A Tight Local Algorithm for the Minimum Dominating Set Problem in Outerplanar Graphs

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Abstract

We show that there is a deterministic local algorithm (constant-time distributed graph algorithm) that finds a $5$-approximation of a minimum dominating set on outerplanar graphs. We show there is no such algorithm that finds a $(5 - \varepsilon)$-approximation, for any $\varepsilon > 0$. Our algorithm only requires knowledge of the degree of a vertex and of its neighbors, so that large messages and unique identifiers are not needed.

2012 ACM Subject Classification Theory of computation → Distributed algorithms; Mathematics of computing → Approximation algorithms

Keywords and phrases Outerplanar graphs, dominating set, LOCAL model, constant-factor approximation algorithm

Digital Object Identifier 10.4230/LIPIcs.DISC.2021.13


Funding Marthe Bonamy: Supported by the ANR project GrR (ANR-18-CE40-0032). Linda Cook: This work was supported by AFOSR grant A9550-19-1-0187, NSF grant DMS-1800053 and the Institute for Basic Science (IBS-R029-C1). Carla Groenland: Supported by the European Research Council Horizon 2020 project CRACKNP (grant agreement no. 853234). Alexandra Wesolek: Supported by the Vanier Canada Graduate Scholarships program.

Acknowledgements We thank the referees for helpful comments which improved the presentation of the paper.

1 Introduction

Given a sparse graph class, how well can we approximate the size of the minimum dominating set (MDS) in the graph using a constant number of rounds in the LOCAL model? A dominating set of a graph $G = (V, E)$ is a set $S \subseteq V$ such that every vertex in $V \setminus S$ has a neighbor in $S$. Given a graph $G$ and an integer $k$, deciding whether $G$ has a dominating set of size at most $k$ is NP-complete even when restricting to planar graphs of maximum degree three [9]. Moreover, the size of the MDS is NP-hard to approximate within a constant factor (for general graphs) [16]. The practical applications of MDS are diverse but almost always involve large networks [3], and it is therefore natural to turn to the the distributed setting. No constant factor approximation of the MDS is possible using a sub-linear number of rounds in the LOCAL model [13], and so various structural restrictions have been considered on the graph classes with the hope of finding more positive results (see [8] for an overview).
Planar graphs are a hallmark case. For planar graphs, guaranteeing that some constant factor approximation can be achieved is already highly non-trivial [7, 14]. The current best known upper-bound is 52 [19], while the best lower-bound is 7 [11]. Substantial work has focused on generalizing the fact that some constant factor approximation is possible to more general classes of sparse graphs, like graphs that can be embedded on a given surface, or more recently graphs of bounded expansion [1, 2, 5, 12]. Tight bounds currently seem out of reach in those more general contexts.

In this paper we focus instead on restricted subclasses of planar graphs. Better approximation ratios can be obtained with additional structural assumptions: 32 if the planar graph contains no triangle [3] and 18 if the planar graph contains no cycle of length four [4]. These bounds are not tight, and in fact we expect they can be improved significantly. We are able to provide tight bounds for a different type of restriction: we consider planar graphs with no $K_{2,3}$-minor or $K_4$-minor\(^1\), i.e. outerplanar graphs. Outerplanar graphs can alternatively be defined as planar graphs that can be embedded so that there is a special face which contains all vertices in its boundary.

Outerplanar graphs are a natural intermediary graph class between planar graphs and forests. A planar graph on \(n\) vertices contains at most \(3n - 6\) edges, and a forest on \(n\) vertices contains at most \(n - 1\) edges; an outerplanar graph on \(n\) vertices contains at most \(2n - 3\) edges. Every planar graph can be decomposed into three forests [15]; it can also be decomposed into two outerplanar graphs [10].

For planar graphs, as discussed above, we are far from a good understanding of how to optimally approximate Minimum Dominating Set in \(O(1)\) rounds. Let us discuss the case of forests, as it is of very relevant to the outerplanar graph case. For forests, a trivial algorithm yields a 3-approximation: it suffices to take all vertices of degree at least 2 in the solution, as well as vertices with no neighbor of degree at least 2 (that is, isolated vertices and isolated edges). The output is clearly a dominating set, and the proof that it is at most three times as big as the optimal solution is rather straightforward. In fact, the trivial algorithm is tight because of the case of long paths. Indeed, no constant-time algorithm can avoid taking all but a sub-linear number of vertices of a long path, while there is a dominating set containing only a third of the vertices.

### Our contribution

We prove that a similarly trivial algorithm (as the one described for forests above) works to obtain a 5-approximation of MDS for outerplanar graphs in the LOCAL model.

**Algorithm 1** A local algorithm to compute a dominating set in outerplanar graphs.

<table>
<thead>
<tr>
<th>Input:</th>
<th>An outerplanar graph (G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result:</td>
<td>A set (S \subseteq V(G)) that dominates (G)</td>
</tr>
<tr>
<td></td>
<td>In the first round, every vertex computes its degree and sends it to its neighbors;</td>
</tr>
<tr>
<td></td>
<td>(S := {\text{Vertices of degree} \geq 4} \cup {\text{Vertices with no neighbor of degree} \geq 4};)</td>
</tr>
</tbody>
</table>

It is easy to check that the algorithm indeed outputs a dominating set. It is significantly harder to argue that the resulting dominating set is at most 5 times as big as one of minimum size. To do that, we delve into a rather intricate analysis of the behavior of a hypothetical counterexample, borrowing tricks from structural graph theory (see Lemma 2).

The proof that the bound of 5 is tight for outerplanar graphs is similar to the proof that the bound of 3 is tight for trees. Every graph in the family depicted in Figure 1 is outerplanar, and every local algorithm that runs in a constant number of rounds selects all

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\(^1\) For any integer \(n \geq 1\), \(K_n\) denotes the complete graph on \(n\) vertices. For integers \(n, m \geq 1\), \(K_{n,m}\) denotes the complete bipartite graph with partite classes of size \(n\) and \(m\).
Definitions and notation

For a vertex set \( A \subseteq V \), let \( G[A] \) denote the induced subgraph of \( G \) with vertex set \( A \). Let \( E(A) \) denote set of edges of \( G[A] \). For vertex sets \( A, B \subseteq G \), let \( E(A,B) \) denote the set of edges in \( G \) with one end in \( A \) and the other end in \( B \). We write \( G \setminus e \) for the graph in which the edge \( e \) is removed from the edge set of \( G \). For a set \( P \subseteq V \) inducing a connected subgraph, we write \( G/P \) for the graph obtained by contracting the set \( P \): we replace the vertices in \( P \) with a new vertex \( v_P \), which is adjacent to \( u \in V \setminus P \) if and only if \( u \) has some neighbor in \( P \). For a set \( X \) of vertices, we let \( N[X] \) denote the set \( X \cup \bigcup_{x \in X} \bigcup_{x \in X} N(x) \) and we let \( N(X) \) denote the set \( N[X] \setminus X \). If \( x_1, x_2, \ldots, x_k \) are the elements of \( X \), we may also denote \( N[X] \) and \( N(X) \) as \( N[x_1, x_2, \ldots, x_k] \) and \( N(x_1, x_2, \ldots, x_k) \), respectively.

Given a graph \( G \), let \( V_4^+(G) \) denote the set of vertices of degree at least 4 in \( G \), and let \( V^*(G) \) denote the set \( V(G) \setminus N[V_4^+(G)] \). In other words, \( V^*(G) \) is the set of vertices of degree at most 3 in \( G \) which only have neighbors of degree at most 3. For a graph \( G \) and a dominating set \( S \) of \( G \), we denote \( V_4^+(G) \setminus S \) by \( B_S(G) \) and we denote \( V^*(G) \setminus S \) by \( D_S(G) \). We additionally let \( A_S(G) \) denote the set \( V(G) \setminus (S \cup D_S(G) \cup B_S(G)) \). In situations where
For any \( p, q \in \mathbb{N} \), there is a planar graph \( G_{p,q} \) which admits a dominating set of size 2 such that \(|\{\text{Vertices with degree} \geq q\}| \geq p\).

Our choice of \( G, S \) is not ambiguous we will simply write \( B, D, A \) for \( B_S(G), D_S(G) \) and \( A_S(G) \), respectively. An overview of the notation is given in Table 1.

Table 1 An overview of the notation used in Section 2.

<table>
<thead>
<tr>
<th>( v ) is an element of ( V_{4+}(G) )</th>
<th>( \deg(v) )</th>
<th>degrees of neighbors of ( v )</th>
<th>further restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_S(G) )</td>
<td>( \geq 4 )</td>
<td>arbitrary</td>
<td>( v \notin S )</td>
</tr>
<tr>
<td>( V^*(G) )</td>
<td>( \leq 3 )</td>
<td>( \leq 3 )</td>
<td>( v \notin S )</td>
</tr>
<tr>
<td>( D_S(G) )</td>
<td>( \leq 3 )</td>
<td>( \leq 3 )</td>
<td>( v \notin S )</td>
</tr>
<tr>
<td>( A_S(G) )</td>
<td>( \leq 3 )</td>
<td>at least one neighbor of degree ( \geq 4 )</td>
<td>( v \notin S )</td>
</tr>
</tbody>
</table>

An outerplanar embedding of \( G \) is an embedding in which a special outer face contains all vertices in its boundary.

We denote by \( H_G(S) \) the multigraph with vertex set \( S \), obtained from \( G \) as follows. For every vertex \( u \) in \( V(G) \setminus S \), we select a neighbor \( s(u) \in N(u) \cap S \), and contract the edge \( \{u, s(u)\} \). Contrary to the contraction operation mentioned earlier, this may create parallel edges, but we delete all self-loops. The resulting multigraph inherits the set \( S \) as its vertex set. We refer to Figure 3 for an example.

Note that \( H_G(S) \) inherits an outerplanar embedding from \( G \). If the graph \( G \) and the dominating set \( S \) are clear, we will write \( H \) for \( H_G(S) \). Lemma 3 provides some intuition as to why the graph \( H \) is useful.

Properties of outerplanar graphs

Here we mention some standard but useful properties of outerplanar graphs. A graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from \( G \) through a series of vertex or edge deletions and edge contractions. Alternatively, an \( H \)-minor of \( G \) consists of a connected set \( X_h \subseteq V(G) \) for each \( h \in V(H) \) and a set of paths \( \{P_{hh'} \mid hh' \in E(H)\} \), where \( P_{hh'} \) is a path in \( G \) between a vertex in \( X_h \) and a vertex in \( X_{h'} \), all of which are pairwise vertex-disjoint except for possibly their ends. Note that any minor of an outerplanar graph is outerplanar. Neither \( K_4 \) nor \( K_{2,3} \) can be drawn in the plane so that all vertices appear on the boundary of a special face. Therefore, outerplanar graphs are \( K_4 \)-minor-free and \( K_{2,3} \)-minor-free.
Any outerplanar graph $G$ satisfies $|E(G)| \leq 2|V(G)| - 3$ by a simple application of Euler’s formula. It follows immediately that every outerplanar graph contains a vertex of degree at most 3, but a standard structural analysis guarantees that every outerplanar graph contains a vertex of degree at most 2.

2  Analysis of the approximation factor

This section is devoted to proving the following result. (An overview of the relevant notation is given in Table 1.)

▶ Lemma 2. For every outerplanar graph $G$, any dominating set $S$ of $G$ satisfies $|S| \geq \frac{1}{4}(|B_S(G)| + |D_S(G)|)$.

We briefly argue that Lemma 2 yields the desired result. Given an outerplanar graph, Algorithm 1 outputs $V^+(G) \cup V^*(G)$ as a dominating set. To argue that it is a 5-approximation of the Minimum Dominating Set problem, it suffices to prove that any dominating set $S$ of $G$ satisfies $|S| \geq \frac{1}{4}(|B_S(G)| + |D_S(G)|)$. For technical reasons, it is easier to bound $S$ as a function of the vertices in $V^+(G) \cup V^*(G)$ that are not in $S$, i.e. $|S| \geq \frac{1}{4}(|B_S(G)| + |D_S(G)|)$, which yields $|S| \geq \frac{1}{4}(|V^+(G) \cup V^*(G)|)$.

We prove the lemma by analyzing the structure of a “smallest” counterexample. A counterexample satisfies $|S| < \frac{1}{4}(|B_S(G)| + |D_S(G)|)$, and we will choose one which minimizes $|S|$ and with respect to that maximizes $|B_S(G)| + |D_S(G)|$. For this, we need that $|B_S(G)| + |D_S(G)|$ is bounded in terms of $|S|$ by some constant, otherwise a counterexample maximizing $|B_S(G)| + |D_S(G)|$ might not exist since $|B_S(G)| + |D_S(G)|$ could be arbitrarily large. We therefore first prove the following much weaker result.

▶ Lemma 3. For every outerplanar graph $G$, any dominating set $S$ of $G$ satisfies $|S| \geq \frac{1}{39}(|B_S(G)| + |D_S(G)|)$.

We did not try to optimize the constant 39 and rather aim to get across some of the main ideas as clearly as possible. The proof shows the importance of the graph $H_G(S)$, which we will also use in the proof of Lemma 2.
Proof of Lemma 3. We may assume that the graph $G$ is connected; otherwise, we can repeat the same argument for each connected component of $G$. We fix an outerplanar embedding of $G$. For each $u \in V(G) \setminus S$ we select an arbitrary neighbor $s(u) \in N(u) \cap S$ that we contract it with (keeping parallel edges but removing self-loops), resulting in the multigraph $H_G(S)$ on the vertex set $S$. The key step in our proof is showing that $H_G(S)$ has bounded edge multiplicity. Indeed, every edge $s_1s_2$ in $H_G(S)$ is obtained from $G$ by contracting at least one vertex or from the edge $s_1s_2$ in $G$. For $i \in \{1, 2\}$, let $V_i$ be the set of vertices contracted to $s_i$ that gave an edge between $s_1$ and $s_2$ in $H_G(S)$. Since there is no $K_{2,3}$-minor in $G$ (as $G$ is outerplanar), we find $|V_1| \leq 2$ and $|V_2| \leq 2$. Any edge between $s_1$ and $s_2$ in $H_G(S)$ can now be associated with an edge between $\{s_1\} \cup V_1$ and $\{s_2\} \cup V_2$ in $G$, and hence edges in $H_G(S)$ have multiplicity at most 9 (this is far from tight).

We derive that $|E(H_G(S))| \leq 9|E(H')|$, where $H'$ is the simple graph underlying $H_G(S)$ (i.e. the simple graph obtained by letting $s_1, s_2 \in S$ be adjacent in $H'$ if and only if there is an edge between them in $H_G(S)$). Note that $H'$ is a minor of $G$. Since outerplanar graphs are closed under taking minors, the graph $H'$ is an outerplanar graph. It follows that $|E(H')| \leq 2|S| - 3$. Combining both observations, we get $|E(H_G(S))| \leq 18|S|$. 

By outerplanarity, we have $|E(H_G(S))| \leq \frac{1}{3}|B_S(G)|$. Indeed, each vertex $u \in B_S(G)$ has at most two common neighbors with $s(u)$ (otherwise there would be a $K_{2,3}$), hence $u$ has at least one neighbor $v$ such that $v \notin N[s(u)]$. The edge $uv$ corresponds to an edge in $E(H_G(S))$, hence each $u \in B_S(G)$ contributes at least half an edge to $E(H_G(S))$ (as $v$ could be also in $B_S(G)$). We derive $\frac{1}{3}|B_S(G)| \leq |E(H_G(S))| \leq 18|S|$. We observe that $|D_S(G)| \leq 3|S|$; indeed, each vertex from $S$ is adjacent to at most 3 vertices from $D_S(G)$, since any vertex adjacent to a vertex in $D_S(G)$ has degree at most 3 by definition. We conclude that $|B_S(G)| + |D_S(G)| \leq (36 + 3)|S| = 39|S|$. ▶

In Lemma 3 we use that edges in $H$ have low multiplicity, from which we then obtain a bound on the size of $S$. In order to improve the bound from Lemma 3, we dive into a deeper analysis of the graph $H$.

Proof of Lemma 2. We will consider a special counterexample $(G,S)$ (satisfying $|S| < \frac{1}{3}(|B_S(G)| + |D_S(G)|)$) so that our counterexample has a structure we can deal with more easily than a general counterexample. In particular we will choose a counterexample $(G,S)$ amongst those that minimize $S$ and with respect to that maximize $B_S(G) \cup D_S(G)$ to maximize and minimize certain other graph parameters.

Namely, we assume that $(G,S)$ in order: minimizes $|S|$; maximizes $|B_S(G) \cup D_S(G)|$; minimizes $|E(B_S(G))|$; maximizes $|E(S,N(S))|$; maximizes $|E(G)|$. Note that this is well-defined since we established $|B_S(G) \cup D_S(G)| \leq 39|S|$, and clearly $|E(S,N(S))| \leq |E(G)| \leq 2|V(G)|$ by outerplanarity. Consequently, if a counterexample exists, then there exists one satisfying all of the above assumptions. More formally, we select a counterexample that is minimal for $$(|S|, 39|S| - |B_S(G) \cup D_S(G)|, |E(B_S(G))|, |E(G)|, 2|V(G)| - |E(S,N(S))|, |E(G)|) \quad (\dagger)$$ in the lexicographic order. Since all the elements in the sextuple are non-negative integers and their minimum is bounded below by zero, this is well-defined. (We remark that the parts indicated in gray were added to ensure the entries are non-negative; minimizing $39|S| - |B_S(G) \cup D_S(G)|$ comes down to maximizing $|B_S(G) \cup D_S(G)|$.)

While this approach is not entirely intuitive, the assumptions will prove to be extremely useful for simplifying the structure of $G$. For example, we can show that in a smallest counterexample that minimizes $(\dagger)$, $S$ is a stable set (Claim 5) and no two vertices in $S$ have
a common neighbor (Claim 7). In general, the Claims 4 to 14 show that such a minimal counterexample $G$ has strong structural properties, by showing that otherwise we could delete some vertices and edges, or contract edges, and find a smaller counterexample.

Informally, for any vertex from $S$ that we remove from the graph, we may decrease $|B_S(G) \cup D_S(G)|$ by $4$ while maintaining $|S| < \frac{1}{4}(|B_S(G) \cup D_S(G)|)$. It is therefore natural to consider what happens when we reduce $|S|$ by one by contracting a connected subset containing two or more vertices from $S$. The result is again an outerplanar graph and we aim to show it is a smaller counterexample (unless the graph has some nice structure). Contracting an edge $uv$ can affect the degrees of the remaining vertices in the graph $G$. Therefore $B_S(G)$ may “lose” additional vertices besides $u$ and $v$ if more than one of its neighbors are contracted and $D_S(G)$ may “lose” additional vertices if a neighbor got contracted, increasing the degree. We remark that vertices from $D_S(G)$ have no neighbors in $B_S(G)$, and therefore removing or contracting them does not affect the set $B_S(G)$.

We note that our minimal counterexample $(G, S)$ is connected and again fix an outerplanar embedding of $G$. By definition, vertices in $D$ can have no neighbors in $B$. In fact, the following stronger claim holds.

\begin{itemize}
  \item Claim 4. Every vertex $d \in D$ satisfies $N(d) \subseteq S$.
\end{itemize}

**Proof.** Let $e = dv \in E(G)$ be such that $d \in D$. Suppose $v \notin S$. We consider the graph $G \setminus e$. Since $v \notin S$, $S$ is a dominating set of $G \setminus e$. We find $|B_S(G \setminus e)| = |B_S(G)|$, since a vertex in $D$ has no neighbor in $B$. Similarly, $|D_S(G \setminus e)| = |D_S(G)|$. Hence we also find that $|S| < \frac{1}{4}(|B_S(G \setminus e)| + |D_S(G \setminus e)|)$. Since $v \notin S$, the number of edges with one end incident to $S$ is the same in $G$ and $G \setminus e$. It follows that $(G \setminus e, S)$ is a counterexample to Lemma 2. Since $v \notin S$, $G$ and $G \setminus e$ have the same number of edges with exactly one end in $S$. Hence since $|E(G \setminus e)| < |E(G)|$, the pair $(G \setminus e, S)$ is smaller with respect to $\updownarrow$, contradicting our choice of $(G, S)$. 

We are now ready to make more refined observations about the structure of $(G, S)$. When considering a pair $(G', S')$ that is smaller than $(G, S)$ with respect to $\updownarrow$ with $V(G') \subseteq V(G)$, it can be useful to refer informally to vertices that belong to $B_S(G)$ but not to $B_{S'}(G')$ as lost vertices (similarly for $D_S(G)$ and $D_{S'}(G')$). The number of lost vertices is an upper bound on $|B_S(G) \cup D_S(G) - |B_{S'}(G') \cup D_{S'}(G')|$.

We need the following notation. Let $P \subseteq E(G)$. We denote the multigraph obtained from $G$ by contracting every edge in $P$ and deleting self-loops by $G/P$. Note $G/P$ remains outerplanar and may contain parallel edges.

\begin{itemize}
  \item Claim 5. The set $S$ is a stable set.
\end{itemize}

**Proof.** Assume towards a contradiction that there are two vertices $u$ and $w$ in $S$ that are adjacent.

Consider the outerplanar graph $G' = G/\{uw\}$ and let $v_{uw}$ be the vertex resulting from the contraction of the edge $uw$. Let $S' = S \setminus \{u, w\} \cup \{v_{uw}\}$. Define $B' = B_{S'}(G')$, and $D' = D_{S'}(G')$. Note that $S'$ dominates $G'$. Since we reduced the size of the dominating set by one, we are allowed to “lose up to four vertices from $B \cup D$", as then we get that

$$|B'| + |D'| \geq |B| + |D| - 4 > 4(|S| - 1) = 4|S'|.$$

We will now show the above inequality holds. If $v \in B \setminus B'$, then $v$ is a common neighbor of $u$ and $w$ (and $u, w$ have at most two such neighbors by outerplanarity). If $v \in D \setminus D'$, then $v$ is a neighbor of $u$ and/or $w$ in $G$ (since no vertex in $V(G) \cap V(G')$ has a higher degree in $G'$ than $G$). We consider three cases, depending on the neighbors of $u$ and of $w$ in $D$. 

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Suppose that $u, w$ have no neighbors in $D$. Then $D' = D$ and we lose only vertices from $B$ which are common neighbors of $u, w$, so at most 2.

Suppose that $u, w$ both have neighbors in $D$. Then $u, w$ are of degree at most 3. Since they are adjacent to each other, they are adjacent to at most 4 other vertices in total. So $|B'| + |D'| \geq |B| + |D| - 4$.

Suppose that $u$ has a neighbor in $D$ and $w$ does not (the other case is analogous). Now $|B| + |D|$ decreases by at most 2, since any “lost” vertex is adjacent to $u$ and distinct from $w$.

In all cases, $S$ has decreased by 1 and $|B| + |D|$ by at most 4, so we indeed find $|B'| + |D'| > 4|S'|$. Since $|S'| < |S|$, this gives a contradiction with the minimality of our choice of $(G, S)$.

We remark that no vertex in $S$ has only neighbors in $D$. Indeed, if some $s \in S$ has only neighbors in $D$ then we remove $s$ and its neighborhood from the graph and we have a smaller counterexample, as we reduced $|S|$ by one and $|D|$ by at most 3. In fact, we will show the following:

\[\triangleright\text{ Claim 6.}\text{ If } d \in D \text{ and } s \in N(d) \cap S, \text{ then } s \text{ has a neighbor in } B.\]

\[\text{Proof.}\text{ Suppose that } d \in D \text{ is adjacent to } s \in S. \text{ Since } d \in D, \text{ we find that } s \text{ has degree at most 3. Say } s \text{ has neighbors } w_1 \text{ and } w_2 \text{ (possibly equal, but both not equal to } d). \text{ We argued above that } s \text{ has a neighbor outside of } D, \text{ so without loss of generality } w_1 \notin D. \text{ Suppose towards a contradiction that } w_1, w_2 \notin B. \text{ By Claim 5, we find } w_1, w_2 \notin S, \text{ and so } w_1, w_2 \text{ have degree at most 3. As } w_1 \notin D, \text{ we find } w_1 \in A.

Suppose first that } w_1 \text{ and } w_2 \text{ together have at most two neighbors outside of } \{w_1, w_2, s\}.\text{ When we remove } N[s] \text{ from the graph, } |S| \text{ goes down by one and } |B \cup D| \text{ goes down by at most four ("counting" the two outside neighbors, } w_2 \text{ and } d), \text{ contradicting the minimality of our counterexample. So } w_1, w_2 \text{ have at least three "outside" neighbors, which implies that } w_2 \text{ exists and that } w_1, w_2 \text{ are non-adjacent (see Figure 4). Moreover, } w_1, w_2 \text{ together have at least three neighbors in } B \text{ by using the same strategy (showing that deleting } N[s] \text{ would give a smaller counterexample). Since } w_1 \text{ has degree at most 3 and is already adjacent to } s, \text{ it has at most two neighbors in } B. \text{ Thus } w_2 \text{ is also not in } D, \text{ since vertices in } D \text{ have no neighbors in } B. \text{ Recalculating now that we know that } w_1, w_2 \notin B \cup D, \text{ if } N(\{w_1, w_2\}) \text{ contains at most three vertices of degree four in } B, \text{ then } |B \cup D| \text{ goes down by at most four in } G \setminus N[s]. \text{ This would be a contradiction as } (G \setminus N[s], S \setminus \{s\}) \text{ would be a smaller counterexample. Hence, } w_1, w_2 \text{ have exactly four neighbors in } B, \text{ all of which are of degree exactly 4.}\]

![Figure 4](image-url) An illustration of the case when $w_1, w_2$ are not in $B$ and together have three neighbors which are not $s, w_1$ or $w_2$. At least one of the wavy edges is present and at least one of $w_1, w_2$ has degree three in the picture. In particular, $w_1$ and $w_2$ are not adjacent.
We show that we may assume that $d$ has degree 1. If $d$ has another neighbor, it is in $S$ by Claim 4. We delete the vertices $s, w_1, w_2$; since $d$ is still in $D$ and dominated, $|B \cup D|$ decreases by at most 4, and hence $(G \setminus \{s, w_1, w_2\}, S \setminus s)$ is a smaller counterexample. Hence we can assume $d$ has degree one and therefore the edge $ds$ is on the outer face.

Let $G'$ be the graph obtained from $G$ by adding the edge $w_1w_2$. Considering the local rotation of the three neighbors of $s$ in an outerplanar embedding of $G$, we note that $w_1, w_2$ are consecutive neighbors of $s$. We can draw the edge $w_1w_2$ close to the path $w_1-s-w_2$ keeping the embedding outerplanar (note that the edge $ds$ is still on the outer face). It follows that $w_1, w_2 \in B_S(G) \setminus B_S(G')$, so $|B_S(G) \cup D_S(G)| < |B_S(G') \cup D_S(G')|$. Thus, $(G', S)$ is a smaller counterexample, a contradiction. 

\begin{claim}
No two vertices in $S$ have a common neighbor.
\end{claim}

\begin{proof}
Assume towards a contradiction that there are two vertices $s_1$ and $s_2$ in $S$ that have a common neighbor $v$. Since $S$ is a stable set (Claim 5), we have $v \not\in S$.

We will consider the outerplanar graph $G' = G/P$ obtained by contracting $P = \{s_1v, vS_2\}$ into a single vertex $v_P$. Let $S' = S \setminus \{s_1, s_2\} \cup \{v_P\}$. We use the abbreviations $B' = B_S(G')$ and $D' = D_S(S')$.

We will again do a case analysis, on the union of the neighbors of $s_1$ and the neighbors of $s_2$ in $D \setminus \{v\}$, to find a smaller counterexample. If $|B'| + |D'| \geq |B| + |D| - 4$, then $(G', S')$ is a smaller counterexample. Note that vertices in $B \setminus B'$ have at least two neighbors in the set $\{s_1, v, s_2\}$.

Suppose first that for some $i \in \{1, 2\}$, $s_i$ is adjacent to at least two vertices in $D \setminus \{v\}$.

Then $v$ is the only other neighbor of $s_i$, so the graph $G''$ obtained from $G$ by deleting $s_i$ and its two neighbors in $D$, satisfies $|B(G'') \cup D(G'')| \geq |B \cup D| - 3$ whereas the set $S'' = S \setminus \{s_i\}$ is dominating. This gives a smaller counterexample.

Suppose that both $s_1, s_2$ are adjacent to a single vertex in $D \setminus \{v\}$. Then both have degree at most 3. Let $d_1, d_2 \in D \setminus \{v\}$ be the neighbors of $s_1, s_2$ respectively (where $d_1, d_2$ might be equal). The graph $G'$ is a smaller counterexample unless we lost two vertices from $B$ besides possibly $v$, that is, $|B'| \leq |B \setminus \{v\}| - 2$. Any vertex lost from $B \setminus \{v\}$ must be adjacent to two vertices among $\{s_1, v, s_2\}$ (as otherwise its degree did not change), and since both $s_1$ and $s_2$ already have two named neighbors, $G'$ is a counterexample unless there is, for each $i \in \{1, 2\}$, a common neighbor $b_i \in B$ of $s_i$ and $v$, and all named vertices are distinct.

Since $d_1 \in D$ and $b_1 \in B$, we find that $b_1d_1$ is not an edge of $G$. Since $s_1$ has three neighbors, $b_1$ and $d_1$ are consecutive neighbors and the edge $b_1d_1$ can be added without making the graph non-planar. Consider adding the edge $b_1d_1$ in $G$ along the path $b_1s_1d_1$, such that there are no vertices in between the edge and the path. This may affect whether $s_1$ is on the outer face, but it does not affect whether $s_2$ is on the outer face. Therefore, after contracting this adjusted graph, the obtained graph $G''$ is still outerplanar. Moreover, $b_1$ has the same degree in $G''$ as in $G$, and so $|B_S(G'') \cup D_S(G'')| \geq |B| + |D| - 4$ and $G''$ is a smaller counterexample.

Suppose that $s_1$ is adjacent to a vertex $d_1$ in $D \setminus \{v\}$ and $s_2$ is not (the symmetric case is analogous). There can be at most three vertices in $B \setminus \{v\}$ which are adjacent to two vertices in $s_1, v, s_2$ (as only one can be adjacent to $s_1$ and $v, s_2$ have at most two common neighbors since the graph is outerplanar). The only way in which $G'$ is not a counterexample, is when there is a common neighbor $b_1$ of $s_1$ and $v$ and two common neighbors $b_2, b_3$ of $s_2$ and $v$ with all named vertices distinct. As before, we may now add the edge $b_1d_1$ in order to obtain a smaller counterexample $G''$. 

\end{proof}
Finally, suppose that \( s_1 \) and \( s_2 \) have no neighbors in \( D \setminus \{ v \} \). By outerplanarity, there are at most four vertices with two neighbors among \( \{ s_1, v, s_2 \} \). Hence \( G' \) is a counterexample unless there are exactly four (the only vertices “lost” from \( B \cup D \) are either \( v \) or among such common neighbors, since \( s_1 \) and \( s_2 \) have no neighbors in \( D \setminus \{ v \} \)). All four vertices are adjacent to \( v \), because otherwise \( G \) contains a \( K_{2,3} \)-minor\(^2\), a contradiction. In particular, \( G' \) is a counterexample unless there are two common neighbors of \( v \) and \( s_1 \) and two common neighbors of \( v \) and \( s_2 \) (and so \( d(v) \geq 6 \) and \( v \in B \)).

Fix a clockwise order \( w_1, w_2, \ldots, w_d \) on the neighbors of \( v \) such that the path \( w_1vw_d \) belongs to the boundary of the outer face. Let \( i \neq j \) such that \( w_i = s_1 \) and \( w_j = s_2 \). After relabelling, we may assume \( i < j \). Since \( s_1 \) and \( s_2 \) both have two common neighbors with \( v \), we find \( i > 1, j < d \) and \( i + 1 < j - 1 \). The vertices adjacent to multiple vertices in \( \{ s_1, v, s_2 \} \) are \( w_{i-1}, w_{i+1}, w_{j-1} \) and \( w_{j+1} \). We create a new graph \( G'' \) by replacing \( v \) with two adjacent vertices \( v_1' \) and \( v_2' \), where \( v_1' \) is adjacent to \( w_1, w_2, \ldots, w_{i+1} \) and \( v_2' \) to \( w_{i+2}, \ldots, w_d \). This graph is outerplanar because both \( v_1' \) and \( v_2' \) have an edge incident to the outer face. Moreover, \( d(v_1') \) and \( d(v_2') \) are both at least 4, since they are adjacent to each other, to either \( s_1 \) or \( s_2 \) and to at least two vertices among \( w_1, \ldots, w_d \). The set \( S \) is still a dominating set, but \( |B(G'') \cup D(G'')| > |B \cup D| \) so this is a smaller counterexample. In all cases, we found a smaller counterexample. This contradiction proves the claim. \(<\)

Since vertices in \( D \) only have neighbors in \( S \), the claim implies in particular that each vertex of \( D \) has degree 1.

With the claims above in hand, we now analyze the structure of \( H = H_G(S) \) as described in the notation section more closely. Note that the for each \( u \in V(G) \setminus S \) the vertex \( s(u) \) is uniquely defined by Claim 7.

Recall that \( H \) is outerplanar. It follows that there is a vertex \( s_1 \in V(H) \) with at most 2 distinct neighbors in \( H \).

We start with an easy observation.

\begin{boxed Observation} \end{boxed Observation}

\textbf{Observation 8.} Let \( b \in B \) and \( s(b) \) be its unique neighbor in \( S \). Then there exists \( w \in N(b) \setminus \{ s(b) \} \), such that its unique neighbor \( s(w) \in S \) is not equal to \( s(b) \).

Indeed, the vertex \( b \) can have at most two common neighbors with \( s(b) \) (otherwise there would be a \( K_{2,3} \), contradicting outerplanarity), and a vertex in \( B \) has degree at least 4 by definition.

Note that the vertex \( s_1 \) has at least one neighbor in \( H \). Indeed, if \( s_1 \) has no neighbor in \( H \), then \( N[s_1] \) is a connected component in \( G \). Since \( G \) is connected, \( G = N[s_1] \). By Observation 8, we have \( B = \emptyset \), so \( |D \cup B| \leq 3 \).

\begin{boxed Claim} \end{boxed Claim}

\textbf{Claim 9.} The vertex \( s_1 \) has precisely two neighbors in \( H \).

\begin{proof} \end{proof}

Assume towards a contradiction that \( s_1 \) has a single neighbor \( s_2 \) in \( H \). Let \( v_1, \ldots, v_k \) be the vertices in \( N[s_1] \) that have a neighbor in \( N[s_2] \), and conversely let \( u_1, \ldots, u_\ell \) be the vertices in \( N[s_2] \) that have a neighbor in \( N[s_1] \). Note that by Claims 5 and 7, all of \( \{ v_1, \ldots, v_6, u_1, \ldots, u_\ell, s_1, s_2 \} \) are pairwise distinct. If \( \ell \geq 3 \), then contracting the connected set \( N[s_1] \) in \( G \) gives a \( K_{2,3} \) on the contracted vertex and \( s_2 \) on one side and \( u_1, u_2, u_3 \) on the other. We derive that \( \ell \leq 2 \), and by symmetry, \( k \leq 2 \). By Observation 8, the only neighbors of \( s_1 \) that belong to \( B \) are in \( \{ v_1, v_2 \} \). As we assumed that \( s_1 \) has degree 1 in \( H \), we have \( N[v_i] \subseteq N[s_1] \cup \{ u_1, u_2 \} \) for \( i \in \{ 1, 2 \} \). We will do a case distinction on \( N[s_1] \cap D \).

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\(^2\) The vertices \( s_1, s_2 \) can have at most one further common neighbor \( v^* \) besides \( v \). If \( v^* \) exists, we contract it with \( s_1 \) and \( s_2 \). We find a \( K_{2,3} \) subgraph with \( v, v^* \) on one side and the three other common neighbors on the other side.
If $s_1$ has no neighbor in $D$, we delete $N[s_1]$, and note that $D_S(G) = D_{S \setminus s_1}(G \setminus N[s_1])$, while $B_S(G) \setminus \{v_1, v_2, u_1, u_2\} \subseteq B_{S \setminus s_1}(G \setminus N[s_1])$. Therefore,

$$|S \setminus \{s_1\}| \geq \frac{1}{4} \cdot (|D_{S \setminus s_1}(G \setminus N[s_1])| + |B_{S \setminus s_1}(G \setminus N[s_1])|).$$

So we have found a smaller counterexample.

Suppose $s_1$ has two neighbors $d_1 \neq d_2$ in $D$. Then $v_2$ does not exist since $d(s_1) \leq 3$ and because $v_1, v_2$ are distinct from $d_1, d_2$ (vertices in $D$ have degree 1). Suppose first that $v_1$ has degree at least 4. Let $x$ be its neighbor distinct from $u_1, u_2, s_1$. By assumption on $s_1$, the vertex $x$ has no neighbor in $S \setminus \{s_1\}$. Therefore, $x$ is adjacent to $s_1$. However, $x$ is distinct from $d_1, d_2$ and $v_1$, which contradicts $d(s_1) \leq 3$. This case is illustrated in Figure 5. Hence $v_1 \notin B$ and removing $N[s_1]$ now gives a smaller counterexample, a contradiction.

**Figure 5** An illustration of the case where $s_1$ has degree one in $H$ and two neighbors $d_1, d_2 \in D$ in $G$. If $v_1 \in B$, then some vertex $x$ exists such that both wavy edges are present in $G$, a contradiction.

Suppose that $s_1$ has a single neighbor $d_1$ in $D$. Removing $N[s_1]$ gives a smaller counterexample again, unless all of $u_1, u_2, v_1, v_2$ exist and belong to $B$. In particular, $v_1, v_2$ both have degree at least 4. Each of $v_1$ and $v_2$ can only have neighbors within $\{s_1, v_1, v_2, u_1, u_2\}$ because a neighbor $x$ not within $\{s_1, v_1, v_2, u_1, u_2\}$ is a neighbor of $s_1$, but $d(s_1) \leq 3$. Therefore, both $v_1$ and $v_2$ are adjacent to $u_1$ and $u_2$. Together with $s_1$, this forms a $K_{2,3}$ subgraph (see Figure 6): a contradiction.

**Figure 6** An illustration of the case where $s_1$ has degree one in $H$ and has exactly one neighbor in $D$ in $G$. We reduce to the case in which the depicted graph is a subgraph of $G$. We find a contradiction since the depicted graph contains a $K_{2,3}$.

So $s_1$ has two neighbors in $H$. Let $s_2, s_3 \in V(H)$ be its neighbors. In $G$, let $w_1, \ldots, w_p$ be the vertices in $N[s_1]$ that have a neighbor in $N[s_3]$, and conversely let $x_1, \ldots, x_q$ be the vertices in $N[s_3]$ that have a neighbor in $N[s_1]$. By the same argument as before for $s_1$ and $s_2$, we obtain $p \leq 2$ and $q \leq 2$ and that all of $\{w_1, w_2, x_1, x_2, s_1, s_3\}$ are pairwise distinct. However, there may be a vertex in $\{w_1, w_2\} \cap \{v_1, v_2\}$; there may not be two such vertices since this would lead to a $K_{2,3}$-minor (with vertices $\{v_1, w_1\}$ and $\{v_2, w_2\}$ in one part, and $s_1, \{s_2, u_1, u_2\}, \{s_3, x_1, x_2\}$ in the other).
Our general approach is to delete $N[s_1]$ and add edges between $\{u_1, u_2\}$ and $\{x_1, x_2\}$ as appropriate so as to mitigate the impact on $|B \cup D|$. If this does not work, we obtain further structure on the graph which we exploit to create a different smaller counterexample. We will repeatedly apply the following Observation 10. Sometimes when deleting vertices and edges from the graph $G$, the result is a disconnected graph, so we can perform the “flipping” operation described below, and connect the different components to get a smaller counterexample $(G', S')$.

**Observation 10 (Flipping).** Let $G$ be the disjoint union of two outerplanar graphs $O_1$ and $O_2$. Consider an outerplanar embedding of $G$, and let $(u_1, u_2, \ldots, u_q)$ denote the outer face of $G[O_2]$ in clockwise order. We can obtain a different outerplanar embedding of $G$ by reversing the order of $O_2$ without modifying the embedding of $O_1$, so that the outer face of $G[O_2]$ is $(u_q, \ldots, u_2, u_1)$ in clockwise order.

![Figure 7](image-url) An illustration of Observation 10.

An example of the observation above is given in Figure 7. Beside Observations 8 and 10, the third useful observation is as follows.

**Observation 11.** $N[v_1, v_2, w_1, w_2] \subseteq N[s_1] \cup \{u_1, u_2, x_1, x_2\}$. Additionally, if $\{v_1, v_2\} \cap \{w_1, w_2\} = \emptyset$, then $N[v_1, v_2] \subseteq N[s_1] \cup \{u_1, u_2\}$ and $N[w_1, w_2] \subseteq N[s_1] \cup \{x_1, x_2\}$.

This observation is argued similarly to Observation 8, we omit the argument.

Since $s_1$ is adjacent to $s_2$ and $s_3$ in $H$, all of $u_1, x_1, v_1$ and $v_1$ exist. We assume that either $\{v_1, v_2\} \cap \{w_1, w_2\} = \emptyset$ or $v_1 = w_1$. Note that $\{u_1, u_2\} \cap \{x_1, x_2\} = \emptyset$ since $s_2$ and $s_3$ do not have common neighbors by Claim 7. See Figure 8 for an illustration. For simplicity, when depicting which edges to add in which cases, we represent “$u_2$ does not exist” as “$u_2$ is possibly equal to $u_1$” (and variations). This means merely that if $u_2$ does not exist then the edges involving $u_2$ involve $u_1$ instead – multiple edges are ignored.

**Claim 12.** One of $u_2$ and $v_2$ exists.

Proof. Suppose that neither $u_2$ nor $v_2$ exists. It is possible that $v_1 = w_1$, and that $u_2$ or $x_2$ do not exist. By Observation 11, if $v_1 \neq w_1$, then $u_1, u_2$ are not adjacent to $w_1$ and $x_1, x_2$ are not adjacent to $v_1$.

The degrees of $x_1, x_2, u_1, u_2$ in $G \setminus N[s_1]$ are at least one less than their degrees in $G$. Every vertex in $V(G) \setminus (N[s_1] \cup \{x_1, x_2, u_1, u_2\})$ has the same degree in $G$ and in $G \setminus N[s_1]$. Let $S' = S \setminus \{s_1\}$, and note that $S'$ dominates $G \setminus N[s_1]$.

Suppose $x_2, u_2$ do not exist. If $v_1$ belongs to $B$, then it needs to have a neighbor which is not $u_1, s_1$ or one of $x_1, v_1$ (depending on whether $v_1 = w_1$), so it shares a neighbor with $s_1$ which is not in $B \cup D$. This implies that if $v_1 \in B$, then $s_1$ can have a neighbor
We would obtain a $s_2, s_3$ in $H$, each of $s_2, s_3$ has at most two neighbors with edges to vertices in $N[s_1]$. Moreover, $s_2$ and $s_3$ may have at most one common neighbor in $N[s_1]$. We draw vertices which may not exist in $G$ as a dotted circle and connect vertices which may be equal with dotted edges. There may be more edges present in that are not drawn.

By Observation 11, the degrees of other vertices are not affected by removing $u_1$ and $x_1$. The degrees of the vertices $u_1, x_1$ are at least as large as their respective degrees in $G$ (the degree of $x_1$ might have dropped if the edge $u_1 x_1$ was already present in $G$). Note that $G'$ is outerplanar and that $S'$ dominates $G'$. Since $G$ is outerplanar, if $u_1 x_1$ is an edge in $G$, then neither $u_1 x_2$ nor $x_2 x_2$ is an edge in $G$. In $G'$, the degrees of the vertices $u_1, u_2, x_2$ are at least as large as their respective degrees in $G$ (the degree of $x_1$ might have dropped if the edge $u_1 x_1$ was already present in $G$). Note that $|\{v_1, u_1, x_1\} \cup (N[s_1] \cap D)| \leq 4$, hence $|B_{S'}(G') \cup D_{S'}(G')| \geq |B \cup D| - 4$, a contradiction.

**Claim 13.** If $w_1 = v_1$, then $v_2$ and $w_2$ exist.

**Proof.** By Claim 12 we can assume $v_2$ does not exist and $w_1 = v_1$. We remove $N[s_1]$ and add edges between $\{u_1, u_2\}$ and $\{x_1, x_2\}$ as above to ensure that for all but at most one of them, the degree does not decrease. To see an illustration of how the edges are added, see Figure 10. We suppose first that there are no edges between $\{u_1, u_2\}$ and $\{x_1, x_2\}$. The edges remedy the degree for $x_1, x_2$, since they only lost $v_1$, and for one of $u_1, u_2$; indeed, it is not possible that both $u_1$ and $u_2$ are adjacent to both $v_1$ and $v_2$ (since we would obtain a $K_{2,3}$ when considering $s_1$ as well).

By Observation 11, the degrees of other vertices are not affected by removing $N[s_1]$. 

**Figure 8** When $s_1$ has exactly two neighbors $s_2, s_3$ in $H$, each of $s_2, s_3$ has at most two neighbors with edges to vertices in $N[s_1]$. Moreover, $s_2$ and $s_3$ may have at most one common neighbor in $N[s_1]$. We draw vertices which may not exist in $G$ as a dotted circle and connect vertices which may be equal with dotted edges. There may be more edges present in that are not drawn.

**Figure 9** The case where $w_2, v_2$ do not exist. The original graph is drawn at the left and the modified graph is drawn at the right. The wavy line indicates there may be an edge between $u_1$ and $x_1$. Edges that may have been added are drawn in blue. Note that $u_2$ may not exist. There may be more edges which are not drawn (for instance $v_1$ might be adjacent to $w_1$) but these edges are not relevant to our argument.
A Tight Local Algorithm for MDS in Outerplanar Graphs

We henceforth assume that a maximum independent set, which in turn depends on multiple applications of Ramsey’s theorem in Figure 1. The argument of [7] builds on a lower bound for local algorithms computing a dominating set in graphs.

They showed that for every local distributed algorithm $A$ on a graph $G$ that does not exist, we can delete three of its edges to get an outerplanar graph $G'$. For $G'$, there exists a minimum dominating set $N$ for which the algorithm $A$ computes a dominating set with approximation $O(1)$. Since in all claims and cases we can show that there is a smaller counterexample, there must exist a dominating set $D$ such that $|D| < |N|$. Since the details of the remaining casework are not particularly illuminating, we will omit them for brevity. Appendix A and the longer arXiv version [6] of our paper both contain the full details.

3 Lower bound for outerplanar graphs

In this section we show that there is no deterministic local algorithm that finds an $\delta$-approximation of a minimum dominating set on outerplanar graphs using $T$ rounds, for any $T \in \mathbb{N}$. To do so we use a result from Czygrinow, Hańckowiak and Wawrzyniak [7, pp. 87–88] who gave a lower bound in the planar case. For $n \equiv 0 \bmod 10$, they consider a graph $G_n$, which is a cycle $v_1, v_2, \ldots, v_n, v_1$ where edges between vertices of distance two are added. They showed that for every local distributed algorithm $A$ and every $\delta > 0$ and $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ for which the algorithm $A$ outputs a dominating set for $G_n$ that is not within a factor of $5 - \delta$ of the optimal dominating set for $G_n$. Their graph $G_n$ is not outerplanar, but we can delete three of its edges to get an outerplanar graph $G'_n$. The graph $G'_n$ is a path $v_1 \ldots v_n$ where all edges between vertices of distance two are added as in Figure 1. The argument of [7] builds on a lower bound for local algorithms computing a maximum independent set, which in turn depends on multiple applications of Ramsey’s theorem.
theorem. A similar approach is used by [11] to obtain the best-known lower-bound for planar graphs. Using the graph $G_n^-$, this approach can also be used to prove our result; the main idea is that since in the middle all the vertices “look the same”, no local algorithm can do better than selecting almost all of them.

Alternatively, we can exploit the result of [7] as follows. For any bound $T \in \mathbb{N}$ on the number of rounds, any vertex in $M = \{v_{2T+1}, \ldots, v_{n-2T-1}\}$ has the same local neighborhood in $G_n$ as in $G_n^-$. Since $G_n$ is rotation symmetric, a potential local algorithm also finds a dominating set $D$ for $G_n$ (for $n \geq 4T + 2$), and with the result of [7] we obtain $|D| \geq (5 - \delta)\gamma(G_n)$. For $n$ sufficiently large with respect to $T$, the set $D$ is the same as the set $D'$ that the algorithm would give for $G_n$ up to at most $\delta n/10$. Since $n \equiv 0 \mod 10$, $\gamma(G_n^-) = \gamma(G_n^-) = \frac{n}{5}$ and we find the desired lower-bound $|D'| \geq (5 - \frac{\delta}{2})\gamma(G_n^-) \geq (5 - \epsilon)\gamma(G_n^-)$ for $\delta$ small enough.

4 Conclusion

Through a rather intricate analysis of the structure of outerplanar graphs, we were able to determine that a very naive algorithm gives a tight approximation for minimum dominating set in outerplanar graphs in $O(1)$ rounds. While there are some highly non-trivial obstacles to extending such work to planar graphs, we believe that similar techniques can be used to vastly improve the state of the art for triangle-free planar graphs and for $C_4$-free planar graphs. In the first case, recall that a 32-approximation is known [3], and there is a simple construction (a large 4-regular grid) showing that 5 is a lower bound. We believe that 5 is the right answer. In the second case, an 18-approximation is known [4], and there is no non-trivial lower bound. We refrain from conjecturing the right bound here – we simply point out that there is no reason yet to think 3 is out of reach. We believe that very similar techniques to the ones developed here can be used to obtain a 9-approximation, and possibly lower.

References

A Remaining casework

This appendix will provide the details of the case analysis from end of the proof of Lemma 2.

We are in the case in which \(|\{v_1, v_2, w_1, w_2\}| \geq 3\). In particular, we may assume that \(s_1\) has no neighbor in \(D\).

\(\blacktriangleright\) Claim 14. We have \(v_1 \neq w_1\).

Proof. If not, then \(v_1 = w_1\). By Claim 13, both \(v_2\) and \(w_2\) exist. Let \(G'\) be the outerplanar graph obtained from \(G\) by splitting the vertex \(v_1\) into two vertices \(v'_1\) and \(w'_1\), both adjacent to \(s_1\) and adjacent to each other, where \(v'_1\) is adjacent to \(N[v_1] \cap N[s_2]\) and \(w'_1\) is adjacent to \(N[v_1] \cap N[s_3]\). This gives three neighbors for both \(v'_1\) and \(w'_1\). Since we can always add the edges \(v'_1v_2\) and \(w'_1w_2\) (which are chords of a cycle, using also that \(N[s_1] \cap (N[s_2] \cup N[s_3]) = \emptyset\)), we find \(|B_S(G')| > |B_S(G)|\), whereas \(D_S(G') = D_S(G)\), \(S\) is still dominating and \(G'\) is outerplanar. We find a contradiction with our assumption of the minimality of \(G\). \(\blacktriangleright\)

We henceforth assume that \(v_1, w_1\) exist and are distinct and at least one of \(v_2, w_2\) exists.

\(\blacktriangleright\) Claim 15. The vertices \(v_2, w_2\) exist.
We have one final case in which we can align the components of the vertices $A$. The case when exactly one of $D$ present in $N$ does not exist. At top we show the case where $N[s_1]$, at most one of $u_1, u_2$ loses two neighbors, and the other (if it exists) loses only a single neighbor. The vertices $x_1, x_2$ can lose only a single neighbor. By Observation 10, after deleting $N[s_1]$, we can align the components of $N[s_2]$ and $N[s_3]$ in such a way that we can add the edges from $\{u_1x_1, u_2x_2, u_2x_2\}$. (For brevity, we handle the cases in which some of $u_2, x_2$ do not exist here as well, in which case we might add less edges.) We lost at most 4 vertices $B \cup D$, namely at most $v_1, v_2, w_1$ and one of the $u_i$ (if one of them lost two neighbors).

A.1 The case where $x_2$ and $u_2$ both exist

Suppose that $x_2$ and $u_2$ both exist. Note that at most one of $u_1, u_2$ and one of $x_1, x_2$ is adjacent to two vertices in $N[s_1]$. Let us assume without loss of generality that $x_1, u_2$ have at most one neighbor in $N[s_1]$. By Observation 10, after deleting $N[s_1]$, we can re-embed the graph in a way that we can add the edges $u_1x_1, u_1x_2, u_2x_2$. This gives a smaller counterexample, since we have “fixed” the degrees of $u_1, u_2, x_1, x_2$ and only lost $N[s_1] \cap (B \cup D)$, which has size at most 4.

A.2 The case when exactly one of $x_2, u_2$ exists

Suppose now that only $u_2$ exists. (The case in which only $x_2$ exists is analogous.) Note that $s_2$ can have at most one neighbor in $D$. See also Figure 11.

Suppose first that $s_2$ has a neighbor $d \in D$. We delete and add edges (if needed) and renumber such that $u_1$ is adjacent to $v_1, v_2$ to $v_2$ and $u_1$ to $u_2$, but no other edges among $\{u_1, u_2, v_1, v_2\}$ are present. Now we can add the chords $u_1s_1, u_2s_1$ to the cycle $s_1v_1, s_1u_2v_2s_1$. We delete $s_2$ and $d$. We have now lost at most four vertices from $B \cup D$: namely at most $v_1, v_2, d$ and one of the $u_i$ (if it was adjacent to $v_1$ and $v_2$ originally).

Figure 11: An illustration of the case $v_1, v_2, w_1, w_2, u_2$ all exist and are all distinct and $x_2$ does not exist. At top we show the case where $s_2$ has a neighbor $d \in D$. Some of the wavy edges may be present in $G$. As usual, there may be other edges present in $G$ that have not been drawn, but they are not relevant to our argument. At the bottom we illustrate the case where $s_2$ has no neighbor in $D$. For $i \in \{1, 2\}$, we add the vertices $z_i$ and edges $z_is_2, z_iu_i$ if needed to make $\deg(u_i) \geq 4$ in $G'$.
A.3 The case when neither $u_2$ nor $x_2$ exists

If both $u_1$ and $x_1$ do not have degree exactly four, then we can remove $N[s_1]$ and add the edge $u_1x_1$; in this case we only lose a subset of $\{v_1, v_2, w_1, w_2\}$ from $B \cup D$. Hence we can assume by symmetry that $u_1$ has degree exactly four.

- We first handle the case in which $s_2$ has no neighbor in $D$. Since $u_1$ has degree exactly 4, after removing $N[s_1]$ we can create a new vertex $v$ and add the edges $u_1v, s_2v, x_1v, u_1x_1$. As a result, we have lost at most $v_1, v_2, w_1, w_2$ from $B \cup D$ and found a smaller counterexample. See Figure 12.

- Suppose now that $s_2$ has only neighbors in $D \cup \{u_1\}$, which we name $d_1, d_2$ (where $d_2$ may or may not exist). We remove $N[s_2] \setminus \{u_1\}$ (at most three vertices), remove the edge $v_1v_2$ and add the edge $u_1s_1$. We again found a smaller counterexample as the only vertices we may have lost from $B \cup D$ are $v_1, v_2, d_1, d_2$.

- Suppose now that $s_2$ has exactly one neighbor $d \in D$. It may have another neighbor $y \neq u_1, d$, which if it exists, is not in $D$. We delete the vertices $s_2, d$ as well as the edges $u_1v_1$ and $v_1v_2$ (if these exist). As $u_1$ was a cut-vertex previously, we can now add the edges $u_1s_1$ and $ys_1$ (say along the path $u_1v_2s_1$) to ensure that the size of the dominating set has dropped by one whereas we lost at most $d, u_1, v_1, v_2$ from $B \cup D$. See Figure 13.