rho_{\perp}$ has a payoff equal to or strictly smaller than a payoff in $P$ and that in case of equality $\text{won}(\rho_{\perp}) = 1$. This shows that $P_{\sigma_0} = P$ and that $\sigma_0$ is a solution to the SPS problem. □

### 3.2 Proof of the FPT Results

We now sketch the proof of Theorem 3 which works by specializing the generic objective $\Omega_P$ to handle reachability and parity SP games. We begin with reachability SP games. We extend the arena $G'$ of the C-P game such that its vertices keep track of the objectives of $G$ which are satisfied along a play. Given an extended payoff $(w, p) \in \{0, 1\} \times \{0, 1\}^V$ and a vertex $v \in V$, we define the payoff update $\text{upd}(w, p, v) = (w', p')$ such that
\[ w' = 1 \iff w = 1 \text{ or } v \in T_0, \]
\[ p'_i = 1 \iff p_i = 1 \text{ or } v \in T_i, \quad \forall i \in \{1, \ldots, t\}. \]

We obtain the extended arena \( G^* \) as follows: (i) its set of vertices is \( V' \times \{0, 1\}^t \), (ii) its initial vertex is \( \bot^* = (\bot, 0, (0, \ldots, 0)) \), and (iii) \( ((m, w, p), (m', w', p')) \) with \( m' = (v', P, W) \) or \( m' = (v', P, (W_l, W_r)) \) is an edge in \( G^* \) if \( (m, m') \in E' \) and \( (w', p') = \text{upd}(w, p, v') \).

We define the zero-sum game \( G^* = (G^*, \Omega^*_p) \) in which the three abstract conditions (1-3) detailed previously are encoded into the following Büchi objective by using the \((w, p)\)-component added to vertices. We define \( \Omega^*_p = \text{Büchi}(B^*) \) with

\[
B^* = \{(v, P, W, w, p) \in V^*_p \mid (W = \{p\} \land w = 1) \lor (W = 0 \land p \in P \land w = 1) \lor (W = 0 \land \exists p' \in P, p < p')\}.
\]

The proof of the next proposition is a consequence of Theorem 4.

\begin{itemize}
  \item \textbf{Proposition 5.} Player 0 has a strategy \( \sigma_0 \) that is solution to the SPS problem in a reachability SP game \( G \) if and only if \( P \) has a winning strategy \( \sigma^*_P \) in \( G^* \).
\end{itemize}

We obtain the following FPT algorithm for deciding the existence of a solution to the SPS problem in a reachability SP game \( G \). First, we construct the zero-sum game \( G^* \) whose number of vertices is linear in the number of vertices in the original game and double exponential in the number \( t \) of objectives of Player 1. Second, by Proposition 5, deciding whether there exists a solution to the SPS problem in \( G \) amounts to solving the zero-sum Büchi game \( G^* \); this can be done in quadratic time in the number of vertices of \( G^* \) [10]. Those two steps are in FPT for parameter \( t \).

We now turn to parity SP games and briefly explain why solving the SPS problem in these games is in FPT, again by reduction to the C-P game. The arena \( G^* \) of the C-P game remains as is and its objective \( \Omega_P \) is replaced by a \textit{Boolean Büchi} objective \( \Omega^*_P \) which encodes the three conditions for parity objectives. Boolean Büchi objectives are Boolean combinations of Büchi objectives and zero-sum games with such objectives are shown to be solvable in FPT in [8]. It follows that the SPS problem is also in FPT.

### 4 Complexity Class of the SPS Problem

In this section, we study the complexity class of the SPS problem and prove its \textsc{NEXPTIME}-completeness for both reachability and parity SP games.

#### 4.1 \textsc{NEXPTIME}-Membership

We first show the membership to \textsc{NEXPTIME} of the SPS problem by providing a nondeterministic algorithm with time exponential in the size of the game \( G \). By \textit{size}, we mean the number \(|V|\) of its vertices and the number \( t \) of objectives of Player 1. Notice that the time complexity of the FPT algorithms obtained in the previous section is too high, preventing us from directly using the C-P game to show a tight membership result. Conversely, the nondeterministic algorithm provided in this section is not FPT as it is exponential in \(|V|\).

\begin{itemize}
  \item \textbf{Theorem 6.} The SPS problem is in \textsc{NEXPTIME} for reachability and parity SP games.
\end{itemize}
We show this membership result by proving that if Player 0 has a strategy which is a solution to the problem, he has one which is finite-memory with at most an exponential number of memory states\(^4\). This yields a NEXPTIME algorithm in which we nondeterministically guess such a strategy and check in exponential time that it is indeed a solution.

While our proof requires some specific arguments to treat both reachability and parity objectives, it is based on the following common principles. We first explain why, when there is a solution \(\sigma_0\) to the SPS problem, there is one that is finite-memory. We consider a fixed set of witnesses \(\text{Wit}_{\sigma_0}\). Figure 3 illustrates the two steps of the following construction. We start by showing the existence of a strategy \(\hat{\sigma}_0\) constructed from \(\sigma_0\), in which Player 0 follows \(\sigma_0\) as long as the current consistent history is prefix of at least one witness in \(\text{Wit}_{\sigma_0}\). Then when a deviation from \(\text{Wit}_{\sigma_0}\) occurs, Player 0 switches to a so-called punishing strategy. A deviation is a history that leaves the set of witnesses \(\text{Wit}_{\sigma_0}\) after a move of Player 1 (this is not possible by a move of Player 0). After such a deviation, \(\hat{\sigma}_0\) systematically imposes that the consistent play either satisfies \(\Omega_0\) or is not \(\sigma_0\)-fixed Pareto-optimal, i.e., it gives to Player 1 a payoff that is strictly smaller than the payoff of a witness in \(\text{Wit}_{\sigma_0}\). This makes the deviation irrational for Player 1. We show that this can be done, both for reachability and parity objectives, with at most exponentially many different punishing strategies, each having a size bounded exponentially in the size of the game. The strategy \(\hat{\sigma}_0\) that we obtain is therefore composed of the part of \(\sigma_0\) that produces \(\text{Wit}_{\sigma_0}\) and a punishment part whose size is at most exponential.

Then, we show how to decompose each witness in \(\text{Wit}_{\sigma_0}\) into at most exponentially many sections that can, in turn, be compacted into finite elementary paths or lasso shaped paths of polynomial length. As \(\text{Wit}_{\sigma_0}\) contains exactly \(|P_{\sigma_0}|\) witnesses \(\rho\), those compact witnesses \(c\rho\) can be produced by a finite-memory strategy with an exponential size for both reachability and parity objectives. This allows us to construct a strategy \(\tilde{\sigma}_0\) that produces the compact witnesses and acts as \(\hat{\sigma}_0\) after any deviation. This strategy is a solution of the SPS problem and has an exponential size as announced.

We can now sketch the proof of Theorem 6, again by giving arguments that work for both reachability and parity objectives. We guess a solution \(\sigma_0\) to the SPS problem that we can assume to be finite-memory, that is, we guess it as a Moore machine \(\mathcal{M}\) with an exponential number of memory states. We then verify that \(\sigma_0\) is indeed a solution by first computing the set \(P_{\sigma_0}\) and then checking that every \(\sigma_0\)-fixed Pareto-optimal play satisfies the objective \(\Omega_0\) of Player 0. To this end, we construct the cartesian product \(G \times \mathcal{M}\) which is an automaton whose infinite paths are exactly the plays consistent with \(\sigma_0\). We then use classical results from automata theory about the emptiness problem for an intersection of reachability (resp. parity) objectives to get the announced exponential complexity of our verifying algorithm.

\(^4\) Recall that to have a solution to the SPS problem, memory may be necessary as shown in Example 2.
4.2 NP-Completeness for Tree Arenas

Before turning to the NEXPTIME-hardness of the SPS problem in the next section, we first want to show that the SPS problem is already \(\text{NP}\)-complete in the simple setting of reachability objectives and arenas that are trees. To do so, we use a reduction from the Set Cover problem (SC problem) which is \(\text{NP}\)-complete [23].

▶ Theorem 7. The SPS problem is \(\text{NP}\)-complete for reachability SP games on tree arenas.

Notice that when the game arena is a tree, it is easy to design an algorithm for solving the SPS problem that is in \(\text{NP}\). First, we nondeterministically guess a strategy \(\sigma_0\) that can be assumed to be memoryless as the arena is a tree. Second, we apply a depth-first search algorithm from the root vertex which accumulates to leaf vertices the extended payoff of plays which are consistent with \(\sigma_0\). Finally, we check that \(\sigma_0\) is a solution.

Let us explain why the SPS problem is \(\text{NP}\)-hard on tree arenas by reduction from the SC problem. We recall that an instance of the SC problem is defined by a set \(C = \{e_1, e_2, \ldots, e_n\}\) of \(n\) elements, \(m\) subsets \(S_1, S_2, \ldots, S_m\) such that \(S_i \subseteq C\) for each \(i \in \{1, \ldots, m\}\), and an integer \(k\) \(\leq m\). The problem consists in finding \(k\) indexes \(i_1, i_2, \ldots, i_k\) such that the union of the corresponding subsets equals \(C\), i.e., \(C = \bigcup_{j=1}^k S_{i_j}\).

Given an instance of the SC problem, we construct a game with an arena consisting of \(n + k \cdot (m + 1) + 3\) vertices. The arena \(G\) of the game is provided in Figure 4 and can be seen as two sub-arenas reachable from the initial vertex \(v_0\). The game is such that there is a solution to the SC problem if and only if Player 0 has a strategy from \(v_0\) in \(G\) which is a solution to the SPS problem. The game is played between Player 0 with reachability objective \(\Omega_0\) and Player 1 with \(n + 1\) reachability objectives. The objectives are defined as follows: \(\Omega_0 = \text{Reach}(\{v_2\})\), \(\Omega_i = \text{Reach}(\{e_i\} \cup \{S_j \mid e_i \in S_j\})\) for \(i \in \{1, 2, \ldots, n\}\) and \(\Omega_{n+1} = \text{Reach}(\{v_2\})\). First, notice that every play in \(G_1\) is consistent with any strategy of Player 0 and is lost by that player. It holds that for each \(i \in \{1, 2, \ldots, n\}\), there is such a play with payoff \((p_1, \ldots, p_{n+1})\) such that \(p_i = 1\) and \(p_j = 0\) for \(j \neq i\). These payoffs correspond to the elements \(e_i\) which are to be covered in the SC problem. A play in \(G_2\) visits \(v_2\) and then a vertex \(c\) from which Player 0 selects a vertex \(S\). Such a play is always won by Player 0 and its payoff is \((p_1, \ldots, p_{n+1})\) such that \(p_{n+1} = 1\) and \(p_r = 1\) if and only if the element \(e_r\) belongs to the set \(S\). It follows that the payoff of such a play corresponds to a set of elements in the SC problem. It is easy to see that the following proposition holds.

▶ Proposition 8. There is a solution to an instance of the SC problem if and only if Player 0 has a strategy from \(v_0\) in the corresponding SP game that is a solution to the SPS problem.
4.3 NEXPTIME-Hardness

Let us come back to regular game arenas and show the NEXPTIME-hardness result thanks to the succinct variant of the SC problem presented below.

Theorem 9. The SPS problem is NEXPTIME-hard for reachability and parity SP games.

Succinct Set Cover Problem. The Succinct Set Cover problem (SSC problem) is defined as follows. We are given a Conjunctive Normal Form (CNF) formula \( \phi = C_1 \land C_2 \land \cdots \land C_p \) over the variables \( X = \{x_1, x_2, \ldots, x_m\} \) made up of \( p \) clauses, each containing some disjunction of literals of the variables in \( X \). The set of valuations of the variables \( X \) which satisfy \( \phi \) is written \( [\phi] \). We are also given an integer \( k \in \mathbb{N} \) (encoded in binary) and an other CNF formula \( \psi = D_1 \land D_2 \land \cdots \land D_q \) over the variables \( X \cup Y \) with \( Y = \{y_1, y_2, \ldots, y_n\} \), made up of \( q \) clauses. Given a valuation \( \text{valy} : Y \to \{0, 1\} \) of the variables in \( Y \), called a partial valuation, we write \( \psi[\text{valy}] \) the CNF formula obtained by replacing in \( \psi \) each variable \( y \in Y \) by its valuation \( \text{valy}(y) \). We write \( [\psi[\text{valy}]] \) the valuations of the remaining variables \( X \) which satisfy \( \psi[\text{valy}] \). The SSC problem is to decide whether there exists a set \( K = \{ \text{valy} \mid \text{valy} : Y \to \{0, 1\} \} \) of \( k \) valuations of the variables in \( Y \) such that the valuations of the remaining variables \( X \) which satisfy the formulas \( \psi[\text{valy}] \) include the valuations of \( X \) which satisfy \( \phi \). Formally, we write this \( [\phi] \subseteq \bigcup_{\text{valy} \in K} [\psi[\text{valy}]] \).

We can show that this corresponds to a set cover problem succinctly defined using CNF formulas. The set \( [\phi] \) of valuations of \( X \) which satisfy \( \phi \) corresponds to the set of elements we aim to cover. Parameter \( k \) is the number of sets that can be used to cover these elements. Such a set is described by a formula \( \psi[\text{valy}] \), given a partial valuation \( \text{valy} \), and its elements are the valuations of \( X \) in \( [\psi[\text{valy}]] \). This is illustrated in the following example.

Example 10. Consider the CNF formula \( \phi = (x_1 \lor \neg x_2) \land (x_2 \lor x_3) \) over the variables \( X = \{x_1, x_2, x_3\} \). The set of valuations of the variables which satisfy \( \phi \) is \( [\phi] = \{(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 0, 1)\} \). Each such valuation corresponds to one element we aim to cover. Consider the CNF formula \( \psi = (y_1 \lor y_2) \land (x_1 \lor y_2) \land (x_2 \lor y_3 \lor y_1) \) over the variables \( X \cup Y \) with \( Y = \{y_1, y_2\} \). Given the partial valuation \( \text{valy} \) of the variables in \( Y \) such that \( \text{valy}(y_1) = 0 \) and \( \text{valy}(y_2) = 1 \), we get the CNF formula \( \psi[\text{valy}] = (0 \lor 1) \land (x_1 \lor 1) \land (x_2 \lor x_3 \lor 0) \). This formula describes the contents of the set identified by the partial valuation (as a partial valuation yields a unique formula). The valuations of the variables \( X \) which satisfy \( \psi[\text{valy}] \) are the elements contained in the set. In this case, these elements are \( [\psi[\text{valy}]] = \{(0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 0, 1), (1, 1, 1)\} \). We can see that this set contains the elements \( \{(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 0, 1)\} \) of \( [\phi] \).

It is easy to see that the SSC problem is in NEXPTIME. Its NEXPTIME-hardness can be obtained by reduction from the Succinct Dominating Set problem, which is NEXPTIME-complete for graphs succinctly encoded using CNF formulas [14].

Theorem 11. The SSC problem is NEXPTIME-complete.
from $v_0$ in $G$ which is a solution to the SPS problem. The arena $G$, provided in Figure 5, can be viewed as three sub-arenas reachable from $v_0$. Sub-arenas $G_1$ and $G_2$ are completely controlled by Player 1. Plays entering these sub-arenas are therefore consistent with any strategy of Player 0. Sub-arena $G_3$ starts with a gadget $Q_k$ whose vertices belong to Player 1 and which provides exactly $k$ different paths from $v_0$ to $v_3$.

**Objectives.** The game is played between Player 0 with reachability objective $\Omega_0$ and Player 1 with $t = 1 + 2 \cdot m + p + q$ reachability objectives. The payoff of a play therefore consists in a single Boolean for objective $\Omega_1$, a vector of $2 \cdot m$ Booleans for objectives $\Omega_{x_1}, \Omega_{\neg x_1}, \ldots, \Omega_{x_m}, \Omega_{\neg x_m}$, a vector of $p$ Booleans for objectives $\Omega_{C_1}, \ldots, \Omega_{C_p}$ and a vector of $q$ Booleans for objectives $\Omega_{D_1}, \ldots, \Omega_{D_q}$. The objectives are defined as follows.

- The target set for objective $\Omega_0$ of Player 0 and objective $\Omega_1$ of Player 1 is $\{v_2, v_3\}$.
- The target set for objective $\Omega_{x_i}$ (resp. $\Omega_{\neg x_i}$) with $i \in \{1, \ldots, m\}$ is the set of vertices labeled $x_i$ (resp. $\neg x_i$) in $G_1$, $G_2$ and $G_3$.
- The target set for objective $\Omega_{C_i}$ with $i \in \{1, \ldots, p\}$ is the set of vertices in $G_1$ and $G_3$ corresponding to the literals of $X$ which make up the clause $C_i$ in $\phi$. In addition, vertex $i_j$ in $G_2$ belongs to the target set of objective $\Omega_{C_i}$ for all $\ell \in \{1, \ldots, p\}$ such that $\ell \neq j$.
- The target set of objective $\Omega_{D_i}$ with $i \in \{1, \ldots, q\}$ is the set of vertices in $G_3$ corresponding to the literals of $X$ and $Y$ which make up the clause $D_i$ in $\psi$. In addition, vertices $v_1$ and $v_2$ satisfy every objective $\Omega_{D_i}$ with $i \in \{1, \ldots, q\}$.

**Payoff of Plays in $G_1$.** Plays in $G_1$ do not satisfy objective $\Omega_0$ of Player 0 nor objective $\Omega_1$ of Player 1. A play in $G_1$ is of the form $v_0, v_1, z_1 \square \cdots \square (z_m)^\omega$ where $z_i$ is either $x_i$ or $\neg x_i$. It follows that a play satisfies the objective $\Omega_{x_i}$ or $\Omega_{\neg x_i}$ for each $x_i \in X$. The vector of payoffs for these objectives corresponds to a valuation of the variables in $X$, expressed as a vector of $2 \cdot m$ Booleans. In addition, due to the way the objectives are defined, objective $\Omega_{C_i}$ is satisfied in a play if and only if clause $C_i$ of $\phi$ is satisfied by the valuation this play corresponds to. The objective $\Omega_{D_i}$ for $i \in \{1, \ldots, q\}$ is satisfied in every play in $G_1$.

**Lemma 12.** Plays in $G_1$ are consistent with any strategy of Player 0. Their payoff are of the form $(0, val, sat(\phi, val), 1, \ldots, 1)$ where $val$ is a valuation of the variables in $X$ expressed as a vector of payoffs for objectives $\Omega_{x_i}$ to $\Omega_{\neg x_m}$ and $sat(\phi, val)$ is the vector of payoffs for objectives $\Omega_{C_i}$ to $\Omega_{C_p}$ corresponding to that valuation. All plays in $G_1$ are lost by Player 0.
Payoff of Plays in $G_2$. Plays in $G_2$ satisfy the objectives $\Omega_0$ of Player 0 and $\Omega_1$ of Player 1. A play in $G_2$ is of the form $v_0 v_2 i_j \square z_1 \square \cdots \square (z_m)^2$ where $z_2$ is either $x_i$ or $\neg x_i$. It follows that a play satisfies either the objective $\Omega_x$ or $\Omega_{\neg x}$ for each $x \in X$ which again corresponds to a valuation of the variables in $X$. The objective $\Omega_{D_i}$ for $i \in \{1, \ldots, q\}$ is satisfied in every play in $G_2$. Compared to the plays in $G_1$, the difference lies in the objectives corresponding to clauses of $\phi$ which are satisfied. In any play in $G_2$, a vertex $i_j$ with $j \in \{1, \ldots, p\}$ is first visited, satisfying all the objectives $\Omega_{C_j}$ with $\ell \in \{1, \ldots, p\}$ and $\ell \neq j$. All but one objective corresponding to the clauses of $\phi$ are therefore satisfied.

Lemma 13. Plays in $G_2$ are consistent with any strategy of Player 0. Their payoff are of the form $(1, \text{val}, \text{vec}, 1, \ldots, 1)$ where $\text{val}$ is a valuation of the variables in $X$ expressed as a vector of payoffs for objectives $\Omega_x$, $\Omega_{\neg x}$ and $\text{vec}$ is a vector of payoffs for objectives $\Omega_{C_i}$ to $\Omega_{C_p}$ in which all of them except one are satisfied. All plays in $G_2$ are won by Player 0.

Plays in $G_2$ are such that their payoff is strictly larger than the payoff of plays in $G_1$ whose valuation of $X$ does not satisfy $\phi$. It is easy to see that, when considering $G_1$ and $G_2$, the only plays in $G_1$ with a Pareto-optimal payoff are exactly those whose valuation satisfies all clauses of $\phi$. The following lemma therefore holds.

Lemma 14. The set of payoffs of plays in $G_1$ that are $\sigma_0$-fixed Pareto-optimal when considering $G_1 \cup G_2$ for any strategy $\sigma_0$ of Player 0 is equal to the set of payoffs of plays in $G_1$ whose valuation of $X$ satisfies $\phi$.

Problematic Payoffs in $G_1$. The plays described in the previous lemma correspond exactly to the valuations of $X$ which satisfy $\phi$ and therefore to the elements we aim to cover in the SSC problem. They are $\sigma_0$-fixed Pareto-optimal when considering $G_1 \cup G_2$ and are lost by Player 0. All other $\sigma_0$-fixed Pareto-optimal payoffs in $G_1 \cup G_2$ are only realized by plays in $G_2$ which are all won by Player 0. It follows that in order for Player 0 to find a strategy $\sigma_0$ from $v_0$ that is solution to the SPS problem, it must hold that those payoffs are not $\sigma_0$-fixed Pareto-optimal when considering $G_1 \cup G_2 \cup G_3$. Otherwise, a play consistent with $\sigma_0$ with a $\sigma_0$-fixed Pareto-optimal payoff is lost by Player 0. We call those payoffs problematic payoffs.

In order for Player 0 to find a strategy $\sigma_0$ which is a solution to the SPS problem, this strategy must be such that for each problematic payoff in $G_1$, there is a play in $G_3$ consistent with $\sigma_0$ and with a strictly larger payoff. Since the plays in $G_3$ are all won by Player 0, this would ensure that the strategy $\sigma_0$ is a solution to the problem. This corresponds in the SSC problem to selecting a series of sets in order to cover the valuations of $X$ which satisfy $\phi$.

Payoff of Plays in $G_3$. Plays in $G_3$ satisfy the objectives $\Omega_0$ of Player 0 and $\Omega_1$ of Player 1. A play in $G_3$ consistent with a strategy $\sigma_0$ is of the form $v_0 \square \cdots \square v_3 r_1 \circ \cdots \circ r_n \square z_1 \square \cdots \square (z_m)^2$ where $r_i$ is either $y_i$ or $\neg y_i$ and $z_1$ is either $x_i$ or $\neg x_i$. Since only the vertices leading to $y$ or $\neg y$ for $y \in Y$ belong to Player 0, it holds that $v_3 r_1 \circ \cdots \circ r_n$ is the only part of any play in $G_3$ which is directly influenced by $\sigma_0$. That part of a play comes after a history from $v_0$ to $v_3$ of which there are $k$, provided by gadget $Q_k$. By definition of a strategy, this can be interpreted as Player 0 making a choice of valuation of the variables in $Y$ after each of those $k$ histories. After this, the play satisfies either the objective $\Omega_x$ or $\Omega_{\neg x}$ for each $x \in X$ which corresponds to a valuation of $X$. Due to the way the objectives are defined, the objective $\Omega_{C_i}$ (resp. $\Omega_{D_i}$) is satisfied if and only if clause $C_i$ of $\phi$ (resp. $D_i$ of $\psi$) is satisfied by the valuation of the variables in $X$ (resp. $X$ and $Y$) the play corresponds to.
**Creating Strictly Larger Payoffs in $G_3$.** In order to create a play with a payoff $r'$ that is strictly larger than a problematic payoff $r$, $\sigma_0$ must choose a valuation of $Y$ such that there exists a valuation of the remaining variables $X$ which together with this valuation of $Y$ satisfies $\psi$ and $\phi$ (since in $r$ every objective $\Omega_{C_i}$ for $i \in \{1, \ldots, p\}$ and $\Omega_{D_i}$ for $i \in \{1, \ldots, q\}$ is satisfied). Since the plays in $G_3$ also satisfy the objective $\Omega_1$ and plays in $G_1$ do not, this ensures that $r < r'$.

We finally briefly explain why the proposed reduction is correct. In case of a positive instance of the SSC problem, by carefully selecting $k$ valuations of $Y$, Player 0 ensures that for each valuation $val_X$ satisfying $\phi$, there is a play in $G_3$ with a valuation $val_Y$ such that $val_X \in [\psi[val_Y]]$. Therefore, when considering the whole arena, no play in $G_1$ is Pareto-optimal and every Pareto-optimal play is won by Player 0. In case of a negative instance, Player 0 is not able to do so and some play in $G_1$ thus has a Pareto-optimal payoff and is lost by Player 0.

**5 Conclusion**

We have introduced in this paper the class of two-player SP games and the SPS problem in those games. We provided a reduction from SP games to a two-player zero-sum game called the C-P game, which we used to obtain FPT results on solving this problem. We showed how the arena and the generic objective of this C-P game can be adapted to specifically handle reachability and parity SP games. This allowed us to prove that reachability (resp. parity) SP games are in FPT when the number $t$ of objectives of Player 1 (resp. when $t$ and the maximal priority according to each priority function in the game) is a parameter. We then turned to the complexity class of the SPS problem and sketched the main arguments used in our proof of its NEXPTIME-membership, which relied on showing that any solution to the SPS problem in a reachability or parity SP game can be transformed into a solution with an exponential memory. We showed the NP-completeness of the problem in the simple setting of reachability SP games played on tree arenas. We then came back to regular game arenas and established the NEXPTIME-hardness of the SPS problem in reachability and parity SP games. This result relied on a reduction from the SSC problem which we proved to be NEXPTIME-complete, a result of potential independent interest.

In future work, we want to study other $\omega$-regular objectives as well as quantitative objectives such as mean-payoff in the framework of SP games and the SPS problem. It would also be interesting to study whether other works, such as rational synthesis, could benefit from the approaches used in this paper.

**References**


