PSPACE-Completeness of the Temporal Logic of Sub-Intervals and Suffixes

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Abstract
In this paper, we establish PSPACE-completeness of the finite satisfiability and model checking problems for the fragment of Halpern and Shoham interval logic with modality $\langle E \rangle$, for the “suffix” relation on pairs of intervals, and modality $\langle D \rangle$, for the “sub-interval” relation, under the homogeneity assumption. The result significantly improves the EXPSPACE upper bound recently established for the same fragment, and proves the rather surprising fact that the complexity of the considered problems does not change when we add either the modality for suffixes ($\langle E \rangle$) or, symmetrically, the modality for prefixes ($\langle B \rangle$) to the logic of sub-intervals (featuring only $\langle D \rangle$).

2012 ACM Subject Classification Theory of computation → Logic and verification

Keywords and phrases Interval temporal logic, Satisfiability, Model checking

Digital Object Identifier 10.4230/LIPIcs.TIME.2021.9

Acknowledgements The open access publication of this article was supported by the Alpen-Adria-Universität Klagenfurt, Austria.

1 Introduction

For a long time, in computer science, interval temporal logics (ITLs) have been considered an attractive, but impractical, alternative to standard point-based ones. On the one hand, they are a natural choice as a specification/representation language in a number of domains; on the other hand, the high undecidability of the satisfiability problem for the most well-known ITLs [8, 14, 16, 20, 30], such as Halpern and Shoham’s modal logic of time intervals (HS for short) [14] and Venema’s CDT [30], discouraged their extensive use (in fact, some restricted variants of them have been applied in formal verification and AI over the years [17, 24, 26]).

The present work finds its place in the framework of the logic HS, which features one modality for each of the 13 Allen’s relations [1], apart from equality. In Table 1, we depict 6 Allen’s relations for ordered pairs of intervals, together with the corresponding HS (existential) modalities; the other 7 relations are their inverses and the equality relation. The recent discovery of a significant number of expressive and computationally well-behaved fragments of HS changed the landscape of ITL research [11, 23]. Meaningful examples are the logic $\mathcal{A}\mathcal{X}$ of the temporal neighborhood [10] (the HS fragment with modalities for the meets relation and its inverse) and the logic $\mathcal{D}$ of (temporal) sub-intervals [9] (the HS fragment with modality $\langle D \rangle$ for the contains relation only) over dense orderings.

Model checking (MC) of (finite) Kripke structures against HS and its fragments has been investigated in a series of papers [3, 5, 17, 18, 19, 21, 22] and shown to be decidable. In this setting, each finite path of a Kripke structure is interpreted as an interval whose
An expressive comparison of MC for homogeneity assumption, than BD that, over finite linear orders and under the homogeneity assumption, asymptotically optimal way the considered problems for in [2] are generalized by an elegant algebraic framework, which allows us to solve in an LTL standard DE of the theoretic approach for solving MC and finite satisfiability under the homogeneity assumption for the maximal fragment BE of HS which features modalities (B) and (E) for prefixes and suffixes. These complexity bounds easily transfer to finite satisfiability, that is, satisfiability over finite linear orders, of BE under the homogeneity assumption. Whether or not these problems can be solved elementarily is a difficult open question. On the other hand, MC and finite satisfiability under the homogeneity assumption for all the fragments of BE are known to be elementarily decidable [2, 3, 7]. In particular, for the fragment D of BE (note that the contains relation $R_D$ can be expressed as $R_B \cup R_E \cup R_B \cdot R_E$), these problems are known to be PSPACE-complete [2].

In a recent contribution [7], we investigated finite satisfiability under the homogeneity assumption for the maximal fragment BD of BE that features modalities (B) and (D) (the other maximal fragment DE of BE with modalities (D) and (E) is completely symmetric, and thus all results for BD immediately transfer to it, and vice versa). The addition of modality (B) makes satisfiability checking for BD more complex than the one for D, as the two relations/modalities may interact in a non-trivial way. We proved EXPSPACE membership of the problem [7] by means of a purely model-theoretic argument, leaving the question of its exact complexity open. In this paper, we answer the question proving that, surprisingly, PSPACE-completeness of D is not affected by the addition of either (B) or (E), and the MC problem for DE (and symmetrically BD) is PSPACE-complete as well. In Figure 1, we add these new MC results to the picture of known MC complexities, showing that they enrich the set of HS “tractable” fragments with two meaningful members. We propose an automata-theoretic approach for solving MC and finite satisfiability under the homogeneity assumption of DE and BD which non-trivially generalizes the one for D [2] and the well-known one for standard LTL [29]. In particular, some important aspects that were not well understood in [2] are generalized by an elegant algebraic framework, which allows us to solve in an asymptotically optimal way the considered problems for DE and BD. In addition, we prove that, over finite linear orders and under the homogeneity assumption, D is less expressive than BD and DE, which in turn are less expressive than BE (in [4], we show that, under the homogeneity assumption, BE and LTL over finite words have the same expressive power).

Table 1 Allen’s relations and corresponding HS modalities.

<table>
<thead>
<tr>
<th>Allen relation</th>
<th>HS</th>
<th>Definition w.r.t. interval structures</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEETS</td>
<td>⟨A⟩</td>
<td>$[x, y]R_A[v, z] \iff y = v$</td>
<td><img src="example1.png" alt="Example" /></td>
</tr>
<tr>
<td>BEFORE</td>
<td>⟨L⟩</td>
<td>$[x, y]R_L[v, z] \iff y &lt; v$</td>
<td><img src="example2.png" alt="Example" /></td>
</tr>
<tr>
<td>STARTED-BY</td>
<td>⟨B⟩</td>
<td>$[x, y]R_B[v, z] \iff x = v \land z &lt; y$</td>
<td><img src="example3.png" alt="Example" /></td>
</tr>
<tr>
<td>FINISHED-BY</td>
<td>⟨E⟩</td>
<td>$[x, y]R_E[v, z] \iff y = z \land x &lt; v$</td>
<td><img src="example4.png" alt="Example" /></td>
</tr>
<tr>
<td>CONTAINS</td>
<td>⟨D⟩</td>
<td>$[x, y]R_D[v, z] \iff [v, z] \subseteq [x, y]$</td>
<td><img src="example5.png" alt="Example" /></td>
</tr>
<tr>
<td>OVERLAPS</td>
<td>⟨O⟩</td>
<td>$[x, y]R_O[v, z] \iff x &lt; y &lt; z$</td>
<td><img src="example6.png" alt="Example" /></td>
</tr>
</tbody>
</table>

Labeling satisfies the homogeneity assumption [27]: a proposition letter holds over an interval if and only if it holds over all its constituent points (states). MC against full HS is at least EXPSPACE-hard [5] and the only known upper bound is non-elementary [21, 6].

1 An expressive comparison of MC for HS and standard point-based temporal logics LTL [25], CTL, and CTL* [13] can be found in [4].
We conclude the introduction by recalling an interesting connection between the finite satisfiability problem for \( \text{BE} \) and its fragments, under the homogeneity assumption, and the non-emptiness problem for generalized \(*\)-free regular expressions [7]. The latter problem has been shown to be non-elementarily decidable by Stockmeyer in [28], and it can be easily proved to be equivalent to finite satisfiability for the interval temporal logic \( C \) of the chop modality [15, 24, 30] under the homogeneity assumption (the chop modality allows one to split the current interval in two parts and to state what is true over the first part and what over the second one). It can be shown that over finite linear orders and under the homogeneity assumption, \( \text{BE} \) (resp., its proper fragments \( \text{BD} \) and \( \text{DE} \)) is equivalent to the weakening of generalized \(*\)-free regular expressions where the concatenation operator is replaced by the weaker \textit{prefix} and \textit{suffix} ones (resp., \textit{prefix} and \textit{infix}, and \textit{infix} and \textit{suffix}) [7]. Note that the infix operator can be expressed in terms of the combination of the prefix and suffix operators. An immediate by-product of the results given in this paper is that the non-emptiness problem for \(*\)-free generalized regular expressions turns out to be elementarily decidable and, precisely, \( \text{PSPACE-complete} \) if one makes use of the \textit{suffix} (resp., \textit{prefix}) operator and the \textit{infix} operator instead of the concatenation operator in the expressions. As for the fragment with both the \textit{prefix} and the \textit{suffix} operators, we only know that its non-emptiness problem is (non-elementarily decidable and) \( \text{EXPSPACE-hard} \) [5].

\[ \text{Figure 1} \] Complexity of the MC problem for \( \text{HS} \) and its fragments.

\section{Preliminaries}

We fix the following notation. For a finite word (or sequence) \( w \) over some finite alphabet \( \Sigma \), we denote by \( |w| \) the length of \( w \). Moreover, for all \( 0 \leq i < |w| \), \( w[i] \) denotes the \((i + 1)^{th}\) letter of \( w \). Given two non-empty finite words \( w, w' \) over \( \Sigma \), we denote by \( w \cdot w' \) the concatenation of \( w \) and \( w' \). Moreover, if the last letter of \( w \) coincides with the first letter of \( w' \), we denote by \( w \ast w' \) the word \( w \cdot w'[1] \ldots w'[n-1] \), where \( n = |w'| \) (i.e. the word obtained by concatenating \( w \) with the word obtained from \( w' \) by erasing the first letter). In particular, when \( |w'| = 1 \), then \( w \ast w' = w \).
Finite automata over finite words. A nondeterministic finite automaton (NFA) is a tuple $\mathcal{N} = (\Sigma, Q, Q_0, \delta, F)$, where $\Sigma$ is a finite alphabet, $Q$ is a finite set of states, $Q_0 \subseteq Q$ is the set of initial states, $\delta : Q \times \Sigma \to 2^Q$ is the transition function, and $F \subseteq Q$ is the set of accepting states. Given a finite word $w$ over $\Sigma$, with $|w| = n$, a run of $\mathcal{N}$ over $w$ is a finite sequence of states $q_0, \ldots, q_n$ such that $q_0 \in Q_0$, and for all $i \in [0, n - 1]$, $q_{i+1} \in \delta(q_i, w[i])$.

The language $L(\mathcal{N})$ accepted by $\mathcal{N}$ consists of the finite words over $\Sigma$ such that there is a run over $w$ ending in some accepting state. A deterministic finite automaton (DFA) is an NFA $\mathcal{D} = (\Sigma, Q, \{q_0\}, \delta, F)$ such that for all $(q, \sigma) \in Q \times \Sigma$, $\delta(q, \sigma)$ is a singleton.

Finite Kripke structures. We fix a finite set $\mathcal{AP}$ of proposition letters which represent predicates over the states of the given system. A finite Kripke structure over $\mathcal{AP}$ is a tuple $\mathcal{K} = (W, s_0, E, \mu)$, where $W$ is a finite set of states, $s_0 \in W$ is the initial state, $E \subseteq W \times W$ is a left-total relation between states, and $\mu : W \to 2^\mathcal{AP}$ is a labelling function assigning to each state the set of propositions that hold at it.

A path of $\mathcal{K}$ is a non-empty finite sequence of states $\rho = s_1 \cdots s_n$ such that (i) the first state $s_1$ coincides with the initial state $s_0$ of $\mathcal{K}$, and (ii) $(s_i, s_{i+1}) \in E$ for $i = 1, \ldots, n - 1$. We extend the labeling $\mu$ to paths of $\mathcal{K}$ in the usual way: for a path $\rho = s_1 \cdots s_n$, $\mu(\rho)$ denotes the word over $2^\mathcal{AP}$ of length $n$ given by $\mu(s_1) \cdots \mu(s_n)$. A trace of $\mathcal{K}$ is a non-empty finite word over $2^\mathcal{AP}$ of the form $\mu(\rho)$ for some path $\rho$ of $\mathcal{K}$.

3 The logics DE and BD under the homogeneity assumption

In this section, we recall the logic $\mathcal{BE}$ of prefix and suffixes corresponding to the linear-time fragment of $\mathcal{HS}$, and we focus our attention on the fragments $\mathcal{DE}$ and $\mathcal{BD}$ of $\mathcal{BE}$ interpreted over finite linear orders under the homogeneity assumption.

Let $S = (S, \langle \rangle)$ be a linear order over the nonempty set $S \neq \emptyset$, and $\leq$ be the reflexive closure of $\langle \rangle$. Given $x, y \in S$ such that $x \leq y$, we denote by $[x, y]$ the (closed) interval over $S$ given by the set of elements $z \in S$ such that $x \leq z$ and $z \leq y$. We denote the set of all intervals over $S$ by $\mathcal{I}(S)$. We focus our attention on three Allen’s relations over intervals:

1. the proper prefix (or started-by) relation $\mathcal{R}_B$ defined as follows: $[x, y] \mathcal{R}_B [x', y']$ if $x = x'$ and $y < y'$.
2. the proper sub-interval (or contains) relation $\mathcal{R}_D$ defined as follows: $[x, y] \mathcal{R}_D [x', y']$ if $x' \geq x$, $y' \leq y$, and $[x, y] \neq [x', y']$ (the proper subset relation over intervals), and
3. the proper suffix (or finished-by) relation $\mathcal{R}_E$ defined as follows: $[x, y] \mathcal{R}_E [x', y']$ if $x < x'$ and $y' = y$.

$\mathcal{BE}$ formulas $\varphi$ are defined by the following syntax:

$$\varphi ::= \top \mid p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle B \rangle \varphi \mid \langle D \rangle \varphi \mid \langle E \rangle \varphi$$

where $p \in \mathcal{AP}$, and $(B)$ (resp., $(D)$, resp., $(E)$) is the existential temporal modality for the Allen’s relation $\mathcal{R}_B$ (resp., $\mathcal{R}_D$, resp., $\mathcal{R}_E$). We also exploit the conjunction connective $\land$, and for any temporal modality $[X]$, with $X \in \{B, D, E\}$, the dual universal modality $[X]$ defined as: $[X] \psi \equiv \neg (X) \neg \psi$. The size $|\varphi|$ of a formula $\varphi$ is the number of distinct sub-formulas of $\varphi$. We focus on the fragments $\mathcal{DE}$ (logic of sub-intervals and suffixes) and $\mathcal{BD}$ (logic of sub-intervals and prefixes) of $\mathcal{BE}$ obtained by disallowing the temporal modalities for the Allen’s relations $\mathcal{R}_B$ and $\mathcal{R}_E$, respectively. We also consider the fragments $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$ defined in the obvious way.
The semantics of the logic BE is given in terms of interval models. An interval model $M$ is a pair $(I(S), \mathcal{V})$, where $\mathcal{V} : \mathcal{A}P \rightarrow 2^{I(S)}$ is a valuation function that assigns to every proposition letter $p$ the set of intervals $\mathcal{V}(p)$ over which $p$ holds. Given an interval model $M = (I(S), \mathcal{V})$, an interval $[x, y] \in I(S)$, and a formula $\phi$, the satisfaction relation $M, [x, y] \models \phi$, meaning that $\phi$ holds over the interval $[x, y]$ of $M$, is inductively defined as follows:

- for every proposition letter $p \in \mathcal{A}P$, $M, [x, y] \models p$ if $[x, y] \in \mathcal{V}(p)$;
- $M, [x, y] \models \neg \phi$ if $M, [x, y] \not\models \phi$;
- $M, [x, y] \models \phi_1 \lor \phi_2$ if $M, [x, y] \models \phi_1$ or $M, [x, y] \models \phi_2$;
- $M, [x, y] \models (X) \phi$ for $X \in \{B, D, E\}$ if there is an interval $[x', y'] \in I(S)$ such that $[x, y] |− M, [x', y'] \models \phi$.

A BE-formula is satisfiable if it holds over some interval of an interval model. In this paper, we restrict our attention to the finite satisfiability problems, which is satisfiability over the class of finite linear orders, for the fragments BD and DE. The problems are known to be undecidable [20] in the general case, but decidability can be recovered by restricting to the class of homogeneous interval models [21]. Formally, an interval model $M = (I(S), \mathcal{V})$ is homogeneous if for every interval $[x, y] \in I(S)$ and every $p \in \mathcal{A}P$, it holds that $[x, y] \in \mathcal{V}(p)$ if and only if $[x', y'] \in \mathcal{V}(p)$ for every $x' \in [x, y]$.

We observe that homogeneous interval models over finite linear orders correspond to non-empty finite words over $2^\mathcal{A}P$. In particular, each non-empty finite word $w$ over $2^\mathcal{A}P$ induces the homogeneous interval model $M(w) = (I(S), \mathcal{V})$ over the finite linear order induced by $w$ defined as follows:

- $\mathcal{S} = \{\{0, \ldots, |w| - 1\}, \prec\}$, and
- for every interval $[i, j]$ of $\mathcal{S}$ (note that $0 \leq i \leq j < |w|$) and $p \in \mathcal{A}P$, $[i, j] \in \mathcal{V}(p)$ if and only if $p \in w[h]$ for all $h \in [i, j]$.

Any fragment $F$ of BE interpreted over homogeneous models is denoted by $F_{\text{Hom}}$. A non-empty finite word $w$ over $2^\mathcal{A}P$ satisfies an $F_{\text{Hom}}$ formula $\phi$, denoted by $w \models \phi$, if $M(w), [0, |w| - 1] \models \phi$. A finite Kripke structure $\mathcal{K}$ over $\mathcal{A}P$ is a model of $\phi$, written $\mathcal{K} \models \phi$, if each trace $w$ of $\mathcal{K}$ satisfies $\phi$. We also consider the model checking problem against $DE_{\text{Hom}}$ (resp., $BE_{\text{Hom}}$) that is the problem of deciding for a given finite Kripke structure $\mathcal{K}$ and $DE_{\text{Hom}}$ (resp., $BD_{\text{Hom}}$) formula $\phi$, whether $\mathcal{K} \models \phi$.

Expressiveness issues. Let $F_1$ and $F_2$ be two logics interpreted over non-empty finite words over $2^\mathcal{A}P$. Given $\psi_1 \in F_1$ and $\psi_2 \in F_2$, $\psi_1$ and $\psi_2$ are equivalent if $\psi_1$ and $\psi_2$ are satisfied by the same non-empty finite words over $2^\mathcal{A}P$. We say that $F_1$ is subsumed by $F_2$, denoted $F_1 \preceq_f F_2$, if for each $F_1$ formula there is an equivalent $F_2$ formula. $F_1$ and $F_2$ have the same expressiveness (resp., are expressively incomparable) if $F_1 \preceq_f F_2$ and $F_2 \preceq_f F_1$ (resp., $F_1 \not\preceq_f F_2$ and $F_2 \not\preceq_f F_1$). Finally, $F_1$ is less expressive than $F_2$, denoted by $F_1 \prec_f F_2$, if $F_1 \preceq_f F_2$ and $F_2 \not\preceq_f F_1$. It is known [4] that $BE_{\text{Hom}}$ has the same expressiveness as standard LTL over finite words. Here, we show that over finite words, the fragment $D_{\text{Hom}}$ is less expressive than the fragments $BD_{\text{Hom}}$ and $DE_{\text{Hom}}$, which in turn are less expressive than $BE_{\text{Hom}}$ or, equivalently, LTL. In particular, the following hold (a proof is in Appendix A).

**Theorem 1.** There exists a $B_{\text{Hom}}$ (resp., $E_{\text{Hom}}$) formula which cannot be expressed in $DE_{\text{Hom}}$ (resp., $BD_{\text{Hom}}$) over finite linear orders. Hence, $D_{\text{Hom}} \prec_f BD_{\text{Hom}} \prec_f BE_{\text{Hom}}$, $D_{\text{Hom}} \prec_f DE_{\text{Hom}} \prec_f BE_{\text{Hom}}$, and $BD_{\text{Hom}}$ and $DE_{\text{Hom}}$ are expressively incomparable.

In order to illustrate the succinctness of the logics $D_{\text{Hom}}$, $BD_{\text{Hom}}$, and $DE_{\text{Hom}}$, we consider a combinatorial requirement. For each $n \geq 1$, let $\mathcal{A}P_n = \{p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_n\}$. The property that “there is a proper infix such for each $i \in [1, n]$, exclusively either $p_i$ holds at some position, or $\bar{p}_i$ holds at some position” can be expressed by the following $D_{\text{Hom}}$ formula $\psi_n$. 
We conjecture that there is no LTL formula equivalent to $\psi_n$ of size polynomial in $n$.

**Encoding $\DE_{\text{Hom}}$ and $\BE_{\text{Hom}}$ in fragments of generalized $*$-free regular expressions.**

In the following, we show how to encode in $\DE_{\text{Hom}}$ (resp., $\BE_{\text{Hom}}$) over finite linear orders the fragment of generalized $*$-free regular expressions where the concatenation operator is replaced by the *infix* and *suffix* ones (resp., the *infix* and *prefix* ones). Recall that a generalized $*$-free regular expression (hereafter, simply called *general expression*) $e$ over a finite alphabet $\Sigma$ is a term of the form:

$$e ::= 0 | a | \lnot e | e + e | e \cdot e,$$

for any $a \in \Sigma$.

We exclude the empty word $\epsilon$ from the syntax as it makes more direct the correspondence between restricted expressions and $\BE_{\text{Hom}}$ fragments (such a simplification is quite common in the literature). A general expression $e$ of the above form defines the language $L(e) \subseteq \Sigma^+$, which is inductively defined as follows: (i) $L(\emptyset) = \emptyset$; (ii) $L(a) = \{a\}$; (iii) $L(\lnot e) = \Sigma^+ \setminus L(e)$; (iv) $L(e_1 + e_2) = L(e_1) \cup L(e_2)$; (v) $L(e_1 \cdot e_2) = \{w_1 \cdot w_2 : w_1, w_2 \in L(e_1), w_2 \in L(e_2)\}$.

In [28], Stockmeyer proves that the problem of deciding non-emptiness of $L(e)$, for a given general expression $e$, is non-elementary hard. Here, we focus our attention on the following class of restricted expressions, called *prefix/suffix expressions*:

$$e ::= 0 | a | \lnot e | e + e | \text{Pre}(e) | \text{Suf}(e) | \text{Inf}(e),$$

where $\text{Pre}(e)$ and $\text{Suf}(e)$ are, respectively, a shorthand for $e \cdot (\emptyset)$ and $(\emptyset) \cdot e$, while $\text{Inf}(e)$ is a shorthand for $\text{Pre}(e) + \text{Suf}(e) + \text{Pre}(\text{Suf}(e))$. An *infix/suffix* (resp., *infix/prefix*) expression is obtained by a prefix/suffix expression by disallowing the prefix operator $\text{Pre}$ (resp., the suffix operator $\text{Suf}$). Assuming that $\Sigma = 2^{\mathbb{N}}$, every suffix/prefix (resp., infix/suffix, resp., infix/prefix) expression $e$ can be mapped into an equivalent formula $\psi_e$ of $\BE_{\text{Hom}}$ (resp., $\DE_{\text{Hom}}$, resp., $\BD_{\text{Hom}}$) by applying the usual constructions for empty language, letters, negation, and union, plus the following three rules: (i) $\psi_{\text{Pre}(e)} = (\mathcal{B}) \psi_e$, (ii) $\psi_{\text{Suf}(e)} = (\mathcal{E}) \psi_e$, and (iii) $\psi_{\text{Inf}(e)} = (\mathcal{D}) \psi_e$. It is well known that LTL over finite words characterizes the class of languages defined by general expressions [12]. Since over finite words, LTL and $\BE_{\text{Hom}}$ have the same expressiveness [4], prefix/suffix expressions and general expressions have the same expressiveness as well. On the other hand, by Theorem 1, infix/suffix (resp., infix/prefix) expressions are less expressive than general expressions.

### 4 Satisfiability and model checking of $\DE_{\text{Hom}}$ over finite linear orders

In this section, we provide an automata-theoretic approach for solving satisfiability and model checking for $\DE_{\text{Hom}}$-formulas over finite linear orders. The proposed approach generalizes in a non-trivial manner the classical automata construction [29] for standard LTL over finite words based on the notion of Hintikka sequence. Given a $\DE_{\text{Hom}}$-formula $\varphi$ and a non-empty finite word $w$ over $2^{\mathbb{N}}$, we associate to each interval $[i, j]$ of $w$ a maximal propositionally consistent set of formulas ($\varphi$-atom) in the syntactical closure $\text{CL}(\varphi)$ of $\varphi$ which, intuitively, represents the set of formulas in $\text{CL}(\varphi)$ which hold at the interval $[i, j]$. The syntactical definition of $\varphi$-atom locally captures the semantics of the Boolean connectives. In order to capture the
semantics of the temporal modalities and the homogeneity assumption, we define syntactical “semi-local” rules which allow (i) to specify in a functional way the atom associated to a non-singleton interval $I$ in terms of the atoms associated to the two proper maximal sub-intervals of $I$, and (ii) to enforce “termination” conditions on the atoms associated with singleton intervals of $w$. Next, for each prefix $w_p$ of $w$, we consider the sequence of $\varphi$-atoms, called row, associated with the suffixes of $w_p$ ordered for increasing values of the length (note that in the automata-theoretic approach for LTL, the notion of row collapses to the atom associated to the current position of the given finite word). The previous syntactical rules guarantee monotonicity properties on the atoms of a row and the existence of a functional relation that given the row of a proper prefix $w_p$ of $w$ and the uniquely determined atom of the singleton interval associated to position $|w_p|$ of $w$, provides the row for the prefix of $w$ leading to position $|w_p|$ (see Section 4.1). As a main technical step (see Section 4.2), by exploiting the monotonicity of rows, we deduce for the given $\text{DE}_{\text{Hom}}$-formula, the existence of an equivalence relation on the set of rows of exponential-size index satisfying three fundamental properties: (i) the equivalence relation preserves the set of atoms visited by a row and their relative ordering along the row, (ii) each equivalence class has a minimal representative whose length is polynomial in the size of the given formula, and (iii) the functional relation crucially preserves the equivalence between rows. The previous three properties (i)–(iii) lead to the construction in singly exponential time of a DFA having as states the set of minimal rows and accepting the non-empty finite words over $2^{AP}$ which satisfy the given formula (see Section 4.3). We now proceed with the technical details.

Given a $\text{DE}$-formula $\varphi$, we define the closure of $\varphi$, denoted by $\text{CL}(\varphi)$, as the set of all sub-formulas $\psi$ of $\varphi$ and of their negations $\neg \psi$ (we identify $\neg \psi$ with $\psi$). A $\varphi$-atom $A$ is a subset of $\text{CL}(\varphi)$ such that:

- for every $\psi \in \text{CL}(\varphi)$, $\psi \in A$ if and only if $\neg \psi \notin A$, and
- for every $\psi_1 \lor \psi_2 \in \text{CL}(\varphi)$, $\psi_1 \lor \psi_2 \in A$ if and only if $A \cap \{\psi_1, \psi_2\} \neq \emptyset$.

We denote the set of all $\varphi$-atoms by $\mathcal{A}_\varphi$; its cardinality is clearly bounded by $2^{|\varphi|}$. We now consider non-empty finite words over $2^{AP}$ equipped with a mapping assigning to each interval a $\varphi$-atom.

**Definition 2** ($\varphi$-word structures and fulfilling $\varphi$-word structures). Let $\varphi$ be a $\text{DE}$-formula. A $\varphi$-word structure $\mathcal{W}$ is a pair $\mathcal{W} = (w, \mathcal{L})$ consisting of a non-empty finite word over $2^{AP}$ and a mapping $\mathcal{L}$ assigning to each interval of $w$ (i.e., an interval in the homogeneous interval model $\mathcal{M}(w)$) a $\varphi$-atom such that for each position $0 \leq i < |w|$, $\mathcal{L}([i, i]) \cap AP = w[i]$. The $\varphi$-word structure $\mathcal{W} = (w, \mathcal{L})$ is fulfilling if for each interval $I$ of $w$ (we also say that $I$ is an interval of $\mathcal{W}$) and for each $\psi \in \text{CL}(\varphi)$ it holds that $\psi \in \mathcal{L}(I)$ if and only if $\mathcal{M}(w), I \models \psi$.

Evidently, for each non-empty finite word $w$ over $2^{AP}$, there exists a unique fulfilling $\varphi$-word structure associated with $w$. Let $\mathcal{W} = (w, \mathcal{L})$ be a $\varphi$-word structure. For each interval $[i, j]$ of $\mathcal{W}$, we write $\mathcal{L}(i, j)$ to mean $\mathcal{L}([i, j])$. For each $0 \leq i < |w|$, the $i$-row of $\mathcal{W}$ is the sequence $row_i$ of $\varphi$-atoms given by $row_i = \mathcal{L}(i, i) \cdots \mathcal{L}(0, i)$, i.e., the sequence of atoms labeling the suffixes of the prefix of $w$ until position $i$ ordered for increasing values of the length.

### 4.1 Characterization of fulfilling $\varphi$-word structures

In this section, for the given $\text{DE}_{\text{Hom}}$-formula $\varphi$, we provide a characterization of fulfilling $\varphi$-word structures $\mathcal{W}$ in terms of a “syntactical” functional relation between adjacent $\mathcal{W}$-rows. For a $\varphi$-atom $A$ and $X \in \{D, E\}$, we consider the following sets:

- $\text{Req}_X(A) := \{\psi \in \text{CL}(\varphi) : (X) \psi \in A\}$ (temporal requests of $A$);
- $\text{Obs}_X(A) := \{\psi \in A : (X) \psi \in \text{CL}(\varphi)\}$ (observables of $A$).
The next proposition, stating that, once the proposition letters of \( A \) and its temporal requests have been fixed, \( A \) gets unambiguously determined, can be easily proved by induction.

\begin{itemize}
\item \textbf{Proposition 3.} Let \( \varphi \) be a DE-formula. Given a set \( R_D \subseteq \{ \psi \mid (D) \psi \in \mathcal{CL}(\varphi) \} \), a set \( R_E \subseteq \{ \psi \mid (E) \psi \in \mathcal{CL}(\varphi) \} \), and a set \( P \subseteq \mathcal{CL}(\varphi) \cap \mathcal{AF} \), there exists a unique \( \varphi \)-atom \( A \) that satisfies \( \text{Req}_D(A) = R_D \), \( \text{Req}_E(A) = R_E \), and \( A \cap \mathcal{AF} = P \).
\end{itemize}

\begin{itemize}
\item \textbf{Definition 4.} Let \( A_B \) and \( A_E \) be two \( \varphi \)-atoms. We denote by \( \text{succ}_\varphi(A_B, A_E) \) the unique \( \varphi \)-atom \( A \) whose sets of propositions and (sub-interval and suffix) temporal requests satisfy:
\begin{enumerate}
\item \( A \cap \mathcal{AF} = A_B \cap A_E \cap \mathcal{AF} \),
\item \( \text{Req}_D(A) = \text{Req}_D(A_B) \cup \text{Obs}_D(A_B) \cup \text{Req}_D(A_E) \cup \text{Obs}_D(A_E) \), and
\item \( \text{Req}_E(A) = \text{Req}_E(A_E) \cup \text{Obs}_E(A_E) \).
\end{enumerate}
\end{itemize}

Definition 4 can be exploited to label a fulfilling \( \varphi \)-word structure \( \mathcal{W} \); namely, to determine the \( \varphi \)-atoms labeling all the intervals \([i, j]\) of \( \mathcal{W} \), starting from the singleton ones. The idea is the following: if two \( \varphi \)-atoms \( A_B \) and \( A_E \) label respectively the greatest proper prefix \([i, j−1]\) and the greatest proper suffix \([i+1, j]\) of the same non-singleton interval \([i, j]\), then the atom \( A \) labeling interval \([i, j]\) is unique, and it is precisely the one given by \( \text{succ}_\varphi(A_B, A_E) \). The next lemma proves that this is the general rule for labeling fulfilling \( \varphi \)-word structures (a proof of Lemma 5 is in Appendix B).

\begin{itemize}
\item \textbf{Lemma 5.} Let \( \mathcal{W} = (w, \mathcal{L}) \) be a \( \varphi \)-word structure. Then \( \mathcal{W} \) is fulfilling if and only if for each interval \([i, j]\) of \( \mathcal{W} \), it holds that \( (i) \mathcal{L}(i, j) = \text{succ}_\varphi(\mathcal{L}(i, j−1), \mathcal{L}(i+1, j)) \), if \( i < j \), and \( (ii) \text{Req}_D(\mathcal{L}(i, j)) = \emptyset \) and \( \text{Req}_E(\mathcal{L}(i, j)) = \emptyset \), if \( i = j \).
\end{itemize}

We now introduce the abstract notion of \( \varphi \)-\textit{rows}, finite sequences of \( \varphi \)-atoms satisfying “syntactical” adjacency requirements which capture the behaviour of \( \mathcal{W} \)-rows in fulfilling \( \varphi \)-word structures \( \mathcal{W} \).

\begin{itemize}
\item \textbf{Definition 6.} A \( \varphi \)-row \( \mathcal{R} \) is a non-empty finite sequence of \( \varphi \)-atoms such that for all \( 0 \leq i < |\mathcal{R}|−1 \): \( \text{(i) } (\mathcal{R}[i] \cap \mathcal{AF}) \supseteq (\mathcal{R}[i+1] \cap \mathcal{AF}) \), \( \text{(ii) } \text{Req}_D(\mathcal{R}[i+1]) \supseteq \text{Req}_D(\mathcal{R}[i]) \cup \text{Obs}_D(\mathcal{R}[i]) \), and \( \text{(iii) } \text{Req}_E(\mathcal{R}[i+1]) = \text{Req}_E(\mathcal{R}[i]) \cup \text{Obs}_E(\mathcal{R}[i]) \).
\end{itemize}

The \( \varphi \)-row \( \mathcal{R} \) is initialized if \( \text{Req}_D(\mathcal{R}[0]) = \emptyset \) and \( \text{Req}_E(\mathcal{R}[0]) = \emptyset \).

We denote by \( \mathcal{R}_{\text{Rows}} \) the set of all possible \( \varphi \)-\textit{rows}. We observe that the sequence of atoms along a \( \varphi \)-row \( \mathcal{R} = A_0 \cdot \ldots A_n \) has a monotonic behaviour, and the number of distinct atoms in \( \mathcal{R} \) is linearly bounded in the size of \( \varphi \). Indeed, by Definition 6, \( \mathcal{R} \) induces three monotonic sequences: \( \text{(i) one concerns the atomic propositions, is decreasing and is given by } (A_0 \cap \mathcal{AF}) \supseteq (A_1 \cap \mathcal{AF}) \supseteq \ldots \supseteq (A_n \cap \mathcal{AF}) \), \( \text{(ii) the second and the third are increasing, concern the temporal requests, and are given by } \text{Req}_D(A_0) \subseteq \text{Req}_D(A_1) \subseteq \ldots \subseteq \text{Req}_D(A_n) \) and \( \text{Req}_E(A_0) \subseteq \text{Req}_E(A_1) \subseteq \ldots \subseteq \text{Req}_E(A_n) \). The number of distinct elements in each sequence is bounded by \( |\varphi| \) (w.l.o.g. we assume that \( |\mathcal{AF}| \leq |\varphi| \), i.e. we can consider only the propositional letters actually occurring in \( \varphi \)). Since a set of requests and a set of proposition letters uniquely determine a \( \varphi \)-atom, any \( \varphi \)-row may feature at most \( 3|\varphi| \) distinct atoms, i.e., \( n \leq 3|\varphi| \).

Since in a fulfilling \( \varphi \)-word structure there are no temporal requests in the atoms labeling the singleton intervals, by Lemma 5, we have the following result.

\begin{itemize}
\item \textbf{Lemma 7.}
\begin{enumerate}
\item The number of distinct atoms in a \( \varphi \)-row \( \mathcal{R} \) is at most \( 3|\varphi| \). Moreover, for all \( 0 \leq i < j < |\mathcal{R}| \), if \( A_i = A_j \), then \( A_k = A_i \) for all \( k \in [i, j] \).
\item Each \( \mathcal{W} \)-row of a fulfilling \( \varphi \)-word structure \( \mathcal{W} \) is initialized \( \varphi \)-row.
\end{enumerate}
\end{itemize}
We now generalize the successor function $\text{succ}_\varphi$ to $\varphi$-rows: given a $\varphi$-row $\text{row}$ and a $\varphi$-atom $A$, we consider the $\varphi$-row of length $|\text{row}| + 1$ and first atom $A$ obtained by a component-wise application of $\text{succ}_\varphi$ starting from $A$ and the first atom of $\text{row}$.

**Definition 8.** Given a $\varphi$-atom $A$ and a $\varphi$-row $\text{row}$ with $|\text{row}| = n$, the $A$-successor of $\text{row}$, denoted by $\text{succ}_\varphi(\text{row}, A)$, is the sequence $B_0 \ldots B_n$ of $\varphi$-atoms defined as follows: $B_0 = A$ and $B_{i+1} = \text{succ}_\varphi(\text{row}[i], B_i)$ for all $i \in [0, n - 1]$.

By Definitions 4 and 8, we can easily derive the following lemma.

**Lemma 9.** Let $\text{row}$ be a $\varphi$-row and $A$ be a $\varphi$-atom. Then, $\text{succ}_\varphi(\text{row}, A)$ is a $\varphi$-row. Moreover, if $\text{row}$ is of the form $\text{row} = \text{row}_1 \cdot \text{row}_2$, then $\text{succ}_\varphi(\text{row}, A) = \text{succ}_\varphi(\text{row}_1, A) \ast \text{succ}_\varphi(\text{row}_2, A_1)$, where $A_1$ is the last $\varphi$-atom of $\text{succ}_\varphi(\text{row}_1, A)$.

By Lemma 5, consecutive rows in fulfilling $\varphi$-word structures respect the successor function. In particular, by Lemmata 5 and 7, we obtain the following characterization result.

**Corollary 10 (Characterization of fulfilling $\varphi$-word structures).** Let $\mathcal{W} = (w, \mathcal{L})$ be a $\varphi$-word structure such that for all $0 \leq i < |w|$, $\text{Req}_D(L(i, i)) = 0$ and $\text{Req}_E(L(i, i)) = 0$. Then $\mathcal{W}$ is fulfilling if and only if, for each $0 \leq j < |w| - 1$, $\text{row}_{j + 1} = \text{succ}_\varphi(\text{row}_j, \text{row}_{j + 1}[0])$, where $\text{row}_i$ is the $i$-row of $\mathcal{W}$ for all $0 \leq i < |w|$.

### 4.2 Finite abstractions of rows

We describe now the core of the proposed automata-theoretic approach to the satisfiability and model checking problems for $\text{DE}_{\text{Rom}}$. Given a $\text{DE}_{\text{Rom}}$ formula $\varphi$, we introduce an equivalence relation $\sim_\varphi$ of finite index over $\text{Rows}_\varphi$, whose number of equivalence classes is singly exponential in the size of $\varphi$ and such that each equivalence class has a representative whose length is polynomial in the size of $\varphi$. As a crucial result we show that the successor function preserves the equivalence between $\varphi$-rows. The equivalence relation $\sim_\varphi$ is based on the notion of uniform factorization of $\varphi$-rows and rank of $\varphi$-atoms. In the following, we denote by $N_{D, \varphi}$ the number of sub-interval temporal requests in $\text{CL}(\varphi)$ plus one, i.e., $|\{\psi \mid (D) \psi \in \text{CL}(\varphi)\}| + 1$, and by $N_{E, \varphi}$ the number of suffix temporal requests in $\text{CL}(\varphi)$ plus one, i.e., $|\{\psi \in \text{CL}(\varphi)\}| + 1$. Note that $1 \leq N_{D, \varphi} \leq |\varphi|$ and $1 \leq N_{E, \varphi} < |\varphi|$.

**Definition 11 (Uniform $\varphi$-rows).** A $\varphi$-row $\text{row}$ is uniform if for all $0 \leq i < |\text{row}| - 1$, $(\text{row}[i] \cap \mathcal{AP}) = (\text{row}[i + 1] \cap \mathcal{AP})$ and $\text{Req}_D(\text{row}[i + 1]) = \text{Req}_D(\text{row}[i])$.

Thus, in a uniform $\varphi$-row $\text{row}$, all the atoms occurring in $\text{row}$ have the same propositional letters and the same sub-interval temporal requests. We represent an arbitrary $\varphi$-row $\text{row}$ in the form $\text{row} = \text{row}_1 \cdot \ldots \cdot \text{row}_k$ (uniform factorization of $\text{row}$) where $\text{row}_1, \ldots, \text{row}_k$ are uniform $\varphi$-rows and $\text{row}_i \cdot \text{row}_i+1[0]$ is not uniform for all $1 \leq i < k$. By the monotonicity properties of a $\varphi$-row $\text{row}$ (see Definition 6 and Lemma 7(1)), the number $k$ of uniform segments in the factorization of $\text{row}$ is linearly bounded in the size of $\varphi$.

**Lemma 12.** The following statements hold:

1. Let $\text{row}$ be a $\varphi$-row with uniform factorization $\text{row}_1 \cdot \ldots \cdot \text{row}_k$. Then $k$ is at most $3|\varphi|$.
2. Let $\text{row}$ be a uniform $\varphi$-row such that $|\text{row}| > N_{E, \varphi}$. Then $\text{row}$ is of the form $\text{row} = \text{row}' \ast B^m$ where $|\text{row}'| = N_{E, \varphi}$, $m \geq 1$, and $B$ is the last atom of $\text{row}'$.
3. Given a $\varphi$-atom $A$ and an integer $n \geq 1$, there is at most one uniform $\varphi$-row such that $\text{row}[0] = A$ and $|\text{row}| = n$. 

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Proof. By Lemma 7(1), the number $k$ of uniform segments in the uniform factorization of the $\varphi$-row $row$ is at most the number of distinct atoms in $row$. Hence, $k$ is at most $3|\varphi|$, and Property (1) directly follows. As for Property (2), let $row$ be a uniform $\varphi$-row such that $|row| > N_{E,\varphi}$. By definitions of $\varphi$-row and uniform $\varphi$-row, for all $0 \leq i < |row| - 1$, either $\text{Req}_\varphi(row[i]) \subseteq \text{Req}_\varphi(row[i + 1])$ or $row[j] = row[i]$ for all $i < j < |row|$. Thus, since for a $\varphi$-atom $A$, $0 \leq |\text{Req}_\varphi(A)| < N_{E,\varphi}$, we obtain that $row$ is of the form $row = row' \cdot B^m$ where $|row'| = N_{E,\varphi}$, $m \geq 1$, and $B$ is the last atom of $row'$.

As for Property (3), it suffices to observe that in a uniform $\varphi$-row all the atoms have the same propositional letters and the same sub-interval temporal requests. Thus, since the suffix temporal requests of a non-first atom in a $\varphi$-row are completely specified by the previous atom along the row, the result follows.

We now introduce the notion of rank of a $\varphi$-atom. We first define the $D$-rank of a $\varphi$-atom $A$, written $\text{rank}_D(A)$, as $N_{D,\varphi} - |\text{Req}_D(A)|$. Clearly, $1 \leq \text{rank}_D(A) \leq |\varphi|$. The rank of a $\varphi$-atom $A$, written $\text{rank}(A)$, is defined as $\text{rank}_D(A) \cdot N_{E,\varphi}$ (i.e. the product of the $D$-rank of $A$ with the increment of the overall number of suffix temporal requests in $\varphi$).

Clearly, $1 \leq \text{rank}(A) \leq |\varphi|^2$. Intuitively, we can see the rank of an atom as the “number of degrees of freedom” that it gives to the atoms that stay “above it”. In particular, by Definition 6, for every $\varphi$-row $row = A_0 \cdots A_n$, we have $\text{rank}_D(A_0) \geq \cdots \geq \text{rank}_D(A_n)$ and $\text{rank}(A_0) \geq \cdots \geq \text{rank}(A_n)$. Moreover, in a uniform $\varphi$-row, all the atoms occurring in row have the same rank, and we write $\text{rank}(row)$ for such a rank.

**Definition 13 (Equivalence relation).** Given two uniform $\varphi$-rows $row$ and $row'$, we say that $row$ and $row'$ are equivalent, written $row \sim_\varphi row'$, if the following conditions hold:

= $row[0] = row'[0]$ (hence $\text{rank}(row) = \text{rank}(row')$), and

= either $|row| = |row'|$ or both $|row|$ and $|row'|$ are strictly greater than $\text{rank}(row)$.

Two arbitrary $\varphi$-rows $row$ and $row'$ with uniform factorizations $row_1 \cdots row_k$ and $row'_1 \cdots row'_k$, respectively, are equivalent, written $row \sim_\varphi row'$, if $k = k'$ and $row_i \sim_\varphi row'_i$ for all $i \in [1, k]$. A minimal $\varphi$-row is a $\varphi$-row whose uniform factorization $row_1 \cdots row_k$ is such that $|row_i| \in [1, \text{rank}(row_i) + 1]$, for each $1 \leq i \leq k$.

By construction and Lemma 12, the number of minimal $\varphi$-rows is finite and each equivalence class of $\sim_\varphi$ contains a unique minimal $\varphi$-row. Thus, the equivalence relation $\sim_\varphi$ has finite index coinciding with the number of minimal $\varphi$-rows. This number is roughly bounded by the number of all the possible uniform factorizations of the form $row_1 \cdots row_k$ where $k \leq 3|\varphi|$ and for all $i \in [1, k]$, $|row_i|$ ranges from 1 to $|\varphi|^2$ and $row_i$ is the unique uniform $\varphi$-row of length $|row_i|$ having as first atom $row_i[0]$. Since the number of possible $\varphi$-atoms is $2^{|\varphi|}$, the number of distinct equivalence classes of $\sim_\varphi$ is bounded by $(2^{|\varphi|} \cdot |\varphi|^2)^3|\varphi| \leq 2^{9|\varphi|^2}$, which is exponential in the length of the input formula $\varphi$. Moreover, each minimal $\varphi$-row has length at most $3|\varphi|^3$. Hence, we obtain the following result.

**Lemma 14.** Each equivalent class of $\sim_\varphi$ contains a unique minimal $\varphi$-row. The length of a minimal $\varphi$-row is at most $3|\varphi|^3$, and the number of minimal $\varphi$-rows is at most $2^{9|\varphi|^2}$.

We observe that if we replace a segment (sub-row) of a $\varphi$-row by an equivalent one, we obtain a $\varphi$-row which is equivalent to the original one (for a proof, see Appendix C).

**Lemma 15.** Let $row_1, row'_1, row_2, row'_2$ be $\varphi$-rows such that $row_1 \sim_\varphi row'_1$ and $row_2 \sim_\varphi row'_2$. If $row_1 * row_2$ and $row'_1 * row'_2$ are defined, then $row_1 * row_2 \sim_\varphi row'_1 * row'_2$.

We now show that the successor function $\text{succ}_\varphi$ on $\varphi$-rows preserves the equivalence of $\varphi$-rows. We first show (Lemma 16) that the result holds for uniform $\varphi$-rows (the proof is provided in Appendix D), and then we generalize Lemma 16 to arbitrary $\varphi$-rows.
Lemma 16. Let \( A \) be a \( \varphi \)-atom. Then the following statements hold:
1. let \( row \) be a uniform \( \varphi \)-row such that \( |row| > \text{rank}(row) \). Then the \( \varphi \)-row \( \text{succ}_\varphi(row, A) \) is of the form \( A \cdot row_1 \cdot \ldots \cdot row_k \) for some \( k \geq 1 \) such that \( row_1, \ldots, row_k \) are uniform \( \varphi \)-rows and \( |row_k| > \text{rank}(row_k) \).
2. Let \( row \) and \( row' \) be two uniform \( \varphi \)-rows such that \( row \sim_\varphi row' \). Then \( \text{succ}_\varphi(row, A) \sim_\varphi \text{succ}_\varphi(row', A) \).

Lemma 17. Let \( A \) be a \( \varphi \)-atom and \( row \) and \( row' \) be two \( \varphi \)-rows such that \( row \sim_\varphi row' \). Then for the \( \varphi \)-rows \( \text{succ}_\varphi(row, A) \) and \( \text{succ}_\varphi(row', A) \), it holds that \( \text{succ}_\varphi(row, A) \sim_\varphi \text{succ}_\varphi(row', A) \).

Proof. The proof is by induction on the number \( N(row) \) of distinct uniform segments in the uniform factorization of \( row \). Being \( row \) and \( row' \) equivalent, \( N(row') = N(row) \).

Base step: \( N(row) = N(row') = 1 \), i.e. \( row \) and \( row' \) are uniform. In this case, the result directly follows from Lemma 16.

Inductive step: \( N(row) = N(row') > 1 \). Hence, being \( row \sim_\varphi row' \), \( row \) (resp., \( row' \)) can be written in the form \( row = row_1 \cdot row_2 \) (resp., \( row' = row'_1 \cdot row'_2 \)) such that \( row_1 \sim_\varphi row'_1 \), \( row_2 \sim_\varphi row'_2 \). \( N(row_1) = N(row'_1) < N(row) = N(row') \), and \( N(row_2) = N(row'_2) < N(row) = N(row') \). Let \( A_1 \) (resp., \( A'_1 \)) be the last atom in \( \text{succ}_\varphi(row_1, A) \) (resp., \( \text{succ}_\varphi(row'_1, A) \)). By the induction hypothesis, \( \text{succ}_\varphi(row_1, A) \sim_\varphi \text{succ}_\varphi(row'_1, A) \), \( A_1 = A'_1 \), and \( \text{succ}_\varphi(row_2, A_1) \sim_\varphi \text{succ}_\varphi(row'_2, A'_1) \) (note that by Lemma 12, two equivalent \( \varphi \)-rows have the same last atom). By Lemma 9, \( \text{succ}_\varphi(row, A) = \text{succ}_\varphi(row_1, A) \sim_\varphi \text{succ}_\varphi(row', A) = \text{succ}_\varphi(row'_1, A) \sim_\varphi \text{succ}_\varphi(row'_2, A'_1) \). Thus, by applying Lemma 15, we obtain that \( \text{succ}_\varphi(row, A) \sim_\varphi \text{succ}_\varphi(row', A) \), and we are done.

4.3 Optimal upper bounds for \( \text{DE}_{\text{Hom}} \) satisfiability and model-checking

In this subsection, by exploiting Corollary 10 and Lemma 17, we derive an asymptotically optimal automata-theoretic approach for satisfiability and model checking of \( \text{DE}_{\text{Hom}} \) over finite linear orders. Given a \( \text{DE}_{\text{Hom}} \)-formula \( \varphi \), we show that it is possible to construct a deterministic finite automaton (DFA) \( \mathcal{D}_\varphi \) over the alphabet \( 2^{2^p} \) having as states the initialized minimal \( \varphi \)-rows which accepts the non-empty finite words over \( 2^{2^p} \) which satisfy formula \( \varphi \).

Definition 18. Let \( row \) be a minimal \( \varphi \)-row and \( A \) an atom. We denote by \( \text{succ}^{\min}_{\varphi}(row, A) \) the unique minimal \( \varphi \)-row in the equivalence class of \( \sim_\varphi \) containing \( \text{succ}_\varphi(row, A) \). Moreover, for a set \( P \subseteq \mathcal{A}_p \) of proposition letters, we denote by \( A(P) \) the unique \( \varphi \)-atom such that \( A(P) \cap \mathcal{A}_p = \mathcal{P} \), \( \text{Req}_D(A(P)) = \emptyset \), and \( \text{Req}_E(A(P)) = \emptyset \).

We associate with the formula \( \varphi \) the DFA \( \mathcal{D}_\varphi = \langle 2^{2^p}, \text{Rows}^{\min}_{\varphi} \cup \{q_0\}, \{q_0\}, \delta, F \rangle \), where \( \text{Rows}^{\min}_{\varphi} \) is the set of initialized minimal \( \varphi \)-rows, and \( \delta \) and \( F \) are defined as follows:

\( \delta(q_0, P) = A(P) \) for all \( P \in 2^{2^p} \),
\( \delta(row, P) = \text{succ}^{\min}_{\varphi}(row, A(P)) \) for all \( P \in 2^{2^p} \) and \( row \in \text{Rows}^{\min}_{\varphi} \),
\( F \) is the set of \( \varphi \)-rows \( row \in \text{Rows}^{\min}_{\varphi} \) such that \( \varphi \in row[n-1] \), where \( n = |row| \).

We now establish the main technical result of this paper.

Theorem 19. Given a \( \text{DE}_{\text{Hom}} \)-formula \( \varphi \), the DFA \( \mathcal{D}_\varphi \) accepts all and only the non-empty finite words over \( 2^{2^p} \) which satisfy \( \varphi \).

Proof. Let \( w \) be a non-empty finite word over \( 2^{2^p} \) and \( n = |w| - 1 \). We need to show that for the homogeneous interval model \( M(w), M(w), [0, n] \models \varphi \) if and only if \( w \in L(\mathcal{D}_\varphi) \).
(⇒). Assume that \( M(w)\models \phi \). Let \( W = (w, \mathcal{L}) \) be the unique fulfilling \( \varphi \)-word structure associated with the word \( w \) and for all \( i \in [0,n] \), let \( row_i \) be the initialized \( \varphi \)-row corresponding to the \( i \)-row of \( W \). By hypothesis \( \varphi \in row_0[n] \), and by construction \( |row_0| = 1 \) and \( row_0[0] = A(w[i]) \) for all \( i \in [0,n] \). Moreover, by Corollary 10, \( row_{i+1} = succ_\varphi(row_i, row_{i+1}[0]) \) for all \( i \in [0,n-1] \). For each \( i \in [0,n] \), let \( row_i^{min} \) be the unique minimal \( \varphi \)-row in the equivalence class \( [row_i]_{\sim_\varphi} \). Note that \( row_0^{min} = row_0 \), the last atom of \( row_n^{min} \) contains \( \varphi \), \( row_i^{min} \) is initialized and \( row_i^{min}[0] = row_i[0] = A(w[i]) \) for all \( i \in [0,n] \). By applying Lemma 17, \( succ_\varphi(row_i^{min}, row_{i+1}[0]) \) is equivalent to \( row_{i+1} = succ_\varphi(row_i, row_{i+1}[0]) \) for all \( i \in [0,n-1] \). Hence, by the definition of \( succ_\varphi^{min} \), we obtain that \( row_i^{min} = succ_\varphi^{min}(row_i^{min}, A(w[i+1])) \) for all \( i \in [0,n-1] \). By Definition 18, it follows that there is an accepting run of \( D_\varphi \) over \( w \), i.e. \( w \in L(D_\varphi) \).

(⇐). Let us assume that \( w \) is accepted by \( D_\varphi \). By Definition 18, there exist \( n+1 \) minimal initialized minimal \( \varphi \)-rows \( row_0^{min}, \ldots, row_n^{min} \) such that \( row_0^{min} = A(w[0]) \), \( \varphi \) belongs to the last atom of \( row_n^{min} \), \( row_i^{min}[0] = A(w[i]) \) for all \( i \in [0,n] \), and \( row_i^{min} = succ_\varphi^{min}(row_i^{min}, row_{i+1}[0]) \) for all \( i \in [0,n-1] \). Let \( row_0, \ldots, row_n \) be the sequence of \( \varphi \)-rows defined as follows: \( row_0 = row_0^{min} \) and \( row_{i+1} = succ_\varphi(row_i, row_{i+1}[0]) \) for all \( i \in [0,n-1] \). By Lemma 17, we have \( row_i \sim_\varphi row_i^{min} \) for all \( i \in [0,n] \). Hence, \( row_i \) is initialized for all \( i \in [0,n] \), and \( \varphi \in row_n[n] \). Let us define the \( \varphi \)-word structure \( W = (w, \mathcal{L}) \) where \( \mathcal{L}(i,j) = row_j[j-i] \) for every \( 0 \leq i \leq j \leq n \). By Corollary 10, \( W \) is fulfilling. Thus, since \( \varphi \in L(0,n) \), we obtain that \( M(w), [0,n] \models \varphi \) and the result follows.

By Theorem 19, satisfiability of a \( \text{DE}_{Hom} \)-formula \( \varphi \) reduces to checking non-emptiness of the DFA \( D_\varphi \) in Definition 18 whose number of states is singly exponential in the size of \( \varphi \) (Lemma 14). For the model-checking problem, given a finite Kripke structure \( K \), for checking that \( K \) is a model of \( \varphi \), we apply the standard model-checking approach taking the synchronous product \( K \times D_{\neg \varphi} \) of \( K \) with the automaton associated with the negation of the formula \( \varphi \): the NFA \( K \times D_{\neg \varphi} \) accepts all and only the traces of \( K \) which violate property \( \varphi \). Hence, \( K \not\models \varphi \) if and only if the language accepted by \( K \times D_{\neg \varphi} \) of \( K \) is empty. Note that the number of states in \( K \times D_{\neg \varphi} \) is linear in the number of \( K \)-states and singly exponential in the size of \( \varphi \), and the automata \( D_\varphi \) and \( K \times D_{\neg \varphi} \) can be constructed “on the fly”. Thus, since non-emptiness of NFA is in \( \text{NLOGSPACE} \), the complexity classes \( \text{NPSPACE} = \text{PSPACE} \) and \( \text{NLOGSPACE} \) are closed under complement, and satisfiability and model checking against the fragment \( \text{D}_{Hom} \) are known to be \( \text{PSPACE} \)-complete [2], we obtain the following result.

**Theorem 20.** Finite satisfiability and model checking for \( \text{DE}_{Hom} \)-formulas are both \( \text{PSPACE} \)-complete. Moreover, for \( \text{DE}_{Hom} \)-formulas of fixed size, model checking is in \( \text{NLOGSPACE} \).

As for the logic \( \text{BD}_{Hom} \) over finite linear orders, we obtain results similar to Theorem 20. Let \( \text{DE}(\varphi) \) be the \( \text{BD}_{Hom} \) formula obtained from a \( \text{BD}_{Hom} \) formula \( \varphi \) by replacing each occurrence of modality \( (B) \) with \( (E) \). For each non-empty finite word \( w \) over \( 2^\mathcal{AP} \), \( w \models \varphi \) iff \( w^R \models \text{DE}(\varphi) \) (\( w^R \) is the reverse of \( w \)). Hence, the automaton \( \mathcal{A}_\varphi \) accepting the models \( w \) of \( \varphi \) corresponds to the “reverse” of the DFA \( D_{\text{DE}(\varphi)} \) of Definition 18 associated with \( \text{DE}(\varphi) \). Note that \( \mathcal{A}_\varphi \) has the same states as \( D_{\text{DE}(\varphi)} \) but it is not deterministic. This is an important difference between \( \text{BD}_{Hom} \) and \( \text{DE}_{Hom} \) in the proposed automata-theoretic approach.

5 Results for \( \text{BD}_{Hom} \) and concluding remarks

For the logic \( \text{BD}_{Hom} \) over finite linear orders, we obtain results similar to Theorem 20. For a \( \text{BD}_{Hom} \) formula \( \varphi \), let \( \text{DE}(\varphi) \) be the \( \text{DE}_{Hom} \) formula obtained by replacing each occurrence of modality \( (B) \) with \( (E) \). Evidently, for each non-empty finite word \( w \) over \( 2^\mathcal{AP} \), \( w \models \varphi \) iff
$w^R \models DE(\varphi)$ ($w^R$ is the reverse of $w$). Hence, the automaton $N_\varphi$ accepting the models $w$ of $\varphi$ corresponds to the “reverse” of the DFA $D_{DE(\varphi)}$ of Definition 18 associated with $DE(\varphi)$. Note that $N_\varphi$ has the same states as $D_{DE(\varphi)}$ but it is not deterministic. On the other hand, $N_\varphi$ is deterministic in the backward-direction. Thus, for the $DE_{\text{Hom}}$ formulas, the associated automata are deterministic in the forward-direction but non-deterministic in the backward-direction. Dually, for the $BD_{\text{Hom}}$ formulas, the equivalent automata are deterministic in the backward-direction but non-deterministic in the forward-direction.

The results obtained for $DE_{\text{Hom}}$ and $BD_{\text{Hom}}$ are particularly interesting when compared with known results for $BE_{\text{Hom}}$, where the latter includes $DE_{\text{Hom}}$ and $BD_{\text{Hom}}$ as proper fragments and, apparently, is quite close to $DE_{\text{Hom}}$ and $BD_{\text{Hom}}$. The complexity of MC for $BE_{\text{Hom}}$ is still unknown: the problem is at least ExpSpace-hard [5], while the only known upper bound is nonelementary [21]. Whether or not this problem can be solved elementarily is a difficult open question. The exact complexity of finite satisfiability for $BE_{\text{Hom}}$ is also an open issue: the same upper/lower bounds can be shown to hold by linear-time reductions to/from the MC problem.

References

In this section we provide a proof of Theorem 1. We establish the following result. Hence, Theorem 1 directly follows.

// Proposition 21. Over finite linear orders, the following holds:
1. there exists a $B_{Hom}$ formula which cannot be expressed in $DE_{Hom}$;
2. there exists an $E_{Hom}$ formula which cannot be expressed in $BD_{Hom}$.
We prove Proposition 21(2). The proof of Proposition 21(1) is similar and we omit the details here. Let $\mathcal{AF} = \{p\}$. We consider the $E_{\text{Hom}}$ formula $\varphi_E$ over $\mathcal{AF}$ defined as follows:

$$\varphi_E := \langle E \rangle (p \wedge \langle E \rangle \top)$$

which asserts the existence of a proper suffix of length at least 2 where $p$ holds. In order to prove that no formula in $\mathcal{BD}_{\text{Hom}}$ is equivalent to $\varphi_E$ over finite linear orders, we define two families $(w_n)_{n \geq 1}$ and $(w'_n)_{n \geq 1}$ of non-empty finite words over $2^\mathcal{P}$ such that:

- $\varphi_E$ distinguishes between $w_n$ and $w'_n$ for each $n \geq 1$, and
- for every $\mathcal{BD}_{\text{Hom}}$ formula $\psi$, there is $n \geq 1$ such that $\psi$ does not distinguish between $w_n$ and $w'_n$.

For each $n \geq 1$, the words $w_n$ and $w'_n$ over $2^\mathcal{P}$ are defined as follows:

$$w_n = (\emptyset \{p\}^n)^{n+2}\{p\} \text{ and } w'_n = (\emptyset \{p\}^n)^{n+2}\emptyset\{p\}$$

Note that for each $n \geq 1$, the property of having a suffix of length at least 2 where $p$ holds is satisfied by $w_n$ but not by $w'_n$. Hence, the following holds:

$\blacktriangleright$ **Lemma 22.** For each $n \geq 1$, $w_n \models \varphi_E$ and $w'_n \not\models \varphi_E$.

For a $\mathcal{BD}_{\text{Hom}}$ formula $\psi$, we denote by $d(\psi)$ the joint nesting depth of the temporal modalities in $\psi$. Proposition 21(2) directly follows from Lemma 22 and the following lemma.

$\blacktriangleright$ **Lemma 23.** Let $n \geq 1$. Then, for each $\mathcal{BD}_{\text{Hom}}$ formula $\psi$ such that $d(\psi) < n$, it holds that $w_n \models \psi$ if and only if $w'_n \models \psi$.

**Proof.** Fix $n \geq 1$. In order to prove Lemma 23, we need some preliminary results. The following claim can be proved by a straightforward induction on $k \geq 1$.

$\blacktriangleright$ **Claim 1.** Let $k \geq 1$. Then for each $\mathcal{BD}_{\text{Hom}}$ formula $\xi$ such that $d(\xi) < k$, it holds that (i) $\{p\}^k \models \xi$ if $\{p\}^{k+1} \models \xi$ and (ii) $\{p\}^j \emptyset\{p\}^k \models \xi$ if $\{p\}^j \emptyset\{p\}^{k+1} \models \xi$ for each $j \geq 0$.

$\blacktriangleright$ **Claim 2.** Let $i, j \geq 0$, $k \geq 1$, and $\nu(j, i, k) = \{p\}^j (\emptyset\{p\}^i)^\emptyset\{p\}^k$. Then for each $\mathcal{BD}_{\text{Hom}}$ formula $\xi$ such that $d(\xi) < k$, $\nu(j, i, k) \models \xi$ if $\nu(j, i, k) \cdot \{p\} \models \xi$.

**Proof of Claim 2.** The proof is by induction on $k \geq 1$. For the base case ($k = 1$), being $d(\xi) < 1$ (hence, $\xi$ does not contain temporal modalities), the result trivially follows. Now, assume that $k > 1$. We proceed by a double induction on $i \geq 0$. If $i = 0$, the result directly follows from Claim 1. Now, let us assume that $i > 0$. By construction, for each proper prefix (resp., proper infix) $v$ of $\nu(j, i, k) \cdot \{p\}$, there is a proper prefix (resp., proper infix) $v'$ of $\nu(j, i, k)$ such that either (i) $v' = v$, or (ii) $v = \{p\}^{k+1}$ and $v' = \{p\}^k$, or (iii) $v = \nu(j', i', k') \cdot \{p\}$ and $v' = \nu(j', i', k')$ for some $j', i' \geq 0$ and $k' \geq 1$ such that either $k' = k$ and $i' < i$, or $k' = k - 1$ and $i' \leq i$. Hence, the result easily follows from the induction hypothesis and Claim 1.

$\blacktriangleright$ **Claim 3.** Let $h \geq 1$, $i, j \geq 0$, $\nu(j, h) = \{p\}^j (\emptyset\{p\}^i)^h$ and $\nu(j, h, i) = \{p\}^j (\emptyset\{p\}^i)^h \emptyset\{p\}^i$. Then, for each $\mathcal{BD}_{\text{Hom}}$ formula $\xi$ such that $d(\xi) < h$, it holds that (i) $\nu(j, h) \models \xi$ if $\nu(j, h + 1) \models \xi$, and (ii) $\nu(j, h, i) \models \xi$ if $\nu(j, h + 1, i) \models \xi$.

**Proof of Claim 3.** The proof is by induction on $h \geq 1$. The base case ($h = 1$) trivially follows, since being $d(\xi) < 1$, $\xi$ does not contains temporal modalities. Now, assume that $h > 1$. Let $w = \nu(j, h)$ (resp., $w = \nu(j, h, i)$) and $w' = \nu(j, h + 1)$ (resp., $w' = \nu(j, h + 1, i)$). We need to prove that $w \models \xi$ if $w' \models \xi$. By construction, the following holds:
for each proper infix of \( v \) (resp., \( v' \)) of \( w \) (resp., \( w' \)), there exists a proper infix of \( v' \) (resp., \( v \)) of \( w' \) (resp., \( w \)) such that \( v' \) (resp., \( v \)) is a prefix of \( w' \) (resp., \( w \)) if \( v' \) (resp., \( v \)) is a prefix of \( w \) (resp., \( w' \)) and either (i) \( v' = v \), or (ii) \( v' = v(j', h) \) and \( v = v(j', h - 1) \) for some \( j' \geq 0 \), or (iii) \( v' = v(j', h, i') \) and \( v = v(j', h - 1, i') \) for some \( i', j' \geq 0 \).

Hence, the result easily follows from the induction hypothesis. \( \triangleright \)

By Claim 3, we easily deduce the following result.

\( \triangleright \) Claim 4. Let \( j \geq 0 \). Then for each \( \text{BD}_{\text{Hom}} \) formula \( \xi \) such that \( d(\xi) < n \), \( \{p\}^j(\{\{p\}^n\}^n + 1) \models \xi \) iff \( \{p\}^j(\{\{p\}^n\}^n + 1) \models \xi \).

We now prove Lemma 23. The proof is by induction on \( n \). The base case \( (n = 1) \) trivially follows, since being \( d(\psi) < n \), \( \psi \) does not contains temporal modalities. Now, assume that \( n > 1 \). We need to show that for each \( \text{BE}_{\text{Hom}} \) formula \( \xi \) such that \( d(\xi) < n - 1 \), the following holds:

- for each proper infix (resp., proper prefix) \( v \) of \( w_n \), there exists a proper infix (resp., proper prefix) \( v' \) of \( w'_n \) such that \( v \models \xi \) iff \( v' \models \xi' \);
- for each proper infix (resp., proper prefix) \( v' \) of \( w'_n \), there exists a proper infix (resp., proper prefix) \( v \) of \( w_n \) such that \( v \models \xi \) iff \( v' \models \xi' \).

The result easily follows from the definition of \( w_n \) and \( w'_n \) and Claims 1–4. \( \triangleright \)

**B Proof of Lemma 5**

\( \triangleright \) Lemma 5. Let \( W = (w, \mathcal{L}) \) be a \( \varphi \)-word structure. Then \( W \) is fulfilling if and only if for each interval \( [i, j] \) of \( W \), it holds that (i) \( L(i, j) = \text{succ}_{\varphi}(L(i, j - 1), L(i + 1, j)) \), if \( i < j \), and (ii) \( \text{Req}_{\text{D}}(L(i, j)) = \emptyset \) and \( \text{Req}_{\text{E}}(L(i, j)) = \emptyset \), if \( i = j \).

**Proof.**

(\( \Rightarrow \)). Assume that \( W \) is fulfilling. Hence, for each interval \( [i, j] \) of \( W \), \( L(i, j) \) is the set of formulas \( \psi \in \text{CL}(\varphi) \) such that \( M(w)[i, j] \models \psi \) (recall that \( M(w) \) is the homogeneous interval model associated with the word \( w \)). Thus, if \( i = j \), then \( \text{Req}_{\text{D}}(L(i, j)) = \emptyset \) and \( \text{Req}_{\text{E}}(L(i, j)) = \emptyset \). Otherwise, \( i < j \) and being \( M(w) \) homogeneous, we have that \( L(i, j) \cap \mathcal{A} = L(i, j - 1) \cap (L(i + 1, j) \cap \mathcal{A}) \). Moreover, by the semantics of \( \text{DE} \), the following holds:

- for each \( (D) \psi \in \text{CL}(\varphi) \), \( (D) \psi \in L(i, j) \) if and only if either \( (D) \psi \in L(i, j - 1) \), or \( \psi \in L(i, j - 1) \), or \( (D) \psi \in L(i + 1, j) \), or \( \psi \in L(i + 1, j) \);
- for each \( (E) \psi \in \text{CL}(\varphi) \), \( (E) \psi \in L(i, j) \) if and only if either \( (E) \psi \in L(i + 1, j) \) or \( \psi \in L(i + 1, j) \).

This means that \( L(i, j) = \text{succ}_{\varphi}(L(i, j - 1), L(i + 1, j)) \), and the result follows.

(\( \Leftarrow \)). Assume that for every interval \( [i, j] \) of \( W \), we have \( L(i, j) = \text{succ}_{\varphi}(L(i, j - 1), L(i + 1, j)) \) if \( i < j \), and \( \text{Req}_{\text{D}}(L(i, j)) = \emptyset \) and \( \text{Req}_{\text{E}}(L(i, j)) = \emptyset \) if \( i = j \). We have to prove that \( W \) is fulfilling. Let \( [i, j] \) be an interval of \( W \) and \( \psi \in \text{CL}(\varphi) \). We prove by induction on the structure of \( \psi \) that \( \psi \in L(i, j) \) if and only if \( M(w)[i, j] \models \psi \). Hence, the result follows.

- \( \psi = p \) with \( p \in \mathcal{A} \): we have to show that \( L(i, j) \cap \mathcal{A} = \bigcap_{h \in [i, j]} L(h, h) \cap \mathcal{A} \). The proof is by a double induction on \( j - i \geq 0 \). If \( i = j \), the property trivially holds. Let us assume now that \( j - i > 0 \). Since \( \text{L}(i, j) = \text{succ}_{\varphi}(\text{L}(i, j - 1), \text{L}(i + 1, j)) \), by Condition (i) of Definition 4 and the induction hypothesis, we obtain that \( L(i, j) \cap \mathcal{A} = \bigcap_{h \in [i, j - 1]} L(h, h) \cap \bigcap_{h \in [i + 1, j]} L(h, h) \cap \mathcal{A} \). Hence, the result directly follows.
\[ \psi = \neg \psi_1 \text{ or } \psi = \psi_1 \lor \psi_2; \] for these cases, the result directly follows from the induction hypothesis and the definition of \( \varphi \)-atom (recall that \( \mathcal{L}(i, j) \) is a \( \varphi \)-atom).

\[ \psi = (D) \psi_1 \text{ or } \psi = (E) \psi_1; \] the proof is by a double induction on \( j - i \geq 0 \). If \( i = j \), then \( \mathcal{M}(w), [i, j] \not\models \psi \), \( \text{Req}_D(\mathcal{L}(i, j)) = \emptyset \), and \( \text{Req}_E(\mathcal{L}(i, j)) = \emptyset \). Hence, the result follows. Now, assume that \( j - i > 0 \). First, let us consider the case where \( \psi = (D) \psi_1 \). Since \( \mathcal{L}(i, j) = \text{succ}_\varphi(\mathcal{L}(i, j - 1), \mathcal{L}(i + 1, j)) \), by Condition (ii) of Definition 4 and the induction hypothesis, we have that \( (D) \psi_1 \in \mathcal{L}(i, j) \) if and only if either \( \mathcal{M}(w), [i, j - 1] \models (D) \psi_1 \), or \( \mathcal{M}(w), [i + 1, j] \models (D) \psi_1 \), or \( \mathcal{M}(w), [i, j] \models \psi_1 \) if and only if \( \mathcal{M}(w), [i, j] \models (E) \psi_1 \).

Now, let us consider the case where \( \psi = (E) \psi_1 \). By Condition (iii) of Definition 4 and the induction hypothesis, we have that \( (E) \psi_1 \in \mathcal{L}(i, j) \) if and only if either \( \mathcal{M}(w), [i, j] \models (E) \psi_1 \) or \( \mathcal{M}(w), [i + 1, j] \models (E) \psi_1 \), and the result follows.

\section*{C Proof of Lemma 15}

\begin{lemma}
Let \( \text{row}_1, \text{row}_1', \text{row}_2, \text{row}_2' \) be \( \varphi \)-rows such that \( \text{row}_1 \sim_\varphi \text{row}_1' \) and \( \text{row}_2 \sim_\varphi \text{row}_2' \). If \( \text{row}_1 \ast \text{row}_2 \) and \( \text{row}_1' \ast \text{row}_2' \) are defined, then \( \text{row}_1 \ast \text{row}_2 \sim_\varphi \text{row}_1' \ast \text{row}_2' \).
\end{lemma}

\begin{proof}
We consider the case where \( \text{row}_1 \) and \( \text{row}_2 \) are uniform, hence, \( \text{row}_1' \) and \( \text{row}_2' \) are uniform as well. The general case easily follows from the considered case. By hypothesis \( \text{row}_1 \ast \text{row}_2 \) and \( \text{row}_1' \ast \text{row}_2' \) are defined. This entails that \( \text{row}_1 \ast \text{row}_2 \) and \( \text{row}_1' \ast \text{row}_2' \) are uniform as well. Thus since \( \text{row}_1 \sim_\varphi \text{row}_1' \) and \( \text{row}_2 \sim_\varphi \text{row}_2' \), by Definition 13, we obtain that \( \text{row}_1 \ast \text{row}_2 \) and \( \text{row}_1' \ast \text{row}_2' \) have the same first atom \( A \) and indicated by \( m \) (resp., \( m' \)) the length of \( \text{row}_1 \ast \text{row}_2 \) (resp., \( \text{row}_1' \ast \text{row}_2' \)), it holds that either \( m = m' \), or both \( m > \text{rank}(A) \) and \( m' > \text{rank}(A) \). Hence, the result follows.
\end{proof}

\section*{D Proof of Lemma 16}

In order to prove Lemma 16, we need a preliminary technical result (Lemma 5) that considers uniform \( \varphi \)-rows of the form \( B^m \) for some \( \varphi \)-atom \( B \).

\begin{lemma}
Let \( A \) and \( B \) be two \( \varphi \)-atoms such that \( \text{rank}_D(\text{succ}_\varphi(B, A)) = \text{rank}_D(B) - h \) for some \( h \geq 0 \) (note that \( h < \text{rank}_D(B) \)). Given \( m > (\text{rank}_D(B) - h) \cdot \text{N}_{E, \varphi} \), if \( B^m \) is a \( \varphi \)-row than the \( \varphi \)-row \( \text{succ}_\varphi(B^m, A) \) is of the form \( A \cdot \text{row}_1 \cdot \ldots \cdot \text{row}_k \) for some \( k \geq 1 \) such that \( \text{row}_1, \ldots, \text{row}_k \) are uniform \( \varphi \)-rows and
\[ \text{rank}_D(\text{row}_i) > \text{rank}_D(\text{row}_{i+1}) \text{ for each } 1 \leq i < k, \]
\[ \text{rank}_D(\text{row}_k) > \text{rank}(\text{row}_k). \]
\end{lemma}

\begin{proof}
Let \( \text{rank}_D(\text{succ}_\varphi(B, A)) = \text{rank}_D(B) - h \) for some \( 0 \leq h < \text{rank}_D(B) \), \( m > (\text{rank}_D(B) - h) \cdot \text{N}_{E, \varphi}, \) and \( B \) be the \( \varphi \)-row of length \( m + 1 \) given by \( \text{succ}_\varphi(B^m, A) \). Since \( \text{row}[0] = A \) and \( \text{row}[i + 1] = \text{succ}_\varphi(B, \text{row}[i]) \) for all \( i \in [0, m - 1] \), by Definition 4, for all \( i \in [1, m] \), the following holds:
\[ \text{row}[i] = B \cap A \cap \varphi; \]
\[ \text{rank}_D(\text{row}[i]) \geq \text{rank}_D(\text{row}[i]) \text{ and } \text{Req}_E(\text{row}[i]) \subseteq \text{Req}_E(\text{row}[i]); \]
\[ \text{if } i < m \text{ and } \text{row}[i] > \text{row}[i + 1], \text{ then } \text{row}[j] = \text{row}[i] \text{ for all } j \geq i. \]

Since by hypothesis \( \text{rank}_D(\text{row}[1]) = \text{rank}_D(B) - h \), we easily deduce that \( \text{row} = \text{succ}_\varphi(B^m, A) \) is of the form
\[ A \cdot \text{row}_1 \cdot \ldots \cdot \text{row}_k \]
for some \( k \geq 1 \) such that \( \text{row}_1, \ldots, \text{row}_k \) are uniform \( \varphi \)-rows and
Lemma 16. Let $A$ be a $\varphi$-atom. Then the following statements hold:

1. Let $row$ be a uniform $\varphi$-row such that $|row| > rank(row)$. Then the $\varphi$-row $\text{succ}_\varphi(row, A)$ is of the form $A \cdot row_1 \ldots \cdot row_k$ for some $k \geq 1$ such that $row_1, \ldots, row_k$ are uniform $\varphi$-rows and $|row_k| > rank(row_k)$.

2. Let $row$ and $row'$ be two uniform $\varphi$-rows such that $row \sim_\varphi row'$. Then $\text{succ}_\varphi(row, A) \sim_\varphi \text{succ}_\varphi(row', A)$.

Proof. 

Proof of Property (1). Let $A$ be a $\varphi$-atom and $row$ be a uniform $\varphi$-row such that $|row| > rank(row)$. We need to show that the length $|\eta|$ of the last uniform segment $\eta$ in the uniform factorization of $\text{succ}_\varphi(row, A)$ satisfies $|\eta| > rank(\eta)$. Since $|row| > rank(row)$ and rank($row$) $\geq N_{E, \varphi}$, by Lemma 12(2), row is of the form $row = row_1 \cdot B^m$ where $m \geq 1$, $|row_1| = N_{E, \varphi}$ and $B$ is the last atom of $row_1$. Let $row'$ be the $\varphi$-row given by $\text{succ}_\varphi(row, A)$. Then $row'$ can be written in the form

$$row' = (A \cdot row'_1) \cdot \text{succ}_\varphi(B^m, B')$$

where $A \cdot row'_1 = \text{succ}_\varphi(row_1, A)$ and $B'$ is the last atom of $row'_1$. In particular, $|row'_1| = N_{E, \varphi}$. Let $B'' = \text{succ}_\varphi(B, B')$. By Definition 4, we have that rank$_D(B'') \leq$ rank$_D(row'_1[0]) \leq$ rank$_D(row)$. We distinguish two cases:

- rank$_D(B'') = rank_D(row)$. In this case, we have that all the atoms in $row'_1 \cdot B''$ have the same sub-interval temporal requests. Moreover, since row is uniform, by Definition 4, all the atoms in $row'_1 \cdot B''$ have the same propositional letters. Hence, $row'_1 \cdot B''$ is a uniform $\varphi$-row. Since $|row'| = N_{E, \varphi}$, by Lemma 12(2), $B''$ coincides with the last atom $B'$ of $row'_1$. Thus, $B' = \text{succ}_\varphi(B, B')$ and $row'_1 = A \cdot row'_1 \cdot (B')^m$ where $row'_1 \cdot (B')^m$ is a uniform $\varphi$-row having the same length and the same rank as row. Thus, since $|row| > rank(row)$, the result in this case holds.

- rank$_D(B'') < rank_D(row) = rank_D(B)$. We have that $m = |row| - N_{E, \varphi} > rank(row) - N_{E, \varphi} = (rank_D(B) - 1) \cdot N_{E, \varphi} \geq rank(B'')$. Since $B'' = \text{succ}_\varphi(B, B')$, by Lemma 5, the length $|\eta|$ of the last uniform segment $\eta$ in the uniform factorization of $\text{succ}_\varphi(B'', B')$ satisfies $|\eta| > rank(\eta)$, and the result follows.

Proof of Property (2). Let $A$ be a $\varphi$-atom and $row$ and $row'$ be two uniform $\varphi$-rows such that $row \sim_\varphi row'$. We need to show that $\text{succ}_\varphi(row, A) \sim_\varphi \text{succ}_\varphi(row', A)$. By hypothesis and Definition 13, there are two cases:

- $row[0] = row'[0]$ and $|row| = |row'|$. Since row and row' are uniform, by Lemma 12(3), $row = row'$, and the result obviously follows.
row[0] = row'[0], |row| ≠ |row'|, |row| > rank(row) and |row'| > rank(row'). Assume that |row| < |row'| (the case where |row'| < |row| being similar). Since row and row' are uniform and row[0] = row'[0], it holds that rank(row) = rank(row'). Moreover, |row| > rank(row) ≥ N_{E, φ}. Applying Lemma 12(2) and Lemma 12(3), we deduce that row is of the form row = row_1 · B^2 and row' = row_1 · B^{k+2} where B is a φ-atom and k = |row'| − |row|. By Property (1) of Lemma 16 the last uniform segment η of succ_{φ}(row, A) satisfies |η| > rank(η) ≥ N_{E, φ}. Thus, by Lemma 12(2), succ_{φ}(row, A) is of the form row' · (B')^2 for a φ-atom B' such that B' = succ_{φ}(B, B'). Since succ_{φ}(row', A) = (row' · (B')^2) · succ_{φ}(B^k, B'), we obtain that succ_{φ}(row', A) = row' · (B')^{2+k}. Thus, since the last uniform segment η in row' · (B')^2 satisfies |η| > rank(η), we deduce that succ_{φ}(row, A) and succ_{φ}(row', A) are equivalent. □