Minimality Notions via Factorization Systems

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Abstract

For the minimization of state-based systems (i.e. the reduction of the number of states while retaining the system's semantics), there are two obvious aspects: removing unnecessary states of the system and merging redundant states in the system. In the present article, we relate the two aspects on coalgebras by defining an abstract notion of minimality.

The abstract notion minimality and minimization live in a general category with a factorization system. We will find criteria on the category that ensure uniqueness, existence, and functoriality of the minimization aspects. The proofs of these results instantiate to those for reachability and observability minimization in the standard coalgebra literature. Finally, we will see how the two aspects of minimization interact and under which criteria they can be sequenced in any order, like in automata minimization.

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1 Introduction

Minimization is a standard task in computer science that comes in different aspects and lead to various algorithmic challenges. The task is to reduce the size of a given system while retaining its semantics, and in general there are two aspects of making the system smaller: 1. merge redundant parts of the system that exhibit the same behaviour (*observability*) and 2. omit unnecessary parts (*reachability*). Hopcroft's automata minimization algorithm [21] is an early example: in a given deterministic automaton, 1. states accepting the same language are identified and 2. unreachable states are removed. Moreover, Hopcroft's algorithm runs in quasilinear time; for an automaton with n states, reachability is computed in $\mathcal{O}(n)$ and observability in $\mathcal{O}(n \log n)$.

Since the reachability is a simple depth-first search, it is straightforward to apply it to other system types. On the other hand, it took decades until quasilinear minimization algorithms for observability were developed for other system types such as transition systems [29], labelled transition systems [14, 34], or Markov chains [12, 35]. Though their differences in complexity, the aspects of observability and reachability have very much in common when modelling state-based systems as coalgebras. Then, observability is the task to find the greatest coalgebra quotient and reachability is the task of finding the smallest subcoalgebra containing the initial state, or generally, a distinguished point of interest.

In the present article, we define an abstract notion of minimality and minimization in a category with an $(\mathcal{E}, \mathcal{M})$-factorization system. Such a factorization systems gives rise to a generalized notion of quotients and subobjects, and the minimization is the task of finding the least quotient resp. subobject. To make this general setting applicable to coalgebras, we show that the category of coalgebras inherits the factorization system from the base category under a mild assumption (namely that the functor preserves $\mathcal{M}$). Dually, a factorization system also lifts to algebras, and even to the Eilenberg-Moore category.

Then, we will present different characterizations of minimality (Figure 4) and then study properties of minimizations, e.g. under which criteria they exist and are unique, rediscovering the respective proofs for reachability and observability for coalgebras in the literature [4, 22]. When combining the two minimization aspects, we discuss under which criteria reachability and observability can be computed in arbitrary order.

The goal of the present work is not only to show the connections between existing minimality notions, but also to provide a series of basic results that can be used when developing new minimization techniques or even new notions of minimality.

Related Work

There is a series of works [10, 9, 11, 30] that studies the minimization of coalgebras by their duality to algebras. In those works, the correspondence between observability in coalgebras and reachability in algebras is used. For instance, Rot [30] relates the final sequence (for observability in coalgebras) with the initial sequence (for reachability in algebras). In the present paper however, we consider both observability and reachability on an abstract level that work for a general factorization system and discuss their instance in coalgebras. The paper is based on Chapter 7 of the author's PhD dissertation [38].

If not included in the main text, detailed proofs of all results can be found in the appendix, both for the standard results recalled in the preliminaries and the new results of the main sections.

2 Preliminaries

In the following, we assume basic knowledge of category theory (cf. standard textbooks [2, 5]).

Given a diagram $D\colon \mathcal{D} \to \mathcal{C}$ (i.e. a functor D from a small category $\mathcal{D}$), we denote its limit by $\lim D$ and colimit by $\operatorname{colim} D$ – if they exist. The limit projections, resp. colimit injections, are denoted by

$$\mathsf{pr}_i\colon \lim D \to Di \qquad \mathsf{inj}_i\colon Di \to \operatorname{colim} D \qquad \text{for } i \in \mathcal{D}.$$

2.1 Coalgebra

We model state-based systems as coalgebras for an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$:

▶ **Definition 2.1.** *An* F-coalgebra *(for an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$) is a pair (C, c) consisting of an object C (of $\mathcal{C}$) and a morphism $c\colon C \to FC$ (in $\mathcal{C}$). An F-coalgebra morphism $h\colon (C,c) \to (D,d)$ between F-coalgebras (C,c) and (D,d) is a morphism $h\colon C \to D$ with $d \cdot h = Fh \cdot c$ (see Figure 1a on p. 4 for the corresponding commuting diagram).*

Intuitively, the *carrier* C of a coalgebra (C, c) is the state space and the morphism $c\colon C \to FC$ sends states to their possible next states. The functor of choice F defines how these possible next states FC are structured.

▶ **Example 2.2.** Many well-known system-types can be phrased as coalgebras:

- Deterministic automata (without an explicit initial state) are coalgebras for the Set-functor $FX = 2 \times X^A$, where A is the set of input symbols. In an F-coalgebra (C, c), the first component of $c(x)$ denotes the finality of the state $x \in C$ and the second component is the transition function $A \to C$ of the automaton.
- Labelled transition systems are coalgebras for the Set-functor $FX = \mathcal{P}(A \times X)$ and the coalgebra morphisms preserve bisimilarity.
- Weighted systems with weights in a commutative monoid $(M, +, 0)$ (and finite branching) are coalgebras for the *monoid-valued functor* [18, Def. 5.1]

  $$M^{(X)} = \{\mu \colon X \to M \mid \mu(x) = 0 \text{ for all but finitely many } x \in X\}$$

  which sends a map $f \colon X \to Y$ to the map

  $$M^{(f)} \colon M^{(X)} \to M^{(Y)} \qquad M^{(f)}(\mu)(y) = \sum\{\mu(x) \mid x \in X, f(x) = y\}$$

  In an $M^{(-)}$-coalgebra (C, c), the transition weight from state $x \in C$ to $y \in C$ is given by $c(x)(y) \in M$. E.g. one obtains real-valued weighted systems as coalgebras for the functor $(\mathbb{R}, +, 0)^{(-)}$.
- The *bag* functor is defined by $\mathcal{B}X = (\mathbb{N}, +, 0)^{(X)}$. Equivalently, $\mathcal{B}X$ is the set of finite multisets on X. Its coalgebras can be viewed as weighted systems or as transition systems in which there can be more than one transition between two states.
- A wide range of probabilistic and weighted systems can be obtained as coalgebras for respective distribution functors, see e.g. Bartels et al. [8].

▶ **Definition 2.3.** *The category of F-coalgebras and their morphisms is denoted by* $\mathsf{Coalg}(F)$.

Intuitively, the coalgebra morphisms preserve the behaviour of states:

▶ **Definition 2.4.** *In* Set*, two states $x, y \in C$ in an F-coalgebra (C, c) are* behaviourally equivalent *if there is a coalgebra homomorphism $h \colon (C, c) \to (D, d)$ with $h(x) = h(y)$.*

▶ **Example 2.5.**

- For deterministic automata ($FX = 2 \times X^A$), states are behaviourally equivalent iff they accept the same language [31, Example 9.5]. Indeed, for sufficiency, if two states x, y in an F-coalgebra are identified by a coalgebra homomorphism, then one can show by induction over input words $w \in A^*$ that either both states or neither of them accepts w. For necessity, note that the map sending states to their semantics $C \to \mathcal{P}(A^*)$ is a coalgebra homomorphism.
- For labelled transition systems ($FX = \mathcal{P}(A \times X)$), states are behaviourally equivalent iff they are bisimilar [1].
- For weighted systems, i.e. coalgebras for $M^{(-)}$, the coalgebraic behavioural equivalence captures weighted bisimilarity [23].
- Further semantic notions can be modelled with coalgebras by changing the base category from $\mathcal{C} = \mathsf{Set}$ to the Eilenberg-Moore [33] or Kleisli category [20] of a monad, to nominal sets [26, 28], or to partially ordered sets [6].

The category of coalgebras inherits many properties from the base-category $\mathcal{C}$. For instance, we have the following standard result:

▶ **Corollary 2.6.** *The forgetful functor $\mathsf{Coalg}(F) \to \mathcal{C}$ creates all colimits. That is, the colimit of a diagram $D \colon \mathcal{D} \to \mathsf{Coalg}(F)$ exists, if $U \cdot D \colon \mathcal{D} \to \mathcal{C}$ has a colimit, and moreover, there is a unique coalgebra structure on $\operatorname{colim}(UD)$ making it the colimit of D and making the colimit injections of $\operatorname{colim} D$ coalgebra morphisms.*

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "h"] & FC \arrow[d, "Fh"] \\ D \arrow[r, "d"] & FD \end{tikzcd}$$

(a) F-Coalgebras.

$$\begin{tikzcd} FA \arrow[r, "a"] \arrow[d, "Fh"] & A \arrow[d, "h"] \\ FB \arrow[r, "b"] & B \end{tikzcd}$$

(b) F-Algebras.

Figure 1 A homomorphism between . . . .

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \\ & \mathsf{Im}(f) \arrow[ur, tail, "m"'] & \end{tikzcd}$$

(a) Factorization.

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists! d"] \\ C \arrow[r, tail, "m"] & D \end{tikzcd}$$

(b) Diagonal fill-in.

Figure 2 Diagrams for Definition 2.10.

On the other hand, we do not necessarily have all limits in $\mathsf{Coalg}(F)$. If F preserves a limit of a diagram $D\colon \mathcal{D} \to \mathcal{C}$, then the limit also exists in $\mathsf{Coalg}(F)$.

Coalgebras model systems with a transition structure, and pointed coalgebras extend this by a notion of initial state:

▶ **Definition 2.7.** *For an object $I \in \mathcal{C}$, an I-*pointed *F-coalgebra (C, c, i_C) is an F-coalgebra (C, c) together with a morphism $i_C\colon I \to C$. A pointed coalgebra morphism $h\colon (C, c, i_C) \to (D, d, i_D)$ is a coalgebra morphism $h\colon (C, c) \to (D, d)$ that preserves the point: $i_D = h \cdot i_C$.*

The category of I-pointed F-coalgebras is denoted by $\mathsf{Coalg}_I(F)$.

▶ **Example 2.8.** For $I := 1$ in Set, a pointed coalgebra (C, c, i_C) for $FX = 2 \times X^A$ is a deterministic automaton, where the initial state is given by the map $i_C\colon 1 \to C$.

The point can also be understood as an algebraic flavour. In general, coalgebras are dual to F-algebras in the following sense.

▶ **Definition 2.9.** *An F-algebra (for a functor $F\colon \mathcal{C} \to \mathcal{C}$) is a morphism $a\colon FA \to A$, an algebra homomorphism $h\colon (A, a) \to (B, b)$ is a morphism $h\colon A \to B$ fulfilling $b \cdot Fh = h \cdot a$ (Figure 1b). The category of F-algebras is denoted by $\mathsf{Alg}(F)$.*

In other words, $\mathsf{Alg}(F) = \mathsf{Coalg}(F^{\mathsf{op}})^{\mathsf{op}}$ for $F^{\mathsf{op}}\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$. The I-pointed coalgebras thus are also algebras for the constant I functor. Most of the results of the present paper also apply to algebras for a functor $F\colon \mathcal{C} \to \mathcal{C}$.

2.2 Factorization Systems

The process of minimizing a system constructs a quotient or subobject of the state space, where the notions of quotient and subobject respectively stem from a factorization system in the category of interest. This generalizes the well-known image factorization of a map into a surjective and an injective map:

▶ **Definition 2.10** [2, Definition 14.1]**.** *Given classes of morphisms $\mathcal{E}$ and $\mathcal{M}$ in $\mathcal{C}$, we say that $\mathcal{C}$ has an $(\mathcal{E}, \mathcal{M})$-*factorization system *provided that:*

1. *$\mathcal{E}$ and $\mathcal{M}$ are closed under composition with isomorphisms.*
2. *Every morphism $f\colon A \to B$ in $\mathcal{C}$ has a factorization $f = m \cdot e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ (Figure 2a). We write $\mathsf{Im}(f)$ for the intermediate object, $\twoheadrightarrow$ for morphisms $e \in \mathcal{E}$, and $\rightarrowtail$ for morphisms $m \in \mathcal{M}$.*
3. *For each commutative square $g \cdot e = m \cdot f$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, there exists a unique diagonal fill-in d with $m \cdot d = g$ and $d \cdot e = f$ (Figure 2b).*

▶ **Example 2.11.** In Set, we have an $(\mathsf{Epi}, \mathsf{Mono})$-factorization system where Epi is the class of surjective maps, and Mono the class of injective maps. The image of a map $f\colon A \to B$ is given by

$$\mathsf{Im}(f) = \{b \in B \mid \text{there exists } a \in A \text{ with } f(a) = b\}.$$

canonically yielding maps $e\colon A \twoheadrightarrow \mathsf{Im}(f)$ and $m\colon \mathsf{Im}(f) \rightarrowtail B$. Note that one can also regard $\mathsf{Im}(f)$ as a set of equivalence classes of A:

$$\mathsf{Im}(f) \cong \big\{\{a' \in A \mid f(a') = f(a)\} \mid a \in A\big\}.$$

Intuitively, the diagonal fill-in property (Definition 2.10.3, also called *diagonal lifting*) provides a way of defining a map d on equivalence classes (given by the surjective map at the top) and with a restricted codomain (given by the injective map at the bottom).

▶ **Example 2.12.** In general, the elements of $\mathcal{E}$ are not necessarily epimorphisms and the elements of $\mathcal{M}$ are not necessarily monomorphisms. In particular, every category has an $(\mathcal{E}, \mathcal{M})$-factorization system with $\mathcal{E} := \mathsf{Iso}$ being the class of isomorphisms and $\mathcal{M} := \mathsf{Mor}$ being the class of all morphisms (and also vice-versa).

▶ **Definition 2.13.** *An $(\mathcal{E}, \mathcal{M})$-factorization system is called* proper *if $\mathcal{E} \subseteq \mathsf{Epi}$ and $\mathcal{M} \subseteq \mathsf{Mono}$.*

These two conditions of properness are independent. In fact, $\mathcal{M} \subseteq \mathsf{Mono}$ is equivalent to every split-epimorphism being in $\mathcal{E}$ [2, Prop. 14.11]. In the literature, it is often required that the factorization system is proper, and in fact a proper factorization system arises in complete or cocomplete categories:

▶ **Example 2.14.** Every complete category has a $(\mathsf{StrongEpi}, \mathsf{Mono})$-factorization system [2, Thm. 14.17 and 14C(d)] and also an $(\mathsf{Epi}, \mathsf{StrongMono})$-factorization system [2, Thm. 14.19, and 14C(f)]. By duality, every cocomplete category has so as well.

▶ Remark 2.15. $(\mathcal{E}, \mathcal{M})$-factorization systems have many properties known from surjective and injective maps on Set [2, Chp. 14]:

1. $\mathcal{E} \cap \mathcal{M}$ is the class of isomorphisms of $\mathcal{C}$.
2. If $f \cdot g \in \mathcal{M}$ and $f \in \mathcal{M}$, then $g \in \mathcal{M}$. If $\mathcal{M} \subseteq \mathsf{Mono}$, then $f \cdot g \in \mathcal{M}$ implies $g \in \mathcal{M}$.
3. $\mathcal{E}$ and $\mathcal{M}$ are respectively closed under composition.
4. $\mathcal{M}$ is stable under pullbacks, $\mathcal{E}$ is stable under pushouts.

The stability generalizes as follows to wide pullbacks and pushouts:

▶ **Lemma 2.16.** *$\mathcal{M}$ is stable under wide pullbacks: for a family $(f_i\colon A_i \to B)_{i\in I}$ and its wide pullback $(\mathsf{pr}_i\colon P \to A_i)_{i\in I}$, a projection $\mathsf{pr}_j\colon P \to A_j$ is in $\mathcal{M}$ if f_i is in $\mathcal{M}$ for all $i \in I \setminus \{j\}$.*

A factorization system also provides notions of subobjects and quotients, generalizing the notions of subset and quotient sets:

▶ **Definition 2.17.** *For a class $\mathcal{M}$ of morphisms, an $\mathcal{M}$-*subobject *of an object X is a pair (S, s) where $s\colon S \rightarrowtail X$ is in $\mathcal{M}$. Two $\mathcal{M}$-subobjects (s, S), (s', S') are called* isomorphic *if there is an isomorphism $\phi\colon S \to S'$ with $\phi \cdot s = s'$. We write $(s, S) \le (s', S')$ if there is a morphism $h\colon S \to S'$ with $s' \cdot h = s$. Dually, an $\mathcal{E}$-*quotient *of X is pair (Q, q) for a morphism $q\colon X \twoheadrightarrow Q$ ($q \in \mathcal{E}$). If $(\mathcal{E}, \mathcal{M})$ is fixed from the context, we simply speak of subobjects and quotients.*

For subobjects, it is often required that $\mathcal{M}$ is a class of monomorphisms [2, Def. 7.77], but many of the results in the present work hold without this assumption. If $\mathcal{M}$ is so, then the subobjects of a given object X form a preordered class. Moreover, the subobjects form a preordered set iff $\mathcal{C}$ is $\mathcal{M}$-wellpowered. This is in fact the definition: $\mathcal{C}$ is $\mathcal{M}$-wellpowered if for each $X \in \mathcal{C}$ there is (up to isomorphism) only a set of $\mathcal{M}$-subobjects. On Set, the isomorphism classes of (Mono-)subobjects of X correspond to subsets of X and the isomorphism classes of (Epi-)quotients of X correspond to partitions on X.

If $(\mathcal{E}, \mathcal{M})$ forms a factorization system, then its axioms provide us with methods to construct and work with subobjects and quotients, e.g. the image factorization means that for every morphism, we obtain a quotient on its domain and a subobject of its codomain. The minimization of coalgebras amounts to the construction of certain subobjects or quotients with respect to a suitable factorization system in the category of coalgebras $\mathsf{Coalg}(F)$.

3 Factorization System for Coalgebras

If we have an $(\mathcal{E}, \mathcal{M})$-factorization system on the base category $\mathcal{C}$ on which we consider coalgebras for $F: \mathcal{C} \to \mathcal{C}$, then it is natural to consider coalgebra morphisms whose underlying $\mathcal{C}$-morphism is in $\mathcal{E}$, resp. $\mathcal{M}$:

▶ **Definition 3.1.** *Given a class of $\mathcal{C}$-morphisms $\mathcal{E}$, we say that an F-coalgebra morphism $h: (C, c) \to (D, d)$ is* $\mathcal{E}$-carried *if $h: C \to D$ is in $\mathcal{E}$.*

This induces the standard notions of subcoalgebra and quotient coalgebras as instances of $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients in $\mathsf{Coalg}(F)$: an $\mathcal{M}$-subcoalgebra of (C, c) is an ($\mathcal{M}$-carried)-subobject of (C, c) (in $\mathsf{Coalg}(F)$), i.e. is represented by an $\mathcal{M}$-carried homomorphism $m: (S, s) \rightarrowtail (C, c)$. Likewise, a *quotient* of a coalgebra (C, c) is an ($\mathcal{E}$-carried)-quotient of (C, c) (in $\mathsf{Coalg}(F)$), i.e. is represented by a coalgebra morphism $e: (C, c) \twoheadrightarrow (Q, q)$ carried by an epimorphism.

Note that for the case where $\mathcal{M}$ is the class of monomorphisms, the monomorphisms in $\mathsf{Coalg}(F)$ coincide with the Mono-carried homomorphisms only under additional assumptions:

▶ **Lemma 3.2.** *If weak kernel pairs exist in $\mathcal{C}$ and are preserved by $F: \mathcal{C} \to \mathcal{C}$, then the monomorphisms in $\mathsf{Coalg}(F)$ are precisely the Mono-carried coalgebra homomorphisms.*

This is a commonly known criterion, and Gumm and Schröder [19, Example 3.5] present a functor not preserving kernel pairs and a monic coalgebra homomorphism that is not carried by a monomorphism.

For the construction of quotient coalgebras and subcoalgebras, it is handy to have the factorization system directly in $\mathsf{Coalg}(F)$. It is a standard result that the image factorization of homomorphisms lifts (see e.g. [27, Lemma 2.5]). Under assumptions on $\mathcal{E}$ and $\mathcal{M}$, Kurz shows that the factorization system lifts to Coalg(F) [25, Theorem 1.3.7] (and to other categories with a forgetful functor to the base category $\mathcal{C}$).

In fact, the factorization system always lifts to $\mathsf{Coalg}(F)$ under the condition that F preserves $\mathcal{M}$. By this condition, we mean that $m \in \mathcal{M}$ implies $Fm \in \mathcal{M}$.

▶ **Lemma 3.3.** *If $F: \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{M}$, then the $(\mathcal{E}, \mathcal{M})$-factorization system lifts from $\mathcal{C}$ to an ($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in $\mathsf{Coalg}(F)$. The factorization of an F-coalgebra homomorphism $h: (C, c) \to (D, d)$ is given by that of the underlying morphism $h: C \to D$.*

Proof. We verify Definition 2.10:

1. The $\mathcal{E}$- and $\mathcal{M}$-carried morphisms are closed under composition with isomorphisms, respectively.
2. Given an F-coalgebra morphism $f: (A, a) \to (B, b)$, consider its factorization $f = m \cdot e$ in $\mathcal{C}$. Since F preserves $\mathcal{M}$, we have $Fm \in \mathcal{M}$ and thus can apply the diagonal fill-in

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "a"] \arrow[rr, bend left, "f"] & \mathsf{Im}(f) \arrow[r, tail, "m"] \arrow[d, dashed, "\exists!d"] & B \arrow[d, "b"] \\
FA \arrow[r, "Fe"] \arrow[rr, bend right, "Ff"'] & F\mathsf{Im}(f) \arrow[r, tail, "Fm"] & FB
\end{tikzcd}$$

(a) Factorization.

$$\begin{tikzcd}
(A,a) \arrow[r, two heads, "e"] \arrow[d, "f"'] & (B,b) \arrow[d, "g"] \\
(C,c) \arrow[r, tail, "m"] & (D,d)
\end{tikzcd}$$

(b) Premise in $\mathsf{Coalg}(F)$.

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists!h"'] \\
C \arrow[r, tail, "m"] & D
\end{tikzcd}$$

(c) Fill-in in $\mathcal{C}$.

Figure 3 Diagrams for the proof of Lemma 3.3.

property (Definition 2.10.3) to the coalgebra morphism square of f (Figure 3a). This defines a unique coalgebra structure d on $\mathsf{Im}(f)$ making e and m coalgebra morphisms.

3. In order to check the diagonal-lifting property of the ($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system, consider a commutative square $g \cdot e = m \cdot f$ in $\mathsf{Coalg}(F)$ with $m \in \mathcal{M}$, $e \in \mathcal{E}$ as depicted in Figure 3b. In $\mathcal{C}$, there exists a unique $h\colon B \to C$ with $h \cdot e = f$ and $m \cdot h = g$ (Figure 3c). We only need to prove that $h\colon B \to C$ is a coalgebra homomorphism $(B,b) \to (C,c)$, i.e. that $c \cdot h = Fh \cdot b$. We prove this equality by showing that both $c \cdot h$ and $Fh \cdot b$ are diagonals in a commutative square of the form of Definition 2.10.3. Indeed, we have the commutative squares:

$$\begin{tikzcd}
A \arrow[rr, two heads, "e"] \arrow[d, "a"'] \arrow[dr, "f"] & & B \arrow[dl, "h"'] \arrow[d, "g"] \arrow[dr, "b"] & \\
FA \arrow[r, phantom, "\text{f hom.}"] \arrow[d, "Ff"'] & C \arrow[r, tail, "m"] \arrow[dl, "c"] \arrow[dr, phantom, "\text{m hom.}"] & D \arrow[r, phantom, "\text{g hom.}"] \arrow[d, "d"] & FB \arrow[dl, "Fg"] \\
FC \arrow[rr, tail, "Fm"] & & FD &
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
A \arrow[rr, two heads, "e"] \arrow[d, "a"'] \arrow[dr, phantom, "\text{e hom.}"] & & B \arrow[dl, "b"] \arrow[d, "b"] \\
FA \arrow[r, "Fe"] \arrow[d, "Ff"'] & FB \arrow[r, phantom, "\text{trivial}"] \arrow[dl, "Fh"] \arrow[dr, "Fg"] & FB \arrow[d, "Fg"] \\
FC \arrow[rr, tail, "Fm"] & & Fm
\end{tikzcd}$$

By the uniqueness of the diagonal in Definition 2.10.3, $c \cdot h = Fh \cdot b$. ◀

▶ Remark 3.4. The condition that F preserves $\mathcal{M}$ is commonly met. For Set and $\mathcal{M}$ being the class of injective maps, it can be assumed wlog for coalgebraic purposes that F preserves injective maps: every set functor preserves injective maps with non-empty domain and only needs to be modified on $\emptyset$ in order to preserve all injective maps [32]. The resulting functor has an isomorphic category of coalgebras.

We have the dual result for F-algebras:

▶ **Lemma 3.5.** *If $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{E}$, then the $(\mathcal{E}, \mathcal{M})$-factorization system lifts from $\mathcal{C}$ to $\mathsf{Alg}(F)$.*

Proof. We have an $(\mathcal{M}, \mathcal{E})$-factorization system in $\mathcal{C}^{\mathsf{op}}$. By Lemma 3.3, this factorization system lifts to $\mathsf{Coalg}(F^{\mathsf{op}})$ since $F^{\mathsf{op}}\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ preserves $\mathcal{E}$. Thus, we have an ($\mathcal{E}$-carried, $\mathcal{M}$-carried)-factorization system in $\mathsf{Alg}(F) = \mathsf{Coalg}(F^{\mathsf{op}})^{\mathsf{op}}$. ◀

This lifting result even holds for Eilenberg-Moore algebras for a monad $T\colon \mathcal{C} \to \mathcal{C}$. The Eilenberg-Moore category of a monad T is a full subcategory of $\mathsf{Alg}(T)$ containing those algebras that interact coherently with the structure of the monad T, see e.g. Awodey [5] for details:

▶ **Lemma 3.6.** *If a monad $T\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{E}$, then the $(\mathcal{E}, \mathcal{M})$-factorization system lifts from $\mathcal{C}$ to the Eilenberg-Moore category of T.*

The factorization system lifts further also to pointed coalgebras:

▶ **Lemma 3.7.** *If $F\colon \mathcal{C} \to \mathcal{C}$ preserves $\mathcal{M}$, then the $(\mathcal{E}, \mathcal{M})$-factorization system lifts from $\mathcal{C}$ to $\mathsf{Coalg}_I(F)$.*

Proof. A combination of Lemma 3.3 and 3.5, using that the constant functor preserves $\mathcal{E}$-morphisms. ◀

4 Minimality in a Category

Having seen multiple categories with an $(\mathcal{E}, \mathcal{M})$-factorization system, we can now define the minimality of objects abstractly.

▶ **Definition 4.1.** *Given a category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system, an object C of $\mathcal{K}$ is called* $\mathcal{M}$-minimal *if every morphism $h\colon D \rightarrowtail C$ in $\mathcal{M}$ is an isomorphism.*

▶ Remark 4.2. Every $(\mathcal{E}, \mathcal{M})$-factorization system on $\mathcal{K}$ is an $(\mathcal{M}, \mathcal{E})$-factorization system on $\mathcal{K}^{\mathsf{op}}$, and thus induces a dual notion of $\mathcal{E}$-minimality: an object C of $\mathcal{K}$ is called *$\mathcal{E}$-minimal* if every $h\colon C \twoheadrightarrow D$ in $\mathcal{E}$ is an isomorphism.

In the following, $\mathcal{K}$ will denote the category in which we consider the minimal objects, e.g. a category of coalgebras for a functor $F\colon \mathcal{C} \to \mathcal{C}$.

▶ **Assumption 4.3.** In the following, assume that the category $\mathcal{K}$ has an $(\mathcal{E}, \mathcal{M})$-factorization system. Whenever we consider a category of coalgebras for a functor $F\colon \mathcal{C} \to \mathcal{C}$ in the following, we achieve this by assuming that $\mathcal{C}$ has an $(\mathcal{E}, \mathcal{M})$-factorization system and that F preserves $\mathcal{M}$.

The leading examples of the minimality notion in the present work are the following two instances in coalgebras:

▶ **Instance 4.4.** For $\mathcal{K} := \mathsf{Coalg}_I(F)$, the ($\mathcal{M}$-carried-)minimal objects are the *reachable* coalgebras, as introduced by Adámek et al. [4]. Concretely, an I-pointed F-coalgebra (C, c, i_C) is reachable if it has no (proper) pointed subcoalgebra, equivalently, if every $\mathcal{M}$-carried coalgebra morphism $m\colon (S, s, i_S) \rightarrowtail (C, c, i_C)$ is necessarily an isomorphism [4].

In Set, this corresponds to the usual notion of reachability: if a state $x \in C$ is contained in a subcoalgebra $m\colon (S, s, i_S) \rightarrowtail (C, c, i_C)$, then all successors of x need to be contained in the subcoalgebra as well, since m is a coalgebra homomorphism. Moreover, the subcoalgebra has to contain the point $i_C\colon I \to C$, and thus also all its successors, and in total all states reachable from i_C in finitely many steps. Hence, (C, c, i_C) is reachable if it is not possible to omit any state in a pointed subcoalgebra (S, s, i_S), i.e. if any such injective m is a bijection.

▶ **Instance 4.5.** For $\mathcal{K} := \mathsf{Coalg}(F)^{\mathsf{op}}$, the ($\mathcal{E}$-carried-)minimal objects are called *simple* coalgebras, as mentioned by Gumm [22]. Usually, a simple coalgebra is defined as a coalgebra that does not have any proper quotient [36].[1]

[1] Gumm [22, p. 34] defines a simple coalgebra as the quotient of a coalgebra on Set modulo behavioural equivalence.

X is $\mathcal{M}$-minimal	X is $\mathcal{E}$-minimal
$\Leftrightarrow$ every $Y \rightarrowtail X$ is an isomorphism	$\Leftrightarrow$ every $X \twoheadrightarrow Y$ is an isomorphism
$\Leftrightarrow$ every $Y \to X$ is in $\mathcal{E}$	$\Leftrightarrow$ every $X \to Y$ is in $\mathcal{M}$
if $\mathcal{E} = \mathsf{Epi}$ and $\mathcal{K}$ has weak equalizers: $\Leftrightarrow$ all parallel $X \rightrightarrows Y$ equal	if $\mathcal{M} = \mathsf{Mono}$ and $\mathcal{K}$ has weak coequalizers: $\Leftrightarrow$ all parallel $Y \rightrightarrows X$ equal ('X subterminal')

Figure 4 Equivalent characterizations of $\mathcal{M}$-minimality and $\mathcal{E}$-minimality in a category $\mathcal{K}$.

In Set, a coalgebra is simple iff all states have different behaviour – this characterization follows directly the following equivalent characterization of minimal objects as we will see in Instance 4.7:

▶ **Lemma 4.6.** *An object C in $\mathcal{K}$ is $\mathcal{M}$-minimal iff every $h\colon D \to C$ is in $\mathcal{E}$.*

Proof. In the 'if' direction, consider some $\mathcal{M}$-morphism $h\colon D \to C$. By the assumption, h is also in $\mathcal{E}$ and thus an isomorphism. In the 'only if' direction, take some morphism $h\colon D \to C$ and consider its $(\mathcal{E}, \mathcal{M})$-factorization $e\colon D \twoheadrightarrow \mathsf{Im}(h)$ and $m\colon \mathsf{Im}(h) \rightarrowtail C$ with $h = m \cdot e$. Since C is $\mathcal{M}$-minimal, m is an isomorphism and thus $h = m \cdot e$ is in $\mathcal{E}$. ◀

▶ **Instance 4.7.** For $\mathcal{K} := \mathsf{Coalg}(F)^{\mathsf{op}}$, an F-coalgebra (C, c) is simple iff every F-coalgebra homomorphism $h\colon (C, c) \to (D, d)$ is $\mathcal{M}$-carried.

In Set, this equivalence shows that the simple coalgebras are precisely those coalgebras for which behavioural equivalence coincides with behavioural equivalence:

- If states $x, y \in C$ are behaviourally equivalent, then there is some $h\colon (C, c) \to (D, d)$ with $h(x) = h(y)$. By Instance 4.7, h must be injective and thus $x = y$.
- Conversely, if all states in (C, c) have different behaviour, then every $h\colon (C, c) \to (D, d)$ is necessarily injective by Definition 2.4. Thus, by Instance 4.7, (C, c) is simple.

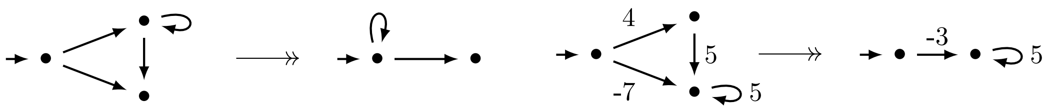
Gumm already noted that in Set, every outgoing coalgebra morphism from a simple coalgebra is injective [22, Hilfssatz 3.6.3] – but it was not used to characterize simplicity. If $\mathcal{E}$, resp. $\mathcal{M}$, happens to be the class of epimorphisms, resp. monomorphisms, another characterization of minimality exists:

▶ **Lemma 4.8.** *Assume $\mathcal{E} = \mathsf{Epi}$ and weak equalizers in $\mathcal{K}$, then X is $\mathcal{M}$-minimal iff there is at most one morphism $u\colon C \to D$ for every $D \in \mathcal{K}$.*

Dually, given $\mathcal{M} = \mathsf{Mono}$ and weak coequalizers, C is $\mathcal{E}$-minimal iff C is subterminal*, that is, iff there is a most one $u\colon D \to C$ for every $D \in \mathcal{K}$.*

The name *subterminal* stems from the fact that if $\mathcal{K}$ has a terminal object, its subobjects are the subterminal objects.

▶ **Instance 4.9.** For $\mathcal{K} := \mathsf{Coalg}(F)^{\mathsf{op}}$, assume $\mathcal{M} = \mathsf{Mono}$ and that F preserves weak kernel pairs and that the base category $\mathcal{C}$ has coequalizers. Hence, the monomorphisms in $\mathsf{Coalg}(F)$ are precisely the Mono-carried homomorphisms (Lemma 3.2) and the assumption of Lemma 4.8 is met. Consequently, the simple coalgebras are precisely the *subterminal* coalgebras. If the final coalgebra exists, then its subcoalgebras are precisely the simple coalgebras. For a non-example, Gumm and Schröder [19, Example 3.5] provide a functor not preserving weak kernel pairs and a subterminal coalgebra that is not simple.



(a) For $FX = \mathcal{P}X$ **(b)** For $FX = (\mathbb{R}, +, 0)^{(X)}$

Figure 5 Examples of simple quotients in F-coalgebras.

We have now established a series of equivalent characterizations of minimality (Figure 4) and will now discuss how to construct minimal objects. This process of minimization – i.e. of constructing the reachable part or the simple quotient of a coalgebra – is abstracted as follows:

▶ **Definition 4.10.** *An* $\mathcal{M}$-minimization *of* $C \in \mathcal{K}$ *is a morphism* $m\colon D \rightarrowtail C$ *in* $\mathcal{M}$ *where* D *is* $\mathcal{M}$*-minimal.*

In fact, we will show that an $\mathcal{M}$-minimization is unique, so we can speak of *the* $\mathcal{M}$-minimization.

▶ **Instance 4.11.** The task of finding an $\mathcal{M}$-minimization of a given $C \in \mathcal{K}$ instantiates to the standard minimization tasks on coalgebras:

- For $\mathcal{K} := \mathsf{Coalg}_I(F)$, an $\mathcal{M}$-minimization of a given pointed coalgebra (C, c, i_C) is called its *reachable subcoalgebra* [4]. This is a subcoalgebra obtained by removing all unreachable states. The explicit definition is: the reachable subcoalgebra of (C, c, i_C) is a (pointed) subcoalgebra $h\colon (R, r, i_R) \rightarrowtail (C, c, i_C)$ where (R, r, i_R) itself has no proper (pointed) subcoalgebras.
- For $\mathcal{K} := \mathsf{Coalg}(F)^{\mathsf{op}}$, an $\mathcal{E}$-minimization of a given coalgebra (C, c) is called the *simple quotient* of (C, c) [22]. The explicit definition is: the simple quotient of (C, c) is a quotient $h\colon (C, c) \twoheadrightarrow (Q, q)$ where (Q, q) itself has no proper quotient coalgebra.
  In Set, this is a quotient in which all behaviourally equivalent states are identified, in other words, the simple quotient of (C, c) is the unique coalgebra structure on $C/{\sim}$. Examples of simple quotients can be found in Figure 5. Since all states in the codomain of the surjective homomorphisms are behaviourally inequivalent, the respective codomains are simple.

▶ **Example 4.12.** For the trivial factorization systems (Example 2.12), we have:

- For the $(\mathsf{Iso}, \mathsf{Mor})$-factorization system, the Iso-minimization of an object X is X itself.
- For the $(\mathsf{Mor}, \mathsf{Iso})$-factorization system on category, if a strict initial object 0 exist, then it is the Mor-minimization of every $X \in \mathcal{C}$. Recall that an initial object 0 is called *strict* if every morphism with codomain 0 is an isomorphism.

It is well-defined to speak of *the* $\mathcal{M}$-minimization of an object C, because it is unique:

▶ **Lemma 4.13.** *Consider* $h\colon M \to C$ *with* $\mathcal{M}$*-minimal* M *and an* $\mathcal{M}$*-subobject* $s\colon S \rightarrowtail C$*. The pullback of* s *along* h *exists iff* h *factors uniquely through* s*, that is, iff there is a unique* $u\colon M \to S$ *with* $s \cdot u = h$*.*

$$\begin{tikzcd} & C & \\ M \arrow[ur, "h"] \arrow[rr, dashed, "\exists! u"'] & & S \arrow[ul, tail, "\forall s \in \mathcal{M}"'] \end{tikzcd} \quad (\text{in } \mathcal{K}) \qquad \begin{tikzcd} & C \arrow[dl, "h"'] \arrow[dr, two heads, "\forall q \in \mathcal{E}"] & \\ M & & Q \arrow[ll, dashed, "\exists! u"'] \end{tikzcd} \quad (\text{in } \mathcal{K}^{\mathsf{op}})$$

Proof. In the "if"-direction, let $d\colon M \to S$ be the unique morphism with $s \cdot d = h$. The pullback is simply given by M itself with projections $\mathsf{id}_M\colon M \to M$ and $d\colon M \to S$. To verify its universal property, consider $e\colon E \to M$, $f\colon E \to S$ with $h \cdot e = s \cdot f$ (Figure 6a). Since M is $\mathcal{M}$-minimal, $e\colon E \to M$ is in $\mathcal{E}$ (Lemma 4.6). Thus, we can apply the diagonal lifting property to $h \cdot e = s \cdot f$ yielding a diagonal u with $s \cdot u = h$ and $u \cdot e = f$. Thus, $d = u$ and $d \cdot e = f$, showing that $e\colon (E, e, f) \to (M, \mathsf{id}_M, d)$ is the mediating cone morphism. Its uniqueness is clear because id_M is isomorphic.

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[dr, dashed, "u"] & C \\ E \arrow[u, two heads, "e"] \arrow[r, "f"] & S \arrow[u, tail, "s"'] \end{tikzcd}$$

(a) "If"-direction.

$$\begin{tikzcd} P \arrow[r, tail, "\phi"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "h"] \\ S \arrow[r, tail, "s"] & C \end{tikzcd}$$

(b) Premise of "only if".

$$\begin{tikzcd} M \arrow[r, dashed, "u"] \arrow[rr, bend left, "\mathsf{id}_M"] \arrow[dr, "v"'] & P \arrow[r, tail, "\phi"] \arrow[d, "d"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "h"] \\ & S \arrow[r, tail, "s"] & C \end{tikzcd}$$

(c) Uniqueness for "only if".

Figure 6 Diagrams for the proof of Lemma 4.13.

In the 'only if'-direction, consider the pullback (P, ϕ, d) (Figure 6b). Since $\mathcal{M}$-morphisms are stable under pullback (Remark 2.15.4), ϕ is in $\mathcal{M}$, too. By the minimality of M, the $\mathcal{M}$-morphism ϕ is an isomorphism and we have $d \cdot \phi^{-1}\colon M \to S$.

In order to see that $d \cdot \phi^{-1}$ is indeed the unique morphism $M \to S$ making the triangle commute, consider an arbitrary $v\colon M \to S$ with $s \cdot v = h = h \cdot \mathsf{id}_M$. Thus, M is a competing cone for the pullback P and thus induces a morphism $u\colon M \to P$ with $d \cdot u = v$ and $\phi \cdot u = \mathsf{id}_M$ (Figure 6c). Since ϕ is an isomorphism, we have $u = \phi^{-1}$ and thus $v = d \cdot \phi^{-1}$ as desired. ◀

▶ **Corollary 4.14.** *If $\mathcal{K}$ has pullbacks of $\mathcal{M}$-morphisms along $\mathcal{M}$-morphisms and if there is an $\mathcal{M}$-minimization M of C, then M is the least $\mathcal{M}$-subobject of C and it is unique (up to unique isomorphism).*

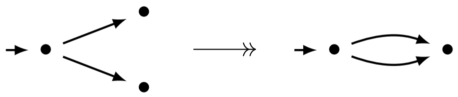
Proof. Consider Lemma 4.13 first for $h \in \mathcal{M}$ and then also with S being $\mathcal{M}$-minimal. ◀

▶ **Instance 4.15.** Not only the result but also the proof instantiates to the uniqueness results in the instances of reachable subcoalgebras and simple quotients:

1. If $\mathcal{C}$ has pullbacks of $\mathcal{M}$-morphisms (i.e. finite intersections) and $F\colon \mathcal{C} \to \mathcal{C}$ preserves them, then $\mathsf{Coalg}_I(F)$ has pullbacks of $\mathcal{M}$-carried homomorphisms. Given a reachable subcoalgebra (D, d, i_D) of (C, c, i_C), then it is the least I-pointed subcoalgebra of (C, c, i_C) (cf. [4, Notation 3.18]) and is unique up to isomorphism.
2. If $\mathcal{C}$ has pushouts of $\mathcal{E}$-morphisms, then $\mathsf{Coalg}(F)$ has pushouts of $\mathcal{E}$-carried homomorphisms. Hence, the simple quotient of a coalgebra (C, c) is the greatest quotient of (C, c) and unique up to isomorphism (e.g. [36, Lemma 2.9]).

There are instances where a minimization M exists, but where a mediating morphism in the sense of Lemma 4.13 is not unique:

▶ **Example 4.16** (Tree unravelling)**.** Let $\mathsf{Coalg}_I(F)_{\mathsf{reach}}$ be the category of reachable pointed F-coalgebras, i.e. the full subcategory $\mathsf{Coalg}_I(F)_{\mathsf{reach}} \subseteq \mathsf{Coalg}_I(F)$ such that $(C, c, i_C) \in \mathsf{Coalg}_I(F)_{\mathsf{reach}}$ iff it is reachable. For simplicity, restrict to $F\colon \mathsf{Set} \to \mathsf{Set}$ with the $(\mathsf{Epi}, \mathsf{Mono})$-factorization system. Thus, all morphisms in $\mathsf{Coalg}_I(F)_{\mathsf{reach}}$ are surjective (Lemma 4.6). Considering the (trivial) $(\mathsf{Mor}, \mathsf{Iso})$-factorization system on $\mathsf{Coalg}_I(F)_{\mathsf{reach}}$, a coalgebra (C, c, i_C) is (Mor-)minimal iff every coalgebra morphism $h\colon (D, d, i_D) \to (C, c, i_C)$ (with (D, d, i_D) also reachable) is an isomorphism. If (C, c, i_C) is Mor-minimal, then it is a tree: to see this, take h to be its tree unravelling (see e.g. Figure 7), and by the Mor-minimality, h is an isomorphism, so (C, c, i_C) is already a tree.



(a) Unravelling of siblings

(b) Unravelling of a loop

Figure 7 Tree unravelling for $FX = \mathcal{B}X$.

This implies that if the (Mor-)minimization of a coalgebra exists, then it is its tree unravelling. For example, for $I = 1$ and the bag functor $FX = \mathcal{B}X$, we have the minimizations as illustrated in Figure 7. It is easy to see that for $FX = \mathcal{P}X$ however, no coalgebra (with at least one transition) has a Mor-minimization, because one can always duplicate successor states.[2]

For $FX = \mathcal{B}X$, all Mor-minimizations exist, but they are not unique up to unique isomorphism. Consider the tree unravelling $m\colon M \twoheadrightarrow X$ in Figure 7a. There is an isomorphism $\phi\colon M \to M$ that swaps the two successors of the initial state. Hence, $\phi \neq \mathsf{id}_B$, but $m \cdot \phi = m \cdot \mathsf{id}_M$, so M is unique up to isomorphism, but not unique up to unique isomorphism.

For proving the existence of an $\mathcal{M}$-minimization, we need to require that $\mathcal{M}$ is a subclass of the monomorphisms in $\mathcal{K}$. Under this assumption, we first establish the converse of Corollary 4.14:

▶ **Lemma 4.17.** *If $\mathcal{M} \subseteq \mathsf{Mono}$ and if the least $\mathcal{M}$-subobject M of X exists, then M is the $\mathcal{M}$-minimization of X.*

Proof. Let $m\colon M \rightarrowtail X$ be the least $\mathcal{M}$-subobject of X, and consider $s\colon S \rightarrowtail M$ in $\mathcal{M}$. Since $m \cdot s \in \mathcal{M}$, there is some $u\colon M \to S$ with $(m \cdot s) \cdot u = m$. Since m is monic, we obtain $s \cdot u = \mathsf{id}_M$. Hence, s is a split-epimorphism, and together with $s \in \mathcal{M} \subseteq \mathsf{Mono}$, s is an isomorphism. ◀

▶ **Proposition 4.18.** *If $\mathcal{M} \subseteq \mathsf{Mono}$, $\mathcal{K}$ has wide pullbacks of $\mathcal{M}$-morphisms, and is $\mathcal{M}$-wellpowered, then every object C of $\mathcal{K}$ has an $\mathcal{M}$-minimization.*

Proof. Since $\mathcal{K}$ is $\mathcal{M}$-wellpowered, all the $\mathcal{M}$-carried morphisms $m\colon M \rightarrowtail C$ form up to isomorphism a set S. The wide pullback of all $m \in S$ exists in $\mathcal{K}$ by assumption, denote it by $\mathsf{pr}_m\colon P \to M$ for $m\colon M \rightarrowtail C$. All $m' \in S$ are in $\mathcal{M}$ and so are all pr_m by Lemma 2.16. Hence, $p := m \cdot \mathsf{pr}_m\colon P \rightarrowtail C$ for an arbitrary $m \in S$ represents an $\mathcal{M}$-subobject, and moreover the least $\mathcal{M}$-subobject of C, as witnessed by the projections pr_m. By Lemma 4.17, P is the minimization of C. ◀

▶ **Instance 4.19.** This proof directly instantiates to the proofs of the existence of the reachable subcoalgebra and simple quotient:

1. In the reachability case, let $\mathcal{M}$ be a subclass of the monomorphisms, let the base category $\mathcal{C}$ have all $\mathcal{M}$-intersections, and let $F\colon \mathcal{C} \to \mathcal{C}$ preserve all intersections. Then the reachable part of a given pointed coalgebra (C, c, i_C) is obtained as the intersection of all pointed subcoalgebras of (C, c, i_C) [4].
   For $\mathcal{C} = \mathsf{Set}$, and $\mathcal{M}$ being the class of injective maps, all intersections exist. The condition that $F\colon \mathsf{Set} \to \mathsf{Set}$ preserves all intersections is mild: all finitary functors preserve all

[2] It can be conjectured that Mor-minimization of reachable F-coalgebras exist if F admits precise factorizations [37, Def. 3.1, 3.4].

intersections ([3, Proof of Lem. 8.8] or [38, Lem. 2.6.10]) and many non-finitary functors do as well, e.g. the powerset functor. An example of a functor that does not preserve all intersections is the filter functor [16, Sect. 5.3].

2. For the existence of simple quotients, let $\mathcal{E}$ be a subclass of the epimorphisms and let the base category $\mathcal{C}$ be cocomplete and $\mathcal{E}$-cowellpowered. Then every F-coalgebra (C,c) has a simple quotient given by the wide pushout of all quotient coalgebras ([4, Proposition 3.7], and [17] for $\mathcal{C} = \mathsf{Set}$).
   Every set has only a set of outgoing surjective maps, so all assumptions are met for $\mathcal{C} = \mathsf{Set}$, $\mathcal{E}$ the surjective maps, and every Set-functor F.

▶ Remark 4.20. All observations on simple quotients also apply to pointed coalgebras: An I-pointed F-coalgebra is simple iff it is $\mathcal{E}$-carried-minimal in $\mathcal{K} := \mathsf{Coalg}_I(F)^{\mathsf{op}}$. The forgetful functor

$$\mathsf{Coalg}_I(F) \longrightarrow \mathsf{Coalg}(F)$$

preserves and reflects simple coalgebras and simple quotients (note that for every pointed coalgebra (C, c, i_C), the slice categories $(C, c, i_C)/\mathsf{Coalg}_I(F)$ and $(C, c)/\mathsf{Coalg}(F)$ are isomorphic). For the sake of simplicity, we will not state the results explicitly for simple coalgebras in $\mathsf{Coalg}_I(F)$.

▶ **Definition 4.21.** *We denote by $J\colon \mathcal{K}_{\text{min}} \hookrightarrow \mathcal{K}$ the full subcategory formed by the $\mathcal{M}$-minimal objects of $\mathcal{K}$.*

In the existence proof of minimal objects (Proposition 4.18) we only required (wide) pullbacks where all morphisms in the diagram are in $\mathcal{M}$. We obtain additional properties if we assume the pullback along $\mathcal{M}$-morphisms, i.e. pullbacks where only one of the two morphisms is in $\mathcal{M}$:

▶ **Proposition 4.22.** *Suppose that pullbacks along $\mathcal{M}$-morphisms exist in $\mathcal{K}$ and that every object of $\mathcal{K}$ has an $\mathcal{M}$-minimization. Then $J\colon \mathcal{K}_{\text{min}} \hookrightarrow \mathcal{K}$ is a coreflective subcategory. Its right-adjoint $R\colon \mathcal{K} \to \mathcal{K}_{\text{min}}$ ($J \dashv R$) sends an object to its $\mathcal{M}$-minimization; in particular, minimization is functorial.*

Proof. The universal property of J follows directly from Lemma 4.13: To this end, it suffices to consider R as an object assignment. Given a morphism $h\colon M \to X$ where M is $\mathcal{M}$-minimal, we need to show that it factorizes uniquely through the $\mathcal{M}$-minimization $s\colon S \rightarrowtail X$ of D, that is $RD := S$. Since the pullback of h along s exists by assumption, Lemma 4.13 yields us the desired unique factorization $u\colon M \to S$ with $s \cdot u = h$. ◀

▶ **Instance 4.23.** For both of our main instances, this adjunction has been observed before:

1. If $F\colon \mathcal{C} \to \mathcal{C}$ preserves inverse images (w.r.t. $\mathcal{M}$), then pullbacks along $\mathcal{M}$-carried homomorphisms exist in $\mathsf{Coalg}_I(F)$. Hence, the reachable I-pointed F-coalgebras form a coreflective subcategory of $\mathsf{Coalg}_I(F)$, where the coreflector maps a pointed coalgebra to its reachable part [39, Thm 5.23]
2. The simple coalgebras form a reflective subcategory of $\mathsf{Coalg}(F)$, and the reflector sends a coalgebra to its simple quotient, under the assumption that the base category has pushouts along $\mathcal{E}$-morphisms. For coalgebras in Set, the adjunction $J \vdash R$ has been shown by Gumm [17, Theorem 2.3].

▶ **Corollary 4.24.** *If pullbacks along $\mathcal{M}$-morphisms exist in $\mathcal{K}$ and all $\mathcal{M}$-minimizations exist, then $\mathcal{M}$-minimal objects are closed under $\mathcal{E}$-quotients.*

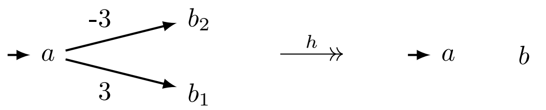
Proof. Consider an $\mathcal{E}$-morphism $e\colon C \twoheadrightarrow D$ where C is $\mathcal{M}$-minimal. Take the adjoint transpose $f\colon C \to RD$ with $m \cdot f = e$ where $m\colon RD \rightarrowtail D$ is the $\mathcal{M}$-minimization of D:

$$\begin{tikzcd} C \arrow[d, "f"'] \arrow[dr, two heads, "e"] & \\ RD \arrow[r, tail, "m"] & D \end{tikzcd}$$

Since RD is $\mathcal{M}$-minimal, f is in $\mathcal{E}$ (Lemma 4.6). Moreover, $f \in \mathcal{E}$ and $m \cdot f \in \mathcal{E}$ imply $m \in \mathcal{E}$ (Remark 2.15.2), hence $m \in \mathcal{E} \cap \mathcal{M}$ is an isomorphism. ◀

▶ **Example 4.25.**

1. If F preserves inverse images, then reachable F-coalgebras are closed under quotients [39, Cor. 5.24]. Note that if F does not preserve inverse images, then a quotient of a reachable F-coalgebra may not be reachable. For example, in (pointed) coalgebras for the monoid-valued functor $(\mathbb{R}, +, 0)^{(-)}$ there is the coalgebra quotient with $h(b_1) = h(b_2) = b$:



   Since transition weights may cancel out each other ($-3 + 3 = 0$), the codomain of h is not reachable even though its domain is.
2. If the base category $\mathcal{C}$ has pushouts along $\mathcal{E}$-morphisms, then simple F-coalgebras are closed under subcoalgebras. For $\mathcal{C} = \mathsf{Set}$, this is obvious: if in a coalgebra (C, c), all states are of pairwise different behaviour, then so they are in every subcoalgebra of (C, c).

5 Interplay of minimality notions

The two main aspects of minimization we have seen – reachability and minimization for observability – are closely connected on an abstract level and also interact well as we see in the following. In order to minimize a pointed coalgebra under both aspects, we have two options: first construct the reachable part and then the simple quotient, or we first form the simple quotient and then construct its reachable part. Given the existence of pullbacks of $\mathcal{M}$-morphisms along arbitrary morphisms, we can show that any order is fine.

In the abstract setting of a category $\mathcal{K}$ with an $(\mathcal{E}, \mathcal{M})$-factorization system we are transforming an object $C \in \mathcal{K}$ into an object C' that is $\mathcal{M}$-minimal in $\mathcal{K}$ and $\mathcal{E}$-minimal in $\mathcal{K}^{\mathsf{op}}$.

▶ **Proposition 5.1.** *Suppose $\mathcal{K}$ has an $(\mathcal{E}, \mathcal{M})$-factorization system such that all $\mathcal{M}$-minimizations in $\mathcal{K}$ and all $\mathcal{E}$-minimizations in $\mathcal{K}^{\mathsf{op}}$ exist. If $\mathcal{K}$ has pullbacks along $\mathcal{M}$-morphisms and pushouts along $\mathcal{E}$-morphisms, then for every C in $\mathcal{K}$ the following two constructions yield the same object:*

1. *The $\mathcal{M}$-minimization of C in $\mathcal{K}$ followed by its $\mathcal{E}$-minimization in $\mathcal{K}^{\mathsf{op}}$.*
2. *The $\mathcal{E}$-minimization of C in $\mathcal{K}^{\mathsf{op}}$ followed by its $\mathcal{M}$-minimization in $\mathcal{K}$.*

Proof. In the first approach, denote the $\mathcal{M}$-minimization of C by $m\colon R \rightarrowtail C$ and its $\mathcal{E}$-minimization by $s\colon R \twoheadrightarrow V$. In the other approach, denote the $\mathcal{E}$-minimization of C by $e\colon C \twoheadrightarrow Q$ and its $\mathcal{M}$-minimization by $t\colon W \rightarrowtail Q$:

$$\begin{tikzcd} R \arrow[rr, tail, "m"] \arrow[d, two heads, "s"'] & & C \arrow[d, two heads, "e"] \\ V & W \arrow[r, tail, "t"] & Q \end{tikzcd}$$

We need to prove that V and W are isomorphic, making the above (then-closed) square commute. The $\mathcal{M}$-minimal objects form a coreflective subcategory (Proposition 4.22), so $e \cdot m$, whose domain is $\mathcal{M}$-minimal, factorizes through the $\mathcal{M}$-minimization of the codomain of $e \cdot m$, i.e. we have $h\colon R \to W$ with $t \cdot h = e \cdot m$. Since Q is $\mathcal{E}$-minimal, its $\mathcal{M}$-subobject W is also $\mathcal{E}$-minimal in $\mathcal{K}^{\mathsf{op}}$ (Corollary 4.24). The $\mathcal{E}$-minimal objects form a reflective subcategory (Proposition 4.22). Applying the reflection to $h\colon R \to W$, we obtain $\phi\colon V \to W$ with $h = \phi \cdot s$. Since V is $\mathcal{E}$-minimal (in $\mathcal{K}^{\mathsf{op}}$), ϕ is in $\mathcal{M}$, and since W is $\mathcal{M}$-minimal, ϕ is in $\mathcal{E}$, and thus ϕ is an isomorphism. ◀

In the concrete case of F-coalgebras, a coalgebra that is both simple and reachable is called a *well-pointed* coalgebra (see [4, Section 3.2]).

▶ **Instance 5.2.** If $F\colon \mathcal{C} \to \mathcal{C}$ fulfils all assumptions from the previous Instance 4.23 (and in particular preserves inverse images), then the construction of the simple quotient and the reachability construction for F-coalgebras can be performed in any order, yielding the same well-pointed coalgebra.

If F does not preserve inverse images, then in the construction of the simple quotient, transitions may cancel out each other and this may affect the reachability of states. We have seen an example for this in Example 4.25.1 where performing reachability first and observability second leads to a simple coalgebra in which states are unreachable, i.e. the result is not well-pointed. Hence, in contrast to the well-known automata minimization procedure, the minimization of a coalgebra in general has to be performed by first computing its simple quotient and secondly computing the reachable part in the simple quotient.

If F preserves inverse images, such as the functor for automata, any order is fine. In sets, the reachability computation is a simple breadth-first search [39], and hence runs in linear time. On the other hand, existing algorithms for computing the simple quotient for many Set-functors run in at least $n \cdot \log n$ time where n is the size of the coalgebra [15, 36]. Hence, the reachability analysis should be done first whenever possible.

6 Conclusions

We have seen a common ground for minimality notions in a category with various instances in a coalgebraic setting. The abstract results about the uniqueness and the existence of the minimization instantiate to the standard results for reachability and observability of coalgebras. Most of the general results even hold if the $(\mathcal{E}, \mathcal{M})$-factorization system is not proper. The tree unravelling of an automaton is an instance of minimization for a non-proper factorization system.

It remains for future work to relate the efficient algorithmic approaches to the minimization tasks: reachability is computed by breadth-first search [39, 7] and observability is computed by partition refinement algorithms [24, 36, 13]. Even though their run-time complexity differs – reachability is usually linear, whereas partition refinement algorithms are quasilinear or slower – they have striking similarities. All these algorithms compute a chain of subobjects resp. quotients on the carrier of the input coalgebra and terminate at the first element of the chain admitting a coalgebra structure compatible with the input coalgebra. It is thus likely that this relation can be made formal. A similar connection between the reachability of algebras and partition refinement on coalgebras is already known [30].

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Appendix: Omitted Proofs

Proof of Corollary 2.6

Let $c_i \colon UDi \to FUDi$ be the coalgebra structure of $Di \in \mathsf{Coalg}(F)$ for every $i \in \mathcal{D}$. For the colimit of $UD \colon \mathcal{D} \to \mathcal{C}$

$$UDi \xrightarrow{\mathsf{inj}_i} \operatorname{colim}(UD) \qquad \text{for every } i \in \mathcal{D}$$

apply F and precompose with c_i, yielding

$$UDi \xrightarrow{c_i} FUDi \xrightarrow{F\mathsf{inj}_i} F\operatorname{colim}(UD) \qquad \text{for every } i \in \mathcal{D}.$$

This is a cocone for the diagram D because for all $h \colon i \to j$ in $\mathcal{D}$ the outside of the following diagram commutes:

$$\begin{tikzcd}
UDi \arrow[r, "c_i"] \arrow[d, "UDh"'] \arrow[dr, phantom, "\begin{array}{c}Dh\text{ coalgebra}\\\text{morphism}\end{array}"] & FUDi \arrow[d, "FUDh"] \arrow[dr, "F\mathsf{inj}_i"] & \\
UDj \arrow[r, "c_j"'] & FUDj \arrow[r, "F\mathsf{inj}_j"'] & F\operatorname{colim}(UD)
\end{tikzcd}$$

Thus we obtain a coalgebra structure $u \colon \operatorname{colim}(UD) \to F\operatorname{colim}(UD)$. Since u is a cocone-morphism, every inj_i is an F-coalgebra morphism.

In order to show that $(\mathrm{colim}(UD), u)$ is the colimit of $D\colon \mathcal{D} \to \mathsf{Coalg}(F)$, consider another cocone $(m_i\colon Di \to (E, e))_{i\in\mathcal{D}}$.

$$\begin{tikzcd}
\mathrm{colim}(UD) \arrow[r, dashed, "w"] \arrow[d, "u"'] & E \arrow[d, "e"] \\
F\,\mathrm{colim}(UD) \arrow[r, "Fw"] & FE
\end{tikzcd}$$

In $\mathcal{C}$, we obtain a cocone morphism $w\colon\ \mathrm{colim}(UD) \to E$. With a similar verification as before, $(e \cdot m_i\colon UDi \to FE)_{i\in\mathcal{D}}$ is a cocone for D, and thus both $d \cdot w$ and $Fw \cdot u\colon\ \mathrm{colim}(UD) \to FE$ are cocone morphisms (for UD). Since $\mathrm{colim}(UD)$ is the colimit, this implies that $d{\cdot}w = Fw{\cdot}u$, i.e. $w\colon (colim(UD), u) \to (E, e)$ is a coalgebra morphism. Since $U\colon \mathsf{Coalg}(F) \to \mathcal{C}$ is faithful, w is the unique cocone morphism, and so $(\mathrm{colim}\, UD, u)$ is indeed the colimit of UD. ◀

Proof of Lemma 2.16

Consider the $(\mathcal{E}, \mathcal{M})$-factorization of pr_j into $e\colon P \twoheadrightarrow C$ and $m\colon C \rightarrowtail A_j$ with $\mathsf{pr}_j = m \cdot e$. On the image, we define a cone structure $(c_i\colon C \to A_i)_{i\in I}$ by $c_j = m$ and for every $i \in I \setminus \{j\}$ by the diagonal fill-in:

$$\begin{tikzcd}
P \arrow[r, two heads, "e"] \arrow[d, "\mathsf{pr}_i"'] & C \arrow[r, tail, "c_j"] \arrow[dl, dashed, "c_i"] & A_j \arrow[d, "f_j"] \\
A_i \arrow[rr, tail, "f_i"] & & B
\end{tikzcd} \qquad \text{for all } i \in I \setminus \{j\}.$$

The diagonal c_i is induced, because $f_i \in \mathcal{M}$ for all $i \in I \setminus \{j\}$. The family $(c_i)_{i\in I}$ forms a cone for the wide pullback, because for all $i, i' \in I$ we have $f_i \cdot c_i = f_j \cdot c_j = f_{i'} \cdot c_{i'}$. This makes e a cone morphism, because $c_i \cdot e = \mathsf{pr}_i$ for all $i \in I$. Moreover, the limiting cone P induces a cone morphism $s\colon C \to P$ and we have $s \cdot e = \mathsf{id}_P$. Consider the commutative diagrams:

$$\begin{tikzcd}
P \arrow[rr, two heads, "e"] \arrow[dd, "e"'] \arrow[dr, "\mathsf{id}_P"] & & C \arrow[dd, "c_j"] \arrow[dl, "s"'] \\
& P \arrow[dl, "e"'] \arrow[dr, "\mathsf{pr}_j"] \arrow[r, phantom, "(*)"] \arrow[d, phantom, "(*)"] & {} \\
C \arrow[rr, tail, "c_j"'] & {} & A_j
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
P \arrow[r, two heads, "e"] \arrow[d, "e"'] & C \arrow[d, "c_j"] \arrow[dl, "\mathsf{id}_C"'] \\
C \arrow[r, tail, "c_j"] & A_j.
\end{tikzcd}$$

The parts marked by $(*)$ commute because e and s are cone morphisms. Since the diagonal fill-in in Definition 2.10.3 is unique, we have $e \cdot s = \mathsf{id}_C$. Thus, e is an isomorphism, and $\mathsf{pr}_j = c_j \cdot e$ is in $\mathcal{M}$, as desired. ◀

Proof of Lemma 3.2

It is clear that every Mono-carried homomorphism is monic in $\mathsf{Coalg}(F)$. Conversely, let $m\colon (C, c) \to (D, d)$ be a monomorphism in $\mathsf{Coalg}(F)$. Let $\mathsf{pr}_1, \mathsf{pr}_2\colon K \to C$ be a weak kernel pair of m. Since F preserves weak kernel pairs, $F\mathsf{pr}_1, F\mathsf{pr}_2\colon FK \to FC$ is a weak kernel pair of $Fm\colon FC \to FD$. This induces some cone morphism $k\colon K \to FK$ making pr_1 and pr_2

coalgebra morphisms $(K,k) \to (C,c)$:

$$\begin{tikzcd}
K \arrow[r, shift left, "\mathsf{pr}_1"] \arrow[r, shift right, "\mathsf{pr}_2"'] \arrow[d, dashed, "k"] & C \arrow[r, "m"] \arrow[d, "c"] & D \arrow[d, "d"] \\
FK \arrow[r, shift left, "F\mathsf{pr}_1"] \arrow[r, shift right, "F\mathsf{pr}_2"'] & FC \arrow[r, "Fm"] & FD
\end{tikzcd}$$

Since m is monic in $\mathsf{Coalg}(F)$, this implies that $\mathsf{pr}_1 = \mathsf{pr}_2$. For the verification that m is a monomorphism in $\mathcal{C}$, consider $f, g\colon X \to C$ with $m \cdot f = m \cdot g$. Since $\mathsf{pr}_1, \mathsf{pr}_2$ is a weak kernel pair, it induces some cone morphism $v\colon X \to K$, fulfilling $f = \mathsf{pr}_1 \cdot v$ and $g = \mathsf{pr}_2 \cdot v$. Since, $\mathsf{pr}_1 = \mathsf{pr}_2$, we find $f = g$ as desired. ◀

Proof of Lemma 3.6

Denote the unit and multiplication of the monad T by $\eta\colon \mathsf{Id} \to T$ and $\mu\colon TT \to T$, respectively. Consider an T-algebra homomorphism $f\colon (A,a) \to (B,b)$ for Eilenberg-Moore algebras (A,a) and (B,b) and denote its image factorization in $\mathsf{Alg}(T)$ by (I,i), with homomorphisms $e\colon (A,a) \twoheadrightarrow (I,i)$ and $m\colon (I,i) \rightarrowtail (B,b)$. We verify that $i\colon TI \to I$ is an Eilenberg-Moore algebra.

- First, we verify $i \cdot \eta_I = \mathsf{id}_I$ by showing that both $i \cdot \eta_I$ and id_I are both diagonals of the following square:

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "e"'] & I \arrow[d, "m"] \arrow[dl, "\mathsf{id}_I"'] \\
I \arrow[r, tail, "m"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, two heads, "e"] \arrow[dr, "\eta_A"] \arrow[dd, "\mathsf{id}_A"'] & & \arrow[d, phantom, "\text{(N)}"] & I \arrow[dl, "\eta_I"'] \arrow[d, "m"] \\
\arrow[r, phantom, "\text{(A)}"] & TA \arrow[r, "Te"] \arrow[dl, "a"] & TI \arrow[d, "Tm"] \arrow[ddll, "i"] \arrow[r, phantom, "\text{(N)}"] & B \arrow[dl, "\eta_B"] \arrow[dd, "\mathsf{id}_B"] \\
A \arrow[d, "e"'] & \text{(H)} & TB \arrow[dr, "b"] \arrow[r, phantom, "\text{(A)}"] & {} \\
I \arrow[rrr, tail, "m"'] & & & B
\end{tikzcd}$$

  The left-hand square commutes trivially, and the right-hand square commutes because η is natural (N), because e and m are T-algebra homomorphisms (H), and because (A,a) and (B,b) are Eilenberg-Moore algebras (A). By the uniqueness of the diagonal lifting property, we obtain $i \cdot \eta_I = \mathsf{id}_I$.
- Next, we verify $i \cdot Ti = i \cdot \mu_i$ by showing that both are diagonals in same square. To this end, let $(s_A, s_I, s_B) \in \{(Ta, Ti, Tb), (\mu_A, \mu_I, \mu_B)\}$, i.e. we obtain one diagonal for (Ta, Ti, Tb) and one for (μ_A, μ_I, μ_B):

$$\begin{tikzcd}
TTA \arrow[rrr, two heads, "TTe"] \arrow[d, "\mu_A"'] \arrow[dr, "s_A"] & & \arrow[d, phantom, "\text{(N)}"] & TTI \arrow[dl, "s_I"'] \arrow[d, "TTm"] \\
TA \arrow[d, "a"'] \arrow[r, phantom, "\text{(A)}"] & TA \arrow[r, "Te"] \arrow[dl, "a"] & TI \arrow[d, dashed, "Tm"] \arrow[ddll, "i"] \arrow[r, phantom, "\text{(N)}"] & TTB \arrow[dl, dashed, "s_B"'] \arrow[d, "\mu_B"] \\
A \arrow[d, "e"'] & \text{(H)} & TB \arrow[dr, "b"] \arrow[r, phantom, "\text{(A)}"] & TB \arrow[d, "b"] \\
I \arrow[rrr, tail, "m"'] & & & B
\end{tikzcd}$$

  In both cases, we have that e and m are T-algebra homomorphisms (H). For $(s_A, s_I, s_B) :=$ (Ta, Ti, Tb), we have that (A,a) and (B,b) are Eilenberg-Moore algebras (A) and that e

and m are homomorphisms (N). For $(s_A, s_I, s_B) := (\mu_A, \mu_I, \mu_B)$, we have that the parts (A) commute trivially, and the parts (N) commute because μ is natural. Since T preserves $\mathcal{E}$, we have $TTe \in \mathcal{E}$, and thus by the uniqueness of the diagonal, we obtain $i \cdot Ti = i \cdot \mu_I$.

Thus, (I, i) fulfils the axioms of an Eilenberg-Moore algebra. The remaining properties of the factorization system hold because the Eilenberg-Moore category is a full subcategory of $\mathsf{Alg}(T)$. ◀

Proof of Lemma 4.8

We verify the postulated equivalence using that C is $\mathcal{M}$-minimal iff every $h\colon B \to C$ is an epimorphism (Lemma 4.6, $\mathcal{E} = \mathsf{Epi}$).

- For 'if', we verify that every $h\colon B \to C$ is an epimorphism: for $u, v\colon C \to D$ with $u \cdot h = v \cdot h$, we directly obtain $u = v$ by assumption. Thus, h is an epimorphism.
- For 'only if', consider $u, v\colon C \to D$ and take a weak equalizer $e\colon E \to C$; hence, $u \cdot e = v \cdot e$. Since e is an epimorphism (by minimality), we obtain $u = v$. ◀