Unique-Neighbor-Like Expansion and Group-Independent Cosystolic Expansion

Tali Kaufman ✉
Department of Computer Science, Bar-Ilan University, Ramat Gan, Israel

David Mass ✉
Department of Computer Science, Bar-Ilan University, Ramat Gan, Israel

──── **Abstract** ────

In recent years, high dimensional expanders have been found to have a variety of applications in theoretical computer science, such as efficient CSPs approximations, improved sampling and list-decoding algorithms, and more. Within that, an important high dimensional expansion notion is *cosystolic expansion*, which has found applications in the construction of efficiently decodable quantum codes and in proving lower bounds for CSPs.

Cosystolic expansion is considered with systems of equations over a group where the variables and equations correspond to faces of the complex. Previous works that studied cosystolic expansion were tailored to the specific group $\mathbb{F}_2$. In particular, Kaufman, Kazhdan and Lubotzky (FOCS 2014), and Evra and Kaufman (STOC 2016) in their breakthrough works, who solved a famous open question of Gromov, have studied a notion which we term "parity" expansion for small sets. They showed that small sets of k-faces have proportionally many $(k+1)$-faces that contain *an odd number* of k-faces from the set. Parity expansion for small sets could then be used to imply cosystolic expansion only over $\mathbb{F}_2$.

In this work we introduce a stronger *unique-neighbor-like* expansion for small sets. We show that small sets of k-faces have proportionally many $(k+1)$-faces that contain *exactly one* k-face from the set. This notion is fundamentally stronger than parity expansion and cannot be implied by previous works.

We then show, utilizing the new unique-neighbor-like expansion notion introduced in this work, that cosystolic expansion can be made *group-independent*, i.e., unique-neighbor-like expansion for small sets implies cosystolic expansion *over any group*.

1 Introduction

High dimensional expanders

High dimensional expanders are the high dimensional analog of expander graphs. A d-dimensional simplicial complex is a hypergraph with hyperedges of size at most $d+1$ which is downwards closed, i.e., if σ is an hyperedge and $\tau \subset \sigma$ then τ is also an hyperedge. A hyperedge of size $k+1$ is called a k-face of the complex.

In recent years, high dimensional expanders have found a variety of applications in theoretical computer science, such as efficient CSPs approximations [2], improved sampling algorithms [5, 4, 3, 8, 7, 6, 16], improved list-decoding algorithms [11, 1], sparse agreement tests [12, 9, 19] and more.

An especially important high dimensional expansion notion is *cosystolic expansion*. It has been shown to be a key ingredient in the construction of efficiently decodable quantum LDPC codes with a large distance [15], and recently it has been used in the construction of explicit 3XOR instances that are hard for the Sum-of-Squares hierarchy [10].

Cosystolic expansion as an expanding system of equations

A simplicial complex can be viewed as forming *a system of equations* over some group G. Consider a d-dimensional simplicial complex and some dimension $k < d$. The variables of the system are the k-faces of the complex, and the equations are defined by the $(k+1)$-faces; each $(k+1)$-face σ corresponds to the equation $\sum_{i=0}^{k+1} \tau_i = 0$, where τ_i are the k-faces contained in σ and the sum is performed over the group (e.g., addition modulo 2 when the group is $\mathbb{F}_2$).

For any assignment of values to the variables which does not satisfy all the equations, there are two measures of interest. One measure is the fraction of unsatisfied equations (out of all the equations), and the second measure is the fraction of variables (out of all the variables) that their value needs to be changed in order to satisfy all the equations. The second measure is also called the *distance* of the assignment from a satisfying assignment.

A system of equations is said to be *expanding* if for any assignment of values to the variables it holds that either all the equations are satisfied or the fraction of unsatisfied equations is proportional to the distance of the assignment from a satisfying assignment. A d-dimensional simplicial complex is said to be a *cosystolic expander* over a group G if for all $k < d$, the system of equations formed by its k-faces is expanding.

As a simple example, consider a 1-dimensional simplicial complex (i.e., a graph) and the field $\mathbb{F}_2$. The variables of the system are the vertices of the graph, and the equations are $v_i + v_j = 0 \pmod 2$ for each edge $\{v_i, v_j\}$. In this case, if the given graph is an expander graph (i.e., each subset of vertices has proportionally many outgoing edges), then the system of equations is expanding. This is true since each assignment of values over $\mathbb{F}_2$ to the vertices can be identified with a subset of vertices, and the unsatisfied equations are exactly the outgoing edges of this set.

Parity expansion for small sets

Kaufman, Kazhdan and Lubotzky [18], and Evra and Kaufman [14] in their breakthrough works proved the existence of cosystolic expanders of every dimension, solving a famous open question of Gromov [17]. In their works they have studied a notion which we term "parity" expansion for small sets: They have shown that certain high dimensional expansion properties imply that small sets of k-faces have proportionally many $(k+1)$-faces that contain *an odd number* of k-faces from the given set. Then they utilized this property in order to imply cosystolic expansion over the group $\mathbb{F}_2$.

δ_1-expansion for small sets

In this work we study a fundamentally stronger "unique-neighbor-like" expansion in simplicial complexes, which we call δ_1-*expansion*. Let X be a d-dimensional simplicial complex and A a set of k-faces in X. We define $\delta_1(A)$ to be the set of $(k+1)$-faces which contain *exactly one* k-face from A. We say that A is δ_1-*expanding* if the fraction of $(k+1)$-faces in $\delta_1(A)$ (out of all the $(k+1)$-faces) is proportional to the fraction of k-faces in A (out of all the k-faces). Our main result is that certain high dimensional expansion properties imply that small sets are δ_1-expanding.

δ_1-expansion and group-independent cosystolic expansion

The strength of our δ_1-expansion can be demonstrated by its relation to cosystolic expansion. As explained above, cosystolic expansion is considered with a system of equations over a group. Hence, when proving cosystolic expansion, one has to take the group into account. For instance, previous works could obtain cosystolic expansion only over $\mathbb{F}_2$, because only over $\mathbb{F}_2$ there is an equivalence between an unsatisfied equation and an equation that contains an odd number of non-zero variables.

The δ_1-expansion property that we study in this work has the interesting property that it can make cosystolic expansion to be *group-independent*, i.e., it implies cosystolic expansion over any group. The key point is that an equation with exactly one non-zero variable must be unsatisfied *regardless of the group*. Thus, even though cosystolic expansion is defined over a group, δ_1-expansion implies it over any group.

We expect that this stronger δ_1-expansion notion may have further implications to quantum codes and CSPs lower bounds.

On the novelty of our work

We would like to provide a general outline of the differences between our work and previous works [18, 14].[1]

One fundamental difference is the object we analyze. The major part of previous works is dedicated to the analysis of the expansion of *arbitrary small sets*. In our work, the main analysis is focused on the expansion of *"structured" small sets* (given by the coboundary of a small set). We use a similar machinery as in previous works, but we leverage the extra structure of the small sets in order to obtain the stronger δ_1-expansion.

We note that it is not trivial how to utilize this extra structure of small sets in order to obtain δ_1-expansion. One cannot just plug it in the proof of previous works and obtain δ_1-expansion. It requires a completely different proof strategy, which we describe next.

Briefly, the proof strategy of [18] and [14] is as follows. Given a small set A of k-faces, they define a notion of "fat" faces, where an ℓ-face, $\ell \leq k$, is considered fat if a large fraction of the k-faces that contain it belongs to A. It is trivial that in dimension $\ell = k$, the set A sits only on fat k-faces, since every k-face that belongs to A is fat (because the only k-face that contains a k-face is itself). It is also trivial that in dimension $\ell = -1$, A sits only on thin (-1)-faces, since the only (-1)-face is the empty set which is contained in all of the k-faces of the complex, and A is a small set of k-faces. Therefore, there must exist a dimension $\ell \leq k$ for which a transition from mostly fat faces to mostly thin faces occurs, i.e., in dimension ℓ, A sits mostly on fat ℓ-faces, and in dimension $\ell - 1$, A sits mostly on thin $(\ell - 1)$-faces. Their argument is then that the fat ℓ-faces contribute to the parity expansion of A, whereas the thin $(\ell - 1)$-faces account for a negligible error term.

The proof strategy in our work is essentially the opposite. A fat face, which contributes to the parity expansion in the works of [18] and [14], does not have a large δ_1, and hence it is impossible to obtain δ_1-expansion from the fat faces. Our main idea is to gain the δ_1-expansion *from the thin faces*. We observe that if a set sits mostly on thin faces of *one dimension below* then it has a large δ_1. Thus, it is crucial for us to show that a small set A

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[1] Informal note: We are aware that the following explanation might seem not entirely clear at this stage of the paper for an unfamiliar reader. Nevertheless, since we have been repeatedly asked for the differences between our work and previous works, it is important for us to point that out as early as possible. We hope that the main ideas are still clear, even if not all the details are.

of k-faces actually sits mostly on thin faces of one dimension below. Using the terminology of previous paragraph, we have to show that the transition from mostly fat faces to mostly thin faces happens in dimension k itself.

This is where the "structure" comes into play. By considering small sets that are obtained as a coboundary of another set, we know that their own coboundary is 0. Without getting too much into the details, it allows us to bound the fraction of fat faces of dimension ℓ by the fraction of fat faces of dimension $\ell - 1$, for every $0 \le \ell \le k - 1$. Thus, since there are no fat faces in dimension -1 (because A is small), we conclude that there are no fat faces at any dimension! Therefore, A sits mostly on thin $(k-1)$-faces and hence has a large δ_1.

1.1   Some basic definitions

Coboundary and cosystolic expansion

For the sake of introduction we formally define coboundary and cosystolic expansions only over the field $\mathbb{F}_2$. The general definitions will be given in section 2.

Recall that a d-dimensional simplicial complex X is a downwards closed $(d+1)$-hypergraph. A k-face of X is a hyperedge of size $k + 1$, and the set of k-faces of X is denoted by $X(k)$. An assignment of values from $\mathbb{F}_2$ to the k-faces, $k \le d$, is called a k-cochain, and the space of all k-cochains over $\mathbb{F}_2$ is denoted by $C^k(X; \mathbb{F}_2)$.

Any assignment to the k-faces $f \in C^k(X; \mathbb{F}_2)$ induces an assignment to the $(k+1)$-faces by the coboundary operator δ. For any $(k+1)$-face $\sigma = \{v_0, \ldots, v_{k+1}\}$, $\delta(f)(\sigma)$ is defined by

$$\delta(f)(\sigma) = \sum_{i=0}^{k+1} f(\sigma \setminus \{v_i\}) \pmod{2}.$$

We can view the complex as inducing a system of equations, where the equations are determined by the coboundary operator; i.e., each $(k+1)$-face $\sigma \in X(k+1)$ defines the equation $\delta(f)(\sigma) = 0$. The assignments that satisfy all the equations are called the k-cocycles and denoted by

$$Z^k(X; \mathbb{F}_2) = \{f \in C^k(X; \mathbb{F}_2) \mid \delta(f) = \mathbf{0}\}.$$

One can check that $\delta(\delta(f)) = \mathbf{0}$ always holds; i.e., every assignment that is obtained as a coboundary of one dimension below satisfies all the equations. These assignments, that are the coboundary of an assignment of one dimension below, are called the k-coboundaries and denoted by

$$B^k(X; \mathbb{F}_2) = \{\delta(f) \mid f \in C^{k-1}(X; \mathbb{F}_2)\}.$$

Note that $B^k(X; \mathbb{F}_2) \subseteq Z^k(X; \mathbb{F}_2) \subseteq C^k(X; \mathbb{F}_2)$.

For a d-dimensional simplicial complex X, let $P_d : X(d) \to \mathbb{R}_{\ge 0}$ be a probability distribution over the d-faces of the complex. For simplicity, we will assume in this work that P_d is the uniform distribution. This probability distribution over the d-faces induces a probability distribution P_k for every dimension $k < d$ by selecting a d-face σ_d according to P_d and then selecting a k-face $\sigma_k \subset \sigma_d$ uniformly at random.

The weight of any k-cochain $f \in C^k(X; \mathbb{F}_2)$ is defined by

$$\|f\| = \Pr_{\sigma \sim P_k}[f(\sigma) \ne 0],$$

i.e., the (weighted) fraction of non-zero elements in f. The distance between two k-cochains $f, g \in C^k(X; \mathbb{F}_2)$ is defined as $\text{dist}(f, g) = \|f - g\|$.

We can now introduce the notions of coboundary and cosystolic expansion. As mentioned, a complex is said to be a cosystolic expander if for any assignment that does not satisfy all the equations it holds that the fraction of unsatisfied equations is proportional to the distance of the assignment from a satisfying assignment. Formally:

▶ **Definition 1.1** (Cosystolic expansion). *A d-dimensional simplicial complex X is said to be an (ε, μ)-cosystolic expander over $\mathbb{F}_2$, if for every $k < d$:*
1. *For any $f \in C^k(X; \mathbb{F}_2) \setminus Z^k(X; \mathbb{F}_2)$ it holds that*

$$\frac{\|\delta(f)\|}{dist(f, Z^k(X; \mathbb{F}_2))} \geq \varepsilon,$$

   where $dist(f, Z^k(X; \mathbb{F}_2)) = \min\{dist(f, g) \mid g \in Z^k(X; \mathbb{F}_2)\}$.
2. *For any $f \in Z^k(X; \mathbb{F}_2) \setminus B^k(X; \mathbb{F}_2)$ it holds that $\|f\| \geq \mu$.*

The second condition in the definition ensures that the complex cannot be split into many small pieces, i.e., any satisfying assignment that is not obtained as a coboundary must be large.

Coboundary expansion has been introduced by Linial and Meshulam [20] and independently by Gromov [17]. It is a similar but stronger notion than cosystolic expansion. The main difference is that the only satisfying assignments in a coboundary expander are coboundaries (unlike cosystolic expansion, where there could be satisfying assignments which are not coboundaries as long as they are large). Formally:

▶ **Definition 1.2** (Coboundary expansion). *A d-dimensional simplicial complex X is said to be an ε-coboundary expander over $\mathbb{F}_2$ if for every $k < d$ and $f \in C^k(X; \mathbb{F}_2) \setminus B^k(X; \mathbb{F}_2)$ it holds that*

$$\frac{\|\delta(f)\|}{dist(f, B^k(X; \mathbb{F}_2))} \geq \varepsilon,$$

where $dist(f, B^k(X; \mathbb{F}_2)) = \min\{dist(f, g) \mid g \in B^k(X; \mathbb{F}_2)\}$.

Local spectral expansion

Another notion of high dimensional expansion, called *local spectral expansion* is concerned with the spectral properties of local parts of the complex.

For every face $\sigma \in X$, its local view, also called its *link*, is a $(d - |\sigma| - 1)$-dimensional simplicial complex defined by $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$. The probability distribution over the top faces of X_σ is induced from the probability distribution of X, where for any top face $\tau \in X_\sigma(d - |\sigma| - 1)$, its probability is the probability to choose $\sigma \cup \tau$ in X conditioned on choosing σ. Since we assume in this work that the probability distribution over the top faces of X is the uniform distribution, it follows that the probability distribution over the top faces of X_σ is the uniform distribution.

We can now introduce the notion of a local spectral expander.

▶ **Definition 1.3** (Local spectral expansion). *A d-dimensional simplicial complex X is called a λ-local spectral expander if for every $k \leq d - 2$ and $\sigma \in X(k)$, the underlying graph[2] of X_σ is a λ-spectral expander.*

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[2] The graph whose vertices are $X_\sigma(0)$ and its edges are $X_\sigma(1)$.

1.2 Summary of main results

Our main result is a "unique-neighbor-like" expansion for non-local small sets, which we call δ_1-expansion. We start with the definition of the δ_1 of a set.

▶ **Definition 1.4** (δ_1)**.** *Let X be a d-dimensional simplicial complex. For any set of k-faces $A \subseteq X(k)$, $0 \leq k \leq d - 1$, we define $\delta_1(A) \subseteq X(k + 1)$ to be the set of $(k + 1)$-faces that contain exactly one k-face from A.*

Towards proving that small sets have a large δ_1 we introduce an intermediate notion of *non-local sets*. Roughly speaking, we say that a set of k-faces is non-local if its "local view" in almost all of the $(k - 1)$-faces resemble the global picture.

In order to define this notion of non-local sets, we first define the *localization* of a set to a link of a face. For any set $A \subseteq X(k)$ and an ℓ-face $\sigma \in X(\ell)$, $\ell < k$, the localization of A to the link of σ is a set of $(k - \ell - 1)$-faces in the link of σ defined by

$$A_\sigma = \{\tau \in X_\sigma(k - \ell - 1) \mid \sigma \cup \tau \in A\}.$$

We also add a useful definition of a mutual weight of two sets. For $\ell < k$ and two sets $A \subseteq X(k), B \subseteq X(\ell)$ we define their mutual weight by

$$\|(A, B)\| = \Pr_{\sigma_k \sim P_k, \sigma_\ell \subset \sigma_k} [\sigma_k \in A \wedge \sigma_\ell \in B],$$

where σ_k is chosen according to the distribution P_k and σ_ℓ is an ℓ-face chosen uniformly from σ_k (i.e., σ_ℓ is chosen according to P_ℓ conditioned on σ_k being chosen). This notion captures how much the sets are related. For instance, if $\|(A, B)\| \approx \|A\|$, it means that A contains mostly faces from B.

We can now define non-local sets.

▶ **Definition 1.5** (Non-local sets)**.** *Let X be a d-dimensional simplicial complex and $0 < \eta, \varepsilon < 1$. For any set of k-faces $A \subseteq X(k)$, $0 \leq k \leq d - 1$, we define the following set of $(k - 1)$-faces:*

$$S_{k-1} = \{\sigma \in X(k - 1) \mid \|A_\sigma\| \leq \eta\}.$$

We say that A is (η, ε)-non-local if $\|(A, S_{k-1})\| \geq (1 - \varepsilon) \|A\|$.

As a simple example of a "local" set, consider a set of edges A composed of all the edges touching a single vertex. In this case, $\|(A, S_0)\| = (1/2) \|A\|$. It can be easily checked that in this example, all triangles contain either 0 or 2 edges. As can be seen from this example, local sets are not necessarily δ_1-expanding.

The first theorem we show is that non-local sets are δ_1-expanding.

▶ **Theorem 1.6** (Non-local sets are δ_1-expanding - informal)**.** *Let X be a d-dimensional local spectral expander. For any $A \subseteq X(k)$, $1 \leq k \leq d - 1$, if A is non-local then $\|\delta_1(A)\| \geq \Omega(\|A\|)$.*

We consider now a bounded degree local spectral expander whose links are coboundary expanders, where a complex is said to be q-bounded degree if every vertex is contained in at most q top faces. We show that every set of unsatisfied equations can be treated as if it is non-local. Specifically, we consider sets of the form $\operatorname{supp}(\delta(f))$ for a k-cochain $f \in C^k(X; G)$, over some group G. We show a procedure that is given a k-cochain f such that $\|\delta(f)\|$ is small, and returns a k-cochain f' which is close to f such that $\delta(f')$ is non-local.

▶ **Theorem 1.7** (Correction algorithm – informal). *Let X be a d-dimensional bounded degree local spectral expander with coboundary expanding links over a group G. For any $f \in C^k(X; G)$, $1 \leq k \leq d - 2$, if $\|\delta(f)\|$ is sufficiently small, then f is close to a k-cochain $f' \in C^k(X; G)$ such that $\delta(f')$ is small and non-local. Furthermore, there is an efficient algorithm that is given f and finds f'.*

We conclude by a similar reduction as in [18] in order to obtain cosystolic expansion over any group. [18] and [14] could obtain cosystolic expansion only over $\mathbb{F}_2$ because their expansion for small sets only guaranteed that they touch many faces of one dimension above an odd number of times. Since we show here δ_1-expansion for such sets, we obtain cosystolic expansion which does not depend on the group.

▶ **Theorem 1.8** (Cosystolic expansion over any group – informal). *Let X be a d-dimensional bounded degree local spectral expander with coboundary expanding links over a group G. Then the $(d-1)$-skeleton[3] of X is a cosystolic expander over G.*

A concrete example of simplicial complexes for which our theorems apply to are the famous Ramanujan complexes [25, 24], which are the high dimensional analog of the celebrated LPS Ramanujan graphs [23]. These complexes are local spectral expanders [14] and their links, called spherical buildings, are coboundary expanders [22]. We note that [22] proved that spherical buildings are coboundary expanders only over $\mathbb{F}_2$, but their proof can be easily generalized to any abelian group by considering localizations with orientations of k-cochains. As for non-abelian groups, [13] proved that spherical buildings are coboundary expanders over non-abelian groups as well. For more on Ramanujan complexes, we refer the reader to [21].

▶ **Corollary 1.9** (Ramanujan complexes are cosystolic expanders over any group). *Let X be a d-dimensional Ramanujan complex. If X is sufficiently thick[4], then the $(d-1)$-skeleton of X is a cosystolic expander over any group G.*

1.3 Organization

In section 2 we provide some required preliminaries. In section 3 we prove the δ_1-expansion and cosystolic expansion results over abelian groups. In section 4 we provide the definitions for cochains over non-abelian groups and we repeat the same process as in section 3, but this time for non-abelian groups. The general strategy is the same for abelian and non-abelian groups, but the details are different, hence we split them into different sections.

2 Preliminaries

Coboundary and cosystolic expansion over abelian groups

Let X be a d-dimensional simplicial complex and G an abelian group[5]. We first consider an ordered version of the complex and denote it by $\vec{X}$, where

$$\vec{X} = \{(v_0, \ldots, v_k) \mid k \leq d, \ \{v_0, \ldots, v_k\} \in X\},$$

i.e., $\vec{X}$ contains all possible orderings of every face in X.

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[3] The complex which contains the faces of X up to dimension $d - 1$.

[4] The explanation of the "thickness" of a Ramanujan complex is out of scope of this paper. It is only important for us that a Ramanujan complex can be made arbitrarily thick in order to satisfy the required criteria.

[5] For simplicity we deal here only with abelian groups. We discuss non-abelian groups in section 4.

A k-cochain over G, $k \leq d$, is an antisymmetric function $f : \vec{X}(k) \to G$, where f is said to be antisymmetric if for any permutation $\pi \in Sym(k+1)$,

$$f((v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(k)})) = sgn(\pi)f((v_0, v_1, \dots, v_k)).$$

The space of all k-cochains over G is denoted by $C^k(X; G)$.

Any k-cochain is an assignment to the k-faces and it induces a $(k+1)$-cochain, i.e., an assignment to the $(k+1)$-faces, by the coboundary operator δ. For any ordered $(k+1)$-face $\sigma = (v_0, \dots, v_{k+1})$, $\delta(f)(\sigma)$ is defined by

$$\delta(f)(\sigma) = \sum_{i=0}^{k+1}(-1)^i f(v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+1}),$$

where the sum is performed over the group. It is not hard to check that for every k and $f \in C^k(X; G)$, $\delta(f)$ is antisymmetric, i.e., a $(k+1)$-cochain.

We can view the complex as inducing a system of equations, where the equations are determined by the coboundary operator; i.e., each $(k+1)$-face $\sigma \in X(k+1)$ defines the equation $\delta(f)(\sigma) = 0$ (note that the ordering of the face does not matter for the satisfaction of the equation). The assignments that satisfy all the equations are called the *k-cocycles* and denoted by

$$Z^k(X; G) = \{f \in C^k(X; G) \mid \delta(f) = \mathbf{0}\}.$$

One can check that $\delta(\delta(f)) = \mathbf{0}$ always holds; i.e., every assignment that is obtained as a coboundary of one dimension below satisfies all the equations. These assignments, that are the coboundary of an assignment of one dimension below, are called the *k-coboundaries* and denoted by

$$B^k(X; G) = \{\delta(f) \mid f \in C^{k-1}(X; G)\}.$$

Note that $B^k(X; G) \subseteq Z^k(X; G) \subseteq C^k(X; G)$.

Recall that the weight of a k-cochain $f \in C^k(X; G)$ is defined by

$$\|f\| = \Pr_{\sigma \sim P_k}[f(\sigma) \neq 0],$$

i.e., the (weighted) fraction of non-zero elements in f. Since the weight of a cochain is dependent only on its non-zero elements, it is often convenient to consider the set $\mathrm{supp}(f)$ (i.e., the set of non-zero elements in f) and define equivalently

$$\|f\| = \|\mathrm{supp}(f)\| = \Pr_{\sigma \sim P_k}[\sigma \in \mathrm{supp}(f)].$$

For simplicity, we might abuse the notation and write $\sigma \in f$ where we mean that $\sigma \in \mathrm{supp}(f)$.

Now, the formal definitions of coboundary and cosystolic expansion over general abelian groups are identical to definitions 1.1 and 1.2 given in the introduction, with the replacement of $\mathbb{F}_2$ by an abelian group G.

Links and localization

Recall that the link of a k-face $\sigma \in X(k)$ is a $(d - k - 1)$-dimensional complex defined by $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$, where the probability distribution over faces of X_σ is induced from the probability distribution over faces of X. Since we assume in this work that P_d is

the uniform distribution over the d-faces of X, it follows that the probability distribution over the top faces of X_σ is the uniform distribution. In the rest of the paper, we will omit the explicit probability distribution when it is clear from the context.

Recall also that cochains over abelian groups are defined on ordered faces of the complex. For convenience sake, we fix an arbitrary ordering of the faces so that for any face $\sigma \in X$ there is a unique corresponding ordered face $\vec{\sigma} \in \vec{X}$.

For two disjoint ordered faces $\vec{\sigma} = (v_0, \dots, v_k)$ and $\vec{\tau} = (u_0, \dots, u_\ell)$ we denote their concatenation by $\vec{\sigma}\vec{\tau} = (v_0, \dots, v_k, u_0, \dots, u_\ell)$. For any k-face $\sigma \in X(k)$ and a $(k + \ell + 1)$-cochain $f \in C^{k+\ell+1}(X; G)$, the *localization* of f to the link of σ is an ℓ-cochain in the link of σ, $f_\sigma \in C^\ell(X_\sigma; G)$ defined as follows. For any ordered ℓ-face $\vec{\tau} \in \vec{X_\sigma}(\ell)$, $f_\sigma(\vec{\tau}) = f(\vec{\sigma}\vec{\tau})$, where $\vec{\sigma}\vec{\tau}$ is the concatenation of $\vec{\sigma}$ (i.e., the unique corresponding ordered face of σ) and $\vec{\tau}$.

By the law of total probability, the weight of any k-cochain $f \in C^k(X; G)$ can be decomposed as a sum of its weight in the links of the ℓ-faces of the complex:

▶ **Lemma 2.1.** *Let X be a d-dimensional simplicial complex and G an abelian group. For every $f \in C^k(X; G)$, $k \le d$ and $\ell < k$,*

$$\|f\| = \sum_{\tau \in X(\ell)} \|(f, \tau)\|,$$

where $\|(f, \tau)\|$ is the mutual weight of $supp(f) \subseteq X(k)$ and $\{\tau\} \subseteq X(\ell)$.

Proof. It follows immediately from the definitions:

$$\|f\| = \Pr_{\sigma \sim P_k}[\sigma \in supp(f)] = \sum_{\tau \in X(\ell)} \Pr_{\sigma \sim P_k, \tau' \subset \sigma}[\sigma \in supp(f) \wedge \tau' = \tau] = \sum_{\tau \in X(\ell)} \|(f, \tau)\|,$$

where the second equality follows from the law of total probability. ◀

Minimal and locally minimal cochains

One of the technical notions we use in this work is the notion of a minimal cochain. We say that a k-cochain $f \in C^k(X; G)$ is *minimal* if its weight cannot be reduced by adding a coboundary to it, i.e., for every $g \in B^k(X; G)$ it holds that $\|f\| \le \|f - g\|$. Recall that the distance of f from the coboundaries is defined by $dist(f, B^k(X; G)) = \min\{dist(f, g) \mid g \in B^k(X; G)\}$. Since $\mathbf{0} \in B^k(X; G)$, it follows that for every $f \in C^k(X; G)$, $\|f\| \ge dist(f, B^k(X; G))$. Hence, f is minimal if and only if $\|f\| = dist(f, B^k(X; G))$.

We also define the notion of a locally minimal cochain, where we say that $f \in C^k(X; G)$ is *locally minimal* if for every vertex v, the localization of f to the link of v is minimal in the link, i.e., f_v is minimal in X_v for every $v \in X(0)$.

Cheeger inequality for graphs

A 1-dimensional simplicial complex X is just a graph. In this case the known Cheeger inequality gives the following (see e.g. [18] for a proof):

▶ **Lemma 2.2.** *Let X be a 1-dimensional simplicial complex which is a λ-spectral expander graph. For any set of vertices $A \subseteq X(0)$ it holds that*
1. $\|E(A, \overline{A})\| \ge 2(1 - \lambda) \|A\| \|\overline{A}\|$,
2. $\|E(A)\| \le (\|A\| + \lambda) \|A\|$,
where $E(A, \overline{A})$ is the set of edges with one endpoint in A and one endpoint in $\overline{A}$, and $E(A)$ is the set of edges with both endpoints in A.

3 Result for abelian groups

3.1 Non-local sets are δ_1-expanding

In this section we show our results for abelian groups.

Our first theorem is that non-local sets in a local spectral expander have δ_1 that is proportional to their size. We prove Theorem 1.6 which we restate here in a formal way.

▶ **Theorem 3.1** (Non-local sets are δ_1-expanding). *Let X be a d-dimensional λ-local spectral expander and $0 < \eta, \varepsilon < 1$. For any $A \subseteq X(k)$, $0 \leq k \leq d-1$, such that A is (η, ε)-non-local it holds that*

$$\|\delta_1(A)\| \geq \left(1 - \binom{k+2}{k}\left(\lambda + \eta + 2\varepsilon\right)\right) \|A\|.$$

Proof. Recall that we denote by S_{k-1} the set of $(k-1)$-faces $\sigma \in X(k-1)$ satisfying $\|A_\sigma\| \leq \eta$. Let us define the following sets of $(k+1)$-faces:

- $\Gamma(A) = \{\tau \in X(k+1) \mid \exists \sigma \in A \text{ s.t. } \sigma \subset \tau\}$.
- $\Gamma(A, \overline{S_{k-1}}) = \left\{\tau \in X(k+1) \mid \exists \sigma \in A, \ \sigma' \in \overline{S_{k-1}} \text{ s.t. } \sigma' \subset \sigma \subset \tau\right\}$.
- $\Upsilon = \{\tau \in X(k+1) \mid \exists \sigma, \sigma' \in A \text{ s.t. } \sigma, \sigma' \subset \tau, \ \sigma \cap \sigma' \in S_{k-1}\}$.

In words: $\Gamma(A)$ is the set of all $(k+1)$-faces that contain a k-face from A, $\Gamma(A, \overline{S_{k-1}})$ is the set of all $(k+1)$-faces that contain a k-face from A which contains a $(k-1)$-face from $\overline{S_{k-1}} = X(k-1) \setminus S_{k-1}$, and Υ is the set of all $(k+1)$-faces that contain two k-faces from A such that their intersection is a $(k-1)$-face from S_{k-1}.

Note that for every $\tau \in \Gamma(A) \setminus \Gamma(A, \overline{S_{k-1}})$ one of the following cases must hold: Either τ contains exactly one k-face from A, i.e., $\tau \in \delta_1(A)$, or τ contains at least two k-faces from A such that their intersection belongs to S_{k-1}, i.e., $\tau \in \Upsilon$. It follows that

$$\|\delta_1(A)\| \geq \left\|\Gamma(A) \setminus (\Gamma(A, \overline{S_{k-1}}) \cup \Upsilon)\right\| \geq \|\Gamma(A)\| - \left\|\Gamma(A, \overline{S_{k-1}})\right\| - \|\Upsilon\|. \tag{1}$$

Let us bound each of the above terms separately. First, by simple laws of probability

$$\begin{aligned}
\|\Gamma(A)\| &= \Pr[\sigma_{k+1} \in \Gamma(A)] \\
&\geq \Pr[\sigma_{k+1} \in \Gamma(A) \wedge \sigma_k \in A] \\
&= \Pr[\sigma_k \in A] \cdot \Pr[\sigma_{k+1} \in \Gamma(A) \mid \sigma_k \in A] \\
&= \Pr[\sigma_k \in A] = \|A\|.
\end{aligned} \tag{2}$$

Second, again by laws of probability

$$\begin{aligned}
\left\|\Gamma(A, \overline{S_{k-1}})\right\| &= \Pr[\sigma_{k+1} \in \Gamma(A, \overline{S_{k-1}})] \\
&= \frac{\Pr[\sigma_{k+1} \in \Gamma(A, \overline{S_{k-1}}) \wedge \sigma_k \in A \wedge \sigma_{k-1} \notin S_{k-1}]}{\Pr[\sigma_k \in A \wedge \sigma_{k-1} \notin S_{k-1} \mid \sigma_{k+1} \in \Gamma(A, \overline{S_{k-1}})]} \\
&\leq (k+2)(k+1) \Pr[\sigma_k \in A \wedge \sigma_{k-1} \notin S_{k-1}] \\
&= (k+2)(k+1) \left\|(A, \overline{S_{k-1}})\right\| \\
&\leq (k+2)(k+1)\varepsilon \|A\|,
\end{aligned} \tag{3}$$

where the first inequality holds since the probability that $\sigma_k \in A$ and $\sigma_{k-1} \notin S_{k-1}$ given that $\sigma_{k+1} \in \Gamma(A, \overline{S_{k-1}})$ is at least $1/((k+2)(k+1))$, and the second inequality follows since A is an (η, ε)-non-local set.

Lastly, consider a $(k+1)$-face $\tau \in \Upsilon$. By definition, τ contains two k-faces $\sigma, \sigma' \in A$ such that $\sigma \cap \sigma' \in S_{k-1}$. Let us denote $\breve{\tau} = \sigma \cap \sigma'$. Note that τ is seen in the link of $\breve{\tau}$ as an edge between two vertices in $A_{\breve{\tau}}$, i.e., $\tau \setminus \breve{\tau} \in E(A_{\breve{\tau}})$. Thus,

$$
\begin{aligned}
\|\Upsilon\| &= \sum_{\tau \in \Upsilon} \Pr[\sigma_{k+1} = \tau] \\
&= \sum_{\tau \in \Upsilon} \frac{\Pr[\sigma_{k+1} = \tau \wedge \sigma_{k-1} = \breve{\tau}]}{\Pr[\sigma_{k-1} = \breve{\tau} \mid \sigma_{k+1} = \tau]} \\
&\leq \sum_{\tau \in \Upsilon} \binom{k+2}{k} \Pr[\sigma_{k+1} = \tau \mid \sigma_{k-1} = \breve{\tau}] \cdot \Pr[\sigma_{k-1} = \breve{\tau}] \\
&\leq \binom{k+2}{k} \sum_{\sigma \in S_{k-1}} \|E(A_\sigma)\| \cdot \Pr[\sigma_{k-1} = \sigma] \\
&\leq \binom{k+2}{k} \sum_{\sigma \in S_{k-1}} (\|A_\sigma\| + \lambda) \|A_\sigma\| \cdot \Pr[\sigma_{k-1} = \sigma] \\
&\leq \binom{k+2}{k} (\eta + \lambda) \sum_{\sigma \in S_{k-1}} \|(A_\sigma, \sigma)\| \leq \binom{k+2}{k} (\eta + \lambda) \|A\|,
\end{aligned}
\tag{4}
$$

where the third inequality follows since X is a λ-local spectral expander, and the fourth inequality follows since $\sigma \in S_{k-1}$.

Substituting (2), (3) and (4) in (1) finishes the proof. ◄

An immediate corollary of Theorem 3.1 is that any non-local cocycle must be zero.

▶ **Corollary 3.2** (Non-local cocycles vanish). *For any $d \in \mathbb{N}$, an abelian group G, and $0 < \lambda, \eta, \varepsilon < 1$ such that $\lambda + \eta + 2\varepsilon \leq 2/(d+1)^2$ the following holds: Let X be a d-dimensional λ-local spectral expander. For any $f \in Z^k(X; G)$, $0 \leq k \leq d-1$, if f is (η, ε)-non-local then $f = \mathbf{0}$.*

Proof. Since f is (η, ε)-non-local, then by Theorem 3.1 it holds that

$$
\|\delta_1(f)\| \geq \frac{\|f\|}{d+1}.
$$

On the other hand, $f \in Z^k(X; G)$ and hence $\|\delta_1(f)\| \leq \|\delta(f)\| = 0$. It follows that $f = \mathbf{0}$ as required. ◄

3.2 The correction procedure

Our aim now is to show a correction procedure for small coboundaries. We show an algorithm that gets a cochain f such that $\|\delta(f)\|$ is small and returns a cochain f' by making a few changes to f such that $\delta(f')$ is non-local.

We start by showing that any small and locally minimal cocycle is non-local. We note that the following proposition is the main technical contribution of our work.

▶ **Proposition 3.3** (Small and locally minimal cocycles are non-local). *For any $d \in \mathbb{N}$, an abelian group G, and $0 < \beta, \varepsilon < 1$ there exist $0 < \lambda, \eta \leq \varepsilon$ such that the following holds: Let X be a d-dimensional λ-local spectral expander with β-coboundary expanding links. For any $f \in Z^k(X; G)$, $1 \leq k \leq d-1$, if $\|f\| \leq \eta^{2^{k+1}-1}$ and locally minimal then f is (η, ε)-non-local.*

In order to prove Proposition 3.3 we need a few more definitions and lemmas. The proofs of the lemmas can be found in the full version of the paper. Let $f \in C^k(X; G)$, $0 \le k \le d-1$. Recall that S_{k-1} is the set of $(k-1)$-faces σ satisfying $\|f_\sigma\| \le \eta$. For any $-1 \le i \le k-2$, we define the following set of i-faces:

$$S_i = \{\sigma \in X(i) \mid (\overline{S_{i+1}})_\sigma \le \eta^{2^{k-i-1}}\}.$$

We first show that if $\|f\|$ is sufficiently small then $\|S_{-1}\| = 1$, i.e., the empty set belongs to S_{-1}. We will use the following lemma:

▶ **Lemma 3.4.** *Let X be a d-dimensional simplicial complex, G an abelian group and $0 < \eta < 1$. For any $f \in C^k(X; G)$, $0 \le k \le d$, if $\|f\| \le \eta^{2^{k+1}-1}$ then $\|S_{-1}\| = 1$.*

Let $\Upsilon \subseteq X(k+1)$ be the set of $(k+1)$-faces which contain two i-faces $\sigma, \sigma' \in \overline{S_i}$ such that $\sigma \cap \sigma' \in S_{i-1}$. We show that $\|\Upsilon\|$ is a negligible fraction of $\|f\|$.

▶ **Lemma 3.5.** *Let X be a d-dimensional λ-local spectral expander, G an abelian group and $0 < \eta < 1$ such that $\lambda \le \eta^{2^{d-1}}$. For any $f \in C^k(X; G)$, $0 \le k \le d-1$, it holds that*

$$\|\Upsilon\| \le \eta \binom{k+2}{2} 2^{k+2} \|f\|.$$

For any $\sigma \in X(i)$, denote by $f \downarrow \sigma$ the set of k-faces $\tau \in f$ which have a sequence of containments of faces from $\overline{S_j}$, $i < j < k$, down to σ. Formally,

$$f \downarrow \sigma = \{\tau \in f \mid \exists \tau_{k-1} \in \overline{S_{k-1}}, \ldots, \tau_{i+1} \in \overline{S_{i+1}} \text{ s.t. } \tau \supset \tau_{k-1} \supset \cdots \supset \tau_{i+1} \supset \sigma\}.$$

We show that for any cocycle $f \in Z^k(X; G)$ and $0 \le i \le k-1$, the fraction of f that sits on i-faces from $\overline{S_i}$ is approximately the fraction of f that sits on $(i-1)$-faces from $\overline{S_{i-1}}$.

▶ **Lemma 3.6.** *Let X be a d-dimensional simplicial complex such that its links are β-coboundary expanders over an abelian group G. For any locally minimal $f \in Z^k(X; G)$, $1 \le k \le d-1$, and $0 \le i \le k-1$ it holds that*

$$\sum_{\sigma \in \overline{S_i}} \|(f \downarrow \sigma, \sigma)\| \le \frac{1}{\beta}\left((k+1-i)(i+1) \sum_{\sigma' \in \overline{S_{i-1}}} \|(f \downarrow \sigma', \sigma')\| + \|\Upsilon\|\right).$$

We can now prove Proposition 3.3.

Proof of Proposition 3.3. Let

$$\eta = \frac{\beta^{d-1}\varepsilon}{2^d((d+1)!)^2} \qquad \text{and} \qquad \lambda = \eta^{2^{d-1}}.$$

Applying Lemma 3.6 on dimension $i = k-1, k-2, \ldots, 0$ step after step yields

$$\begin{aligned}
\left\|(f, \overline{S_{k-1}})\right\| &= \sum_{\sigma \in \overline{S_{k-1}}} \|(f \downarrow \sigma, \sigma)\| \\
&\le \frac{1}{\beta}\|\Upsilon\| + \frac{1}{\beta} \cdot 2 \cdot k \sum_{\sigma \in \overline{S_{k-2}}} \|(f \downarrow \sigma, \sigma)\| \\
&\le \frac{1}{\beta}\|\Upsilon\| + \frac{1}{\beta^2} \cdot 2 \cdot k \|\Upsilon\| + \frac{1}{\beta^2} \cdot 2 \cdot k \cdot 3 \cdot (k-1) \sum_{\sigma \in \overline{S_{k-3}}} \|(f \downarrow \sigma, \sigma)\| \qquad (5) \\
&\le \left(\frac{1}{\beta} + \frac{1}{\beta^2} \cdot 2 \cdot k + \cdots + \frac{1}{\beta^k}(k!)^2\right)\|\Upsilon\| + \frac{1}{\beta^k}(k!)^2(k+1) \sum_{\sigma \in \overline{S_{-1}}} \|(f \downarrow \sigma, \sigma)\| \\
&= \left(\frac{1}{\beta} + \frac{1}{\beta^2} \cdot 2 \cdot k + \cdots + \frac{1}{\beta^k}(k!)^2\right)\|\Upsilon\| \le k\beta^{-k}(k!)^2\|\Upsilon\|
\end{aligned}$$

Substituting Lemma 3.5 in (5) completes the proof.   ◀

Now, the idea of the correction algorithm is to make $\delta(f)$ locally minimal by correcting f in a few local parts. The algorithm runs in iterations, where at every iteration it does the following one step of correction.

▶ **Lemma 3.7** (One step of correction). *Let X be a d-dimensional simplicial complex and G an abelian group. For any $f \in C^k(X; G)$, $1 \leq k \leq d - 1$, if f is not locally minimal then there exists a vertex $v \in X(0)$ and a $(k-1)$-cochain $g \in C^{k-1}(X; G)$ such that $\|g\| \leq k \|v\|$ and $\|f - \delta(g)\| < \|f\|$.*

Proof. Since f is not locally minimal, there exists a vertex $v \in X(0)$ such that f_v is not minimal in X_v. By definition there exists a $(k-2)$-cochain $h \in C^{k-2}(X_v; G)$ in the link of v such that $\|f_v - \delta(h)\| < \|f_v\|$. Define $g \in C^{k-1}(X; G)$ by

$$g(\sigma) = \begin{cases} h(\tau) & \sigma = v\tau, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g_v = h$, therefore $\|f - \delta(g)\| < \|f\|$. Furthermore, since $g(\sigma) = 0$ for every σ which does not contain v it follows that

$$\|g\| = \Pr[\sigma_{k-1} \in g] = \frac{\Pr[\sigma_{k-1} \in g \wedge \sigma_0 = v]}{\Pr[\sigma_0 = v \mid \sigma_{k-1} \in g]} \leq k \|v\|. \qquad \blacktriangleleft$$

We use Lemma 3.7 iteratively to prove Theorem 1.7 which we restate here in a formal way.

▶ **Theorem 3.8** (Correction algorithm). *For any $d, q \in \mathbb{N}$, an abelian group G, and $0 < \beta, \varepsilon < 1$ there exist constants $0 < \lambda, \eta \leq \varepsilon$ such that the following holds: Let X be a d-dimensional q-bounded degree λ-local spectral expander with β-coboundary expanding links. For any $f \in C^k(X; G)$, $1 \leq k \leq d - 2$, if $\|\delta(f)\| \leq \eta^{2^{k+2}-1}$ then there exists $f' \in C^k(X; G)$ such that $dist(f, f') \leq q\binom{d}{k+1} \|\delta(f)\|$, $\|\delta(f')\| \leq \|\delta(f)\|$, and $\delta(f')$ is (η, ε)-non-local.*

Proof. Let λ and η be as in Proposition 3.3. Apply Lemma 3.7 for $\delta(f)$ step by step until no more corrections are possible. Since at every step the weight decreases, this process terminates after some $r \geq 0$ steps. Denote by $v_1, v_2, \ldots, v_r$ the vertices and by $g_1, g_2, \ldots, g_r$ the k-cochains given by applying Lemma 3.7 for r steps, where at step i we apply it for $\delta(f - g_1 - \cdots - g_{i-1})$.

Let $f' = f - g_1 - g_2 - \cdots - g_r$. Since the norm of $\delta(f)$ decreases at every step of correction, it follows that $\|\delta(f')\| \leq \|\delta(f)\| \leq \eta^{2^{k+2}-1}$. Furthermore, since no more corrections are possible, it must be that $\delta(f')$ is locally minimal. Thus, by Proposition 3.3, $\delta(f')$ is (η, ε)-non-local.

It is left to show that $\|f - f'\|$ is proportional to $\|\delta(f)\|$. By definition, for any $\sigma \in X(k+1)$ it holds that $\|\sigma\| \geq \left(|X(d)|\binom{d+1}{k+2}\right)^{-1}$, hence $r \leq |X(d)|\binom{d+1}{k+2} \|\delta(f)\|$. Thus,

$$dist(f, f') = \|g_1 + g_2 + \cdots + g_r\| \leq \sum_{i=1}^{r} (k+1) \|v_i\|$$

$$\leq |X(d)|\binom{d+1}{k+2}(k+1)\frac{q}{|X(d)|(d+1)} \|\delta(f)\| \leq \binom{d}{k+1} q \|\delta(f)\|$$

which finishes the proof. ◀

3.3 Cosystolic expansion

We use a similar reduction as in [18] in order to show that δ_1-expansion of small sets implies cosystolic expansion over any abelian group. Recall that a complex is a cosystolic expander if the following two properties hold: (1) The systems of equations are expanding, i.e., any assignment that does not satisfy all the equations has a large fraction of unsatisfied equations (proportional to the distance from a satisfying assignment). (2) Every cocycle which is not a coboundary is large. We use the following lemmas which their proofs can be found in the full version of the paper.

▶ **Lemma 3.9** (The systems of equations are expanding). *For any $d, q \in \mathbb{N}$, an abelian group G, and $0 < \beta < 1$ there exist $0 < \lambda, \eta < 1$ such that the following holds: Let X be a d-dimensional q-bounded degree λ-local spectral expander with β-coboundary expanding links over G. For any $f \in C^k(X; G) \setminus Z^k(X; G)$, $1 \le k \le d - 2$, it holds that*

$$\|\delta(f)\| \ge \min\left\{\eta^{2^{k+2}-1}, \frac{1}{q\binom{d}{k+1}}\right\} \cdot dist(f, Z^k(X; G)).$$

▶ **Lemma 3.10** (Every cocycle which is not a coboundary is large). *For any $d \in \mathbb{N}$, an abelian group G, and $0 < \beta < 1$, there exists $0 < \lambda, \eta < 1$ such that the following holds: Let X be a d-dimensional λ-local spectral expander with β-coboundary expanding links over G. For any $f \in Z^k(X; G) \setminus B^k(X; G)$, $0 \le k \le d - 1$, it holds that $\|f\| \ge \eta^{2^d-1}$.*

Theorem 1.8, which we restate here in a formal way, follows immediately from the above two lemmas.

▶ **Theorem 3.11** (Cosystolic expansion over any abelian group). *For any $d, q \in \mathbb{N}$, an abelian group G, and $0 < \beta < 1$ there exist $0 < \lambda, \eta < 1$ such that the following holds: Let X be a d-dimensional q-bounded degree λ-local spectral expander with β-coboundary expanding links over G. Then the $(d-1)$-skeleton of X is an (ε, μ)-cosystolic expander over G, where*

$$\varepsilon = \min\left\{\eta^{2^d-1}, \frac{1}{qd^{d/2}}\right\} \qquad and \qquad \mu = \eta^{2^d-1}.$$

Proof. Immediate from Lemmas 3.9 and 3.10. ◀

4 Result for non-abelian groups

4.1 Non-abelian groups

When the group is non-abelian, the coboundary operator is defined only in dimensions 0 and 1, and its definition is more delicate. Let G be a group with a multiplicative operation. The coboundary of a 0-cochain $f \in C^0(X; G)$ is a 1-cochain $\delta(f)$ defined by

$$\delta(f)(u, v) = f(u)f(v)^{-1}.$$

The coboundary of a 1-cochain $g \in C^1(X; G)$ is a 2-cochain $\delta(g)$ defined by

$$\delta(g)(u, v, w) = g(u, v)g(v, w)g(w, u).$$

One can check that for $f \in C^i(X; G)$, $i \in \{0, 1\}$, $\delta(f)$ is an antisymmetric functions, i.e., $\delta(f)$ is an $(i+1)$-cochain.

The distance between two cochains $f, g \in C^i(X; G)$ is defined by $\mathrm{dist}(f, g) = \left\| gf^{-1} \right\|$, where $gf^{-1}(\sigma) = g(\sigma)f(\sigma)^{-1}$ for every $\sigma \in \vec{X}(i)$.

Similar to the abelian case, we say that $f \in C^i(X; G)$ is a cocycle if $\delta(f) = \mathbf{1}$.[6] The distance of a cochain $f \in C^i(X; G)$ from the i-cocycles is defined by

$$\mathrm{dist}(f, Z^i(X; G)) = \min\{\mathrm{dist}(f, g) \mid g \in Z^i(X; G)\}.$$

In order to measure the distance of a 1-cochain from the 1-coboundaries, we define an action of $C^0(X; G)$ on $C^1(X; G)$, where for $f \in C^0(X; G)$ and $g \in C^1(X; G)$, $f.g(u, v)$ is defined by

$$f.g(u, v) = f(u)g(u, v)f(v)^{-1}.$$

Now, the distance of g from the 1-coboundaries is defined by

$$\mathrm{dist}(g, B^1(X; G)) = \min\{\mathrm{dist}(g, f.g) \mid f \in C^0(X; G)\}.$$

4.2 Weakly-non-local sets and cosystolic expansion

In the case of a non-abelian group, we cannot get the same non-local property as in abelian groups, rather we get a slightly weaker notion which we call *weakly-non-local*. Roughly speaking, a set of k-faces in a given complex is weakly-non-local if its k-faces are evenly distributed on their $(k-2)$-subfaces.

▶ **Definition 4.1** (Weakly-non-local sets). *Let X be a d-dimensional simplicial complex and $0 < \eta, \varepsilon, \alpha < 1$. For any set of k-faces $A \subseteq X(k)$, $1 \leq k \leq d - 1$, we define the following set of $(k-2)$-faces:*

$$S_{k-2} = \{\sigma \in X(k-2) \mid \|A_\sigma\| \leq \eta\}.$$

We say that A is $(\eta, \varepsilon, \alpha)$-weakly-non-local if $\|S_{k-2}\| \geq 1 - \varepsilon \|A\|$ and for every $\tau \in X(k-1)$ it holds that $\|A_\tau\| \leq 1 - \alpha$.

We show that this weakly-non-local property also implies that the set is δ_1-expanding.

▶ **Theorem 4.2** (Weakly-non-local sets are δ_1-expanding). *Let X be a d-dimensional λ-local spectral expander and $0 < \eta, \varepsilon, \alpha < 1$. There exists a constant $c = c(d, \lambda, \eta, \varepsilon, \alpha)$ such that for any $A \subseteq X(k)$, $1 \leq k \leq d - 1$, if A is $(\eta, \varepsilon, \alpha)$-weakly-non-local then*

$$\|\delta_1(A)\| \geq c \|A\|.$$

In particular, if $\varepsilon \leq \alpha/3d^3$, $\lambda \leq \varepsilon^2$ and $\eta \leq \varepsilon^3$ then $\|\delta_1(A)\| \geq \alpha \|A\|$.

The rest of the steps are similar to the abelian case (with modifications for non-abelian groups) and can be found in the full version of the paper. We just state here the final theorem for cosystolic expansion over non-abelian groups.

▶ **Theorem 4.3** (Cosystolic expansion over any group). *For any group G, $q \in \mathbb{N}$ and $0 < \beta < 1$ there exist $0 < \lambda, \eta < 1$ such that the following holds: Let X be a 3-dimensional q-bounded degree λ-local spectral expander with β-coboundary expanding links over G. Then the 2-skeleton of X is an (ε, μ)-cosystolic expander over G, where*

$$\varepsilon = \min\left\{\frac{\beta\eta}{2}, \frac{1}{2q}\right\} \qquad and \qquad \mu = \frac{\beta\eta}{2}.$$

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[6] It is common to denote the identity element of a multiplicative group by 1 and not by 0 as in an additive group.

References

1 V. L. Alev, F. G. Jeronimo, D. Quintana, S. Srivastava, and M. Tulsiani. List decoding of direct sum codes. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1412–1425. SIAM, 2020.

2 V. L. Alev, F. G. Jeronimo, and M. Tulsiani. Approximating constraint satisfaction problems on high-dimensional expanders. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 180–201, 2019.

3 V. L. Alev and L. C. Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 1198–1211, 2020.

4 N. Anari, K. Liu, and S. O. Gharan. Spectral independence in high-dimensional expanders and applications to the hardcore model. *arXiv preprint*, 2020. `arXiv:2001.00303`.

5 N. Anari, K. Liu, S. O. Gharan, and C. Vinzant. Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 1–12, 2019.

6 Z. Chen, A. Galanis, D. Štefankovič, and E. Vigoda. Rapid mixing for colorings via spectral independence. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1548–1557. SIAM, 2021.

7 Z. Chen, K. Liu, and E. Vigoda. Optimal mixing of glauber dynamics: Entropy factorization via high-dimensional expansion. *arXiv preprint*, 2020. `arXiv:2011.02075`.

8 Z. Chen, K. Liu, and E. Vigoda. Rapid mixing of glauber dynamics up to uniqueness via contraction. *arXiv preprint*, 2020. `arXiv:2004.09083`.

9 Y. Dikstein and I. Dinur. Agreement testing theorems on layered set systems. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1495–1524. IEEE, 2019.

10 I. Dinur, Y. Filmus, P. Harsha, and M. Tulsiani. Explicit SoS lower bounds from high-dimensional expanders. *arXiv preprint*, 2020. `arXiv:2009.05218`.

11 I. Dinur, P. Harsha, T. Kaufman, I. L. Navon, and A. Ta Shma. List decoding with double samplers. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2134–2153. SIAM, 2019.

12 I. Dinur and T. Kaufman. High dimensional expanders imply agreement expanders. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 974–985, 2017.

13 Irit Dinur and Roy Meshulam. Near coverings and cosystolic expansion–an example of topological property testing. *arXiv preprint*, 2019. `arXiv:1909.08507`.

14 S. Evra and T. Kaufman. Bounded degree cosystolic expanders of every dimension. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 36–48, 2016.

15 S. Evra, T. Kaufman, and G. Zemor. Decodable quantum LDPC codes beyond the $\sqrt{n}$ distance barrier using high dimensional expanders. In *61st Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 2020. to appear.

16 W. Feng, H. Guo, Y. Yin, and C. Zhang. Rapid mixing from spectral independence beyond the boolean domain. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1558–1577. SIAM, 2021.

17 M. Gromov. Singularities, Expanders and Topology of Maps. Part 2: from Combinatorics to Topology Via Algebraic Isoperimetry. *Geometric And Functional Analysis*, 20(2):416–526, 2010.

18 T. Kaufman, D. Kazhdan, and A. Lubotzky. Ramanujan Complexes and Bounded Degree Topological Expanders. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 484–493, 2014.

19   T. Kaufman and D. Mass. Local-To-Global Agreement Expansion via the Variance Method. In *11th Innovations in Theoretical Computer Science Conference (ITCS)*, volume 151, pages 74:1–74:14, 2020.

20   N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

21   A. Lubotzky. Ramanujan complexes and high dimensional expanders. *Japanese Journal of Mathematics*, 9(2):137–169, 2014.

22   A. Lubotzky, R. Meshulam, and S. Mozes. Expansion of building-like complexes. *Groups, Geometry, and Dynamics*, 10(1):155–175, 2016.

23   A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

24   A. Lubotzky, B. Samuels, and U. Vishne. Explicit constructions of ramanujan complexes of type $\widetilde{A}_d$. *European Journal of Combinatorics*, 26(6):965–993, 2005.

25   A. Lubotzky, B. Samuels, and U. Vishne. Ramanujan complexes of type $\widetilde{A}_d$. *Israel Journal of Mathematics*, 149(1):267–299, 2005.