Abstract

Let $G$ be a graph on $n$ vertices. For $i \in \{0, 1\}$ and a connected graph $G$, a spanning forest $F$ of $G$ is called an $i$-perfect forest if every tree in $F$ is an induced subgraph of $G$ and exactly $i$ vertices of $F$ have even degree (including zero). An $i$-perfect forest of $G$ is proper if it has no vertices of degree zero. Scott (2001) showed that every connected graph with even number of vertices contains a (proper) 0-perfect forest. We prove that one can find a 0-perfect forest with minimum number of edges in polynomial time, but it is NP-hard to obtain a 0-perfect forest with maximum number of edges. We also prove that for a prescribed edge $e$ of $G$, it is NP-hard to obtain a 0-perfect forest containing $e$, but we can find a 0-perfect forest not containing $e$ in polynomial time. It is easy to see that every graph with odd number of vertices has a 1-perfect forest. It is not the case for proper 1-perfect forests. We give a characterization of when a connected graph has a proper 1-perfect forest.

1 Introduction

In this paper all graphs are finite, undirected, have no parallel edges or loops. We use standard terminology and notation, see e.g. [5]. The number of vertices (edges, respectively) of a graph $G$ is called its order (size, respectively). The degree of a vertex $x$ in a graph $G$ is denoted by $d_G(x)$. A vertex $x$ of a graph $G$ is a cut-vertex if $G - x$ has more connected components than $G$. A maximal connected subgraph of a graph $G$ without a cut-vertex is called a block. Thus, every block of $G$ is either a maximal 2-connected subgraph or a bridge (including its vertices) or an isolated vertex, implying that a block of odd order in a connected graph of order at least 3, must be a maximal 2-connected subgraph.

A spanning forest $F$ of $G$ is called a semiperfect forest if every tree of $F$ is an induced subgraph of $G$. Let $G$ be a graph and let $f: V(G) \to \{0, 1\}$ be a function such that $\sum_{v \in V(G)} f(v)$ is even (we will call such a function even-sum). A subgraph $H$ in $G$ where $d_H(x) \equiv f(x) \mod 2$ for all $x \in V(G)$, is called an $f$-parity subgraph. Note that the requirement that $f$ is even-sum is necessary as otherwise an $f$-parity subgraph does not exist. An $f$-parity subgraph $H$ of $G$ is called an $f$-parity perfect forest if $H$ is a semiperfect forest.

For $i \in \{0, 1\}$ and a graph $G$, an $f$-parity perfect forest is called an $i$-perfect forest if $f(x) = 1$ for all vertices of $G$ for $i = 0$, and for all vertices of $G$ apart from one for $i = 1$. An $i$-perfect forest of $G$ is proper if it has no vertices of degree zero. Note that every 0-perfect forest (called a perfect forest in [3, 9] and a pseudo-matching in [18]) is proper. For examples of 0-perfect and 1-perfect forests, see Figures 1 and 2.
Perfect Forests in Graphs and Their Extensions

(a): $G$  (b): A 0-perfect forest of $G$

Figure 1 A graph $G$ is shown in (a) and a 0-perfect forest of $G$ is shown in (b) (as all degrees are odd and the trees are induced in $G$).

(a): $H$  (b): A 1-perfect forest of $H$  (c): A proper 1-perfect forest of $H$

Figure 2 The graph $H$ is shown in (a), a (non-proper) 1-perfect forest of $H$ is shown in (b), and a proper 1-perfect forest of $H$ is shown in (c).

Clearly, every connected graph with a 0-perfect forest is of even order. Scott [17] proved that somewhat surprisingly the opposite implication is also true.

▶ Theorem 1. Every connected graph of even order contains a 0-perfect forest.

The proof of Theorem 1 in [17] is graph-theoretical and relatively long. A short proof using basic linear algebra is obtained in [9] and two short graph-theoretical proofs are given in [3]. All the proofs of Theorem 1 are constructive and yield polynomial algorithms for finding 0-perfect forests. Intuitively, it is clear that a 0-perfect forest can provide a useful structure in a graph and, in particular, this notion was used by Sharan and Wigderson [18] to prove that the perfect matching problem for bipartite cubic graphs belongs to the complexity class $\mathcal{NC}$. Semiperfect forests were used in the proofs of three theorems in [7]. Gutin and Yeo [11] studied extensions of a 0-perfect forest to directed graphs.

Since a 0-perfect forest is a generalization of a matching, it is natural to study the following two problems for a connected graph $G$ of even order $n$:

1. Find a 0-perfect forest of $G$ of minimum size. (Clearly, the minimum size is $n/2$ if and only if $G$ has a perfect matching.)
2. Find a 0-perfect forest of $G$ of maximum size. (This is of interest in matching-like edge-decompositions of $G$.)

The following theorem solves the first problem.

▶ Theorem 2. In polynomial time, we can find a 0-perfect forest of minimum size.

Theorem 2 follows immediately from the next theorem by letting $f(x) = 1$ for all $x \in V(G)$. Theorem 3 shows usefulness of extending Problem 1 to $f$-parity perfect forests. Theorem 3 is proved in Section 2.

▶ Theorem 3. Let $G$ be a connected graph and let $f: V(G) \rightarrow \{0, 1\}$ be an even-sum function. We can in polynomial time find an $f$-parity perfect forest $H$ in $G$, such that $d_H(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$ and $|E(H)|$ is minimized.

As the following theorem shows, the second problem cannot be solved in polynomial time unless $P=NP$. 
Theorem 4. It is \( \text{NP}-\text{hard} \) to find a 0-perfect forest of maximum size.

Let \( n = |V(G)| \). Theorem 4 follows from the next result proved in Section 3. Theorem 5 is optimal in the following sense. The problem of finding a 0-perfect forest of size at least \( n - 1 \) is polynomial-time solvable because \( G \) has a 0-perfect forest of size at least \( n - 1 \) if and only if \( G \) is a tree in which every vertex is of odd degree.

Theorem 5. It is \( \text{NP}-\text{hard} \) to decide whether a connected graph contains a 0-perfect forest with at least \( n - 2 \) edges.

It is easy to show that Theorem 5 holds if we replace \( n - 2 \) by \( n - k \) for any integer \( k \geq 2 \). Indeed, add two new vertices \( x \) and \( y \) to a graph \( G \) as well as two edges \( xy \) and \( yu \), where \( u \) is any vertex in \( G \). The resulting graph is denoted by \( G' \). Observe that there is a 0-perfect forest of size \( |V(G)| - k \) in \( G \) if and only if there is a 0-perfect forest of size \( |V(G')| - (k + 1) \) in \( G' \).

Since the problem of finding a 0-perfect forest of maximum size is \( \text{NP}-\text{hard} \), it is natural to study its parameterized complexity using appropriate parameterizations e.g. the parameterization below the tight upper bound \( n - 1 \) and the parameterization above the tight upper bound \( n/2 \). In other words, we can ask whether there is a 0-perfect forest of size at least \( n - k \) \((n/2 + k) \), respectively), where \( k \) is the parameter. (Above-tight-lower-bound and below-tight-upper-bound parameterizations were studied for many graph-theoretical and constraint satisfaction problems, see e.g. \([1, 4, 10, 13, 14]\).) Theorem 5 shows that the parameterization \( n - k \) is \text{para-NP}-complete (for an introduction to \text{para-NP}-completeness, see e.g. \([6]\)). We do not know the answer to the following question. Is the parameterization \( n/2 + k \) fixed-parameter tractable?\(^1\)

Here is another pair of natural problems on 0-perfect forests. They both are clearly polynomial-time solvable when restricted to perfect matchings. For a graph \( G \) of even order and an edge \( e \) in \( G \),

\[ \begin{align*}
\text{(1’)} & \text{ find a 0-perfect forest containing } e; \\
\text{(2’)} & \text{ find a 0-perfect forest not containing } e.
\end{align*} \]

For Problem 1’, we prove the following result in Section 4.

Theorem 6. The following problem is \( \text{NP}-\text{hard} \). Given a connected graph \( G \) and an edge \( e \in E(G) \), decide whether \( G \) has a 0-perfect forest containing \( e \).

For Problem 2’, we have the next result, which follows immediately from Theorem 8, by letting \( f(x) = 1 \) for all \( x \) in \( G \). Theorem 8 again demonstrates usefulness of \( f \)-parity perfect forests. It is proved in Section 5.

Theorem 7. Given a graph \( G \) and an edge \( e \in E(G) \) we can in polynomial time decide whether \( G \) has a 0-perfect forest not containing \( e \).

Theorem 8. The following problem is polynomial time solvable. Given a graph \( G \), an edge \( e \in E(G) \) and an even-sum function \( f : V(G) \to \{0, 1\} \), decide whether \( G \) has an \( f \)-parity perfect forest not containing \( e \).

\(^1\) While working on the final version of this paper, we obtained a proof that the parameterized problem is \( \text{W}[1] \)-hard. We will include the proof in a journal version of the paper.
Since an odd order connected graph cannot have a 0-perfect forest, it is natural to ask whether every connected graph of odd order has a 1-perfect forest (recall that a 1-perfect forest has only one vertex of even degree). The answer is positive and the proof is trivial. In fact, it is not hard to show the following strengthening of this observation, which will be useful in the proof of Theorem 10.

**Proposition 9.** Let \( x \) be an arbitrary vertex of a connected graph \( G \) of odd order. Then \( G \) has a 1-perfect forest \( F \) such that \( d_F(x) \) is even.

**Proof.** Create a new graph \( H \) by adding a new vertex \( y \) to \( G \) and adding the edge \( xy \). By Theorem 1, \( H \) has a 0-perfect forest, \( F_H \). Deleting the vertex \( y \) from \( F_H \), results in the desired 1-perfect forest of \( G \) where \( x \) is the only vertex of even degree. 

Note that not every connected graph of odd order has a proper 1-perfect forest. For example, no complete graph of odd order has such a forest. Thus, a more interesting question with a potentially more useful answer is when a connected graph of odd order has a proper 1-perfect forest? This question is answered in the following characterization proved in Section 6.

**Theorem 10.** Let \( B \) be the set of all connected graphs where every block is a complete graph of odd order. If \( G \) is a connected graph of odd order \( n \geq 3 \) then \( G \) contains a proper 1-perfect forest if and only if \( G \not\in B \).

Using this theorem and a linear-time algorithm for computing biconnected components in a graph [12], in polynomial time we can decide whether a connected graph \( G \) of odd order contains a proper 1-perfect forest. If \( G \not\in B \), the proof by induction of Theorem 2 yields a polynomial-time recursive algorithm to construct a proper 1-perfect forest.

Our proof of Theorem 10 is graph-theoretical and so are the proofs of Theorem 1 in [17] and [3]. Recall that Gutin [9] gave a linear-algebraic proof of Theorem 1. It would interesting to see whether Theorem 10 can be proved using a linear-algebraic approach, too.

## 2 Proof of Theorem 3

**Lemma 11.** Let \( G \) be a connected graph and let \( f : V(G) \to \{0, 1\} \) be an even-sum function. If \( H \) is an \( f \)-parity subgraph of \( G \) of minimum size, then \( H \) is an \( f \)-parity perfect forest.

**Proof.** Assume that \( H \) is an \( f \)-parity subgraph with minimum possible \( |E(H)| \). Clearly \( H \) contains no cycles, as removing the edges of a cycle would contradict the minimality of \( |E(H)| \). Assume that some tree \( T \) of \( H \) is not an induced tree in \( G \). Let \( xy \) be an edge of \( G \), not belonging to \( T \) but with \( \{x, y\} \subseteq V(T) \). Remove the unique \((x, y)\)-path in \( T \) from \( H \) and add the edge \( xy \) to \( H \). This decreases the number of edges in \( H \) without changing the parity of the degree of any vertex, contradicting the minimality of \( |E(H)| \). Therefore \( H \) is indeed an \( f \)-parity perfect forest.

Lemma 11 implies the following:

**Theorem 12.** Let \( G \) be a connected graph and let \( f : V(G) \to \{0, 1\} \) be an even-sum function. Then there exists an \( f \)-parity perfect forest \( F \) in \( G \).

**Proof.** Let \( x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k \) be the vertices in \( G \) with \( f \)-value equal to one. Let \( P_i \) be any \((x_i, y_i)\)-path in \( G \) for all \( i = 1, 2, \ldots, k \), which exists as \( G \) is connected. Let \( H \) be the spanning subgraph of \( G \) such that an edge \( e \in E(G) \) belongs to \( H \) if and only if \( e \)
Conversely if we will add edges between with 0 

Let \( v \in V(G) \). Observe that \( d_H(x) \) is odd if and only if \( x \) is incident with an odd number of edges in \( \bigcup_{i=1}^{k} E(P_i) \), which is if and only if \( x \) is the endpoint of one of the paths i.e. \( f(x) = 1 \). Thus, \( H \) is an \( f \)-parity subgraph of \( G \). Lemma 11 now implies that if \( H \) is the \( f \)-parity subgraph of \( G \) of minimum size, then \( H \) is an \( f \)-parity perfect forest.

Note that Theorem 12 generalizes Theorem 1: set \( f(x) = 1 \) for all \( x \in V(G) \). Thus, Theorem 12 provides an alternative proof of Theorem 1.

**Theorem 3.** Let \( G \) be a connected graph and let \( f : V(G) \to \{0, 1\} \) be an even-sum function. We can in polynomial time find an \( f \)-parity perfect forest \( H \) in \( G \), such that \( d_H(x) \equiv f(x) \) (mod 2) for all \( x \in V(G) \) and \( |E(H)| \) is minimized.

**Proof.** Let \( G \) be a connected graph and let \( f : V(G) \to \{0, 1\} \) be an even-sum function. Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \). We will construct a weighted auxiliary graph \( H \) as follows. Let \( V(H) = \bigcup_{i=1}^{n} X_i \), where for every \( i \in [n] \), \( |X_i| \in \{n-1, n\} \) and \(|X_i| \equiv f(v_i) \) (mod 2). For all \( 1 \leq i < j \leq n \) and all \( v \in X_i \) and \( v \in X_j \), we let \( w \in E(H) \) if and only if \( v,v \in E(G) \). Finally add a matching \( M_i = \{e_1^i, e_2^i, \ldots, e_{|X_i|/2}^i\} \) to \( X_i \) for all \( i \in [n] \). Let the weight of all the edges within each \( X_i \) (i.e. the edges in \( M_i \)) be zero and let all edges between different \( X_i \)'s have weight one.

We first show that \( H \) contains a perfect matching. As \( \sum_{v \in V(G)} f(v) \) is even we may assume that \( \{v_1, v_2, \ldots, v_{2k}\} \) are the vertices of \( G \) with an \( f \)-value of one for some integer \( k \) with \( 0 \leq k \leq n/2 \). Assume that \( y_i \in X_i \) is the unique vertex in \( X_i \) that is not saturated by \( M \) for all \( i \in [2k] \) and start of by letting \( M \) be the matching containing all \( M_i \)'s.

Let \( P_i = v_{i}v_{i+1} \cdots v_{i+\frac{|X_i|}{2}} \) be any path in \( G \) from \( v_i \) to \( v_{i+k} \) where \( i \in [k] \). It is not difficult to see that there exists an \( M \)-augmenting path, \( Q_i \), in \( H \) starting in \( y_i \) and ending in \( y_{i+k} \) and containing exactly the edges \( e_1^i, e_2^i, \ldots, e_{|X_i|/2}^i \) from \( M_i \). Also observe that \( Q_1, Q_2, \ldots, Q_k \) are vertex disjoint, which implies that we can use all \( Q_i \) to increase the matching \( M \) thereby obtaining a perfect matching in \( H \).

We will now show the following claim. The size of a multiset \( S \) is the total number of elements in \( S \), where if an element \( e \in S \) is of multiplicity \( r \), then \( e \) is counted \( r \) times.

\[
\triangleq \text{Claim A.}
\]

(a) If there exists a perfect matching in \( H \) with weight \( w^* \) then there exists a multiset of edges \( E^* \) in \( G \) of size \( w^* \), such that \( d_{E^*}(x) \equiv f(x) \) (mod 2) for all \( x \in V(G) \).

(b) Conversely if \( E^* \) is a multiset of edges in \( G \) of size \( w^* \), such that \( d_{E^*}(x) \equiv f(x) \) (mod 2) for all \( x \in V(G) \), then there exists a perfect matching in \( H \) with weight at most \( w^* \).

**Proof of Claim A.** First assume that we have a multiset of edges \( E^* \) in \( G \) of size \( w^* \leq W_{\text{max}} \), such that \( d_{E^*}(x) \equiv f(x) \) (mod 2) for all \( x \in V(G) \). Let \( M^* = \emptyset \). For every \( v_i v_j \in E^* \) we will add edges between \( X_i \) and \( X_j \) to \( M^* \) as follows: if \( v_i v_j \) is of multiplicity \( r \) in \( E^* \), then we add an edge between \( X_i \) and \( X_j \) to \( M^* \) if and only if \( r \) is odd. Since we will add \( 2k_i + f(v_i) \) edges that are incident to \( X_i \) for each \( i \in [n] \) (where \( k_i \) is some integer), we can add these edges such that their endvertices are \( V(e_1^i) \cup V(e_2^i) \cup \cdots \cup V(e_{k_i}^i) \) if \( f(v_i) = 0 \) and \( \{y_i\} \cup V(e_1^i) \cup V(e_2^i) \cup \cdots \cup V(e_{k_i}^i) \) if \( f(v_i) = 1 \) for each \( i \in [n] \), where \( V(e_j^i) \) denotes the pair of endvertices of \( e_j^i \). We can now extend \( M^* \) to a perfect matching by adding \( M_i \setminus \{e_1^i, e_2^i, \ldots, e_{k_i}^i\} \) for each \( i \in [n] \). This gives us a perfect matching in \( H \) with weight at most \( |E^*| \) as desired.

Conversely assume that there exists a perfect matching \( M^* \) in \( H \) with weight \( w^* \). Initially let \( E^* = \emptyset \). For every \( xy \in M^* \) with weight one (i.e. \( x \in X_i \) and \( y \in X_j \) for some \( i \neq j \)), add \( v_i v_j \) to \( E^* \). This gives us the desired multiset \( E^* \), thereby completing the proof of Claim A.
We have proved that $H$ has a perfect matching. Let $M_{\text{min}}$ be a minimum weight perfect matching in $H$ which can be determined in polynomial time using Edmonds’ blossom algorithm as a subroutine, see e.g. [15]. Let $W_{\text{min}}$ be the weight of $M_{\text{min}}$. By Claim A(a), using $M_{\text{min}}$, in polynomial time we can find a multiset of edges $E^*$ in $G$ of size $W_{\text{min}}$, such that $d_{E^*}(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$. By Claim A(b), since $W_{\text{min}}$ is the minimum weight of a perfect matching in $H$, $W_{\text{min}}$ is minimum size of a multiset of edges $E^{**}$, such that $d_{E^{**}}(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$.

Note that no edge is in $E^*$ more than once, since if some edge, $e$, appears twice, then we can delete two copies of $e$ from $E^*$, thereby contradicting the minimality of $|E^*|$. Let $F$ be the spanning subgraph of $G$ with edge set $E^*$. By Lemma 11 we note that $F$ is an $f$-parity perfect forest, which completes the proof of the theorem.

## 3 Proof of Theorem 5

We will reduce from the not-all-equal 3-SAT problem, abbreviated to NAE-3-SAT, which is the problem of determining whether an instance of 3-SAT has a truth assignment to its variables such that every clause contains both a true and a false literal. If this is the case we say that the instance is NAE-satisfied. NAE-3-SAT is known to be NP-hard to solve [16]. Let $I$ be an instance of NAE-3-SAT with clauses $C_1, C_2, \ldots, C_m$ and variables $v_1, v_2, \ldots, v_n$. We will construct a graph $G$ such that $G$ contains a 0-perfect forest with at least $n - 2$ edges if and only if $I$ is NAE-satisfied.

We first create a gadget $H_i$ for each $i = 1, 2, \ldots, n$ as follows. Let

$$V(H_i) = \{x_1^i, z_1^i, y_1^i, x_2^i, z_2^i, y_2^i\}$$

and add all possible edges to $H_i$, except $x_1^i y_1^i$ and $x_2^i y_2^i$. For all $i = 1, 2, \ldots, n - 1$ we then add all edges between $\{y_1^i, y_2^i\}$ and $\{x_1^{i+1}, x_2^{i+1}\}$. Now add a pendent edge to each vertex in $V(H_i) \setminus \{x_1^i, x_2^i, y_1^i, y_2^i\}$ for all $i = 1, 2, \ldots, n$. See Figure 3 for an illustration of this part of $G$, which is denoted by $Q$. We will now complete our construction of $G$.

![Figure 3](image)

**Figure 3** The gadgets $H_1, H_2, \ldots, H_n$ and the edges connecting these. The resulting graph is denoted by $Q$.

Let $V(G) = V(Q) \cup \{c_1, c_2, \ldots, c_m\} \cup \{c'_1, c'_2, \ldots, c'_m\}$. For each $j = 1, 2, \ldots, m$ we will add an edge from both $c_j$ and $c'_j$ to $y_1^j$ if and only if $v_i$ is a literal in the clause $C_j$. We will furthermore add an edge from both $c_j$ and $c'_j$ to $y_1^i$ if and only if $\overline{v_i}$ is a literal in the clause $C_j$. This completes the construction of $G$. See Figure 4 depicting $G$ for $I = (v_1, v_2, \overline{v_3})$.

We will now show that $G$ contains a 0-perfect forest of size at least $n - 2$ if and only if $I$ is NAE-satisfied. First assume that $I$ is NAE-satisfied and consider a truth assignment $\tau$ NAE-satisfying $I$. We will construct two vertex-disjoint induced trees, $T_1$ and $T_2$, in $G$, such that all degrees in the trees $T_i$ are odd for $i \in [2]$. If $v_i$ is true in $\tau$ then add the vertices in
\{x_1^i, z_1^i, y_1^i\} \to T_1 \text{ and the vertices in } \{x_2^i, z_2^i, y_2^i\} \to T_2. \text{ Conversely, if } u_i \text{ is false in } \tau \text{ then add}
\text{the vertices in } \{x_1^i, z_1^i, y_1^i\} \to T_2 \text{ and the vertices in } \{x_2^i, z_2^i, y_2^i\} \to T_1. \text{ We furthermore add}
\text{all vertices of degree one to the same tree as their neighbour. Note that the vertices we have}
\text{added so far to } T_i \text{ (for } i \in [2]) \text{ induce a tree in } G, \text{ where every vertex has odd degree in } T_i.

\text{Finally as } I \text{ is NAE-satisfied we note for } j \in [m], \\text{each of } c_j \text{ and } c'_j \text{ has one edge into one of the } T_i \text{'s and two edges into the other } T_i. \\text{Add each of } c_j \text{ and } c'_j \text{ to the } T_i \text{ with which it is}
\text{only connected by one edge. We note that after this operation the vertices we have added}
\text{so far to } T_i \text{ (for } i \in [2]) \text{ still induce a tree in } G \text{ where every vertex has odd degree in } T_i.
\text{After doing the above operation for all } j \in [m] \text{ we have obtained the desired trees } T_1 \text{ and } T_2
\text{whose union form a 0-perfect forest in } G \text{ with } |V(G)| - 2 \text{ edges. See Figure 5 for the found}
\text{T}_1 \text{ and } T_2 \text{ if the instance of NAE-3-SAT is } I = (v_1, v_2, \overline{v}_3) \text{ and the truth assignment is to set}
\text{all variables equal to true.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image4.png}
\caption{The graph } G \text{ if } I = (v_1, v_2, \overline{v}_3).
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image5.png}
\caption{The trees } T_1 \text{ and } T_2 \text{ if } I = (v_1, v_2, \overline{v}_3) \text{ and } v_1 = v_2 = v_3 = \text{true}.
\end{figure}

Conversely, assume that } G \text{ contains a 0-perfect forest with at least } |V(G)| - 2 \text{ edges. As } G \text{ is not a tree this implies that } G \text{ contain two vertex-disjoint trees } T_1 \text{ and } T_2 \text{ such that each}
\text{T}_i \text{ is an induced tree in } G \text{ of order at least 2, all degrees in each } T_i \text{ are odd, and } V(T_1) \text{ and}
\text{V}(T_2) \text{ partition } V(G). \text{ We will now prove the following claims where Claim C completes the proof of the theorem.}

\begin{itemize}
\item[\textbf{Claim A.}] For each } i \in [n] \text{ one of the following cases hold.}
\begin{enumerate}
\item[\textbf{A.1.}] \{x_1^i, z_1^i, y_1^i\} \in V(T_1) \text{ and } \{x_2^i, z_2^i, y_2^i\} \in V(T_2).
\item[\textbf{A.2.}] \{x_1^i, z_2^i, y_1^i\} \in V(T_1) \text{ and } \{x_2^i, z_1^i, y_2^i\} \in V(T_2).
\item[\textbf{A.3.}] \{x_1^i, z_1^i, y_1^i\} \in V(T_2) \text{ and } \{x_2^i, z_2^i, y_2^i\} \in V(T_1).
\item[\textbf{A.4.}] \{x_1^i, z_2^i, y_1^i\} \in V(T_2) \text{ and } \{x_2^i, z_1^i, y_2^i\} \in V(T_1).
\end{enumerate}
\end{itemize}

\textbf{Proof of Claim A.} As the only two non-edges in } H_i \text{ are } x_1^i y_1^i \text{ and } x_2^i y_2^i \text{ we note that there exist a 4-cycle on every set of 4 vertices in } H_i. \text{ Therefore } |V(T_j) \cap V(H_i)| \geq 4 \text{ is not possible}
\text{for any } j \in [2] \text{ and } i \in [n]. \text{ Therefore } |V(T_j) \cap V(H_i)| = 3 \text{ for } j \in [2] \text{ and } i \in [n].
As there is no 3-cycle in \( G[V(T_j)] \) for \( j \in [2] \) we note that \( x_1^j \) and \( y_1^j \) must belong to one of the trees, say \( T_j \), and \( x_2^j \) and \( y_2^j \) must belong to the other tree, \( T_{3-j} \). So if \( x_1^j \in V(T_1) \) then \( y_1^j \in V(T_1) \) and \( \{x_2^j, y_2^j\} \subseteq V(T_2) \) and we are in case A.1 or A.2. On the other hand if \( x_1^j \in V(T_2) \) then \( y_1^j \in V(T_2) \) and \( \{x_2^j, y_2^j\} \subseteq V(T_1) \) and we are in case A.3 or A.4. This completes the proof of Claim A.

▷ Claim B. For \( i = 1, 2 \), \( G[V(Q) \cap V(T_i)] \) is a tree where all vertices have odd degree.

Proof of Claim B. Any vertex in \( G \) with degree one must belong to the same tree, \( T_j \), as its neighbour, as both \( T_1 \) and \( T_2 \) have order at least two. By Claim A, we therefore note that \( G[V(Q) \cap V(T_i)] \) is a path of length 3\( n \) with a pendent edge attached to each non-endpoint of the path. This implies that \( G[V(Q) \cap V(T_i)] \) is a tree where all vertices have odd degree (as all degrees are either 1 or 3). This completes the proof of Claim B.

▷ Claim C. The instance \( I \) is NAE-satisfiable.

Proof of Claim C. Assume that the vertex \( c_j \) belongs to \( T_1 \). First suppose that \(|N_G(c_j)| \cap V(T_1)| = 0 \). In this case \( c_j \) has no neighbours in \( T_1 \), a contradiction, as \( T_1 \) is a tree with order at least two. So \(|N_G(c_j)| \cap V(T_1)| \geq 1 \). Assume that \(|N_G(c_j)| \cap V(T_1)| \geq 2 \). As \( T_1 \) is an induced tree in \( G \), \( c_j \) must have at least two neighbours, say \( x \) and \( y \), in \( T_1 \). However, by Claim B, there exists a \((x, y)\)-path in \( T_1 \) using only vertices from \( V(Q) \), which implies that there is a cycle in \( T_1 \), a contradiction. Therefore \(|N_G(c_j)| \cap V(T_1)| = 1 \).

Analogously, we can show that \(|N_G(c_j)| \cap V(T_2)| = 1 \), whenever \( c_j \in V(T_2) \). So each clause \( C_j \) (\( j \in [m] \)) has either exactly one literal that is false (if \( c_j \in V(T_1) \)) or exactly one literal that is true (if \( c_j \in V(T_2) \)). This implies that \( I \) is NAE-satisfiable, which completes the proof of Claim C and the theorem.

4 Proof of Theorem 6

To prove Theorem 6, we will use the following result. The proof of Theorem 4 follows the same approach as the proof that it is NP-hard to determine whether there is an induced cycle of odd length through a prescribed vertex, given in [2] by Bienstock. The proof is not given here but can be found in the appendix of [8].

▷ Theorem 4. It is NP-hard to determine whether a graph contains an induced cycle through two given edges.

▷ Theorem 6. The following problem is NP-hard. Given a connected graph \( G \) and an edge \( e \in E(G) \), decide whether \( G \) has a 0-perfect forest containing \( e \).

Proof. Let \( G \) be a graph and let \( e_1 = u_1v_1 \) and \( e_2 = u_2v_2 \) be distinct edges of \( G \). We will construct an auxiliary graph \( H \) with an edge \( e'_2 \in E(H) \), such that \( H \) contains a 0-perfect forest containing \( e'_2 \) if and only if \( G \) contains an induced cycle, \( C \), such that \( e_1, e_2 \in E(C) \). This will complete the proof by Theorem 4.

Let \( H \) be obtained from \( G \) by adding a pendent edge to each vertex in \( V(G) \setminus \{u_1, v_1\} \) and deleting the edge \( e_1 \). Let \( E_P \) denote the set of all the pendent edges we just added to \( G \). Let \( e_2' = u_2v_2 \) and note that \( e_2' \in E(H) \). This completes the construction of \( H \).

Assume that there exists an induced cycle, \( C \), in \( G \) such that \( e_1, e_2 \in E(C) \). Let \( E' = E_P \cup E(C) \setminus e_1 \). Note that the edges in \( E' \) induce a 0-perfect forest in \( H \) containing the edge \( e_2' \).

Conversely assume that there is a 0-perfect forest, \( F \), in \( H \) containing \( e_2' \). Clearly \( F \) contains all edges in \( E_P \) as each pendent edge is incident with a vertex of degree one. Let \( Q \) be the subgraph of \( H \) induced by the edges in \( E(F) \setminus E_P \). Note that \( Q \) is a perfect forest
where \( u_1 \) and \( v_1 \) have odd degree and all other vertices have even degree. As \( Q \) is a perfect forest all components are induced trees, and as \( u_1 \) and \( v_1 \) are the only vertices of odd degree, this implies that \( Q \) is an induced path between \( u_1 \) and \( v_1 \). Adding the edge \( e_1 \) to \( Q \) gives us an induced cycle in \( G \) containing both \( e_1 \) and \( e_2 \) (as \( e_2' \in E(F) \)).

Therefore we have proven that \( H \) contains a 0-perfect forest containing \( e_2' \) if and only if \( G \) contains an induced cycle, \( C \), such that \( e_1, e_2 \in E(C) \), as desired. □

5 Proof of Theorem 8

Let \( G \) be a graph and \( e = wv \) an edge of \( G \). Let \( f : V(G) \to \{0, 1\} \) be an even-sum function. Our polynomial-time algorithm will follow from the four claims proved below. At the end of the proof, we briefly discuss how the claims are used in the algorithm.

▷ Claim A. Suppose that \( G \) contains a cut-vertex \( x \), which may or may not belong to \( \{u, v\} \). Let \( C \) be the component in \( G - x \) intersecting \( \{u, v\} \) (there is exactly one such component as \( uv \in E(G) \)) and let \( G' = G[V(C) \cup \{x\}] \). Let \( f'(w) = f(w) \) for all \( w \in V(C) \) and define \( f'(x) \in \{0, 1\} \) such that \( \sum_{z \in V(G_i)} f'(z) \) is even. Then \( G \) has an \( f \)-parity perfect forest not containing \( e \) if and only if \( G' \) has an \( f' \)-parity perfect forest not containing \( e \).

Proof of Claim A. Let \( G \) contain a cut-vertex \( x \) and let \( C_1, C_2, \ldots, C_k \) be the components in \( G - x \). Without loss of generality, assume that \( C_1 \) is the component intersecting \( \{u, v\} \). Let \( G_i = G[V(C_i) \cup \{x\}] \) for all \( i \in [k] \).

For each \( i \in [k] \) we will let \( f_i : V(G_i) \to \{0, 1\} \) be defined such that \( f_i(w) = f(w) \) for all \( w \in V(G_i) \) and \( \sum_{z \in V(G_i)} f_i(z) \) is even (this defines the value of \( f_i(x) \)). We will show that \( G \) has an \( f \)-parity perfect forest not containing \( e \) if and only if \( G_1 \) has an \( f_1 \)-parity perfect forest not containing \( e \), which will complete the proof of Claim A.

First assume that \( G_1 \) has an \( f_1 \)-parity perfect forest \( F_1 \) not containing \( e \). By Theorem 12 there exists an \( f_i \)-parity perfect forest, \( F_i \), in \( G_i \) for all \( i = 2, 3, \ldots, k \). Now \( F_1 \cup F_2 \cup \cdots \cup F_k \) is an \( f \)-parity perfect forest of \( G \) not containing \( e \), as desired.

Conversely assume that \( G \) has an \( f \)-parity perfect forest \( F \) not containing \( e \). If we restrict \( F \) to \( V(G_1) \), then we obtain an \( f_1 \)-parity perfect forest of \( G_1 \) not containing \( e \). □

▷ Claim B. If \( G \) is 2-connected and \( f(u) = 0 \) or \( f(v) = 0 \) then \( G \) has an \( f \)-parity perfect forest not containing \( e \).

Proof of Claim B. Assume without loss of generality that \( f(u) = 0 \). As \( G \) is 2-connected \( G - u \) is connected and \( \sum_{z \in V(G - u)} f(z) \) is even. Therefore, by Theorem 12, there exists an \( f \)-parity perfect forest in \( G - u \), which is also an \( f \)-parity perfect forest in \( G \) not containing the edge \( e \). □

▷ Claim C. If \( G \) is 2-connected and \( f(u) = f(v) = 1 \) then \( G \) has a \( f \)-parity perfect forest if and only if \( \sum_{z \in V(G)} f(z) \geq 4 \).

Proof of Claim C. Let \( S = \sum_{z \in V(G)} f(z) \). As \( f \) is even-sum, \( S \) is even. Since \( f(u) = f(v) = 1 \), we have \( S \geq 2 \). If \( S = 2 \) and \( F \) is an \( f \)-parity perfect forest in \( G \), then \( u \) and \( v \) must be leaves of the same tree in \( F \) (as they are the only vertices with an \( f \)-value of one). Therefore \( e \in E(F) \), as otherwise the tree containing \( u \) and \( v \) is not induced in \( G \). So, if \( S = 2 \) then \( G \) has no \( f \)-parity perfect forest \( F \) in \( G \) with \( e \notin E(F) \).

We may therefore assume that \( S \geq 4 \) and let \( w \in V(G) - \{u, v\} \) have \( f(w) = 1 \). As \( G \) is 2-connected there exists a \((u, w)\)-path, \( P \), in \( G \) with \( w \in V(P) \). (To see it, consider two internally disjoint paths from \( w \) to \( w' \) where \( w' \) is a new vertex added to \( G \) such that
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We now create a spanning tree $T$ in $G$, such that $E(P) \subseteq E(T)$ and $d_T(w) = 2$, as follows. Initially let $T = P$. While $V(T) \neq V(G)$ let $q \in V(G) \setminus V(T)$ be arbitrary such that $q$ has an edge into $V(T) \setminus \{w\}$ (which exists as $G$ is 2-connected). Add $q$ and an edge from $q$ into $V(T) \setminus \{w\}$ to $T$. When $V(T)$ becomes equal to $V(G)$ we have our desired tree $T$.

Let $T_1$ and $T_2$ be the two trees in $T - w$ (there are exactly two trees in $T - w$ as $d_T(w) = 2$). Let $S_1 = \sum_{z \in V(T_1)} f(z)$ and let $S_2 = \sum_{z \in V(T_2)} f(z)$. As $f(w) = 1$ and $V(T_1) \cup V(T_2) = V(G) \setminus \{w\}$, we note that $S_1 + S_2$ is odd. If $S_i$ is odd then add $w$ to $T_i$ ($i \in [2]$), using the edge from $w$ to $V(T_i)$ in $T$. This results in two trees, say $T'_1$ and $T'_2$, where $\sum_{z \in V(T')} f(z)$ is even for $i \in [2]$. Furthermore, as $w \in V(P)$ and $E(P) \subseteq E(T)$, we note that $u$ and $v$ do not belong to the same tree $T'_i$. By Theorem 12 there exists an $f$-parity perfect forest, $F'_{i'}$ of $G[V(T'_i)]$ for $i \in [2]$ (as $T'_i$ is a spanning tree in $G[V(T'_i)]$, $G[V(T'_i)]$ is connected). Now $F'_1 \cup F'_2$ is an $f$-parity perfect forest of $G$ not containing $e$. This completes the proof of Claim C.

It is easy to see that the following algorithm is of polynomial time. Keep reducing the graph (see Claim A) as long as there exists a cut-vertex and when there are no more cut-vertices then the answer is “no” if the endpoints of $e$ have an $f$-value of one and all other vertices have an $f$-value of zero and “yes”, otherwise (see Claims B and C). See Figure 6 for an illustration of the algorithm.

![Figure 6](image-url) An illustration of the algorithm given in Theorem 8, where the values on the nodes indicate the $f$-values. As in the final graph the endpoints of $e$ have an $f$-value of one and all other vertices have an $f$-value of zero there is no $f$-parity perfect forest in $G'$ avoiding the edge $e$ and therefore not in $G$ either.

## 6 Proof of Theorem 10

Theorem 10 follows from Theorem 2 and Lemma 3 proved in this section. To prove Theorem 2, we will use the following:

**Lemma 4.** Let $G$ be a connected graph of even order and let $xy \in E(G)$ such that $G - \{x, y\}$ is connected. If $G - x \in \mathcal{B}$ and $G - y \in \mathcal{B}$ then $N[x] = N[y].$

**Proof.** Let $G$ be a connected graph of even order and let $xy \in E(G)$ be chosen such that $G - \{x, y\}$ is connected. Let $G_x = G - x$ and let $G_y = G - y$ and assume that $G_x \in \mathcal{B}$ and $G_y \in \mathcal{B}$. Let $C_{x}^{1}, C_{x}^{2}, \ldots, C_{x}^{0}$ be the blocks of $G_x$ and without loss of generality assume that $x \in V(C_{x}^{1})$. Analogously, let $C_{y}^{1}, C_{y}^{2}, \ldots, C_{y}^{0}$ be the blocks of $G_y$ and without loss of generality assume that $y \in V(C_{y}^{0}).$
We now note that $\text{Theorem 2.}$ and let $G$ degree is the vertex of even degree in $G$ of odd order. Let $\langle x, y \rangle$ be a path of order 3, which is a proper 1-perfect forest. Now we assume that $G - \{x, y\}$ is connected. This contradiction implies that all vertices in $N_{G_x}(x)$ belong to the same block of $G_x$.

Therefore, $N_{G_x}[x] \subseteq V(C^*_1)$ as $x$ is not a cut-vertex in $G_x$ (as $G - \{x, y\}$ is connected) and hence $x$ only belongs to one block of $G_x$. As $G_x \in \mathcal{B}$ we note that $C^*_1$ is a complete graph of odd order. As $|V(C^*_1)| \geq 3$ (as all blocks contain at least two vertices, and $|V(C^*_1)|$ is odd) and $x$ is not a cut-vertex in $G_x$ we note that $C^*_1 - x$ is a block in $G - \{x, y\}$. This completes the proof of Claim A.

We now return to the proof of the lemma. By Claim A we note that $C^*_1 - y$ is a block in $G - \{x, y\}$ which furthermore is a complete graph of even order. If $C^*_1 - x$ and $C^*_1 - y$ are different blocks in $G - \{x, y\}$, then $C^*_1 - y$ is a block of even order in $G_x$, a contradiction to $G_x \in \mathcal{B}$. So, $C^*_1 - x$ and $C^*_1 - y$ are the same block in $G - \{x, y\}$. By Claim A, we have the following chain of equalities, which completes the proof of the lemma.

$$N_G[x] = V(C^*_1 - x) \cup \{x, y\} = V(C^*_1 - y) \cup \{x, y\} = N_G[y]$$

$\triangleright$ Theorem 2. Every connected graph, $G \notin \mathcal{B}$, of odd order $n \geq 3$ contains a proper 1-perfect forest.

Proof. The proof is by induction over odd integers $n \geq 3$. For $n = 3$, we have $G \cong P_3$, the path of order 3, which is a proper 1-perfect forest. Now we assume that $G$ is a connected graph of odd order $n \geq 5$ such that $G \notin \mathcal{B}$. Let us consider two cases.

Case 1: $G$ is not 2-connected. Assume that $G$ has a cut-vertex $x$ such that $G - x$ has a component $C_1$ of even order. Let $G_1 = G[V(C_1) \cup \{x\}]$ and let $G_2 = G - V(C_1)$. Note that both $G_1$ and $G_2$ are connected graphs of odd order. Furthermore the set of blocks of $G$ is exactly the union of the blocks in $G_1$ and $G_2$. $G \notin \mathcal{B}$ (and therefore some block in $G$ is not a complete graph of odd order) we note that either $G_1 \notin \mathcal{B}$ or $G_2 \notin \mathcal{B}$ (or both).

Let $i \in \{1, 2\}$ be defined such that $G_i \notin \mathcal{B}$ and let $j = 3 - i$. By induction hypothesis, there exists a proper 1-perfect forest $F_i$ in $G_i$. By Theorem 9 there also exists a (not necessarily proper) 1-perfect forest, $F_j$, in $G_j$, where $x$ is the vertex of even degree in $F_j$. We now note that $F_i \cup F_j$ is a proper 1-perfect forest of $G$, where the only vertex of even degree is the vertex of even degree in $F_i$. Thus, we may assume that $G$ has no cut-vertex $x$ such that some component in $G - x$ is of even order.

Now assume that $G$ contains a cut-vertex $x$. By the previous assumption, all components in $G - x$ are of odd order, and let $C_1$ be a component of $G - x$. Let $G_1 = G[V(C_1) \cup \{x\}]$ and let $G_2 = G - V(C_1)$. Note that both $G_1$ and $G_2$ are connected graphs of even order. By Theorem 1 there exists a 0-perfect forest $F_1$ in $G_1$ and a 0-perfect forest $F_2$ in $G_2$. Note that $F_1 \cup F_2$ is now a proper 1-perfect forest of $G$, where the only vertex of even degree is $x$. 

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Case 2: $G$ is 2-connected.

**Definition A.** As $G \notin \mathcal{B}$ and $G$ has odd order, we note that $G$ is not a complete graph. Therefore there exists an induced path $p_1p_2p_3$ in $G$ (that is, $p_1p_2,p_2p_3 \in E(G)$ and $p_1p_3 \notin E(G)$). Let $C_1,C_2, \ldots, C_l$ be the components in $G$−$\{p_2,p_3\}$, such that $p_1 \in C_1$.

Assume first that $|V(C_1)|$ is odd. By Theorem 9 there exists a 1-perfect forest $F_1$ in $C_1$, such that $p_1$ (see Definition A) is the vertex of even degree in $F_1$. Let $G' = G − V(C_1)$ and note that $G'$ is connected and of even order. Therefore, by Theorem 1, there exists a 0-perfect forest, $F'$, in $G'$.

If $d_{F'}(p_1) > 0$ then $F_1 \cup F'$ is a proper 1-perfect forest in $G$. Now consider the case when $d_{F'}(p_1) = 0$. As $N(p_1) \cap V(G') = \{p_2\}$ (as $p_1p_2p_3$ is an induced path in $G$) we note that adding the edge $p_1p_2$ to $F_1 \cup F'$ gives us a proper 1-perfect forest in $G$ (where $p_2$ is the only vertex of even degree). Thus, in the rest of the proof, we may assume that $|V(C_1)|$ is even. Let $G' = G[V(C_1) \cup \{p_2,p_3\}]$ and note that $G$ is connected and of even order. Furthermore $G' − \{p_2,p_3\}$ is connected (as $G' − \{p_2,p_3\} = C_1$). As $p_1$ is adjacent to $p_2$ but not to $p_3$ we note that $N_{G'}[p_2] \neq N_{G'}[p_3]$. By Lemma 4 we must therefore have $G' − p_2 \notin \mathcal{B}$ or $G' − p_3 \notin \mathcal{B}$. Let $i \in \{2,3\}$ be chosen such that $G' − p_i \notin \mathcal{B}$, which by induction hypothesis implies that there is a proper 1-perfect forest $F_1$ in $G' − p_i$.

As $G$ is 2-connected, we note that $p_{i-1}$ is not a cut-vertex of $G$. Therefore every component in $G − \{p_2,p_3\}$ has an edge to $p_i$, which implies that $G − V(F_1)$ is connected and of even order (as both $G$ and $F_1$ are of odd order). By Theorem 1 there exists a 0-perfect forest, $F_2$, in $G − V(F_1)$. Now $F_1 \cup F_2$ is a proper 1-perfect forest in $G$. This completes the proof. ◀

A semiperfect forest $F$ of $G$ is called a 2-perfect forest if exactly two vertices of $F$ have even degree.

**Lemma 3.** If $G$ is a connected graph of odd order and $G \in \mathcal{B}$ then $G$ does not contain a proper 1-perfect forest.

**Proof.** Let $G$ be a connected graph of odd order and let $G \in \mathcal{B}$. We will prove that $G$ contains no proper 1-perfect forest. We will prove this using induction on the number of blocks in $G$.

If $G$ contains only one block then $G$ is a complete graph of odd order. In this case, any forest where all trees are induced, can only contain trees of order 2 (and 1 if we allow isolated vertices). This implies that $G$ cannot contain a proper 1-perfect forest as $G$ has odd order. This completes the base case.

Now assume that $G$ contains at least two blocks, which implies that $G$ contains a cut-vertex, $x$. Let $C_1,C_2, \ldots, C_l$ be the components in $G − x$ and let $G_i = G[V(C_i) \cup \{x\}]$ for $i \in [l]$. For the sake of contradiction suppose that $G$ contains a proper 1-perfect forest $F$ and let $F_i$ denote $F$ restricted to $G_i$ for $i \in [l]$. As $F$ is a proper 1-perfect forest we note that $d_{F}(x) \geq 1$. Without loss of generality, assume that $d_{F_1}(x) \geq 1$. This implies that $F_1$ is a proper $i$-forest in $G_1$ where $i \in \{0,1,2\}$. However as $|V(G_1)|$ is odd (as $G \in \mathcal{B}$) this implies that $F_1$ is a proper 1-perfect forest in $G_1$. This is a contradiction to $G_1 \in \mathcal{B}$ (as $G \in \mathcal{B}$). ◀
References


