# Interval Edge Coloring of Bipartite Graphs with Small Vertex Degrees

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## Abstract

An edge coloring of a graph $G$ is called *interval edge coloring* if for each $v \in V(G)$ the set of colors on edges incident to $v$ forms an interval of integers. A graph $G$ is *interval colorable* if there is an interval coloring of $G$. For an interval colorable graph $G$, by the *interval chromatic index* of $G$, denoted by $\chi'_i(G)$, we mean the smallest number $k$ such that $G$ is interval colorable with $k$ colors. A bipartite graph $G$ is called $(\alpha, \beta)$-biregular if each vertex in one part has degree $\alpha$ and each vertex in the other part has degree $\beta$. A graph $G$ is called $(\alpha^*, \beta^*)$-bipartite if $G$ is a subgraph of an $(\alpha, \beta)$-biregular graph and the maximum degree in one part is $\alpha$ and the maximum degree in the other part is $\beta$.

In the paper we study the problem of interval edge colorings of $(k^*, 2^*)$-bipartite graphs, for $k \in \{3, 4, 5\}$, and of $(5^*, 3^*)$-bipartite graphs. We prove that every $(5^*, 2^*)$-bipartite graph admits an interval edge coloring using at most 6 colors, which can be found in $O(n^{3/2})$ time, and we prove that an interval edge 5-coloring of a $(5^*, 2^*)$-bipartite graph can be found in $O(n^{3/2})$ time, if it exists. We show that every $(4^*, 2^*)$-bipartite graph admits an interval edge 4-coloring, which can be found in $O(n)$ time. The two following problems of interval edge coloring are known to be $\mathcal{NP}$-complete: 6-coloring of $(6, 3)$-biregular graphs (Asratian and Casselgren (2006)) and 5-coloring of $(5^*, 5^*)$-bipartite graphs (Giaro (1997)). In the paper we prove $\mathcal{NP}$-completeness of 5-coloring of $(5^*, 3^*)$-bipartite graphs.

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## 1 Introduction

We use standard definitions and notations of graph theory. In the following, by a graph we mean a nonempty simple graph (i.e., without multiple edges or loops), and by a multigraph we mean a multigraph with possible multiple edges, but without loops.

Let $G$ be a multigraph with vertex set $V(G)$ and edge set $E(G)$. For some technical reasons, we assume that $V(G) \cap \mathbb{N} = \emptyset$. For each vertex $v \in V(G)$, by $N_G(v)$ we mean the set of neighbours of $v$ in $G$, and by $E_G(v)$ we mean the set of edges incident with $v$. The degree of vertex $v$ in $G$, denoted by $\deg_G(v)$, is the number $|E_G(v)|$. By $n(G), m(G), \Delta(G)$ and $\delta(G)$ we denote the number of vertices of $G$, the number of edges of $G$, the maximum and the minimum degree of a vertex of $G$, respectively. By *isolated vertex* we mean a vertex

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of degree 0, and by *pendant vertex* a vertex of degree 1. The set of all pendant vertices of $G$ we denote by $P(G)$. By $G[A]$, where $A \subset V(G)$, we denote a subgraph of $G$ induced by set $A$, and by $G \setminus A$ we mean the graph $G[V \setminus A]$. We write $H \subset G$ if and only if $H$ is a subgraph of $G$, and $H \sqsubset G$ if and only if $H$ is an induced subgraph of $G$, i.e., $H = G[V(H)]$.

A set of integers $[a, b] = \{a, a+1, \ldots, b-1, b\}$, where $a, b \in \mathbb{N}$ and $a \leq b$, is said to be an *interval* of integers. Let $X \underline{\times} Y = \{\{r, s\} \colon r \in X \wedge s \in Y\}$.

A bipartite graph $G$ is called $(\alpha, \beta)$-biregular if all vertices in one part of $G$ have degree $\alpha$ and all vertices in the other part have degree $\beta$. If $G$ is a subgraph of an $(\alpha, \beta)$-biregular graph with the maximum degree in one part $\alpha$ and the maximum degree in the other part $\beta$, it is called an $(\alpha^*, \beta^*)$-bipartite graph. If all vertices in the first part of an $(\alpha^*, \beta^*)$-bipartite graph have the same degree $\alpha$, then it is called an $(\alpha, \beta^*)$-bipartite graph. Analogously, we define $(\alpha^*, \beta)$-bipartite graphs. If we take the partition $(A, B)$ of $V(G)$ of an $(\alpha^*, \beta^*)$-bipartite graph $G$, we mean that the vertices from set $A$ are of degree $\alpha$ or less, and the vertices from set $B$ are of degree $\beta$ or less.

## 1.1 Interval coloring and interval $\chi'_i$-coloring problems

Let $G$ be a graph. Let $c \colon E(G) \to \mathbb{N}$ be an edge coloring, i.e., a function assigning different colors to adjacent edges. By an *interval edge coloring* we mean an edge coloring $c$ such that for each $v \in V(G)$, the set $c(E_G(v))$ is an interval of integers. An interval edge coloring $c$ such that $c(E(G)) = \{1, 2, \ldots, k\}$ is called *interval $k$-coloring*. We say that graph $G$ is *interval colorable* (*$k$-colorable*) if there is an interval coloring ($k$-coloring) of $G$. If $G$ is interval $l$-colorable for some $l \leq k$, then we say that $G$ is interval $k^*$-colorable or there is an interval $k^*$-coloring of $G$. The problem of interval coloring of graphs is the problem of veryfing if an arbitrary graph $G$ is interval colorable. If $G$ is interval colorable, then by *interval chromatic index* of $G$, denoted by $\chi'_i(G)$, we mean the smallest number $k$ such that $G$ is interval $k$-colorable. The problem of interval $\chi'_i$-coloring in the class of interval colorable graphs is to find an interval $\chi'_i(G)$-coloring for an arbitrary interval colorable graph $G$. Let $\mathcal{A}$ be an interval edge coloring algorithm for some class $\mathcal{C}$ of interval colorable graphs. We say that $\mathcal{A}$ is $k^*$-algorithm for class $\mathcal{C}$ if for every graph $G \in \mathcal{C}$ it gives an interval $k^*$-coloring of $G$, and we say that $\mathcal{A}$ is $(\chi'_i + k)^*$-algorithm for class $\mathcal{C}$ if for each graph $G \in \mathcal{C}$ it gives an interval $(\chi'_i(G) + k)^*$-coloring of $G$ (i.e., $\mathcal{A}$ is an additive $k$-approximation algorithm for the interval $\chi'_i$-coloring problem).

The problem of finding school timetables without idle times for both teachers and classes, which may be modelled by edge colorings of bipartite graphs, probably motivated Asratian and Kamalian to introduce in [2, 3] the concept of interval edge coloring of graphs. The open shop scheduling models with unit time operations, *no wait&idle* criterion, and some special bipartite scheduling graphs were considered in [10, 9], where the authors studied schedules with the minimum makespan which corresponds to the interval $\chi'_i$-coloring problem of bipartite scheduling graphs.

In general, not every graph has an interval coloring, since interval colorable graphs are $\Delta$-colorable [2]. Moreover, the problem of determining whether a bipartite graph has an interval coloring turned out to be $\mathcal{NP}$-complete [19], and the smallest known maximum degree of a bipartite graph without an interval coloring is 11 [16].

Interval colorable graphs, which are known to be interval $\chi'_i$-colored in a polynomial time, are regular bipartite graphs (by König theorem) (in $O(n\Delta \log \Delta)$ time with $\chi'_i(G) = \Delta(G)$ colors, for a regular graph $G$) [5], trees [14, 15, 11, 3] (in $O(n)$ time with $\chi'_i(T) = \Delta(T)$ colors, for a tree $T$) and complete bipartite graphs (in $O(m)$ time with $\chi'_i(K_{a,b}) = a + b - \gcd(a, b)$ colors, for a complete bipartite graph $K_{a,b}$) [14, 15, 11, 12]. In [8] the authors constructed an $O(n)$ time algorithm for interval $\chi'_i$-coloring of a grid $G$ with $\chi'_i(G) = \Delta(G)$ colors.

In [9] the authors proposed the linear time algorithm for interval coloring of any outerplanar bipartite graph, but the complexity of the interval $\chi'_i$-coloring problem of outerplanar bipartite graphs seems to be open. In [10] the authors proposed the linear time algorithm giving an interval coloring of bipartite cacti graph $G$ with $\Delta(G)+1$ colors, which is an $(\chi'_i+1)^*$-algorithm.

In [11] the author proved that any $(3^*,3^*)$-bipartite graph has an interval 4-coloring, which can be constructed in $O(n)$ time. In [7] the author proved that an interval $\alpha$-coloring of $(\alpha^*,\alpha^*)$-bipartite graph can be found in $O(n^{3/2})$ time (if it exists), for $\alpha \in \{3,4\}$.

In [12] the authors proved that if an $(\alpha,\beta)$-biregular graph has an interval $k$-coloring, then $k \geq \alpha+\beta-\gcd(\alpha,\beta)$. In [11] the author proved that every $(2\alpha,2)$-biregular graph admits an interval $2\alpha$-coloring (i.e., $\chi'_i$-coloring), for each $\alpha \geq 1$, and this construction can be done in $O(n\Delta\log\Delta)$ time. In [13] the authors proved that every $(2\alpha+1,2)$-biregular graph admits an interval $(2\alpha+2)$-coloring (i.e., $\chi'_i$-coloring), for every $\alpha \geq 1$, and to the best of our knowledge this construction can be done in $O(n^{3/2}\Delta^2)$ time.

The following problems of interval $\chi'_i$-coloring are $\mathcal{NP}$-complete: 6-coloring of $(6,3)$-biregular graphs [1] and 5-coloring of $(5^*,5^*)$-bipartite graphs [7].

In [4] the authors proved that every $(6,3)$-biregular graph admits an interval 7-coloring. The problems of interval coloring of $(4,3)$-biregular and $(5,3)$-biregular graphs are still open.

In the paper we solve the interval $\chi'_i$-coloring problem for $(\alpha^*,2^*)$-bipartite graphs, for $\alpha \in \{3,4,5\}$. We show that if $G$ is a $(5^*,2^*)$-bipartite graph, then $\chi'_i(G) \leq 6$, and the interval $\chi'_i$-coloring problem for $(5^*,2^*)$-bipartite graphs can be solved in $O(n^{3/2})$ time. If $G$ is a $(4^*,2^*)$-bipartite graph, then $\chi'_i(G) \leq 4$, and the interval $\chi'_i$-coloring problem for $(4^*,2^*)$-bipartite graphs can be solved in $O(n)$ time. In section 3 we prove $\mathcal{NP}$-completeness of the interval 5-coloring of $(5^*,3^*)$-bipartite graphs.

## 1.2 General and interval factor problem

We introduce the *general factor problem* [17, 6] as follows: let $G$ be a graph and let $\mathcal{F}\colon V(G) \to 2^{\mathbb{N}} \setminus \{\emptyset\}$, where $\mathcal{F}(v) \subset \{0,\ldots,\deg_G(v)\}$. Does $G$ admit an $\mathcal{F}$*-factor*, i.e., a set $F \subset E(G)$ such that for each vertex $v \in V(G)$, $|\{e\colon v \in e \wedge e \in F\}| \in \mathcal{F}(v)$? Lóvasz [17] proved that the general factor problem is $\mathcal{NP}$-complete for general graphs and mappings taking values $\{0,3\}$ for some vertices. In [6] Cornuéjols showed that the general factor problem is $\mathcal{NP}$-complete for planar bipartite graphs and mappings taking values $\{0,3\}$ for some vertices. A finite set $A \subset \mathbb{N}$ is said to have a *gap of length* $k \geq 1$ if $a, a+k+1 \in A$ and $a+1,\ldots,a+k \notin A$, for some $a$. A finite set $A \subset \mathbb{N}$ has no gaps if and only if it is an interval. Thus, the general factor problem is $\mathcal{NP}$-complete for planar bipartite graphs and mappings taking values, i.e., sets, having gaps of length of at least two. In [6] the author proved the conjecture of Lóvasz [17]: there is a polynomial time algorithm for deciding whether a graph $G$ has an $\mathcal{F}$-factor, where the sets $\mathcal{F}(v)$ have no gap of length of two or more. The complexity of the proposed algorithm is $O(n^4)$ [6].

If for each $v \in V(G)$, set $\mathcal{F}(v)$ is an interval, then $\mathcal{F}$-factor is called an *interval factor*. A special case of interval factors is $\mathcal{F}$-factor, where $\mathcal{F} \equiv \{k\}$, $k \in \mathbb{N}$, which we denote by $k$-factor, e.g., perfect matching is 1-factor.

Let us assume that there is an $O(\phi(m,n))$ time algorithm ($\phi(m,n) = \Omega(m+n)$ and $\phi(m,n) = O(mn^{1/2})$ [18]) solving perfect matching problem in the class of connected bipartite graphs with at most $m$ edges and $n$ vertices. Combining the idea of replacing an edge with a complete bipartite graph [6] and the idea of doubling a graph [7] we prove the following theorem.

▶ **Theorem 1.** *There is an $O(\phi(m\Delta, m))$ time algorithm solving the interval factor problem in the class of connected bipartite graphs with at most $m$ edges and the degree bounded by $\Delta$.*

**Proof.** Let $G$ be a connected bipartite graph and let $\mathcal{F}\colon V(G) \to 2^{\mathbb{N}} \setminus \{\emptyset\}$, such that $\mathcal{F}(v) = [a_v, b_v]$, $0 \le a_v \le b_v \le \deg_G(v)$. We construct a bipartite graph $H$ with $\Delta(H) \le \Delta(G) + 1$, $n(H) \le 8m(G)$ and $m(H) \le 4m(G)(\Delta(G) + 1) + n(G)\Delta(G)$ such that there is 1-factor in $H$ if and only if there is $\mathcal{F}$-factor in $G$.

For each vertex $v \in V(G)$, we denote $d_v = \deg_G(v)$, $p_v = b_v - a_v$ and $q_v = d_v - a_v$. Obviously, $b_v \le d_v$ and $p_v \le q_v$. In the first step we construct graph $H_1$ from graph $G$, by replacing each vertex $v \in V(G)$ with the complete bipartite graph isomorphic to $K_{d_v,q_v}$ with parts $A^1_v = \{v^1_u \colon u \in N_G(v)\}$ and $B^1_v = \{v^1_1, \dots, v^1_{q_v}\}$, and by replacing each edge $\{u, v\} \in E(G)$ with an edge $\{u^1_v, v^1_u\}$. In the second step, we take the resultant graph $H_1$ and its isomorphic copy $H_2$, where $v^1_*$ and $v^2_*$ are corresponding vertices (under isomorphism), and, for each $v \in V(G)$, we add edges $\{v^1_1, v^2_1\}, \dots, \{v^1_{p_v}, v^2_{p_v}\}$.

Let $V(H) = V(H_1) \cup V(H_2)$ and $E(H) = E(H_1) \cup E(H_2) \cup E^*$ , where $V(H_i) = \bigcup_{v\in V(G)} A^i_v \cup B^i_v$, $E(H_i) = \bigcup_{v\in V(G)} A^i_v \underline{\times} B^i_v \cup F^i$, and $F^i = \bigcup_{\{u,v\}\in E(G)} \{\{u^i_v, v^i_u\}\}$, for $i \in \{1, 2\}$, and $E^* = \bigcup_{v\in V(G)} \bigcup_{j\in\{1,\dots,p_v\}} \{\{v^1_j, v^2_j\}\}$.

We prove that there is an $\mathcal{F}$-factor in $G$ iff there is a 1-factor in $H$.

($\Rightarrow$) Let $F$ be $\mathcal{F}$-factor in $G$.

Let us define $F_v = \{u \in N_G(v) \colon \{v, u\} \in F\}$, $\hat{F}_v = N_G(v) \setminus F_v$ and $f_v = |F_v|$, for each $v \in V(G)$. Since $F$ is $\mathcal{F}$-factor, then for each $v \in V(G)$, $a_v \le f_v \le b_v$. Hence, $d_v - f_v \le q_v$ and $0 \le f_v - a_v \le p_v$. Let $v \in V(G)$. Let $F^i_v = \{v^i_u \in V(H_i) \colon u \in F_v\}$ and $\hat{F}^i_v = A^i_v \setminus F^i_v$, for $i \in \{1, 2\}$. Observe that $|\hat{F}^i_v| = d_v - f_v$, and $v^i_u \in F^i_v \iff u^i_v \in F^i_u$. Since $H_i[A^i_v \cup B^i_v] \simeq K_{d_v,q_v}$, there is a 1-factor in $H_i[\hat{F}^i_v \cup \{v^i_{f_v-a_v+1}, \dots, v^i_{d_v-a_v}\}]$, denote it by $\hat{E}^i_v$. Thus, $Q = \bigcup_{\{v,u\}\in F} \{\{v^1_u, u^1_v\}, \{v^2_u, u^2_v\}\} \cup \bigcup_{v\in V(G)} \hat{E}^1_v \cup \hat{E}^2_v \cup E^{12}_v$, where $E^{12}_v = \{\{v^1_1, v^2_1\}, \dots, \{v^1_{f_v-a_v}, v^2_{f_v-a_v}\}\}$, is a 1-factor in $H$. This construction can be done in $O(m(H))$ time.

From definition of $H$, $\Delta(H) \le \Delta(G) + 1$, $n(H) = 2n(H_1) \le 2 \sum_{v\in V(G)} 2d_v = 8m(G)$, and $m(H) \le 4m(G) + \sum_{v\in V(G)} (2d_v(d_v - a_v) + b_v - a_v) \le 4m(G) + \sum_{v\in V(G)} (2d_v^2 + d_v) \le 4m(G) + \Delta(G) \sum_{v\in V(G)} (2d_v + 1) \le 4m(G)(\Delta(G) + 1) + n(G)\Delta(G)$. Since $G$ is bipartite, $H_1$ and $H_2$ are bipartite, and hence $H$ is bipartite. Since there is an $O(\phi(m(H), n(H)))$ time algorithm finding 1-factor in $H$, we get the thesis.

($\Leftarrow$) Let $Q$ be a 1-factor in $H$.

Let $v \in V(G)$ and $i \in \{1, 2\}$. Since $|\{\{v^1_1, v^2_1\}, \dots, \{v^1_{p_v}, v^2_{p_v}\}\} \cap Q| \le p_v$, $q_v - p_v \le |(A^i_v \underline{\times} B^i_v) \cap Q| \le q_v$. Hence, $a_v \le |\{u \in N_G(v) \colon \{v^i_u, u^i_v\} \in Q\}| \le b_v$. Thus, $F = \{\{v, u\} \in E(G) \colon \{v^1_u, u^1_v\} \in Q\}$ is an $\mathcal{F}$-factor in $G$. This construction can be done in $O(m(H))$ time. ◀

In Fig. 1 there is a graph $G$ with an interval factor and in Fig. 2 the constructed graph $H$ with a 1-factor corresponding to the interval factor of $G$. White and black vertices form the partition of a bipartite graph.

Since the problem of 1-factor in the class of bipartite graphs with bounded degrees can be solved in $O(n^{3/2})$ time [18], we have the following corollary.

▶ **Corollary 2.** *There is an $O(n^{3/2})$ time algorithm solving the interval factor problem in the class of bipartite graphs with bounded degree.*

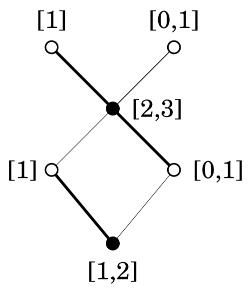


**Figure 1** A bipartite graph $G$ with an interval factor (bold edges).

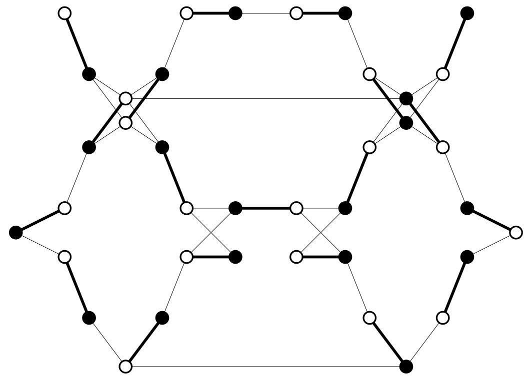

**Figure 2** A bipartite graph $H$ with a 1-factor (bold edges) corresponding to the interval factor of $G$.

## 2 Interval $\chi'_i$-coloring problem of $(\alpha^*, 2^*)$-bipartite graphs

In this section we construct polynomial time algorithms for the interval $\chi'_i$-coloring problem for $(\alpha^*, 2^*)$-bipartite graphs, for $\alpha \in \{3, 4, 5\}$, and give some other minor results.

### 2.1 Introductory properties

▶ **Observation 3.** *Let $G$ be an interval colorable graph, and let $H$ be an induced subgraph of $G$ such that for each $v \in V(H)$, $\deg_H(v) = \deg_G(v)$ or $\deg_H(v) = 1$. Then, for any interval edge coloring $c$ of $G$, the coloring $c' = c|_{E(H)}$ is an interval edge coloring of $H$.*

Let $G$ be an $(\alpha^*, 2^*)$-bipartite graph with partition $(X, Y)$ of $V(G)$ and let $P_2(G) = P(G) \cap Y$. Let $G'$ be the graph obtained from $G$ by adding and joining the unique vertex $x_v$ to each pendant vertex $v \in Y$. Formally, $V(G') = V(G) \cup \{x_v : v \in P_2(G)\}$ and $E(G') = E(G) \cup \{\{x_v, v\} : v \in P_2(G)\}$. In the following, this operation (transformation) we denote by $G \to_p G'$. Since the extension of an interval $k$-coloring of $G$ to an interval $k$-coloring of $G'$ is trivial, by Observation 3 we get the properties.

▶ **Proposition 4.** *Let $\alpha \in \mathbb{N}, \alpha \geq 1$. Let $G$ be an interval colorable $(\alpha^*, 2^*)$-bipartite graph and let $G'$ be the graph obtained by the operation $G \to_p G'$. Then, $G'$ is an interval colorable $(\alpha^*, 2)$-bipartite graph and $\chi'_i(G) = \chi'_i(G')$.*

Observe that for each $(\alpha^*, 2^*)$-bipartite graph $G$ we have $m(G) \leq 2n(G)$.

▶ **Proposition 5.** *Let $k, \alpha \in \mathbb{N}, \alpha \geq 1$. If there is an $O(\phi(n))$ time $k^*$-algorithm for $(\alpha^*, 2)$-bipartite graphs with at most $n$ vertices, then there is an $O(\phi(n))$ time $k^*$-algorithm for $(\alpha^*, 2^*)$-bipartite graphs with at most $n$ vertices.*

Let $G$ be an $(\alpha, 2^*)$-bipartite graph with the partition $(X, Y)$ and let $G'$ be an isomorphic copy of $G$. Let us denote by $v'$ the image of $v$ under isomorphism. Let $P_2(G) = P(G) \cap Y$, $q = |P_2(G)|$, and for each $v \in P_2(G)$, let $p_G(v)$ be the only one neighbour of $v$ in $G$. Let $W = \{w_v : v \in P_2(G)\}$, such that $W \cap (V(G) \cup V(G')) = \emptyset$ and $|W| = q$. Let $H$ be a graph defined as follows: $V(H) = (V(G) \cup V(G')) \setminus (P_2(G) \cup P_2(G')) \cup W$, and $E(H) = E(G \setminus P_2(G)) \cup E(G' \setminus P_2(G')) \cup \bigcup_{v \in P_2(G)} \{\{w_v, p_G(v)\}, \{w_v, p_{G'}(v')\}\}$. In the following, this operation (transformation) is denoted by $G \to_d H$. Note that $G \simeq G_W = H[(V(G) \setminus P_2(G)) \cup W]$, hence $G_W \simeq G'_W = H[(V(G') \setminus P_2(G')) \cup W]$. Thus, if $G_W$ is an interval colorable graph, then for any interval $k$-coloring $c$ of $G_W$, we can extend the coloring $c$ to the coloring of the whole graph $H$ by defining a coloring $c'$ of $G'_W$ as follows: $c'(e') = c(e) + 1$, where $e$ and $e'$ are isomorphic edges. Thus, by Observation 3 we get the following properties.

▶ **Proposition 6.** *Let $\alpha \in \mathbb{N}, \alpha \geq 1$. Let $G$ be an interval colorable $(\alpha, 2^*)$-bipartite graph and let $H$ be the graph obtained by the operation $G \to_d H$. Thus, $H$ is an interval colorable $(\alpha, 2)$-biregular graph and $\chi'_i(G) \leq \chi'_i(H) \leq \chi'_i(G) + 1$.*

▶ **Proposition 7.** *Let $k, \alpha \in \mathbb{N}, \alpha \geq 1$. If there is an $O(\phi(n))$ time $k^*$-algorithm for $(\alpha, 2)$-biregular graphs with at most $n$ vertices, then there is an $O(\phi(n))$ time $k^*$-algorithm for $(\alpha, 2^*)$-bipartite graphs with at most $n$ vertices.*

Let us recall that $\chi'_i(G) = 2\alpha$ if $G$ is an $(2\alpha, 2)$-biregular graph [11], and $\chi'_i(G) = 2\alpha + 2$ if $G$ is an $(2\alpha + 1, 2)$-biregular graph [13].

▶ **Corollary 8.** *Let $\alpha \in \mathbb{N}, \alpha \geq 1$. If there is an $O(\phi(n))$ time $\chi'_i$-algorithm for $(2\alpha, 2)$-biregular graphs with at most $n$ vertices, then there is an $O(\phi(n))$ time $\chi'_i$-algorithm for $(2\alpha, 2^*)$-bipartite graphs with at most $n$ vertices, and for every $(2\alpha, 2^*)$-bipartite graph $G$, $\chi'_i(G) = 2\alpha$.*

▶ **Corollary 9.** *Let $\alpha \in \mathbb{N}$. If there is an $O(\phi(n))$ time $\chi'_i$-algorithm for $(2\alpha + 1, 2)$-biregular graphs with at most $n$ vertices, then there is an $O(\phi(n))$ time $(\chi'_i + 1)^*$-algorithm for $(2\alpha + 1, 2^*)$-bipartite graphs with at most $n$ vertices, and for every $(2\alpha + 1, 2^*)$-bipartite graph $G$, $\chi'_i(G) \leq 2\alpha + 2$.*

## 2.2 Operations on multigraphs and pom-graphs

Let $H$ be a multigraph. Since $H$ may have multiple edges incident with the same two vertices, we introduce the notation $e_i(x, y)$ or $e(x, y, i)$, where $i$ is an identifier, to distinguish two or more edges incident with $x$ and $y$, e.g., $e_1(x, y)$ and $e_2(x, y)$. Let $u, v \in V(H)$ such that there is no edge in $E(H)$ incident with $u$ and $v$. We say that a multigraph $H'$ is obtained from $H$ by *contracting* vertices $v$ and $u$ if vertices $v, u$ are replaced by a new vertex $w(u, v)$ and each edge $e_i(x, t) \in E(H)$, where $x \in \{u, v\}, t \in V(H)$, is replaced with $e_i(w(u, v), t)$ (we say further that $e_i(x, t)$ and $e_i(w(u, v), t)$ are corresponding edges). Formally, $H' = ((V(H) \setminus \{v, u\}) \cup \{w(u, v)\}, E(H \setminus \{u, v\}) \cup \{e_i(w(u, v), t) \colon x \in \{u, v\} \wedge t \in V(H) \wedge e_i(x, t) \in E(H)\}$.

In the following by a multidigraph we mean a multidigraph without loops. Let $D$ be a multidigraph. If $a = (x, y)$ is an arc, then $y$ is said to be the *head* and $x$ the *tail* of the arc a. Since $D$ may have multiple arcs with the same head and the same tail, we introduce the notation $a_i(x, y)$ or $a(x, y, i)$, to distinguish two or more arcs with the same head and tail, e.g., $a_1(x, y)$ and $a_2(x, y)$. By $\text{indeg}_D(v)$ we mean the number of arcs with the head at $v$, and by $\text{outdeg}_D(v)$ we mean the number of arcs with the tail at $v$ in $D$. We say that $v \in V(D)$ is a pendant vertex in $D$ if and only if $\text{indeg}_D(v) + \text{outdeg}_D(v) = 1$.

Let $G$ be an $(\alpha^*, 2)$-bipartite graph with partition $(X, Y)$ of $V(G)$, $\alpha \in \mathbb{N}, \alpha \geq 1$. Let $H$ be the multigraph obtained from $G$ by replacing each two edges $\{u, v\}$ and $\{w, v\}$, where $v \in Y$, with one new edge $e_v(u, w)$ joining $u$ and $w$ (we allow multiple edges between $u$ and $w$). Formally, $V(H) = X$, and $E(H) = \{e_v(u, w) \colon v \in Y\}$. In the following, the multigraph $H$ is said to be the *contraction multigraph* of $G$, which we denote by $G \to_{cn} H$. Let $c$ be an interval $k$-coloring of $G$. We replace each edge $e_v(u, w) \in E(H)$ with the arc $a^c_v$, where $a^c_v$ has the tail at $u$, if $c(\{u, v\})$ is an odd number, otherwise $a^c_v$ has the tail at $w$. Since

$c$ is an interval edge coloring and $\deg_G(v) = 2$, then only one of $c(\{u, v\})$ and $c(\{w, v\})$ is an odd number. Formally, $D_c(G)$ is the directed multigraph (multidigraph) with vertex set $V(D_c(G)) = V(H)$ and arc set $A(D_c(G)) = \{a_v^c : v \in Y\}$.

By a *partially oriented multigraph* or a pom-*graph* $P = (V, E \cup A)$ we mean the union of a multigraph $G = (V, E)$ and a directed multigraph $D = (V, A)$ on the same vertex set $V$, which we denote by $P = G \cup D$. In the following, by $\mathrm{Gr}(P)$ we mean the multigraph $G$, by $\mathrm{Di}(P)$ we mean the multidigraph $D$, by $E(P)$ we mean $E(G)$, and by $A(P)$ we mean $A(D)$. The underlying multigraph of a *pom*-graph $P$, denoted by $\mathrm{Un}(P)$, is the multigraph obtained from $P$ by replacing each arc $a_i(x, y) \in A(P)$ with a new edge $e_i(x, y)$. Formally, $\mathrm{Un}(P) = (V(P), E(P) \cup E_A(P))$, where $E_A(P) = \{e_i(x, y) : a_i(x, y) \in A(P)\}$. For each $e \in E(\mathrm{Un}(P))$, by $o(e)$ we mean $e$, if $e \in E(P)$, or $a_i(x, y)$, if $e \in E_A(P)$ and $e = e_i(x, y)$. Let $H = \mathrm{Un}(P)$ and let $v \in V(P)$. By $EA_P(v)$ we mean $\{o(e) : e \in E_H(v)\}$. By $\deg_P(v)$ we mean $\deg_H(v)$, and hence $\Delta(P) = \Delta(H)$. Let $B \subset V(P)$, by $P[B]$ we denote a *pom*-graph $G[B] \cup D[B]$, and by $P \setminus B$ we mean the *pom*-graph $P[V \setminus B]$. We say that two vertices $u, v \in V(P)$ are *neighbours* in $P$ if and only if $E(\mathrm{Un}(P[\{u, v\}]))$ is a non-empty set.

Let $P$ be a *pom*-graph and let $a \le b \le \Delta(P)$, $a, b \in \mathbb{N}$. By $V_{a,b}(P)$ we denote the set $\{v \in V(P) : \deg_P(v) \in [a, b]\}$ and by $G_{a,b}(P)$ we mean $\mathrm{Gr}(P[V_{a,b}(P)])$. If $a = b$, then we write $G_a(P)$ instead of $G_{a,a}(P)$.

Let $P'$ be the multigraph obtained from *pom*-graph $P$ by adding new vertices on each edge and each arc. If we add vertex $w_e$ on an edge $e = e_i(x, y)$, then we add two new edges $e_i(w_e, x)$ and $e_i(w_e, y)$. If we add vertex $w_a$ on an arc $a = a_i(x, y)$, then we add an edge $e_i(x, w_a)$ and an arc $a_i(w_a, y)$. Formally, $V(P') = V(P) \cup \{w_e : e \in E(P)\} \cup \{w_a : a \in A(P)\}$, $E(P') = \{e_i(w_e, x), e_i(w_e, y) : e \in E(P) \wedge e = e_i(x, y)\} \cup \{e_i(x, w_a) : a \in A(P) \wedge a = a_i(x, y)\}$, $A(P') = \{a_i(w_a, y) : a \in A(P) \wedge a = a_i(x, y)\}$. In the following, the *pom*-graph $P'$ is said to be the *subdivision* pom-*graph* of *pom*-graph $P$, which we denote by $P \to_{sd} P'$.

## 2.3 Interval $\chi_i'$-coloring problem of $(\alpha^*, 2^*)$-bipartite graphs for $\alpha \in \{3, 4\}$

▶ **Theorem 10.** *Let $G$ be a $(3^*, 2^*)$-bipartite graph. Then, $\chi_i'(G) = 3$ if and only if each connected component of $G$ contains at most one cycle. The construction of interval 3-coloring can be done in linear time.*

**Proof.** Let $G$ be a $(3^*, 2^*)$-bipartite graph with partition $(X, Y)$ of $V(G)$, and let $P_2(G) = P(G) \cap Y$. By definition $\Delta(G) = 3$.

($\Rightarrow$) Let us assume that $\chi_i'(G) = 3$, and let $G$ contain at least one cycle.

Let $G'$ be the graph obtained from $G$ by the operation $G \to_p G'$. By Proposition 4, $G'$ is $(3^*, 2)$-bipartite graph and $G'$ is interval 3-colorable if and only if $G$ is interval 3-colorable.

Let $c$ be an interval 3-coloring of $G'$, and let $D = D_c(G')$. Obviously, $\mathrm{indeg}_D(v) \le 1$, for each $v \in V(D)$. Let $D'$ be a digraph obtained from $D$ by successively removing pendant vertices. Hence, for each $v \in V(D')$, $2 \le \mathrm{indeg}_{D'}(v) + \mathrm{outdeg}_{D'} \le 3$, and $\mathrm{indeg}_{D'}(v) \le 1$. Thus, $\mathrm{outdeg}_{D'}(v) \ge 1$. Since $\sum_{v \in V(D')} \mathrm{indeg}_{D'}(v) = \sum_{v \in V(D')} \mathrm{outdeg}_{D'}(v)$, each component of $D'$ is a directed cycle. Thus, each component of $G$ contains at most one cycle.

($\Leftarrow$) Let us assume that each connected component of $G$ has at most one cycle. First, we color each cycle with colors 1 and 2, alternately. Next, for each vertex $v$ of degree 3 that belongs to a colored cycle, color edge $\{v, u\}$ with 3, where $u$ does not belong to the colored cycle. In the last step, color the remaining trees in a greedy way using 3 colors, preserving intervals at vertices. Thus, we get an interval 3-coloring of $G$ in linear time. ◀

▶ **Theorem 11.** *Let $G$ be a $(4^*, 2)$-bipartite graph. Then, $\chi'_i(G) = 4$. The construction of an interval 4-coloring of $G$ can be done in linear time.*

**Proof.** Let $G$ be a $(4^*, 2)$-bipartite graph. The construction proceeds in two crucial stages: first, we construct a *pom*-graph $P$ from graph $G$, then we use the structure of $P$ to build the interval 4-coloring of graph $G$. In the first stage we apply to $G$ the sequence of transformations $G \to_{cn} P_0 \dashrightarrow_1 P_1 \dashrightarrow_2 P_2 \to_{sd} P$, where $P_0$ is the contraction multigraph of $G$, $P_1$ and $P_2$ are some *pom*-graphs, and $P$ is the subdivision *pom*-graph of $P_2$. In the second stage we start from an edge&arc 4-coloring of $P$, preserve this coloring on the underlying graph $G^* = \mathrm{Un}(P)$, and in the final step we transform the edge colored graph $G^*$ to an interval edge colored initial graph $G$, by contracting vertices that come from vertices splitted in the transformations $\dashrightarrow_1$ or $\dashrightarrow_2$.

($P_0 \dashrightarrow_1 P_1$) Initially, let $H' = P_0, D' = (V(P_0), \emptyset)$, and let $P' = H' \cup D'$ be a *pom*-graph. We proceed with the following successive steps in a loop until there is no cycle in $G_{3,4}(P')$.

1. (*find a cycle*) Let $C$ be a subgraph of $G_{3,4}(P')$, which is a cycle. Let $V(C) = \{v_1, \ldots, v_k\}$ and let $E(C) = \{e(v_1, v_2, i_1), \ldots, e(v_{k-1}, v_k, i_{k-1}), e(v_k, v_1, i_k)\}$. Note that $C$ may have two vertices.
2. (*orient the cycle*) Remove $E(C)$ from $E(P')$ and add $k$ arcs $a(v_1, v_2, i_1)$, …, $a(v_{k-1}, v_k, i_{k-1})$, $a(v_k, v_1, i_k)$ to $A(P')$.
3. (*split vertices of the cycle*) For each $v \in V(C)$ such that $\deg_{P'}(v) = 3$, or $\deg_{P'}(v) = 4$ and $\mathrm{indeg}_{D'}(v) = 2$, *split* $v$ into vertices $v^i$ and $v^o$ as shown in Fig. 3 and 4. Note that in this case each dashed line is an arc with the tail at $v$ or $v^o$. Formally, add new vertices $v^i$ and $v^o$ to $V(P')$, for each arc $a_d(u, v)$ add new arc $a_d(u, v^i)$, for each arc $a_d(v, x)$ add new arc $a_d(v^o, x)$, and for each edge $e_d(v, x)$ add new edge $e_d(v^o, x)$, and remove $v$ from $V(P')$.

Knowing that there is no cycle in $G_{3,4}(P')$, let $P_1 = P'$.

▷ Claim 12. For each $v \in V(P_1)$,

- if $\deg_{P_1}(v) = 1$, then $v$ is incident with an edge or an arc with the head at $v$,
- if $\deg_{P_1}(v) = 2$, then $v$ is incident with two edges or an edge and an arc with the tail at $v$, or two arcs with tails at $v$, or two arcs with heads at $v$,
- if $\deg_{P_1}(v) = 3$, then $v$ is incident with three edges,
- if $\deg_{P_1}(v) = 4$, then $v$ is incident with four edges or two edges and two arcs, one with the tail at $v$ and one with the head at $v$.

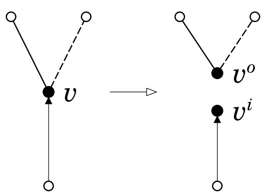


**Figure 3** Splitting vertex $v$ of $\deg_{P'}(v) = 3$.

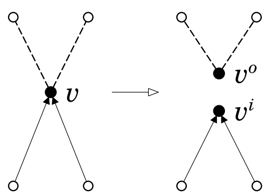


**Figure 4** Splitting vertex $v$ of $\deg_{P'}(v) = 4$.

($P_1 \dashrightarrow_2 P_2$) Let $P_1 = H_1 \cup D_1$, where $H_1$ is a multigraph and $D_1$ is a multidigraph. Initially, let $P' = P_1$, $H' = H_1$ and $D' = D_1$. We proceed with the following successive steps in a loop until there is no path in $G_{3,4}(P')$. Note that if $V(G_{3,4}(P')) \neq \emptyset$, then there is a path in $G_{3,4}(P')$ with at least one vertex, and by Claim 12, $\deg_{H'}(v) \geq 2$, for each $v \in V(G_{3,4}(P'))$.

1. (*find a maximal path*) Let $T$ be a subgraph of $G_{3,4}(P')$, which is a maximal path. Let $V(T) = \{v_1, \ldots, v_k\}$ and let $E(T) = \{e(v_1, v_2, i_1), \ldots, e(v_{k-1}, v_k, i_{k-1})\}$. Note that $T$ may have one vertex.
2. (*orient the path*) Since $T$ is a maximal path in $G_{3,4}(P')$ and $\deg_{H'}(v) \geq 2$, for each $v \in V(G_{3,4}(P'))$, where $H' = \mathrm{Gr}(P')$, there is a vertex $v \in V(P')$ with $\deg_{P'}(v) \leq 2$ such that $e_d(v, v_1) \in E(P')$. Remove $E(T) \cup \{e_d(v, v_1)\}$ from $E(P')$ and add $k$ arcs $a(v, v_1, i_1), \ldots, a(v_{k-1}, v_k, i_{k-1})$ to $A(P')$.
3. (*split vertices of the path*) For each $v \in V(T)$ such that $\deg_{P'}(v) = 3$, or $\deg_{P'}(v) = 4$ and $\mathrm{indeg}_{D'}(v) = 2$, *split* $v$ into vertices $v^i$ and $v^o$ as shown in Fig. 3 and 4. Note that in this case each dashed line may be an edge or an arc with the tail at $v$ or $v^o$. Formally, add new vertices $v^i$ and $v^o$ to $V(P')$, for each arc $a_d(u, v)$ add new arc $a_d(u, v^i)$, for each arc $a_d(v, x)$ add new arc $a_d(v^o, x)$, and for each edge $e_d(v, x)$ add new edge $e_d(v^o, x)$, and remove $v$ from $V(P')$.

Knowing that there is no path in $G_{3,4}(P')$, let $P_2 = P'$ and let $P_2 = H_2 \cup D_2$, where $H_2$ is a multigraph and $D_2$ is a multidigraph.

▷ Claim 13. For each $v \in V(P_2)$, $\deg_{P_2}(v) \leq 2$. If $\deg_{P_2}(v) = 2$ and there is an arc with the head at $v$, then $v$ is incident with two arcs with heads at $v$.

Proof. If $v \in V(P_2)$ is a vertex such that $v = w^o$ or $v = w^i$, for some $w \in V(P_0)$, then $\deg_{P_2}(v) \leq 2$. Let $v \in V(P_1)$ and $\deg_{P_1}(v) > 2$. Then, $v \in V(G_{3,4}(P_1))$, and by Claim 12, there is $u \in V(P_1)$ such that $e_d(u, v) \in E(P_1)$. Thus, $v \in V(T)$ for some path $T$ while applying step (1) in the transformation $P_1 \dashrightarrow_2 P_2$. After orienting the path $T$ in step (2), vertex $v$ is splitted in the next step (3) or its degree is 4 and there are two edges incident with $v$. In the latter case, $v \in V(T')$ for some other path $T'$ while applying step (1). Thus, after orienting the path $T'$ in step (2), $v$ is splitted in the successive step (3).

Let $\deg_{P_2}(v) = 2$, for some $v \in V(P_2)$, and let us assume that there is an arc with the head at $v$. If $v \in V(P_0)$, then there is no arc with the head at $v$ in *pom*-graph $P_2$. Thus, $v = w^i$ for some splitted vertex $w$, and $v$ is incident with two arcs with heads at $v$. ◁

Let $P$ be the *pom*-graph obtained by the transformation $P_2 \to_{sd} P$, and let $H = \mathrm{Gr}(P)$ and $G^* = \mathrm{Un}(P)$. Obviously, $H$ and $G^*$ are simple graphs, and by Claim 13, $\Delta(P) \leq 2$.

▷ Claim 14. Let $T \sqsubset H$ be a maximal path. Let $v, u \in V(T)$ such that $\deg_H(v) = \deg_H(u) = 1$. If there are arcs $(v, x), (u, y) \in A(P)$, then $|E(T)|$ is even.

▷ Claim 15. The graph $G^*$ is bipartite and each component of $G^*$ is a path of length of at least 2 or a cycle of length at least 4.

We define $c' \colon E(P) \cup A(P) \to \{1, 2, 3, 4\}$ separately for each $P' \subset P$ such that $P^*$ is a connected component of $G^*$, where $P^* = \mathrm{Un}(P')$. By Claim 15, $P^*$ is a path or a cycle. Let $V(P') = \{v_1, \ldots, v_l\}$ and let $k = |E(P') \cup A(P')|$. If $P^*$ is a cycle, let us denote $v_{k+1} = v_1$ and $v_{k+2} = v_2$. Let us assume that for each $i \in \{1, \ldots, k\}$, $v_i$ and $v_{i+1}$ are neighbours in $P'$. For each $i \in \{1, \ldots, k\}$, let $o_i = o(\{v_i, v_{i+1}\})$, where $\{v_i, v_{i+1}\} \in E(P^*)$.

First, color arcs in $A(P')$. For each $i \in \{1, \ldots, k\}$, if $o_i$ is an arc with the head at $v_i$, then let $c'(o_i) = 1$, and if $o_i$ is an arc with head at $v_{i+1}$, then let $c'(o_i) = 4$. Next, we color edges in $E(P')$. If $o_1$ is an edge, then let $c'(o_1) = 3$. For each $i \in \{1, \ldots, k-1\}$ (if $P^*$ is a cycle, also for $i = k$), if $o_i$ is an arc and $o_{i+1}$ is an edge, then by Claim 13, $o_i$ has the head at $v_i$. Since $c'(o_i) = 1$, let $c'(o_{i+1}) = 2$. We extend $c'$ to the rest of uncolored edges of $E(P')$, coloring them with colors 2 and 3 such that no two adjacent edges have the same color.

Observe that if for some $i \in \{1, \ldots, k\}$, $o_i$ is an edge and $o_{i+1}$ is an arc with the tail at $v_{i+1}$, then $c'(o_{i+1}) = 4$, and by Claim 14, $c'(o_i) = 3$. If $o_i$ and $o_{i+1}$ are arcs, then by Claim 13, $o_i$ and $o_{i+1}$ have heads at $v_{i+1}$, hence $c'(o_i) = 4$ and $c'(o_{i+1}) = 1$.

Let us define $c^* \colon E(G^*) \to \{1, 2, 3, 4\}$ as follows: $c^*(e) = c'(o(e))$, where $o(e) \in E(P) \cup A(P)$. By the above construction of $c'$, $c^*$ is an edge 4-coloring.

Now, we define $c \colon E(G) \to \{1, 2, 3, 4\}$. We contract all the pairs of vertices from $V(G^*)$ that come from vertices splitted in the transformations $\dashrightarrow_1$ or $\dashrightarrow_2$ and we preserve the colors on the corresponding edges, and thus we get the initial graph $G$ that is edge colored. Formally, for each $v^i, v^o \in V(G^*)$, where $v \in V(P_0)$ (see Fig. 3 and 4), we contract vertices $v^i, v^o$ and we preserve the colors on the corresponding edges $\{v^*, x\}$ and $\{v, x\}$, i.e., $c(\{v, x\}) = c^*(\{v^*, x\})$, where $v^*$ is $v^i$ or $v^o$. Since $c^*(E_{G^*}(v^i))$ is equal to $\{1\}$, $\{4\}$ or $\{1, 4\}$ and $c^*(E_{G^*}(v^o)) = \{2, 3\}$, we get $c(E_G(v)) = \{1, 2, 3\}$ or $c(E_G(v)) = \{2, 3, 4\}$, or $c(E_G(v)) = \{1, 2, 3, 4\}$. Thus, $c$ is an interval 4-coloring of $G$.

Since the transformation of $G$ into $P_2$ can be done in linear time, and the coloring of $P$ can be done in linear time, the final construction of the coloring of $G$ can be done in linear time. ◀

By Theorem 11, and Propositions 4 and 5 we get the following theorem.

▶ **Theorem 16.** *Let $G$ be a $(4^*, 2^*)$-bipartite graph. Then, $\chi'_i(G) = 4$. The construction of an interval 4-coloring of $G$ can be done in linear time.*

## 2.4 Interval $\chi'_i$-coloring problem of $(5^*, 2^*)$-bipartite graphs

Let $G$ be a $(5^*, 2^*)$-bipartite graph. Let $\mathcal{F}_5 \colon V(G) \to 2^{\mathbb{N}} \setminus \{\emptyset\}$ be defined as follows: if $\deg_G(v) = 2i + 1$, for $i \in \{0, 1\}$, then let $\mathcal{F}_5(v) = \{i, i + 1\}$, if $\deg_G(v) = 2i$, for $i \in \{1, 2\}$, then let $\mathcal{F}_5(v) = \{i\}$, and if $\deg_G(v) = 5$, then let $\mathcal{F}_5(v) = \{2\}$.

Let $G$ be a $(5^*, 2^*)$-bipartite graph. If $c$ is an interval 5-coloring of $G$, then $F = \{e \in E(G) \colon c(e) \in \{2, 4\}\}$ is an $\mathcal{F}_5$-factor of $G$.

▶ **Theorem 17.** *Let $G$ be a $(5^*, 2^*)$-bipartite graph with $n$ vertices. Then, $\chi'_i(G) = 5$ if and only if $G$ admits an $\mathcal{F}_5$-factor. The construction of an interval 5-coloring can be done in $O(n^{3/2})$ time.*

▶ **Theorem 18.** *Let $G$ be a $(5^*, 2)$-bipartite graph. Then, $5 \leq \chi'_i(G) \leq 6$ and the construction of an interval $6^*$-coloring of $G$ can be done in $O(n^{3/2})$ time.*

By Theorems 17 and 18, and by Propositions 4 and 5 we get

▶ **Theorem 19.** *Let $G$ be a $(5^*, 2^*)$-bipartite graph. Then, $\chi'_i(G) \leq 6$ and the construction of an interval $\chi'_i$-coloring of $G$ can be done in $O(n^{3/2})$ time.*

# 3 $\mathcal{NP}$-completeness results

▶ **Theorem 20.** *The problem of interval 5-coloring of $(5^*, 3^*)$-bipartite graphs is $\mathcal{NP}$-complete.*

The Table 1 contains the state-of-art and our results presented in this paper, and some open problems for further research.

**Table 1** The complexity of the algorithms for the interval $\chi'_i$-coloring problem.

| Interval edge $\chi'_i$-coloring problem for $(\alpha^*,\beta^*)$-bipartite or $(\alpha,\beta)$-biregular graphs | | | |
|---|---|---|---|
| **Graphs** | $\chi'_i$ | **Complexity** | **Comments** |
| $(\alpha^*, 1^*)$ | $k$ | $O(n)$ | stars |
| $(2^*, 2^*)$ | 2 | $O(n)$ | paths and cycles |
| $(3^*, 2^*)$ | 3 or 4 | $O(n)$ | Thm. 10 ($\chi'_i = 3$), [11] ($\chi'_i \leq 4$) |
| $(4^*, 2^*)$ | 4 | $O(n)$ | Thm. 16 |
| $(5^*, 2^*)$ | 5 or 6 | $O(n^{3/2})$ | Thm. 17 ($\chi'_i = 5$), Thm. 19 ($\chi'_i \leq 6$) |
| $(6^*, 2^*)$ | ? | ? | *interval coloring problem* is open |
| $(3^*, 3^*)$ | 3 or 4 | $O(n^{3/2})$ or $O(n)$ | [7] ($\chi'_i = 3$) [11] ($\chi'_i \leq 4$) |
| $(4, 3)$ | ? | ? | *interval coloring problem* is open |
| $(5, 3)$ | ? | ? | *interval coloring problem* is open |
| $(6, 3)$ | $\leq 7$ | $O(n)$ | [4] ($(\chi'_i + 1)^*$-algorithm) |
| $(5^*, 3^*)$ | 5 | $\mathcal{NP}$-complete | Thm. 20 |
| $(6, 3)$ | 6 | $\mathcal{NP}$-complete | [1] |
| $(2\alpha, 2)$ | $2\alpha$ | $O(n\Delta \log \Delta)$ | [11] |
| $(2\alpha+1, 2)$ | $2\alpha+2$ | $O(n^{3/2}\Delta^2)$ | [13] (compl. of 2-factor by Thm. 1) |
| $(2\alpha, 2^*)$ | $2\alpha$ | $O(n\Delta \log \Delta)$ | [11] and Cor. 8 |
| $(2\alpha+1, 2^*)$ | $\leq 2\alpha+2$ | $O(n^{3/2}\Delta^2)$ | [13] and Cor. 9 ($(\chi'_i + 1)^*$-algorithm) |

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