Finitely Tractable Promise Constraint Satisfaction Problems

Kristina Asimi
Department of Algebra, Faculty of Mathematics and Physics, Charles University, Prague, Czechia

Libor Barto
Department of Algebra, Faculty of Mathematics and Physics, Charles University, Prague, Czechia

Abstract

The Promise Constraint Satisfaction Problem (PCSP) is a generalization of the Constraint Satisfaction Problem (CSP) that includes approximation variants of satisfiability and graph coloring problems. Barto [LICS '19] has shown that a specific PCSP, the problem to find a valid Not-All-Equal solution to a 1-in-3-SAT instance, is not finitely tractable in that it can be solved by a trivial reduction to a tractable CSP, but such a CSP is necessarily over an infinite domain (unless P=NP). We initiate a systematic study of this phenomenon by giving a general necessary condition for finite tractability and characterizing finite tractability within a class of templates – the "basic" tractable cases in the dichotomy theorem for symmetric Boolean PCSPs allowing negations by Brakensiek and Guruswami [SODA'18].

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1 Introduction

Many computational problems, including various versions of logical satisfiability, graph coloring, and systems of equations can be phrased as Constraint Satisfaction Problems (CSPs) over fixed templates (see [5]). One of the possible formulations of the CSP is via homomorphisms of relational structures: a *template* $\mathbb{A}$ is a relational structure with finitely many relations and the CSP over $\mathbb{A}$, written CSP($\mathbb{A}$), is the problem to decide whether a given finite relational structure $\mathbb{X}$ (similar to $\mathbb{A}$) admits a homomorphism to $\mathbb{A}$.

The complexity of CSPs over finite templates (i.e., those templates whose domain is a finite set) is now completely classified by a celebrated dichotomy theorem independently obtained by Bulatov [10] and Zhuk [19, 20]: every CSP($\mathbb{A}$) is either tractable (that is, solvable in polynomial-time) or NP-complete. The landmark results leading to the complete classification include Schaefer's dichotomy theorem [18] for CSPs over Boolean structures (i.e., structures with a two-element domain), Hell and Nešetřil's dichotomy theorem [15] for CSPs over graphs, and Feder and Vardi's thorough study [13] through Datalog and group theory. The latter paper also inspired the development of a mathematical theory of finite-template CSPs [16, 9, 6], the so called *algebraic approach*, that provided guidance and tools for the general dichotomy theorem by Bulatov and Zhuk.

The algebraic approach has been successfully applied in many variants and generalizations of the CSP such as the infinite-template CSP [7] or valued CSP [17]. This paper concerns a recent vast generalization of the basic CSP framework, the Promise CSP (PCSP).

A *template* for the PCSP is a pair $(\mathbb{A}, \mathbb{B})$ of similar structures such that $\mathbb{A}$ has a homomorphism to $\mathbb{B}$, and the PCSP over $(\mathbb{A}, \mathbb{B})$, written $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$, is the problem to distinguish between the case that a given finite structure $\mathbb{X}$ admits a homomorphism to $\mathbb{A}$ and the case that $\mathbb{X}$ does not have a homomorphism to $\mathbb{B}$ (the promise is that one of the cases takes place). This framework generalizes that of CSP (take $\mathbb{A} = \mathbb{B}$) and additionally includes important problems in approximation, e.g., if $\mathbb{A} = \mathbb{K}_k$ (the clique on k vertices) and $\mathbb{B} = \mathbb{K}_l$, $k \leq l$, then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is a version of the approximate graph coloring problem, namely, the problem to distinguish graphs that are k-colorable from those that are not l-colorable, a problem whose complexity is open after more than 40 years of research. On the other hand, the basics of the algebraic approach to CSPs can be generalized to PCSPs [1, 8, 11, 3].

The approximate graph coloring problem shows that a full classification of the complexity of PCSPs over graph templates is still open and so is the analogue of Schaefer's Boolean CSP, PCSPs over pairs of Boolean structures. However, strong partial results have already been obtained. Brakensiek and Guruswami [8] proved a dichotomy theorem for all symmetric Boolean templates allowing negations, i.e., templates $(\mathbb{A}, \mathbb{B})$ such that $\mathbb{A} = (\{0,1\}; R_0, R_1, \dots)$, $\mathbb{B} = (\{0,1\}; S_0, S_1, \dots)$, each relation R_i, S_i is invariant under permutations of coordinates, and $R_0 = S_0$ is the binary disequality relation $\neq$. Ficak, Kozik, Olšák, and Stankiewicz [14] later generalized this result to all symmetric Boolean templates. These templates play a central role in this paper.

To prove tractability or hardness results for PCSPs, a very simple but useful reduction is often applied: If $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{A}', \mathbb{B}')$ are similar PCSP templates and there exist homomorphisms $\mathbb{A}' \to \mathbb{A}$ and $\mathbb{B} \to \mathbb{B}'$, then the trivial reduction (which does not change the instance) reduces $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ to $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$; we say that $(\mathbb{A}', \mathbb{B}')$ is a *homomorphic relaxation* of $(\mathbb{A}, \mathbb{B})$. In fact, all the tractable symmetric Boolean PCSPs can be reduced in this way to a tractable CSP over a structure with a *possibly infinite domain*.

An interesting example of a PCSP that can be naturally reduced to a tractable CSP over an infinite domain is the following problem. An instance is a list of triples of variables and the problem is to distinguish instances that are satisfiable as positive 1-in-3-SAT instances from those that are not even satisfiable as Not-All-Equal-3-SAT instances. This computational problem is essentially the same as $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ where $\mathbb{A}$ consists of the ternary 1-in-3 relation over $\{0,1\}$ and $\mathbb{B}$ consists of the ternary not-all-equal relation over $\{0,1\}$. It is easy to see that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ where $\mathbb{C}$ is the relation "$x+y+z=1$" over the set of all integers. Therefore $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is reducible (by means of the trivial reduction) to $\mathrm{PCSP}(\mathbb{C}, \mathbb{C}) = \mathrm{CSP}(\mathbb{C})$ which is a tractable problem. The main result of [2] is that no finite structure can be used in place of $\mathbb{C}$ for this particular template – this PCSP is not finitely tractable in the sense of the following definition.

▶ **Definition 1.** *We say that* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is* finitely tractable *if there exists a finite relational structure* $\mathbb{C}$ *such that* $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ *and* $\mathrm{CSP}(\mathbb{C})$ *is tractable. Otherwise we call* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ not finitely tractable*. (We assume $P \neq NP$ throughout the paper.)*

In this paper, we initiate a systematic study of this phenomenon. As the main technical contribution, we determine which of the "basic tractable cases" in Brakensiek and Guruswami's classification [8] are finitely tractable. It turns out that finite tractability is quite rare, so the infinite nature of the 1-in-3 versus Not-All-Equal problem is not exceptional at all.

1.1 Symmetric Boolean PCSPs allowing negations

We now discuss the classification of symmetric Boolean templates allowing negations from [8]. It will be convenient to describe these templates by listing the corresponding relation pairs, that is, instead of $(\mathbb{A} = (\{0,1\}; R_1, \dots, R_n), \mathbb{B} = (\{0,1\}; S_1, \dots, S_n))$ we describe this template by the list (R_1, S_1), ..., (R_n, S_n). Recall that the template is *symmetric* if all the involved relations are symmetric, i.e., invariant under any permutation of coordinates, and the template *allows negations* if $(\neq, \neq)$ is among the relation pairs, where $\neq = \{(0,1),(1,0)\}$ is the disequality relation.

It may be also helpful to think of an instance of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ as a list of constraints of the form R_i(variables) and the problem is to distinguish between instances where each constraint is satisfiable and those which are not satisfiable even when we replace each R_i by the corresponding "relaxed version" S_i. Allowing negations then means that we can use constraints $x \neq y$ – we can effectively negate variables.

The following relations are important for the classification.

- odd-in-$s = \{\mathbf{x} \in \{0,1\}^s : \sum_{i=1}^{s} x_i \text{ is odd}\}$, even-in-$s = \{\mathbf{x} \in \{0,1\}^s : \sum_{n=1}^{s} x_i \text{ is even}\}$
- r-in-$s = \{\mathbf{x} \in \{0,1\}^s : \sum_{n=1}^{s} x_i = r\}$
- $\leq r$-in-$s = \{\mathbf{x} \in \{0,1\}^s : \sum_{i=1}^{s} x_i \leq r\}$, $\geq r$-in-$s = \{\mathbf{x} \in \{0,1\}^s : \sum_{i=1}^{s} x_i \geq r\}$
- not-all-equal-$s = \{\mathbf{x} \in \{0,1\}^s : \sum_{i=1}^{s} x_i \notin \{0,s\}\}$

The next theorem lists some of the tractable cases of the classification, which are "basic" in the sense explained below.

▶ **Theorem 2** ([8]). $\mathrm{PCSP}((P,Q),(\neq,\neq))$ *is tractable if* (P,Q) *is equal to*

(a) $(\textit{odd-in-}s, \textit{odd-in-}s)$*, or* $(\textit{even-in-}s, \textit{even-in-}s)$*, or*
(b) $(\leq r\textit{-in-}s, \leq(2r-1)\textit{-in-}s)$ *and* $r \leq s/2$*, or*
$(\geq r\textit{-in-}s, \geq(2r-s+1)\textit{-in-}s)$ *and* $r \geq s/2$*, or*
(c) $(r\textit{-in-}s, \textit{not-all-equal-}s)$

for some positive integers r, s.

It follows from the results in [8] (namely Theorem 2.1 and a simple analysis of compatible relations) that every tractable symmetric Boolean PCSP allowing negations can be obtained by

- taking any number of $(\neq, \neq)$ and any number of relation pairs from a fixed item in Theorem 2,
- adding any number of "trivial" relation pairs (P,Q) such that $P \subseteq Q$, and Q is the full relation or P contains only constant tuples, and
- taking a homomorphic relaxation of the obtained template.

In this sense, Theorem 2 provides building blocks for all tractable templates.

1.2 Contributions

Some of the cases in Theorem 2 are finitely tractable: templates in item (a) are tractable CSPs (they can be decided by solving systems of linear equations of the two-element field), templates in item (c) for r odd and s even are homomorphic relaxations of (odd-in-s, odd-in-s), and templates in item (b) for $r = 1$ or $r = s-1$ as well as all templates with $s \leq 2$ are tractable CSPs (reducible to 2-SAT) [18, 5]. Our main theorem proves that all the remaining cases are not finitely tractable. In fact, we prove this property even for some relaxations of these templates:

▶ **Theorem 3.** *The PCSP over any of the following templates is not finitely tractable.*

(1) $(r\textit{-in-}s, \leq(2r-1)\textit{-in-}s), (\neq,\neq)$ *where* $1 < r < s/2$*,*
$(r\textit{-in-}s, \geq(2r-s+1)\textit{-in-}s), (\neq,\neq)$ *where* $s/2 < r < s-1$
(2) $(\leq r\textit{-in-}s, \leq(2r-1)\textit{-in-}s), (\neq,\neq)$ *where* s *is even,* $1 < r = s/2$
$(\geq r\textit{-in-}s, \geq(2r-s+1)\textit{-in-}s), (\neq,\neq)$ *where* s *is even,* $1 < r = s/2$

(3) $(r\text{-}in\text{-}s, \leq(2r-1)\text{-}in\text{-}s), (\neq, \neq)$ *where s is even, $1 < r = s/2$, and r is even*
$(r\text{-}in\text{-}s, \geq(2r-s+1)\text{-}in\text{-}s), (\neq, \neq)$ *where s is even, $1 < r = s/2$, and r is even*

(4) $(r\text{-}in\text{-}s, not\text{-}all\text{-}equal\text{-}s)$ *where $s > r$, $s > 2$, and r is even or s is odd*

Note that the templates in the last item do not contain the disequality pair; the special case with $r = 1$ and $s = 3$ is the main result of [2]. Disequalities in the other items are necessary, since otherwise the templates are homomorphic relaxations of CSPs over one-element structures.

In Theorem 18 we provide a general necessary condition for finite tractability of an arbitrary finite-template PCSP in terms of so called h1 identities. Showing that templates in Theorem 3 do not satisfy this necessary condition forms the bulk of the paper.

The necessary condition in Theorem 18 seems very unlikely to be sufficient for finite tractability. Nevertheless, we observe in Theorem 12 that finite tractability *does* depend only on h1 identities, just like standard tractability [11], see Theorem 10 and the discussion following the theorem.

2 Preliminaries

2.1 PCSP

We use the notation $[n] = \{1, 2, \ldots, n\}$ throughout the paper.

A *relational structure* (of finite signature) is a tuple $\mathbb{A} = (A; R_1, R_2, \ldots, R_n)$ where A is a set, called the *domain*, and each R_i is a relation on A of arity $\mathrm{ar}(R_i) \geq 1$, that is, $R_i \subseteq A^{\mathrm{ar}(R_i)}$. The structure $\mathbb{A}$ is finite if A is finite. Two relational structures $\mathbb{A} = (A; R_1, R_2, \ldots, R_n)$ and $\mathbb{B} = (B; S_1, S_2, \ldots, S_n)$ are *similar* if they have the same number of relations and $\mathrm{ar}(R_i) = \mathrm{ar}(S_i)$ for each $i \in [n]$. In this case, a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a mapping $f : A \to B$ such that $(f(a_1), f(a_2), \ldots, f(a_k)) \in S_i$ whenever $i \in [n]$ and $(a_1, a_2, \ldots, a_k) \in R_i$ where $k = \mathrm{ar}(R_i)$. If there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, we write $\mathbb{A} \to \mathbb{B}$, and if there is none, we write $\mathbb{A} \not\to \mathbb{B}$.

▶ **Definition 4.** *A* PCSP template *is a pair* $(\mathbb{A}, \mathbb{B})$ *of similar relational structures such that* $\mathbb{A} \to \mathbb{B}$.

The PCSP *over* $(\mathbb{A}, \mathbb{B})$*, written* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$*, is the following problem. Given a finite relational structure* $\mathbb{X}$ *similar to* $\mathbb{A}$ *(and* $\mathbb{B}$*), output "Yes." if* $\mathbb{X} \to \mathbb{A}$ *and output "No." if* $\mathbb{X} \not\to \mathbb{B}$.

We define $\mathrm{CSP}(\mathbb{A}) = \mathrm{PCSP}(\mathbb{A}, \mathbb{A})$.

▶ **Definition 5.** *Let* $(\mathbb{A}, \mathbb{B})$ *and* $(\mathbb{A}', \mathbb{B}')$ *be similar* PCSP *templates. We say that* $(\mathbb{A}', \mathbb{B}')$ *is a* homomorphic relaxation *of* $(\mathbb{A}, \mathbb{B})$ *if* $\mathbb{A}' \to \mathbb{A}$ *and* $\mathbb{B} \to \mathbb{B}'$.

Recall that if $(\mathbb{A}', \mathbb{B}')$ is a homomorphic relaxation of $(\mathbb{A}, \mathbb{B})$, then the trivial reduction, which does not change the input structure $\mathbb{X}$, reduces $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ to $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$.

2.2 Polymorphisms

A crucial concept for the algebraic approach to (P)CSP is a polymorphism.

▶ **Definition 6.** *Let* $\mathbb{A} = (A; R_1, \ldots, R_m)$ *and* $\mathbb{B} = (B; S_1, \ldots, S_m)$ *be two similar relational structures. A function* $c : A^n \to B$ *is a* polymorphism *from* $\mathbb{A}$ *to* $\mathbb{B}$ *if for each relation* R_i *in* $\mathbb{A}$ *with* $k_i = arity(R_i)$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k_i 1} \end{pmatrix} \in R_i, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k_i 2} \end{pmatrix} \in R_i \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{k_i n} \end{pmatrix} \in R_i \Rightarrow \begin{pmatrix} c(a_{11}, a_{12}, \ldots, a_{1n}) \\ c(a_{21}, a_{22}, \ldots, a_{23}) \\ \vdots \\ c(a_{k_i 1}, a_{k_i 2}, \ldots, a_{k_i n}) \end{pmatrix} \in S_i.$$

We denote the set of all polymorphisms from $\mathbb{A}$ to $\mathbb{B}$ by $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ and define $\mathrm{Pol}(\mathbb{C}) = \mathrm{Pol}(\mathbb{C}, \mathbb{C})$.

The computational complexity of a PCSP depends only on the set of polymorphisms of its template [8]. We note that tractability of the PCSPs in Theorem 2 stems from nice polymorphisms: parities (item (a)), majorities (item (b)), and alternating thresholds (item (c)).

The set of polymorphisms is an algebraic object named minion in [11], which we define in Definition 8 below.

▶ **Definition 7.** *An n-ary function $f^\pi : A^n \to B$ is called a* minor *of an m-ary function $f : A^m \to B$ given by a map $\pi : [m] \to [n]$ if*

$$f^\pi(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(m)})$$

for all $x_1, \ldots, x_n \in A$.

▶ **Definition 8.** *Let $\mathcal{O}(A, B) = \{f : A^n \to B : n \geq 1\}$. A* minion *on (A, B) is a non-empty subset $\mathcal{M}$ of $\mathcal{O}(A, B)$ that is closed under taking minors. For fixed $n \geq 1$, let $\mathcal{M}^{(n)}$ denote the set of n-ary functions from $\mathcal{M}$.*

As mentioned, $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is always a minion and the complexity of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ depends only on $\mathcal{M}$. This result was strengthened in [11, 3] (generalizing the same result for CSPs [6]) as follows.

▶ **Definition 9.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two minions. A mapping $\xi : \mathcal{M} \to \mathcal{N}$ is called a* minion homomorphism *if it preserves arities and preserves taking minors, i.e., $\xi(f^\pi) = (\xi(f))^\pi$ for every $f \in \mathcal{M}^{(m)}$ and every $\pi : [m] \to [n]$.*

▶ **Theorem 10.** *Let $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{A}', \mathbb{B}')$ be PCSP templates. If there exists a minion homomorphism $\mathrm{Pol}(\mathbb{A}', \mathbb{B}') \to \mathrm{Pol}(\mathbb{A}, \mathbb{B})$, then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is log-space reducible to $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$.*

An *h1 identity* (h1 stands for height one) is a meaningful expression of the form function(variables) $\approx$ function(variables), e.g., if $f : A^3 \to B$ and $g : A^4 \to B$, then $f(x, y, x) \approx g(y, x, x, z)$ is an h1 identity. Such an h1 identity is *satisfied* if the corresponding equation holds universally, e.g., $f(x, y, x) \approx g(y, x, x, z)$ is satisfied if and only if $f(x, y, x) = g(y, x, x, z)$ for every $x, y, z \in A$.

Every minion homomorphism $\xi : \mathcal{M} \to \mathcal{N}$ preserves h1 identities in the sense that if functions $f, g \in \mathcal{M}$ satisfy an h1 identity, then so do their ξ-images $\xi(f), \xi(g) \in \mathcal{N}$. In fact, an arity-preserving ξ between minions is a minion homomorphism if and only if it preserves h1 identities (see [6] for details). In this sense, Theorem 10 shows that the complexity of a PCSP depends only on h1 identities satisfied by polymorphisms.

2.3 Notation for tuples

Repeated entries in tuples will be indicated by $\times$, e.g. $(2 \times a, 3 \times b)$ stands for the tuple (a, a, b, b, b).

The i-th *cyclic shift* of a tuple $(x_1, \dots, x_m)$ is the tuple $(x_{(m-i \mod m)+1}, \dots, x_m, x_1, \dots, x_{(m-i-1 \mod m)+1})$. A *cyclic shift* is the i-th cyclic shift for some i. We will use cyclic shifts both for tuples of zeros and ones and tuples of variables.

We will often use special p-tuples and $n = p^2$-tuples of zeros and ones as arguments for Boolean functions, where p will be a fixed prime number. For $0 \leq k \leq p$, $0 \leq l \leq p^2$, and $0 \leq k^1, \dots, k^p \leq p$ we write

$$\langle k \rangle_p = (k \times 1, (p-k) \times 0) = (\underbrace{1, 1, \dots, 1}_{k}, \underbrace{0, 0, \dots, 0}_{p-k}), \quad \langle l \rangle_n = (\underbrace{1, 1, \dots, 1}_{l}, \underbrace{0, 0, \dots, 0}_{n-l})$$

and

$$\langle k^1, \dots, k^p \rangle_p = \langle k^1 \rangle_p \langle k^2 \rangle_p \dots \langle k^p \rangle_p$$

for the concatenation of $\langle k^1 \rangle_p$, ..., $\langle k^p \rangle_p$. (Note here that the "i" in k^i is an index, not an exponent.) The subscripts p and n in $\langle\rangle_p$ and $\langle\rangle_n$ will be usually clear from the context and we omit them. We will sometimes need to shift n-ary tuples $\langle k^1, \dots, k^p \rangle$ blockwise, e.g., to $\langle k^2 \dots, k^p, k^1 \rangle$. In such a situation we talk about a *p-ary cyclic shift* to avoid confusion.

It will be often convenient to think of an n-tuple $\mathbf{k} = \langle k^1, \dots, k^p \rangle$ as a $p \times p$ zero-one matrix with columns $\langle k^1 \rangle$, ..., $\langle k^p \rangle$. For example, the ones in $\langle p \times 5 \rangle$ form a $5 \times p$ "rectangle" and $\langle (p-2) \times 5, 2 \times 4 \rangle$ is "almost" a $5 \times p$ rectangle – the bottom right 1×2 corner is removed. A p-ary cyclic shift of $\mathbf{k}$ corresponds to cyclic permutation of columns.

The area of a zero-one n-tuple $\mathbf{k}$ is defined as the fraction of ones and is denoted $\lambda(\mathbf{k})$.

$$\lambda(\mathbf{k}) = \left(\sum_{i=1}^{n} k_i \right) / p^2$$

The area of $\langle k^1, \dots, k^p \rangle$ is thus $(k^1 + \dots + k^p)/p^2$.

If t is a p-ary function we simply write $t\langle k \rangle$ instead of $t(\langle k \rangle)$. Similar shorthand is used for n-ary functions and tuples $\langle k^1, \dots, k^p \rangle_p$.

3 Finitely tractable PCSPs

3.1 Finite tractability depends only on h1 identities

We start by observing that finite tractability also depends only on h1 identities satisfied by polymorphisms, just like standard tractability (recall the discussion about h1 identities and minion homomorphisms below Theorem 10). This result, Theorem 12, is an immediate consequence of the following lemma and Theorem 10.

▶ **Lemma 11.** *Let $(\mathbb{A}, \mathbb{B})$ be a PCSP template. Then the following are equivalent.*

- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is finitely tractable.*
- *There exists a finite relational structure $\mathbb{C}$ such that $\mathrm{CSP}(\mathbb{C})$ is solvable in polynomial time and there exists a minion homomorphism $\mathrm{Pol}(\mathbb{C}) \to \mathrm{Pol}(\mathbb{A}, \mathbb{B})$.*

Proof. This lemma is a consequence of known results and we only sketch the argument here. In Section II.B of [2] it is argued that the first item is equivalent to the claim that a finite tractable template $(\mathbb{C}, \mathbb{C})$ pp-constructs $(\mathbb{A}, \mathbb{B})$. The latter claim is equivalent to the second item by Theorem 4.12 in [3]. ◀

▶ **Theorem 12.** *Let* $(\mathbb{A}, \mathbb{B})$ *and* $(\mathbb{A}', \mathbb{B}')$ *be PCSP templates. If there exists a minion homomorphism* $\mathrm{Pol}(\mathbb{A}', \mathbb{B}') \to \mathrm{Pol}(\mathbb{A}, \mathbb{B})$ *and* $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ *is finitely tractable, then so is* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$.

3.2 Necessary condition for finite tractability

In this subsection, we derive the necessary condition for finite tractability that will be used to prove Theorem 3. A cyclic polymorphism is a starting point for the condition.

▶ **Definition 13.** *A function* $c : A^p \to B$ *is called* cyclic *if it satisfies the h1 identity*

$$c(x_1, x_2, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1).$$

Cyclic polymorphisms can be used [4] to characterize the borderline between tractable and NP-complete CSPs proposed in [9] and confirmed in [10, 19, 20]. We only state the direction needed in this paper.

▶ **Theorem 14** ([4])**.** *Let* $\mathbb{C}$ *be a CSP template over a finite domain* C*. If* $\mathrm{CSP}(\mathbb{C})$ *is not NP-complete, then* $\mathbb{C}$ *has a cyclic polymorphism of arity* p *for every prime number* $p > |C|$.

Polymorphism minions of CSP templates are closed under arbitrary composition (cf. [5]). In particular, if $\mathrm{CSP}(\mathbb{C})$ is not NP-complete, then $\mathrm{Pol}(\mathbb{C})$ contains the function

$$\begin{aligned} &t(x_{11}, x_{21}, \ldots, x_{p1},\ x_{12}, x_{22}, \ldots, x_{p2},\ \ldots,\ x_{1p}, x_{2p}, \ldots, x_{pp}) \\ &\quad = c(c(x_{11}, x_{21}, \ldots, x_{p1}), c(x_{12}, x_{22}, \ldots, x_{p2}), \ldots, c(x_{1p}, x_{2p}, \ldots, x_{pp})), \end{aligned} \tag{1}$$

where c is a p-ary cyclic function and $p > |C|$. Such a function satisfies strong h1 identities which are not satisfied by the templates in Theorem 3. We now (in two steps) describe one such collection of strong enough identities.

▶ **Definition 15.** *A function* $t : A^{p^2} \to B$ *is* doubly cyclic *if it satisfies every identity of the form* $t(\mathbf{x}_1, \ldots, \mathbf{x}_p) \approx t(\mathbf{y}_1, \ldots, \mathbf{y}_p)$*, where* $\mathbf{x}_i$ *is a* p*-tuple of variables and* $\mathbf{y}_i$ *is a cyclic shift of* $\mathbf{x}_i$ *for every* $i \in [p]$*, and every identity of the form* $t(\mathbf{x}_1, \mathbf{x}_2 \ldots, \mathbf{x}_p) \approx t(\mathbf{x}_2, \ldots, \mathbf{x}_p, \mathbf{x}_1)$*, where each* $\mathbf{x}_i$ *is a* p*-tuple of variables.*

Observe that t from Equation (1) is doubly cyclic – the first type of identities come from the cyclicity of the inner c while the second type from the outer c. It will be also useful for us to observe in Lemma 22 that, after rearranging the arguments (we read them row-wise), t is a cyclic function of arity p^2. From the finiteness of the domain C we get one more property of function t. In the next definition, by an *x/y-tuple* we mean a tuple containing only variables x and y.

▶ **Definition 16.** *A doubly cyclic function* $t : A^{p^2} \to B$ *is* b-bounded *if there exists an equivalence relation* $\sim$ *on the set of all* p*-ary* x/y*-tuples with at most* b *equivalence classes such that* t *satisfies every identity of the form* $t(\mathbf{u}_1, \ldots \mathbf{u}_p) \approx t(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ *where* $\mathbf{u}_i$ *and* $\mathbf{v}_i$ *are* x/y*-tuples such that* $\mathbf{u}_i \sim \mathbf{v}_i$ *for every* $i \in [p]$.

▶ **Lemma 17.** *Let* $c : C^p \to C$ *be a cyclic function. Then the function* t *defined by Equation* (1) *is a* b*-bounded doubly cyclic function for* $b = |C|^{|C|^2}$.

Proof. We define $\sim$ by declaring two p-ary x/y-tuples $\mathbf{u}$ and $\mathbf{v}$ $\sim$-equivalent if $c(\mathbf{u}) \approx c(\mathbf{v})$. As there are $b = |C|^{|C|^2}$ binary functions $C^2 \to C$, this equivalence has at most b equivalence classes. By definitions, t is then b-bounded and doubly cyclic. ◀

The promised necessary condition for finite tractability is now a simple consequence:

▶ **Theorem 18.** *Let $(\mathbb{A}, \mathbb{B})$ be a finite PCSP template that is finitely tractable. Then there exists b such that $(\mathbb{A}, \mathbb{B})$ has a p^2-ary b-bounded doubly cyclic polymorphism for every sufficiently large prime p.*

Proof. If $(\mathbb{A}, \mathbb{B})$ is finitely tractable, then, by Lemma 11, there exists a minion homomorphism $\xi : \mathrm{Pol}(\mathbb{C}) \to \mathrm{Pol}(\mathbb{A}, \mathbb{B})$, where $\mathbb{C}$ is finite and $\mathrm{CSP}(\mathbb{C})$ is tractable. By Theorem 14, $\mathbb{C}$ has a p-ary cyclic polymorphism for every sufficiently large prime. Then, by Lemma 17, the polymorphism t of $\mathbb{C}$ defined by Equation (1) is a b-bounded and doubly cyclic (with the appropriate b). As ξ preserves h1 identities, $\xi(t)$ is a b-bounded doubly cyclic polymorphism of $(\mathbb{A}, \mathbb{B})$. ◀

3.3 Proof of Theorem 3

Finally, we are ready to start proving Theorem 3. Without loss of generality, we consider only templates on the first lines of Cases (1)–(3) of Theorem 3 (in particular, $r \leq s/2$) and assume that $r \leq s/2$ in Case (4) (the remaining templates can be obtained by swapping zero and one in the domains). We fix such a template $(\mathbb{A}, \mathbb{B})$.

Striving for a contradiction, suppose that $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is finitely tractable. By Theorem 18 there exists b such that $(\mathbb{A}, \mathbb{B})$ has a p^2-ary b-bounded doubly cyclic polymorphism t for every sufficiently large arity p^2. We fix such a b and t, where p is fixed to a sufficiently large prime p congruent to 1 modulo s (which is possible by the Dirichlet prime number theorem). How large must p be will be seen in due course. We denote $n = p^2$ and observe that $n \equiv 1 \pmod{s}$ as well.

Using the cyclicity in Section 4 and double cyclicity in Section 5 we will show that certain evaluations $t(\mathbf{z})$ of t are tame in that $t(\mathbf{z}) = t\langle 0 \rangle$ (recall here the notation in Subsection 2.3) iff the area of $\mathbf{z}$ is below a threshold θ. The *threshold* is defined as $\theta = 1/2$ for all the templates but the (r-in-s, not-all-equal-s) template in Case (4), where we set $\theta = r/s$ (observe that $\theta = r/s$ also in Case (2) and (3)). We restate the definition of tameness for convenience.

▶ **Definition 19.** *A tuple $\mathbf{z} \in \{0,1\}^n$ is* tame *if*

$$t(\mathbf{z}) = \begin{cases} t\langle 0 \rangle_n & \text{if } \lambda(\mathbf{z}) < \theta \\ 1 - t\langle 0 \rangle_n & \text{if } \lambda(\mathbf{z}) > \theta \end{cases}$$

(Note here that $\lambda(\mathbf{z})$ is never equal to θ since n is odd and $n \equiv 1 \pmod{s}$.)

The evaluations that we use are called near-threshold almost rectangles defined as follows.

▶ **Definition 20.** *A tuple $\mathbf{z} \in \{0,1\}^n$ is an* almost rectangle *if it is a p-ary cyclic shift of a tuple of the form $\langle z^1, \ldots, z^1, z^2, \ldots, z^2 \rangle_p$, where $0 \leq z^1, z^2 \leq p$, the number of z^1's is arbitrary, and $|z^1 - z^2| < 5b$. The quantity $\Delta z = |z^1 - z^2|$ is referred to as the* step size*. We say that $\mathbf{z}$ is* near-threshold *if $|\lambda(\mathbf{z}) - \theta| < 1/s^{\Delta z + 3}$.*

The proof can now be finished by using the tameness of near-threshold almost rectangles (that will be established in Lemma 25) together with the b-boundedness of t as follows.

Let $m = (p-1)/2$ and choose positive integers $z^{2,1}$ and $z^{2,2}$ so that $\theta p - 2b < z^{2,1} < z^{2,2} < \theta p$ and the x/y-tuples $(z^{2,1} \times x, (p - z^{2,1}) \times y)$ and $(z^{2,2} \times x, (p - z^{2,2}) \times y)$ are $\sim$-equivalent (see Definition 16 of boundedness). This is possible by the pigeonhole principle since there are more than b integers in the interval and $\sim$ has at most b classes.

By the choice of $z^{2,1}$ and $z^{2,2}$, for any meaningful choice of z^1, we have $t(\mathbf{z}_1) = t(\mathbf{z}_2)$ where $\mathbf{z}_i = \langle m \times z^1, (p-m) \times z^{2,i} \rangle_p$, $i = 1, 2$. We choose z^1 as the maximum number such that $\lambda(\mathbf{z}_1) < \theta$. (Note here that for $z^1 = p$ the area of $\mathbf{z}_1$ can be made arbitrarily close to

$(1+\theta)/2 > \theta$ by choosing a sufficiently large p, so we may assume $z^1 < p$.) From $m < p/2$ it follows that increasing $z^{2,1}$ by one makes the area of $\mathbf{z}_1$ greater than increasing z^1 by one, therefore $\lambda(\mathbf{z}_2) > \theta$.

Note that $z^1 > p\theta$ since otherwise the area of $\mathbf{z}_2$ is less than θ. On the other hand, $z^1 < p\theta + 3b$, otherwise the area of $\mathbf{z}_1$ is greater (assuming $p > 5$):

$$\lambda(\mathbf{z}_1) = \frac{mz^1 + (p-m)z^{2,1}}{p^2} \geq \frac{\frac{p-1}{2}(p\theta+3b) + \frac{p+1}{2}(p\theta-2b)}{p^2} = \frac{p^2\theta + \frac{b(p-5)}{2}}{p^2} > \theta.$$

It follows that the step size of both $\mathbf{z}_1$ and $\mathbf{z}_2$ is less than $5b$, so both $\mathbf{z}_i$ are almost rectangles. By choosing a sufficiently large p, the difference $\lambda(\mathbf{z}_2) - \lambda(\mathbf{z}_1)$ can be made arbitrarily small, and since $\lambda(\mathbf{z}_1) < \theta < \lambda(\mathbf{z}_2)$ both $\mathbf{z}_i$ are then near-threshold.

Now the tameness of near-threshold almost rectangles (Lemma 25) gives us $t(\mathbf{z}_1) = t\langle 0\rangle_n \neq 1 - t\langle 0\rangle_n = t(\mathbf{z}_2)$. On the other hand, we also have $t(\mathbf{z}_1) = t(\mathbf{z}_2)$, a contradiction.

4 Step size at most one

In this section we prove the following lemma.

▶ **Lemma 21.** *Every near-threshold almost rectangle of step size at most one is tame.*

We will use the cyclicity of an operation obtained from t by an appropriate rearrangnment of its arguments, stated in the following lemma. Its proof is in Appendix A.

▶ **Lemma 22.** *Let $t : A^{p^2} \to B$ be a doubly cyclic function. Then the function t^σ defined by*

$$t^\sigma \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} = t \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{p1} \\ x_{12} & x_{22} & \cdots & x_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1p} & x_{2p} & \cdots & x_{pp} \end{pmatrix}$$

is a cyclic function.

Observe that an almost rectangle $\mathbf{z} = \langle z^2+1, \dots, z^2+1, z^2, \dots, z^2\rangle_p$ regarded as a $p \times p$ matrix is, when read row-wise, equal to a sequence of consecutive ones, followed by zeros. In other words, using the notation t^σ from Lemma 22, we have $t(\mathbf{z}) = t^\sigma\langle k\rangle_n$ for some k. Also note that every almost rectangle of step size at most one has a p-ary cyclic shift of this form. Finally, notice that if $\mathbf{z}$ is near-threshold, then $k \leq 2\lfloor\theta n\rfloor$. In order to prove Lemma 21, it is therefore enough to verify the following lemma.

▶ **Lemma 23.** *Denote $a = \lfloor\theta n\rfloor$. For every $0 \leq k \leq 2a$, we have*

$$t^\sigma\langle k\rangle_n = \begin{cases} t^\sigma\langle 0\rangle_n & \text{if } 0 \leq k \leq a \\ 1 - t^\sigma\langle 0\rangle_n & \text{if } 1+a \leq k \leq 2a \end{cases}$$

The rest of this section is devoted to proving this lemma. We require an additional definition. We say that an s-tuple of evaluations $\langle k_1\rangle_n, \dots, \langle k_s\rangle_n$, where $0 \leq k_i \leq n$, is *plausible* if $\sum_{i=1}^{s} k_i = rn$ in Cases (1), (3), (4) and $\sum_{i=1}^{s} k_i \leq rn$ in Case (2). The following lemma is a consequence of the fact that t^σ is a polymorphism (as t is) which is, additionally, cyclic by Lemma 22. No other properties of t are needed in this section.

▶ **Lemma 24.** *If an s-tuple $\langle k_1\rangle, \dots, \langle k_s\rangle$ is plausible, then $(t^\sigma\langle k_1\rangle, \dots, t^\sigma\langle k_s\rangle) \in Q$ (recall here that (P,Q) is introduced in the statement of Theorem 3).*

Moreover, in Cases (1), (2), and (3), we have $t^\sigma\langle n-k\rangle = 1 - t^\sigma\langle k\rangle$ for every $0 \leq k \leq n$.

Proof. For the first part, let $\langle k_1\rangle, \ldots, \langle k_s\rangle$ be plausible. Form an $s \times rn$ matrix M whose first row is $\langle k_1\rangle_{rn}$ and the j-th row is the $(\sum_{l=1}^{j-1} k_l)$-th cyclic shift of $\langle k_j\rangle_{rn}$ for $j \in \{2, \ldots, s\}$. Note that each of the first $\sum k_i$ columns of M contains exactly 1 one and the remaining columns are all zero (the latter only applies in Case (2)). Split this matrix into r-many $s \times n$ blocks $M^1, M^2, \ldots, M^r$. Their sum $X = \sum_{j=1}^{r} M^j$ is an $s \times n$ zero-one matrix whose each column contains exactly r ones in Cases (1), (3), and (4), and at most r ones in Case (2). Moreover, for all $j \in [s]$, the j-th row of X is a cyclic shift of $\langle k_j\rangle$, therefore its t^σ-image is $t^\sigma\langle k_j\rangle$ by cyclicity of t^σ. Each column belongs to the relation P, therefore, as t^σ is a polymorphism, we get that t^σ applied to the rows gives a tuple in Q. This implies the first claim.

For the second part, we take $\langle k\rangle$ together with the k-th cyclic shift of $\langle n-k\rangle$ and use the fact that t^σ preserves the disequality relation pair. ◀

We now consider Cases (1)–(4) separately. Case (2) is the simplest. If $0 \leq k \leq a$ then $\langle k\rangle, \langle k\rangle, \ldots, \langle k\rangle$ is a plausible tuple. By Lemma 24, the tuple $(t^\sigma\langle k\rangle, t^\sigma\langle k\rangle, \ldots, t^\sigma\langle k\rangle)$ is in Q; therefore $t^\sigma\langle k\rangle = 0$. For the remaining values $2a \geq k \geq a+1$ we apply the second part of this lemma and get $t^\sigma\langle k\rangle = 1$.

For Case (1) we prove $t^\sigma\langle k\rangle = 0$ and $t^\sigma\langle n-k\rangle = 1$ for any $0 \leq k \leq a$ by induction on $i = a-k$, $i = 0, 1, \ldots, a$. For the first step, $k = (n-1)/2$, we apply Lemma 24 to the plausible s-tuple $2r \times \langle k\rangle, \langle r\rangle, (s-2r-1) \times \langle 0\rangle$. Since Q contains no p-tuple with more than $(2r-1)$ ones, we get $t^\sigma\langle k\rangle = 0$. Then also $t^\sigma\langle n-k\rangle = 1$ by the second part of the lemma. For the induction step, we use the tuple

$$r \times \langle k\rangle, r \times \langle n-k-1\rangle, \langle r\rangle, (s-2r-1) \times \langle 0\rangle$$

in a similar way, additionally using that $t^\sigma\langle n-k-1\rangle = 1$ by the induction hypothesis.

We proceed to Case (4). We will prove, starting from the left, the following chain of disequalities.

$$t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle \neq t^\sigma\langle a-1\rangle \neq t^\sigma\langle a+2\rangle \neq t^\sigma\langle a-2\rangle \neq \ldots \neq t^\sigma\langle 2a\rangle \neq t^\sigma\langle 0\rangle$$

This will imply $t^\sigma\langle a\rangle = t^\sigma\langle a-1\rangle = \cdots = t^\sigma\langle 0\rangle \neq t^\sigma\langle a+1\rangle = t^\sigma\langle a+2\rangle = \cdots = t^\sigma\langle 2a\rangle$. We start with the first disequality $t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle$. The sequence of arguments

$$(s-r) \times \langle a\rangle,\ r \times \langle a+1\rangle$$

has length s and is plausible as $(s-r)a + r(a+1) = sa + r$ and $sa + r$ is equal to rn. (Indeed, $n \equiv 1 \pmod{s}$, so $n = ms + 1$ for some integer m; then $a = mr$ and $sa + r = smr + r = (n-1)r + r = rn$.) By Lemma 24, $t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle$ since Q does not contain all-equal tuples in Case (4). The remaining disequalities are proved in Appendix B.

Case (3) can be done similarly as Case (4) with an additional reasoning that we now explain. Consider, e.g., the proof that $t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle$ using the sequence $(s-r) \times \langle a\rangle, r \times \langle a+1\rangle$. We cannot directly conclude that $t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle$ since relation Q contains the all-zero tuple – we can only conclude that $t^\sigma\langle a\rangle$ and $t^\sigma\langle a+1\rangle$ are not both ones. However, we can also prove in the same way that $t^\sigma\langle n-a\rangle$ and $t^\sigma\langle n-(a+1)\rangle$ are not both ones by using the "complementary" tuple $(s-r) \times \langle n-a\rangle,\ r \times \langle n-(a+1)\rangle$. The claim $t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle$ then follows from the second part of Lemma 24.

The proof of Lemma 23 is concluded.

5 Arbitrary step size

The entire section is devoted to the proof of the following lemma.

▶ **Lemma 25.** *Every near-threshold almost rectangle is tame.*

We start by redefining plausibility.

We say that an m-tuple of evaluations $\mathbf{k}_1 = \langle k_1^1, \ldots, k_1^p \rangle, \ldots, \mathbf{k}_m = \langle k_m^1, \ldots, k_m^p \rangle$, where $m \in [s]$, is *plausible* if $\sum_{j=1}^m k_j^i = rp$ for all $i \in [p]$ (note that we do not make exception for Case (2) here). In other words, by arranging the integers defining $\mathbf{k}_1$, $\mathbf{k}_2$, ..., $\mathbf{k}_m$ as rows of an $m \times p$ matrix, we get a matrix whose every column sums up to rp. Note that the sum of the areas of the evaluations is then equal to r.

The following lemma is a "2-dimensional analogue" of Lemma 24. The proof applies the first type of doubly cyclic identities from Definition 15, it is given in Appendix C.

▶ **Lemma 26.** *If a tuple $\mathbf{k}_1, \ldots, \mathbf{k}_s$ is plausible, then $(t(\mathbf{k}_1), \ldots, t(\mathbf{k}_s)) \in Q$.*

Moreover, in Cases (1), (2), and (3), we have $t\langle p - k^1, \ldots, p - k^p \rangle = 1 - t\langle k^1, \ldots, k^p \rangle$ for any evaluation $\langle k^1, \ldots, k^p \rangle$.

The next lemma will be applied to produce plausible sequence of evaluations. The proof uses the other type of doubly cyclic identities. It is given in Appendix D, here we provide a brief sketch. (In the statement, note that $r/\theta = s$ except for Case (1) where $r/\theta = 2r$.)

▶ **Lemma 27.** *Let $\mathbf{z}$ be an almost rectangle of step size $\Delta z \geq 2$ with $|\lambda(\mathbf{z}) - \theta| \leq 1/s^3$ and let p be sufficiently large. Then*

- *there exists a plausible r/θ-tuple $\mathbf{k}_1$, $\mathbf{k}_2, \ldots$, $\mathbf{k}_{r/\theta-1}$, $\mathbf{l}$ of almost rectangles such that $t(\mathbf{z}) = t(\mathbf{k}_1) = t(\mathbf{k}_2) = \cdots = t(\mathbf{k}_{r/\theta-1})$, $\lambda(\mathbf{z}) = \lambda(\mathbf{k}_1) = \cdots = \lambda(\mathbf{k}_{r/\theta-1})$, and $\mathbf{l}$ has the same step size Δz as $\mathbf{z}$;*
- *there exists a plausible r/θ-tuple $\mathbf{k}_1$, $\mathbf{k}_2, \ldots$, $\mathbf{k}_{r/\theta-2}$, $\mathbf{l}_1, \mathbf{l}_2$ of almost rectangles such that $t(\mathbf{z}) = t(\mathbf{k}_1) = t(\mathbf{k}_2) = \cdots = t(\mathbf{k}_{r/\theta-2})$, $\lambda(\mathbf{z}) = \lambda(\mathbf{k}_1) = \cdots = \lambda(\mathbf{k}_{r/\theta-2})$, both $\mathbf{l}_1$ and $\mathbf{l}_2$ have step size strictly smaller than Δz, and $|\lambda(\mathbf{l}_1) - \lambda(\mathbf{l}_2)| \leq 1/p$.*

Proof sketch. We can assume that $\mathbf{z} = \langle c \times z^1, d \times z^2 \rangle$ for some c, d, z^1, z^2. For the first item, we consider the $(r/\theta - 1) \times p$ matrix X whose first row is $\mathbf{z}$ and the i-th row is the c-th cyclic shift of the $(i-1)$-st row for each $i \in \{2, \ldots, r/\theta - 1\}$. Let Y be the $r/\theta \times p$ matrix obtained from X by adding a row $(l^1, \ldots, l^p)$ so that each column sums up to rp and we define $\mathbf{k}_1, \ldots, \mathbf{k}_m, \mathbf{l}$ as the n-tuples determined by the rows of Y via $\langle\rangle$, e.g., $\mathbf{l} = \langle l^1, \ldots, l^p \rangle$. The inequality $|\lambda(\mathbf{z}) - \theta| \leq 1/s^3$ (and p being sufficiently large) ensures that $\mathbf{l}$ is correctly defined (i.e., all the l^i are between 0 and p), the construction gives that $\mathbf{l}$ is an almost rectangle with step size Δz and that $\mathbf{z}$ and $\mathbf{k}_i$ have equal areas, and the double cyclicity of t implies $t(\mathbf{z}) = t(\mathbf{k}_i)$. For the second item we additionally split the l row in two roughly equal rows. This will guarantee the two properties of $\mathbf{l}_1$ and $\mathbf{l}_2$. ◀

Equipped with these lemmata we are ready to prove Lemma 25. The proof is by induction on the step size. Step sizes zero and one are dealt with in Lemma 21, so we assume that $\mathbf{z}$ is a near-threshold almost rectangle of step size $2 \leq \Delta z < 5b$.

We will consider Case (4) in detail and discuss the adjustments for the other cases afterwards. Assume first that $\lambda(\mathbf{z})$ is not *too close to* θ, say, $|\lambda(\mathbf{z}) - \theta| \geq 1/s^{5b+4}$. We apply the second item in Lemma 27 and get a plausible s-tuple $\mathbf{k}_1$, ..., $\mathbf{k}_{s-2}, \mathbf{l}_1, \mathbf{l}_2$ such that $\mathbf{z}$, $\mathbf{k}_1$, ..., $\mathbf{k}_{s-2}$ all have the same t-images and areas, and $\mathbf{l}_1$ and $\mathbf{l}_2$ are almost rectangles with step sizes strictly smaller than Δz, whose areas differ by at most $1/p$.

The average area of almost rectangles $\mathbf{k}_1, \ldots, \mathbf{k}_{s-2}, \mathbf{l}_1, \mathbf{l}_2$ is $r/s = \theta$, the first $s-2$ of them have the same area as $\mathbf{z}$, bounded away from θ by a constant (namely $1/s^{5b+4}$), and the last two have almost the same area (the difference is at most $1/p$). By choosing a large enough p we get $\operatorname{sgn}(\lambda(\mathbf{l}_1)-\theta) = \operatorname{sgn}(\lambda(\mathbf{l}_2)-\theta) \neq \operatorname{sgn}(\lambda(\mathbf{z})-\theta)$ and $|\lambda(\mathbf{l}_i)-\theta| \leq s \cdot |\lambda(\mathbf{z})-\theta|$; in particular, both $\mathbf{l}_i$ are near-threshold since $s \cdot |\lambda(\mathbf{z})-\theta| \leq 1/s^{\Delta z+3-1} \leq 1/s^{\Delta l_i+3}$. By the induction hypothesis, both $\mathbf{l}_i$ are tame. By Lemma 26, the values $t(\mathbf{k}_1), \ldots, t(\mathbf{k}_{s-2})$, $t(\mathbf{l}_1)$, and $t(\mathbf{l}_2)$ are not all equal. But $t(\mathbf{z}) = t(\mathbf{k}_1) = \cdots = t(\mathbf{k}_{s-2})$, $t(\mathbf{l}_1) = t(\mathbf{l}_2)$, and $\operatorname{sgn}(\lambda(\mathbf{z})-\theta) \neq \operatorname{sgn}(\lambda(\mathbf{l}_1)-\theta)$ so it follows that $\mathbf{z}$ is tame, as required.

It remains to deal with the case that $\lambda(\mathbf{z})$ is too close to θ. In this case we will find an almost rectangle $\mathbf{l}$ with the same step size as $\mathbf{z}$ such that $t(\mathbf{l}) = 1 - t(\mathbf{z})$ and $\lambda(\mathbf{l}) - \theta = -s'(\lambda(\mathbf{z})-\theta)$, where s' is such that $2 \leq s' \leq s$. If $\lambda(\mathbf{l})$ is already not too close to the threshold θ, then we observe that $\mathbf{l}$ is near-threshold (indeed, $|\lambda(\mathbf{l})-\theta| \leq s|\lambda(\mathbf{z})-\theta| \leq s/s^{5b+4} \leq 1/s^{\Delta z+3}$) and apply to $\mathbf{l}$ the first part of the proof, thus obtaining that $\mathbf{l}$ is tame and, consequently, $\mathbf{z}$ is tame as well. If $\lambda(\mathbf{l})$ is still too close to θ, then we simply repeat the process until we get a rectangle that is not too close.

To find such an almost rectangle $\mathbf{l}$ we apply the first item of Lemma 27 and get a plausible s-tuple $\mathbf{k}_1, \ldots, \mathbf{k}_{s-1}, \mathbf{l}$ such that $t(\mathbf{z}) = t(\mathbf{k}_1) = \cdots = t(\mathbf{k}_{s-1})$ and $\mathbf{l}$ is an almost rectangle of the same step size as $\mathbf{z}$. Since the area of each $\mathbf{k}_i$ is equal to $\lambda(\mathbf{z})$ and the average area in the plausible s-tuple is θ, we get that $\lambda(\mathbf{l}) - \theta = -(s-1)(\lambda(\mathbf{z})-\theta)$. By Lemma 26, $t(\mathbf{l})$ and $t(\mathbf{z})$ are not equal. This concludes the construction of $\mathbf{l}$ and the proof of Lemma 25 for Case (4).

The remaining cases (1), (2), and (3) require a modification that is similar to the modification for Case (3) in the proof of Lemma 23. Consider the situation that $\lambda(\mathbf{z})$ is not too close to θ. In Cases (2) and (3) Lemma 27 is applied not only to $\mathbf{k}_1, \ldots, \mathbf{k}_{2r-2}, \mathbf{l}_1, \mathbf{l}_2$ but also to the tuple formed by "complementary" almost rectangles, which have different t-images by the second part of Lemma 26. In Case (1) we additionally complete the two $2r$ tuples to s-tuples by adding $s-2r$ zeros. The other situation, that $\lambda(\mathbf{z})$ is too close, is adjusted in an analogous fashion.

6 Conclusion

We have characterized finite tractability among the basic tractable cases in the Brakensiek–Guruswami classification [8] of symmetric Boolean PCSPs allowing negations. A natural direction for future research is an extension to all the tractable cases (not just the basic ones), or even to all symmetric Boolean PCSPs [14], not only those allowing negations. An obstacle, where our efforts have failed so far, is already in relaxations of the basic templates (P, Q) with disequalities. For example, which $(P, Q), (\neq, \neq)$, with P a subset of $\leq r$-in-s and Q a superset of $\leq(2r-1)$-in-s, give rise to finitely tractable PCSPs?

Another natural direction is to better understand the "level of tractability." For the finitely tractable templates $(\mathbb{A}, \mathbb{B})$ considered in this paper, it is always possible to find a tractable CSP($\mathbb{C}$) with $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ and such that $\mathbb{C}$ is two-element. Is it so for all symmetric Boolean templates? For general Boolean templates, the answer is "No": [12] presents an example that requires a three-element $\mathbb{C}$. However, it is unclear whether there is an upper bound on the size of $\mathbb{C}$ for finitely tractable (Boolean) PCSPs, and if there is, how it could be computed. There are also natural concepts beyond finite tractability, still stronger than standard tractability. We refer to [2] for some questions in this direction.

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A Doubly cyclic functions are cyclic

In this appendix we prove Lemma 22, which we restate here for convenience.

▶ **Lemma 22.** *Let $t : A^{p^2} \to B$ be a doubly cyclic function. Then the function t^σ defined by*

$$t^\sigma \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} = t \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{p1} \\ x_{12} & x_{22} & \cdots & x_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1p} & x_{2p} & \cdots & x_{pp} \end{pmatrix}$$

is a cyclic function.

Proof. By cyclically shifting the arguments we get the same result:

$$\begin{aligned}
&t^\sigma(x_{21}, x_{31}, \ldots, x_{p1}, x_{12}, x_{22}, x_{32}, \ldots, x_{p2}, x_{13}, \ldots, x_{2p}, x_{3p}, \ldots, x_{pp}, x_{11}) \\
&= t^\sigma \begin{pmatrix} x_{21} & \cdots & x_{2,p-1} & x_{2p} \\ \vdots & \ddots & \vdots & \vdots \\ x_{p1} & \cdots & x_{p,p-1} & x_{pp} \\ x_{12} & \cdots & x_{1p} & x_{11} \end{pmatrix} = t \begin{pmatrix} x_{21} & \cdots & x_{p1} & x_{12} \\ \vdots & \ddots & \vdots & \vdots \\ x_{2,p-1} & \cdots & x_{p,p-1} & x_{1p} \\ x_{2p} & \cdots & x_{pp} & x_{11} \end{pmatrix} \\
&= t \begin{pmatrix} x_{21} & \cdots & x_{p1} & x_{11} \\ \vdots & \ddots & \vdots & \vdots \\ x_{2,p-1} & \cdots & x_{p,p-1} & x_{1,p-1} \\ x_{2p} & \cdots & x_{pp} & x_{1p} \end{pmatrix} = t \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{p1} \\ x_{12} & x_{22} & \cdots & x_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1p} & x_{2p} & \cdots & x_{pp} \end{pmatrix} \\
&= t^\sigma \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} \\
&= t^\sigma(x_{11}, x_{21}, \ldots, x_{p1}, x_{12}, x_{22}, \ldots, x_{p2}, \ldots, x_{1p}, x_{2p}, \ldots, x_{pp}).
\end{aligned}$$

◀

B Step size one, Case (4)

In this section we finish the proof of Lemma 23 for Case (4). We state the lemma for convenience.

▶ **Lemma 23.** *Denote $a = \lfloor \theta n \rfloor$. For every $0 \le k \le 2a$, we have*

$$t^\sigma \langle k \rangle_n = \begin{cases} t^\sigma \langle 0 \rangle_n & \text{if } 0 \le k \le a \\ 1 - t^\sigma \langle 0 \rangle_n & \text{if } 1 + a \le k \le 2a \end{cases}$$

Recall that we want to prove, starting from the left, the chain of disequalities

$$t^\sigma\langle a\rangle \neq t^\sigma\langle a+1\rangle \neq t^\sigma\langle a-1\rangle \neq t^\sigma\langle a+2\rangle \neq t^\sigma\langle a-2\rangle \neq \ldots \neq t^\sigma\langle 2a\rangle \neq t^\sigma\langle 0\rangle$$

and that we have already verified the first one.

For the second disequality $t^\sigma\langle a+1\rangle \neq t^\sigma\langle a-1\rangle$, as well as for the further disequalities we need to distinguish two cases: Case (4a) r and s have the same parity and Case (4b) r is even and s is odd. In Case (4a) we directly use the sequence

$$(s-r)/2 \times \langle a-1\rangle,\ (s+r)/2 \times \langle a+1\rangle$$

and derive $t^\sigma\langle a+1\rangle \neq t^\sigma\langle a-1\rangle$ using Lemma 24 as before. In Case (4b) we first use

$$(s-1) \times \langle a\rangle,\ \langle a+r\rangle$$

to deduce $t^\sigma\langle a+r\rangle \neq t^\sigma\langle a\rangle$ (so $t^\sigma\langle a+1\rangle = t^\sigma\langle a+r\rangle$) and then

$$(s-1)/2 \times \langle a-1\rangle,\ (s-1)/2 \times \langle a+1\rangle,\ \langle a+r\rangle$$

to deduce $t^\sigma\langle a-1\rangle \neq t^\sigma\langle a+1\rangle$.

To prove $t^\sigma\langle a-i+1\rangle \neq t^\sigma\langle a+i\rangle$ for $i \in \{2,3,\ldots,a\}$, we observe that, by the already established disequalities, we have $t^\sigma\langle a-i+1\rangle = \cdots = t^\sigma\langle a\rangle$, and then use

- $(s+r)/4 \times \langle a+i\rangle,\ (s-r)/2 \times \langle a-1\rangle,\ (s+r)/4 \times \langle a-i+2\rangle$ in Case (4a) and $(s+r)/2$ is even;
- $(s+r+2)/4 \times \langle a+i\rangle,\ (s-r-2)/2 \times \langle a-1\rangle,\ 2 \times \langle a-i+1\rangle,\ (s+r-6)/4 \times \langle a-i+2\rangle$ in Case (4a) and $(s+r)/2$ is odd;
- $r/2 \times \langle a+i\rangle,\ (s-r) \times \langle a\rangle,\ r/2 \times \langle a-i+2\rangle$ in Case (4b).

Finally, for proving $t^\sigma\langle a+i\rangle \neq t^\sigma\langle a-i\rangle$ we use

- $(s-r)/2 \times \langle a-i\rangle,\ (s-r)/2 \times \langle a+i\rangle,\ r \times \langle a+1\rangle$ in Case (4a) and
- $(s-1)/2 \times \langle a-i\rangle,\ (s-1)/2 \times \langle a+i\rangle,\ 1 \times \langle a+r\rangle$ in Case (4b).

This completes the proof for Case (4).

C Proof of Lemma 26

▶ **Lemma 26.** *If a tuple $\mathbf{k}_1,\ldots,\mathbf{k}_s$ is plausible, then $(t(\mathbf{k}_1),\ldots,t(\mathbf{k}_s)) \in Q$.*

Moreover, in Cases (1), (2), and (3), we have $t\langle p-k^1,\ldots,p-k^p\rangle = 1-t\langle k^1,\ldots,k^p\rangle$ for any evaluation $\langle k^1,\ldots,k^p\rangle$.

Proof. Let $\mathbf{k}_1,\ldots,\mathbf{k}_s$ be a plausible tuple. Fix, for a while, an arbitrary $i \in [p]$. Form a $s \times rp$ matrix M_i whose first row is $\langle k_1^i\rangle_{rp}$ and j-th row is the $(\sum_{l=1}^{j-1} k_l^i)$-th cyclic shift of $\langle k_j^i\rangle_{rp}$ for $j \in \{2,\ldots,s\}$. Split this matrix into r-many $s \times p$ blocks $M_i^1, M_i^2, \ldots, M_i^r$. Their sum $X_i = \sum_{j=1}^r M_i^j$ is an $s \times p$ matrix whose each column contains exactly r ones. Moreover, for all $j \in [s]$, the j-th row of the matrix X_i is a cyclic shift of $\langle k_j^i\rangle_p$. Put the matrices X_1, $\ldots$, X_p aside to form an $s \times n$ matrix Y. Its rows have the same t-images as $\mathbf{k}_1,\ldots,\mathbf{k}_s$, respectively, because t is doubly cyclic. Each column belongs to the relation P, therefore, as t is a polymorphism, we get that t applied to the rows gives a tuple in Q. This tuple is equal to $(t(\mathbf{k}_1),\ldots,t(\mathbf{k}_s))$.

The second part can be proved in a similar way as the second part of Lemma 24 using the disequality relation pair. ◀

D Proof of Lemma 27

▶ **Lemma 27.** *Let $\mathbf{z}$ be an almost rectangle of step size $\Delta z \geq 2$ with $|\lambda(\mathbf{z}) - \theta| \leq 1/s^3$ and let p be sufficiently large. Then*

- *there exists a plausible r/θ-tuple $\mathbf{k}_1$, $\mathbf{k}_2, \ldots$, $\mathbf{k}_{r/\theta-1}$, $\mathbf{l}$ of almost rectangles such that $t(\mathbf{z}) = t(\mathbf{k}_1) = t(\mathbf{k}_2) = \cdots = t(\mathbf{k}_{r/\theta-1})$, $\lambda(\mathbf{z}) = \lambda(\mathbf{k}_1) = \cdots = \lambda(\mathbf{k}_{r/\theta-1})$, and $\mathbf{l}$ has the same step size Δz as $\mathbf{z}$;*
- *there exists a plausible r/θ-tuple $\mathbf{k}_1$, $\mathbf{k}_2, \ldots$, $\mathbf{k}_{r/\theta-2}$, $\mathbf{l}_1, \mathbf{l}_2$ of almost rectangles such that $t(\mathbf{z}) = t(\mathbf{k}_1) = t(\mathbf{k}_2) = \cdots = t(\mathbf{k}_{r/\theta-2})$, $\lambda(\mathbf{z}) = \lambda(\mathbf{k}_1) = \cdots = \lambda(\mathbf{k}_{r/\theta-2})$, both $\mathbf{l}_1$ and $\mathbf{l}_2$ have step size strictly smaller than Δz, and $|\lambda(\mathbf{l}_1) - \lambda(\mathbf{l}_2)| \leq 1/p$.*

Proof. Without loss of generality we can assume that $\mathbf{z} = \langle c \times z^1, d \times z^2\rangle$ for some c, d and $z^1 > z^2$. Let $m = r/\theta - 1$ for the first item and $m = r/\theta - 2$ for the second one. We define an integer $m \times p$ matrix X so that the first row is $(c \times z^1, d \times z^2)$ and the i-th row is the c-th cyclic shift of the $(i-1)$-st row for each $i \in \{2, \ldots, m\}$. Let Y be the $(m+1) \times p$ matrix obtained from X by adding a row $(l^1, \ldots, l^p)$ so that each column sums up to rp. It is easily seen by induction on $i \leq m$ that the sum of the first i rows is a cyclic shift of a tuple of the form $(e, \ldots, e, e', \ldots, e')$, where $|e - e'| = \Delta z$ and the "step down" is at position $ci \mod p$ (when columns are indexed from 0). It follows that $(l^1, \ldots, l^p)$ is also a cyclic shift of a tuple of the form $(e, \ldots, e, e', \ldots, e')$ where e and e' differ by Δz.

Next we observe that each $l^i > 0$ if p is sufficiently large. Indeed, note that since $|z^1 - z^2|/p$ can be made arbitrarily small (recall $|z^1 - z^2| < 5b$), we have $p(\lambda(\mathbf{z}) - \epsilon) < z^1, z^2 < p(\lambda(\mathbf{z}) + \epsilon)$, where $\epsilon > 0$ can be made arbitrarily small. We then have $l^i > rp - mp(\lambda(\mathbf{z}) + \epsilon) \geq rp - (r/\theta - 1)p(\theta + 1/s^3 + \epsilon) = p(\theta - (r/\theta - 1)(1/s^3 + \epsilon)) > p(\theta - r/\theta(1/s^3 + \epsilon))$, which is, for a sufficiently small ϵ, greater than 0 since $r/\theta s^3 \leq 1/s^2 < \theta$. Similarly, each $l^i < 2\theta \leq p$ if $m = r/\theta - 1$ and $l^i < 3\theta$ if $m = r/\theta - 2$.

Now we can finish the proof of the first item. We set $\mathbf{k}_1, \ldots, \mathbf{k}_m, \mathbf{l}$ to be the n-tuples determined by the rows of Y via $\langle\rangle$, e.g., $\mathbf{l} = \langle l^1, \ldots, l^p\rangle$. The inequalities $0 \leq l^i \leq p$ guarantee that $\mathbf{l}$ is correctly defined and we see, using also the double cyclicity of t (for $t(\mathbf{z}) = t(\mathbf{k}_1) = \ldots$), that these n-tuples have all the required properties.

To finish the proof of the second item, we define the $\mathbf{k}_i$ as above and set $\mathbf{l}_1 = \langle\lfloor l^1/2\rfloor, \ldots, \lfloor l^p/2\rfloor\rangle, \mathbf{l}_2 = \langle\lceil l^1/2\rceil, \ldots, \lceil l^p/2\rceil\rangle$. Since $0 \leq l^i \leq 3\theta/2 < p$, these tuples are correctly defined almost rectangles. Their areas clearly differ by at most $1/p$. As $\Delta z \geq 2$, their step sizes are strictly smaller than Δz, and we are done in this case as well. ◀