



Smaller ACC0 Circuits for Symmetric Functions

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Abstract

What is the power of constant-depth circuits with MOD_m gates, that can count modulo m ? Can they efficiently compute MAJORITY and other symmetric functions? When m is a constant prime power, the answer is well understood. In this regime, Razborov and Smolensky proved in the 1980s that MAJORITY and MOD_m require super-polynomial-size MOD_q circuits, where q is any prime power not dividing m . However, relatively little is known about the power of MOD_m gates when m is not a prime power. For example, it is still open whether every problem decidable in exponential time can be computed by depth-3 circuits of polynomial-size and only MOD_6 gates.

In this paper, we shed some light on the difficulty of proving lower bounds for MOD_m circuits, by giving new upper bounds. We show how to construct MOD_m circuits computing symmetric functions with non-prime power m , with size-depth tradeoffs that beat the longstanding lower bounds for $\text{AC}^0[m]$ circuits when m is a prime power. Furthermore, we observe that our size-depth tradeoff circuits have essentially optimal dependence on m and d in the exponent, under a natural circuit complexity hypothesis.

For example, we show that for every $\varepsilon > 0$, every symmetric function can be computed using MOD_m circuits of depth 3 and 2^{n^ε} size, for a constant m depending only on $\varepsilon > 0$. In other words, depth-3 CC^0 circuits can compute any symmetric function in *subexponential* size. This demonstrates a significant difference in the power of depth-3 CC^0 circuits, compared to other models: for certain symmetric functions, depth-3 AC^0 circuits require $2^{\Omega(\sqrt{n})}$ size [Håstad 1986], and depth-3 $\text{AC}^0[p^k]$ circuits (for fixed prime power p^k) require $2^{\Omega(n^{1/6})}$ size [Smolensky 1987]. Even for depth-2 $\text{MOD}_p \circ \text{MOD}_m$ circuits, $2^{\Omega(n)}$ lower bounds were known [Barrington Straubing Thérien 1990].

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1 Introduction

We consider constant-depth circuits in which every (unbounded fan-in) gate (called a MOD_m gate) determines whether the sum of its inputs is divisible by a small constant integer m . Although the model looks rather peculiar, constant-depth circuits with constant moduli gates (a.k.a. CC^0 circuits, a.k.a. pure-ACC circuits [39]) have been a longstanding and fundamental roadblock in the way of improved circuit complexity lower bounds. Since their identification

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over 30 years ago [8, 11], scant progress has been made on lower bounds against CC^0 circuits, and their close cousin ACC^0 which includes AND and OR in the gate basis. Some exceptions include work focusing on special cases of the problem (e.g., [9, 21, 15, 16]), uniform lower bounds [2], and work proving strong lower bounds but only for functions whose complexity is in QuasiNP or higher (e.g., [38, 18, 29, 17]). If there has ever been a “circuit complexity winter”, CC^0 circuits are at least partly to blame.

Besides our own ignorance, could there be deeper reasons why CC^0 circuits have been so difficult for showing limitations? In this paper, we explore the possibility that CC^0 circuits may be powerful, focusing on the natural class of *symmetric Boolean functions* whose output depends only on the number of ones in the input. Although it has been conjectured for many years that the AND function does not have polynomial-size CC^0 circuits ([7, 36, 35])² our results show that low-depth MOD_m circuits with arbitrary but fixed modulus m can actually compute arbitrary symmetric Boolean functions (such as MAJORITY) much more efficiently than low-depth circuits with AND, OR, and MOD_q gates, when q is a prime power.

It is well-known that AC^0 circuits, which consist of AND, OR, NOT gates and have constant-depth, require $\exp(\Omega(n^{1/(d-1)}))$ size to compute arbitrary symmetric functions in depth d [25]. In recent work, Oliveira, Santhanam, and Srinivasan [32] have shown that PARITY gates (a.k.a. MOD_2 gates) can help compute symmetric functions more efficiently than what AND, OR, NOT can accomplish in constant depth. In particular, they show that $AC^0[2]$ circuits (with AND, OR, and PARITY) of depth 4 can compute MAJORITY in $\exp(\Theta(n^{1/4}))$ size, depth $d \geq 5$ can compute symmetric functions in size $\exp(\tilde{O}(n^{\frac{2}{3(d-4)}}))$, and they show a size lower bound of $\exp(\Omega(n^{1/(2d-4)}))$ for the MAJORITY function, improving [33, 34].

Smaller MOD_m Circuits

Could even smaller circuits for symmetric functions be achieved using MOD_m gates, for other composite m ? It turns out that this is possible. In fact, even in depth three, any symmetric function can be computed with a MOD_m circuit of size 2^{n^ε} for any desired $\varepsilon > 0$.

► **Theorem 1.** *For every $\varepsilon > 0$, there is a modulus $m \leq (1/\varepsilon)^{2/\varepsilon}$ such that every symmetric function on n bits can be computed by depth-3 MOD_m circuits of $\exp(O(n^\varepsilon))$ size. In fact, the circuits have the form $MOD_{p_1} \circ MOD_{p_2 \dots p_r} \circ MOD_{p_1}$, where p_1, \dots, p_r are distinct primes.*³

That is, without any AND/OR gates, we can obtain CC^0 circuits with a substantially smaller number of gates than the longstanding lower bounds for $AC^0[q]$ circuits computing symmetric functions (mentioned two paragraphs ago), for prime power q .

It has been known for decades [10] that *depth-2* $MOD_p \circ MOD_m$ circuits (and $CC^0[p] \circ MOD_m$ circuits) require $2^{\Omega(n)}$ size to compute the AND function, where p is a prime and m is an arbitrary composite, and that only certain restricted symmetric functions could be computed in subexponential-size and depth-2 [21]. Theorem 1 shows that one additional layer of MOD_p gates makes such circuits much more powerful.

² Hansen and Koucký [23] give an interesting counterpoint, showing that *probabilistic* CC^0 circuits can compute AND efficiently. Thus the $AND \in CC^0$ problem is equivalent to a derandomization question.

³ The $G \circ H \circ I$ notation means that the output gate has type G , on the middle layer there are gates only of type H , and on the bottom layer (nearest the inputs) there are only gates of type I .

It is well-known that for distinct primes p, q , every symmetric function on n bits has a MOD_{pq} circuit of size $\exp(O(n^\varepsilon))$ and depth $O(1/\varepsilon)$.⁴ Our result shows the depth can always be made 3, at the cost of increasing the modulus to a large enough constant. Hansen [22], building on Bhatnagar, Gopalan, and Lipton [14], shows that for m which is the product of r primes, and sufficiently small ℓ (smaller than each of the prime factors of m), the MOD_ℓ function can be represented by a polynomial over \mathbb{Z}_m of degree $O(n^{1/r})$. As a corollary of Hansen's work, Gopalan observed [20] that for every $\varepsilon > 0$ there is an m such that the MOD_2 function has depth-3 MOD_m circuits of size 2^{n^ε} . This naturally suggests the question of whether every symmetric function admits such a circuit, which is answered by our Theorem 1.

As we allow larger depths, we can obtain MOD_m circuits with an interesting size-depth tradeoff.

► **Theorem 2.** *Let $d \geq 3$ be an integer, and let m be a product of $r \geq 2$ distinct primes. Then every symmetric function on n bits can be computed by depth- d MOD_m circuits of size $\exp(\tilde{O}(n^{1/(r+d-3)}))$.*

To contrast, recall that the lower bounds for AC^0 are $\exp(\Omega(n^{1/(d-1)}))$ size for depth d [25], and the lower bounds for $\text{AC}^0[p^k]$ (with AND, OR, and MOD_{p^k} gates) are $\exp(\Omega(n^{1/(2d)}))$ [33, 34] for prime power p^k (where the constant factor depends on p^k). Thus for constant moduli m with enough prime factors, one can beat both lower bounds with MOD_m gates.

For large enough depth d , we can achieve even smaller circuits with a size bound of the form $\exp(n^{O(1)/(r \cdot d)})$, multiplying r and d in the denominator, instead of adding them.

► **Theorem 3.** *There is a universal constant $c \geq 1$ such that, for all sufficiently large depths d , and all composite m with r prime factors, every symmetric function can be computed by a MOD_m gate circuit of depth d and size $\exp(O(n^{c/((d-c)(r-1))}))$.*

We remark that the constant c in the above construction is not terribly small. (Our c is at least 6; this matters if one cares about very small d and r .) In concurrent (very recently released) work, [26] give a circuit construction with a similar tradeoff (but better constants) for the special case of the AND function, building on the polynomials of [9].

Even Smaller Circuits in ACC^0

Allowing AND and OR gates in our circuit, the size of our circuit constructions can be even further improved. Say that a product m of primes q_1, \dots, q_r is **good** if every prime factor of $\phi(m)$ divides m , where ϕ is Euler's totient function. We note that the primorial $m = p_r\#$, the product of the first r primes, is good.⁵

► **Theorem 4.** *Let m be a good product of r primes. For every symmetric function f on n inputs and every depth $d \geq 4$ congruent to 1 modulo 3, there exists an $\text{AC}^0[m]$ circuit of depth d and size $\exp(\tilde{O}(n^{3/((r+3)(d-1)-3)}))$ computing f .*

In the proof of Theorem 4, we make use of several tools from the recent $\text{AC}^0[2]$ circuits of [32] (circuits for elementary symmetric polynomials and circuits for the coin problem), along with known results on computing elementary symmetric polynomials modulo a prime.

⁴ The authors don't know the origin of this construction. It follows from the fact that every function on k bits has a depth-2 MOD_{pq} circuit of size $2^{O(k)}$, and that symmetric functions can be easily "decomposed" into smaller functions (as in [3]).

⁵ Indeed, for all $i = 1, \dots, r$, the prime factors of $q_i - 1$ are contained in $\{q_1, \dots, q_{i-1}\}$, all of which divide $m = p_r\#$.

Applying standard tricks (seen in [24, 37, 32]), Theorem 4 extends to linear threshold functions.

► **Corollary 5.** *Let m be a good product of r primes. For every linear threshold function f on n inputs and every depth $d \geq 4$ congruent to 1 modulo 3, there exists an $\text{AC}^0[m]$ circuit of depth $d + 2$ and size $\exp(\tilde{O}(n^{3/((r+3)(d-1)-3)}))$ computing f .*

The corollary follows directly from the fact that every linear threshold function can be written as an OR of $\text{poly}(n)$ ANDs of $\text{poly}(n)$ symmetric functions on n -bit inputs [24]. In general, Theorem 4 implies that TC^0 circuits (composed of MAJORITY and NOT gates) with small fan-in also have a nontrivial simulation.

► **Corollary 6.** *Every TC^0 circuit of depth e in which every gate has fan-in at most s has an equivalent MOD_m circuit of depth $d \cdot e$ and size at most $\exp(\tilde{O}(s^{3/((r+3)(d-1)-3)}))$, where m is a good product of r primes.*

The corollary follows from direct substitution of each MAJORITY gate with depth- d circuits from our Theorem 4. Note that such depth- e TC^0 circuits (where every gate has fan-in at most s) have at most $O(s^{e-1})$ gates: a depth-1 circuit has $O(1)$ gates, a depth-2 circuit has $O(s)$ gates, and so on. (We do not count inputs as gates.)

Can't you do any better?

Theorem 4 shows that for certain m which are products of r primes, one can compute arbitrary symmetric functions in depth d and size $\exp(n^{\frac{c}{r \cdot d}})$ where $c > 0$ is a constant. We give evidence that it may be difficult to improve asymptotically on the dependence of r and d in the exponent of n , based on a natural hypothesis regarding TC^0 circuits, which are constant-depth circuits composed of MAJORITY and NOT gates. (Of course it is difficult to prove anything unconditional here, because as far as we know, polynomial-size depth-3 MOD_6 circuits could compute every EXP function. Thus we settle for conditional hardness.)

Recall that a $\text{SYM} \circ \text{AND}$ circuit is a depth-2 circuit where the output is a symmetric function and the bottom layer computes ANDs of input variables and negations. The hypothesis is that subexponential-size $\text{SYM} \circ \text{AND}$ circuits cannot compute TC^0 circuits in which each gate has linear fan-in.

► **Hypothesis 7 (SYM \circ AND Hypothesis).** *There are constants $c, k > 1$ such that for sufficiently large n , there is a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ computable by TC^0 circuits of depth c with at most $\tilde{O}(n)$ gates where each gate has fan-in $\tilde{O}(n)$, such that f does not have an $\exp(O(n^{1/k}))$ size $\text{SYM} \circ \text{AND}$ circuit.*

A well-known result in circuit complexity is that every ACC^0 circuit of size s can be simulated by a $\text{SYM} \circ \text{AND}$ circuit of size $s^{\text{poly}(\log s)}$ [13, 18]. Therefore, the $\text{SYM} \circ \text{AND}$ Hypothesis is a strengthening of the longstanding hypothesis that $\text{TC}^0 \not\subseteq \text{ACC}^0$: the $\text{SYM} \circ \text{AND}$ Hypothesis implies exponential lower bounds for simulating TC^0 circuits with ACC^0 circuits. Indeed, the hypothesis implies that our ACC^0 circuits for symmetric functions are nearly size-optimal in their dependence on depth and modulus.

► **Theorem 8 (Near-Optimality Modulo a Conjecture).** *Assuming the $\text{SYM} \circ \text{AND}$ Hypothesis, there is a fixed $\alpha > 0$ such that for every m and d , every depth- d $\text{ACC}^0[m]$ circuit computing the MAJORITY function on n inputs requires size at least $\exp(n^{\frac{\alpha}{r \cdot d}})$ for sufficiently large n , where r is the number of distinct prime factors of m .*

The proof of Theorem 8 is in Appendix A. Therefore, we view size bounds of the form $\exp(n^{1/\Theta(rd)})$ (as seen in our results) as a natural barrier to better upper bounds on MOD_m circuits: any function with significantly smaller MOD_m circuit complexity (as a function of n , r , and d) would also yield a highly non-trivial $\text{SYM} \circ \text{AND}$ circuit simulation of TC^0 . In order to achieve significantly smaller circuits as a function of n , d , and r , one has to at least refute the hypothesis. Of course, even assuming Hypothesis 7, our circuits can probably be improved by constant factors in the exponents.

1.1 Intuition

Our circuit constructions use several tricks from the literature in new ways. Here, we give a short high-level exposition of subexponential-size depth-3 circuits for computing symmetric functions that use only MOD_m gates for composite m (Theorem 1). For simplicity, we will focus on computing EMAJ (“exact majority”) functions, which output 1 on a vector (x_1, \dots, x_n) if and only if

$$\sum_i x_i = T$$

for some target $T \in \{0, 1\}$.

Our first idea is to split the set of input variables into many parts; this is taken from the folklore depth-3 AC^0 circuits for symmetric functions of size $2^{\tilde{O}(\sqrt{n})}$ (although we will beat that size bound considerably). Letting $\delta \in (0, 1)$ be a parameter, we partition the inputs x_1, \dots, x_n into $t := \lceil n^\delta \rceil$ groups G_1, \dots, G_t of $O(n^{1-\delta})$ inputs each. Our circuit will try all possible $T_1, \dots, T_t \in \{0, 1, \dots, n/t\}$ such that $\sum_j T_j = T$, outputting 1 if the circuit finds T_j ’s such that for all $j = 1, \dots, t$, the sum of all variables in group G_j equals T_j . In a depth-3 AC^0 circuit, we can set $\delta = 1/2$ and obtain a circuit of $2^{\tilde{O}(\sqrt{n})}$ size: in particular, we take an OR over all $2^{\tilde{O}(\sqrt{n})}$ choices (T_1, \dots, T_t) , take an AND over all groups $j = 1, \dots, t$, then determine whether the sum of variables in G_j equals T_j using a CNF of size $2^{O(\sqrt{n})}$. Thus we have a circuit of type

$$\text{OR} \circ \text{AND} \circ \text{OR}$$

which computes the EMAJ function in size $2^{\tilde{O}(\sqrt{n})}$.

Using MOD_m gates where m has many distinct prime factors, we can do much better. Applying Lucas’ Theorem (Theorem 11) in a new way, we construct a polynomial P_{T_j} of degree $O(n^{(1-\delta)/r})$ that can determine whether the sum of variables in group G_j equals T_j , where r is the number of distinct prime factors of m . Theorem 16 demonstrates this claim in the case of $r = 2$. (We also need another technical condition on m for this to work, but we ignore that issue here.) This polynomial can be directly simulated by a depth-2 circuit of MOD_{pm} gates (where p is a new prime that does not divide m) and $\exp(\tilde{O}(n^{(1-\delta)/r}))$ size, which can determine whether or not the sum of variables in group G_j equals the target T_j .

Substituting this depth-2 circuit into the depth-3 AC^0 circuit described above (for each group j), we would have a circuit of the form:

$$\text{OR of } \exp(\tilde{O}(n^\delta)) \text{ ANDs of } \tilde{O}(n^\delta) \text{ MOD}_{pm} \text{ of } \exp(\tilde{O}(n^{(1-\delta)/r})) \text{ MOD}_{pm}.$$

Then, setting δ so that $\delta = (1-\delta)/r$, we obtain $\delta = 1/(r+1)$ and a circuit of $\exp(\tilde{O}(n^{1/(r+1)}))$ size. To reduce the depth down to three and use only MOD_{pm} gates, we observe that at most one of the wires into the OR can be true, and we apply another known translation (Proposition 10) to convert the AND of $\tilde{O}(n^\delta) \text{ MOD}_{pm}$ gates into a linear sum modulo pm

of $\exp(\tilde{O}(n^\delta)) \text{MOD}_{pm}$ gates, collapsing the $\text{OR} \circ \text{AND} \circ \text{MOD}_{pm} \circ \text{MOD}_{pm}$ circuit into a depth-3 circuit of only MOD_{pm} gates and $\exp(\tilde{O}(n^{1/(r+1)}))$ size. Setting m so that r is arbitrarily large, we obtain Theorem 1.

To obtain our stronger and more general results (Theorems 2, 3, and 4), we rely on even more tools and tricks, most of which are recorded in the next section. (Throughout the paper, we try to clearly state which ideas are due to prior work, and which are new.)

2 Preliminaries

We assume basic familiarity with computational complexity [6] and Boolean circuit complexity [27], although we have tried to keep the paper relatively self-contained.

Notation

For a binary vector \mathbf{x} , we use $|\mathbf{x}|_1$ to denote the ℓ_1 -norm, i.e., the number of ones in \mathbf{x} .

Besides AC^0 , ACC^0 , CC^0 , and TC^0 , we also use the following additional notation for various circuit types, all of which is standard:

- A circuit of type SYM is simply a symmetric Boolean function.
- An **EMAJ function** outputs 1 on an input $(x_1, \dots, x_n) \in \{0, 1\}$ if and only if $\sum_i x_i = T$ for a fixed target T . It is not hard to see that, by substituting 0/1 constants appropriately, such a function can always be implemented by a gate of $O(n)$ inputs which outputs 1 if and only if $\sum_i x_i = n/2$. This is why such gates are called “exact majority” (EMAJ).
- A circuit of type $G_1 \circ \dots \circ G_d$ is a circuit where the output gate has type G_1 , the next layer of gates all have type G_2 , and so on, and the bottom layer (nearest the inputs) only contains gates of type G_d .

We also make the following standard assumptions:

- We allow all gates to include 0/1 constants in their inputs.
- In our circuit model, we do not count inputs as gates, but rather as incoming wires. Thus, for example, a depth-1 circuit has precisely 1 gate.

The following basic fact is useful to keep in mind; we will apply it frequently.

► **Proposition 9.** *For all positive $m, n \in \mathbb{Z}$, any MOD_m gate of fan-in t can be simulated by a MOD_{mn} of fan-in nt .*

Proof. For any positive integer t , $m \mid t$ if and only if $nm \mid nt$. So $\text{MOD}_m(x_1, \dots, x_t) = \text{MOD}_{mn}(n \cdot x_1, \dots, n \cdot x_t)$. ◀

Tools

We make use of several known results. First, we note that AND circuits of small fan-in have efficient depth-2 MOD_m circuits. A version was first used in [7] in the context of MOD circuits, and more recently a strengthening was used to reduce the size-depth tradeoff for simulating ACC^0 circuits with $\text{SYM} \circ \text{AND}$ circuits [18]. (Chen and Papakonstantinou [18] call this “linearization”.)

► **Proposition 10** ([7, 18]). *Let $a, b \geq 2$ be fixed integers with $\gcd(a, b) = 1$. Every AND of k MOD_b gates can be represented by an $\text{MOD}_a \circ \text{MOD}_b$ circuit of $O(b^k)$ gates. Furthermore, on all k -bit inputs, the sum of the inputs to the output gate of the circuit is always 0 (mod a) or 1 (mod a).*

Our next tool is an old number-theoretic theorem on elementary symmetric polynomials modulo p , masterfully applied by Beigel, Barrington, and Rudich [9] in their non-trivial degree polynomials for the OR functions over composite moduli.

► **Theorem 11** (Lucas' Theorem [28]). *For all primes p and natural numbers n ,*

$$\binom{n}{p^i} \bmod p$$

is the i -th digit in the p -ary representation of n .

Lucas' theorem has the following direct consequence for polynomial representations of Boolean functions.

► **Lemma 12** ([9]). *Let p be a prime, let n be a natural number, and let $e_i(\mathbf{x})$ denote the i -th elementary symmetric polynomial on n variables. For a binary vector \mathbf{x} , let*

$$\sum y_i \cdot p^i = |\mathbf{x}|_1$$

be the p -ary expansion of $|\mathbf{x}|_1$. Then for every i , $e_{p^i}(\mathbf{x}) \equiv y_i \bmod p$.

In order to apply the elementary symmetric polynomials, our construction also involves arithmetic circuits over prime fields. These circuits will be translated into Boolean circuits with MOD_m gates. Such circuits were also used by [32] in their improved $\text{AC}^0[2]$ circuits for MAJORITY.

► **Lemma 13** ([19, 32]). *Let p be a prime, let $n, i \in \mathbb{N}$, and let $d \geq 2$ be even. There is an arithmetic circuit over \mathbb{F}_p of depth d and size $n^{O(i^{2/d})}$ computing the i -th elementary symmetric polynomial (over \mathbb{F}_p) on n inputs, where the output gate is a \times gate.*

We also use AC^0 circuits for the coin problem. These were also used by [32] in their improved $\text{AC}^0[2]$ circuits for symmetric functions.

In the following, we let $i, j \in \{0, 1, \dots, n\}$, and let $D_{i,j}$ be any partial function satisfying the properties:

$$D_{i,j}(\mathbf{x}) = 1 \text{ if } |\mathbf{x}|_1 = i, \text{ and}$$

$$D_{i,j}(\mathbf{x}) = 0 \text{ if } |\mathbf{x}|_1 = j.$$

► **Lemma 14** ([31, 5, 32]). *Let $d \geq 2$ and n be natural numbers, and let $i \neq j$. Then there is an AC^0 circuit of depth d and size $\exp(O(d(n/|i-j|)^{1/(d-1)}))$ computing $D_{i,j}$ on n inputs, where the output gate is an AND.*

Intuitively, Lemma 14 will be useful when $|i-j|$ is “large”.

3 CC0 Circuits for Symmetric Functions

We begin by giving efficient depth-3 CC^0 circuits for symmetric functions.

► **Reminder of Theorem 1.** *For every $\varepsilon > 0$, there is a modulus $m \leq (1/\varepsilon)^{2/\varepsilon}$ such that every symmetric function on n bits can be computed by depth-3 MOD_m circuits of $\exp(O(n^\varepsilon))$ size. In fact, the circuits have the form $\text{MOD}_{p_1} \circ \text{MOD}_{p_2 \dots p_r} \circ \text{MOD}_{p_1}$, where p_1, \dots, p_r are distinct primes.*

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After that, we will generalize the result to a size-depth tradeoff in the next subsection. That tradeoff will be further improved in Section 4 when we allow the use of AND and OR gates.

As a warm-up, we first consider the special case where $\varepsilon > 1/3$ and $m = 30$.

► **Theorem 15.** *Every symmetric Boolean function on n variables has a depth-3 circuit of the form $MOD_5 \circ MOD_6 \circ MOD_5$, of size $\exp(O(n^{1/3} \log n))$. Furthermore, the output gate is a linear sum which always evaluates to either 0 or 1 modulo 5.*

Note that the upper bound of Theorem 15 already beats the well-known lower bounds for depth-3 AC^0 [25]. The remainder of this section is devoted to the proof. A key component is a low-degree multivariate polynomial over \mathbb{Z}_6 that vanishes on a Boolean vector if and only if the sum of the ones in the vector equals a particular value.

► **Theorem 16.** *For every $n \in \mathbb{N}$ and every $T \in \{0, 1, \dots, n\}$, there is a polynomial $P_T(x_1, \dots, x_n)$ of degree at most $3\sqrt{n}$ such that for all $a \in \{0, 1\}^n$, $P_T(a) = 0 \pmod 6$ if and only if $\sum_i a_i = T$.*

Proof. We want a polynomial p on n variables such that for all $y_1, \dots, y_n \in \{0, 1\}$ and $T \in \{0, 1, \dots, n\}$,

$$p(y_1, \dots, y_n) \equiv 0 \pmod 6 \iff \sum_i y_i = T.$$

For the elementary symmetric polynomial $e_J(y_1, \dots, y_n)$ of degree J , and for all $a_1, \dots, a_n \in \{0, 1\}$,

$$e_J(a_1, \dots, a_n) = \binom{\sum_i a_i}{J}.$$

Thus by Lucas' Theorem (Theorem 11), $e_{p^i}(a_1, \dots, a_n) \pmod p$ equals the i -th digit in the p -ary representation of $\sum_i a_i$.

Let s and t be integers so that $2\sqrt{n} \geq 2^s > \sqrt{n}$ and $3\sqrt{n} \geq 3^t > \sqrt{n}$.

Suppose when we write $T \in \{0, 1, \dots, n\}$ in binary notation, the s low order bits are b_{s-1}, \dots, b_0 . Furthermore, when we write T in ternary notation, the t low order trits are c_{t-1}, \dots, c_0 .

Define the polynomials

$$p_2(y_1, \dots, y_n) := 1 - \prod_{j=0}^{s-1} (1 - (b_j - e_{2^j}(y))) \pmod 2$$

and

$$p_3(y_1, \dots, y_n) := 1 - \prod_{j=0}^{t-1} (1 - (c_j - e_{3^j}(y)))^2 \pmod 3.$$

Note the degrees of p_2 and p_3 are $O(\sqrt{n})$. In particular, $\deg(p_2) = \sum_{j=0}^{s-1} 2^j = 2^s - 1$ and

$$\deg(p_3) = \sum_{j=0}^{t-1} (2 \cdot 3^j) = 2(3^t - 1)/2 = 3^t - 1.$$

We observe a few properties of the polynomials p_2 and p_3 :

► **Proposition 17.** *For all $a \in \{0, 1\}^n$, $p_2(a) \equiv 0 \pmod 2$ if and only if the binary representation of $\sum_i a_i$ equals $b_{s-1} \cdots b_0$ in the last s bits. Analogously, $p_3(a) \in \{0, 1\} \pmod 3$, and $p_3(a) \equiv 0 \pmod 3$ if and only if the ternary representation of $\sum_i a_i$ equals $c_{t-1} \cdots c_0$ in the last t trits.*

Proof. We prove the proposition for p_3 ; the case of p_2 is analogous. Let $a \in \{0, 1\}^n$. Each difference $(c_j - e_{3^j}(a))^2$ is either 0 or 1 modulo 3, and it is 0 if and only if $c_j = e_{3^j}(a)$. Thus the product $\prod_{j=0}^{t-1} (1 - (c_j - e_{3^j}(a))^2)$ equals 1 if and only if $c_j = e_{3^j}(a)$ for all $j = 0, \dots, t-1$, hence $p_3(a) \equiv 0 \pmod 3$ if and only if $c_j \equiv e_{3^j}(a) \pmod 3$ for all $j = 0, \dots, t-1$. Recalling that $(e_{3^j}(a) \pmod 3)$ equals the j -th trit of $\sum_i a_i$, the result follows. ◀

We note that in general, working modulo a prime q , we may construct a polynomial with degree $(q^t - 1)$ of the form

$$p_q(y_1, \dots, y_n) = 1 - \prod_{j=0}^{t-1} (1 - (c_j - e_{q^j}(y))^{q-1}). \quad (1)$$

By the above proposition, it follows that for all $a \in \{0, 1\}^n$,

$$p_2(a) \equiv 0 \pmod 2 \iff \sum_i a_i \equiv T \pmod{2^s}$$

and

$$p_3(y) \equiv 0 \pmod 3 \iff \sum_i a_i \equiv T \pmod{3^t}.$$

Since $\sum_i a_i$ and T are both in $\{0, \dots, n\}$ and $2^s \cdot 3^t > n$, by the Chinese Remainder Theorem we have

$$\begin{aligned} \sum_i a_i = T &\iff \left(\sum_i a_i \equiv T \pmod{2^s}\right) \wedge \left(\sum_i a_i \equiv T \pmod{3^t}\right) \\ &\iff (p_2(y) \equiv 0 \pmod 2) \wedge (p_3(y) \equiv 0 \pmod 3) \iff 3p_2(y) + 2p_3(y) \equiv 0 \pmod 6. \end{aligned}$$

Thus $3p_2(y) + 2p_3(y)$ is a polynomial of degree $O(\sqrt{n})$ which equals 0 mod 6 if and only if $\sum_i y_i = T$. This completes the proof of Theorem 16. ◀

We now proceed with the proof of Theorem 15.

Proof. Let f be a symmetric function and let $g : \{0, 1, \dots, n\} \rightarrow \{0, 1\}$ be its companion function. That is, for every \mathbf{x} , $f(\mathbf{x}) = g(|\mathbf{x}|_1)$.

The output gate will be a MOD₅ gate that

- (a) sums over possible choices of $T \in \{0, 1, \dots, n\}$ such that $g(T) = 1$ and
- (b) sums over all ways to partition T into a sum of $t = \lceil n^{1/3} \rceil$ parts $T_1, \dots, T_t \in \{0, 1, \dots, T\}$.

There are $2^{O(n^{1/3} \log n)}$ choices over (a) and (b). We associate each part T_i with a disjoint set S_i of at most $\lceil n^{2/3} \rceil$ variables from the input. For each of the choices from (a) and (b), we wish to verify that, for all $i = 1, \dots, t$, the sum of all variables in S_i equals T_i . Note that there is *at most* one choice from (a) and from (b) that could possibly be consistent with the given input, so we can use a *modulo-5 sum* (not just a MOD₅ gate) to sum over these choices. This modulo-5 sum will always be either 0 or 1 modulo 5.

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By our construction of EMAJ polynomials, each sum over the set S_i of $n^{2/3}$ variables can be checked with a MOD_6 gate of $2^{O(n^{1/3})}$ fan-in, where each input to the MOD_6 gate is the output of an AND of fan-in $O(n^{1/3})$. Putting these $\text{MOD}_6 \circ \text{AND}$ circuits below each wire of the modulo-5 sum, at this point, we have a modulo-5 sum of $2^{O(n^{1/3} \log n)}$ ANDs of fan-in $O(n^{1/3})$ of MOD_6 of fan-in $2^{O(n^{1/3})}$ of ANDs of fan-in $O(n^{1/3})$.

To eliminate the AND gates, we apply Proposition 10, yielding that an AND of $f \text{ MOD}_q$ gates can be represented by a modulo- p sum of $O(q^f)$ MOD_q gates, as long as $\gcd(p, q) = 1$. In particular, for the “middle” ANDs we set $p = 5$ and $q = 6$, and for the “bottom” ANDs we set $p = 6$ and $q = 5$. We obtain a modulo-5 sum of $2^{O(n^{1/3} \log n)}$ MOD_6 of fan-in $2^{O(n^{1/3})}$ of MOD_5 of fan-in $O(n^{1/3})$. ◀

The above construction has several interesting corollaries; here is one.

► **Corollary 18.** *Every circuit of the form $\text{MOD}_5 \circ \text{SYM}$ of size $2^{O(n^{1/3} \log n)}$ can be simulated by a depth-3 $\text{MOD}_5 \circ \text{MOD}_6 \circ \text{MOD}_5$ circuit of size $2^{O(n^{1/3} \log n)}$.*

Proof. We simply replace each SYM gate (which takes n inputs) in the $\text{MOD}_5 \circ \text{SYM}$ circuit with a modulo-5 sum of $\text{MOD}_6 \circ \text{MOD}_5$ as in the previous theorem. ◀

We are now ready to generalize to Theorem 1.

► **Reminder of Theorem 1.** *For every $\varepsilon > 0$, there is a modulus $m \leq (1/\varepsilon)^{2/\varepsilon}$ such that every symmetric function on n bits can be computed by depth-3 MOD_m circuits of $\exp(O(n^\varepsilon))$ size. In fact, the circuits have the form $\text{MOD}_{p_1} \circ \text{MOD}_{p_2 \cdots p_r} \circ \text{MOD}_{p_1}$, where p_1, \dots, p_r are distinct primes.*

Proof. Let $\varepsilon > 0$, and let f be a symmetric function. Take $k := \lceil 1 + 1/\varepsilon \rceil$, let m be the product of the first k primes, and let $m' = m/2$.

We use a similar construction as in Theorem 15 to get a $\text{MOD}_2 \circ \text{MOD}_{m'} \circ \text{MOD}_2$ circuit for f .

The differences are that we partition the target $T \in \{0, 1, \dots, n\}$ into a sum of $\lfloor n^{1/k} \rfloor$ parts where each part is over $v := n^{1-1/k}$ variables, and by using $k - 1$ primes instead of two, we can obtain a polynomial for EMAJ on v variables of degree $O(v^{1/(k-1)}) \leq O(n^{1/k})$ in an analogous way.

More precisely, let $T' \in \{0, 1, \dots, v\}$ be a target value. For the first $k - 1$ odd primes q_1, \dots, q_{k-1} , we take $k - 1$ polynomials $p_{q_1}(x), \dots, p_{q_{k-1}}(x)$ as defined in (1) such that each $p_{q_i}(x)$ has degree $(q_i)^{t_i} - 1$, where the t_i are chosen such that for all $i \in [k]$,

- $(q_i)^{t_i} = \Theta(v^{1/(k-1)})$,
- $T' \leq v < \prod_i (q_i)^{t_i}$, and
- for all $a \in \{0, 1\}^v$ we have

$$p_{q_i}(a) = 0 \pmod{q_i} \iff \sum_i a_i \equiv T' \pmod{(q_i)^{t_i}}.$$

By the Chinese Remainder Theorem, and similar reasoning as in Theorem 15, there are fixed coefficients $M_i \in [m']$ such that

$$\sum_{i=1}^{k-1} M_i \cdot p_{q_i}(a) \equiv 0 \pmod{m'} \iff \bigwedge_{i=1}^{k-1} [p_{q_i}(a) \equiv 0 \pmod{q_i}] \iff \sum_i a_i = T'.$$

Thus we have a polynomial of degree $O(v^{1/(k-1)})$ that vanishes modulo m' precisely when the sum of v variables equals the target T' . Naturally we might write this polynomial as a

$\text{MOD}_{m'} \circ \text{AND}$ circuit of $\exp(\tilde{O}(v^{1/(k-1)}))$ size; by replacing each AND with a modulo- m' sum of MOD_2 gates (Proposition 10), we can express it as a $\text{MOD}_{m'} \circ \text{MOD}_2$ circuit. Our final circuit has the form $\text{MOD}_2 \circ \text{MOD}_{m'} \circ \text{MOD}_2$ and size $\exp(\tilde{O}(n^{1/k})) \leq \exp(O(n^\epsilon))$. ◀

3.1 Size-Depth Tradeoff with CC0

Allowing depth d circuits for $d > 3$, the size of the above construction can be improved as a function of the number of distinct primes r in the modulus. Here we only briefly describe the construction, as the size bound will be improved significantly (as a function of d and r) in the following section.

▶ **Reminder of Theorem 2.** *Let $d \geq 3$ be an integer, and let m be a product of $r \geq 2$ distinct primes. Then every symmetric function on n bits can be computed by depth- d MOD_m circuits of size $\exp(\tilde{O}(n^{1/(r+d-3)}))$.*

Proof. Let p and q be the smallest prime factors of m . We prove by induction on d that there are circuits of size $\exp(O(n^{1/(r+d-3)} \log n))$, and we prove additionally that the output gate is a MOD_p gate when d is odd and a MOD_q gate when d is even.

For $d = 3$, we use the construction of Theorem 1 to obtain a $\text{MOD}_p \circ \text{MOD}_{m/p} \circ \text{MOD}_p$ circuit of depth 3 and size $\exp(O(n^{1/r} \log n))$.

For the inductive step, we proceed similarly to the proof of Theorem 1, except that we use a MOD_p or MOD_q gate as the output gate (depending on the parity of d). We partition the target T into a sum of $t = \lceil n^{1/(r+d-3)} \rceil$ parts, where each part contains at most $\lceil n^{(r+d-4)/(r+d-3)} \rceil$ variables, and our circuit sums over all $\exp(O(t \log n))$ choices for the number of true variables in each part. Since EMAJ is a symmetric function, we can inductively compute each EMAJ on $\lceil n^{(r+d-4)/(r+d-3)} \rceil$ variables with a circuit of depth $d-1$, as guaranteed by the inductive hypothesis. These circuits have size

$$\exp(O((n^{(r+d-4)/(r+d-3)})^{1/(r+d-4)} \log n)) \leq \exp(O(n^{1/(r+d-3)} \log n)),$$

and their output gates have fan-in $\exp(O(n^{1/(r+d-3)} \log n))$. WLOG assume d is odd. Then the depth- $(d-1)$ circuits for EMAJ described above have the form

$$\text{MOD}_q \circ \dots \circ \text{MOD}_p \circ \text{MOD}_{m/p} \circ \text{MOD}_p,$$

and our entire circuit has the form

$$\text{MOD}_p \circ \text{AND} \circ \text{MOD}_q \circ \dots \circ \text{MOD}_p \circ \text{MOD}_{m/p} \circ \text{MOD}_p,$$

where the ANDs have fan-in t . As before, each $\text{AND} \circ \text{MOD}_q$ can be replaced by a modulo- p sum of MOD_q gates using Proposition 10, which only increases the circuit size by a factor of $2^{O(t)}$.

Our final circuit has depth d and size $\exp(O(n^{1/(r+d-3)} \log n))$. ◀

A Better Dependence on Depth and Modulus

We can give a CC^0 circuit construction with a better asymptotic tradeoff (in the double-exponent). We will keep the description of this construction brief and to the point, as its size will be further improved (replaced by better constants) in the next section, using OR and AND gates.

▶ **Reminder of Theorem 3.** *There is a universal constant $c \geq 1$ such that, for all sufficiently large depths d , and m which is the product of the first r prime factors, every symmetric function can be computed by a MOD_m gate circuit of depth d and size $\exp(O(n^{c/((d-c)(r-1)}))$.*

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Proof. First, we recall that every symmetric function f on n variables can be expressed as a MAJORITY of $O(n)$ MAJORITY gates over the n variables (see for example [12] for a reference). Thus it suffices to give a circuit for MAJORITY.

Allender and Koucky [3, Theorem 3.8] give a downward self-reduction for the MAJORITY function: they prove that there is a universal constant $a \geq 1$ such that for every $k \geq 1$, the MAJORITY function on n bits can be computed by a TC^0 circuit of depth at most ak where each MAJORITY gate has fan-in at most $O(n^{1/k})$. Applying these circuits to the depth-2 TC^0 circuits described in the previous paragraph, we obtain an analogous circuit of depth $2ak$ for any given symmetric function f .

Replace each MAJORITY gate of fan-in at most $O(n^{1/k})$ with a depth-3 MOD_m circuit of size at most

$$\exp(\tilde{O}(n^{1/(k(r-1))})),$$

as provided by Theorem 2. (Note that each NOT gate can always be replaced by a single MOD_m gate, if we do not want to allow NOT gates in our CC^0 circuit.) This results in a circuit of depth $6ak$ and size

$$\exp(\tilde{O}(n^{1/(k(r-1))})) \leq \exp(\tilde{O}(n^{1/(k(r-1))})).$$

Thus for depths $d = 6ak$ where k is a positive integer, the size bound is at most $\exp(n^{6a/(d(r-1))})$. For depths d that are not divisible by $6a$, we can simply use the construction for $d' = 6ak$ where $d' < d < 6a(k+1)$, which has size at most $\exp(n^{6a/(d'(r-1))}) < \exp(n^{6a/((d-6a)(r-1)})$. \blacktriangleleft

The above construction is not useful for $d < 6$ and small r , which are of interest. In the next section, we will show that much better constants are obtainable in the ACC^0 setting.

4 Size-Depth Tradeoff With ACC^0

We now turn to showing how adding AND and OR gates can help improve the circuits even further. We begin with a result using the concrete modulus 42.

► **Theorem 19.** *For every symmetric function f on n inputs and every depth d with $d \equiv 2 \pmod{6}$, there is an $\text{AC}^0[42]$ circuit of depth d and size $\exp(\tilde{O}(n^{1/33(\frac{6}{d-2})}))$ computing f .*

Observe that, for sufficiently large d , the circuit size of Theorem 19 already drops below Smolensky's $\text{AC}^0[p^k]$ depth- d lower bound of $\exp(\Omega(n^{1/(2d)}))$ size [34] for computing MOD_q when $\text{gcd}(p, q) = 1$.

We build on the results of Oliveira et al. [32] for computing symmetric functions in $\text{AC}^0[2]$. At a high level, we note that every symmetric function can be written as an OR of AND of (partial) functions of the form $D_{i,j}$, where

$$D_{i,j}(\mathbf{x}) = 1 \text{ if } |\mathbf{x}|_1 = i, \text{ and}$$

$$D_{i,j}(\mathbf{x}) = 0 \text{ if } |\mathbf{x}|_1 = j,$$

recalling that $|\mathbf{x}|_1$ is the number of 1's in \mathbf{x} . Note that $D_{i,j}$ could have *arbitrary* behavior on any other Boolean inputs.

When $|i - j|$ is large, the function $D_{i,j}$ can be simulated by the standard Coin Problem, for which there are known AC^0 circuits (see the Preliminaries). When $|i - j|$ is small, we give a new construction of $\text{AC}^0[42]$ circuits for $D_{i,j}$.

We will utilize arithmetic circuits for elementary symmetric polynomials. To that end, the following lemma shows how to generically translate low-depth arithmetic circuits over \mathbb{F}_p into $\text{AC}^0[p(p-1)]$ circuits, in a way that only increases the circuit depth by a $3/2$ multiplicative factor. (Getting some constant factor increase is not too difficult; Agrawal, Allender, and Datta [1] first showed a correspondence between ACC^0 and arithmetic circuits over finite fields. Their representation of field elements in Boolean circuits does not preserve depth as well as ours, however.)

► **Lemma 20.** *Let p be prime, and let C be an arithmetic circuit over \mathbb{F}_p of size s and depth $2d$ (with alternating layers of $+$ and \times gates, with $+$ at the output) on n inputs, such that for every $\mathbf{x} \in \{0, 1\}^n$, $C(\mathbf{x}) \in \{0, 1\}$. Then C is equivalent to an $\text{AC}^0[p(p-1)]$ circuit C' of size $O(s \cdot p)$ and depth $3d$.*

Proof. We represent an element x of \mathbb{F}_p in unary, by p indicator bits

$$b_0(x), b_1(x), \dots, b_{p-1}(x),$$

where $b_0(x) = 0$ iff $x = 0$, and for $i \neq 0$, we let $b_i(x) = 1$ iff $x = i$. (We treat the 0-th indicator bit as a special case to make later constructions easier.) We now obtain C' by replacing each gate in C with a small $\text{AC}^0[p(p-1)]$ gadget circuit.

For each addition gate of C computing

$$x = \sum_{j=1}^k x_j,$$

we replace that gate with p parallel MOD_p gates, so that

$$b_i(x) = 1 \iff (p-i) + \sum_{j=1}^k \sum_{i'=1}^{p-1} i' \cdot b_{i'}(x_j) \equiv 0 \pmod{p}.$$

(As a special case, we output the negation of the right hand side in the case of $b_0(x)$.) To see why this works, we observe that the inner sum computes x_j , and so the outer sum computes x . Now $x + (p-i) \equiv 0 \pmod{p}$ precisely when $x = i$.

Take g to be a generator of the multiplicative group \mathbb{F}_p^* of \mathbb{F}_p , and let $\log_g(n)$ denote the discrete logarithm base g in \mathbb{F}_p (i.e., $g^{\log_g(n)} = n \pmod{p}$). For each multiplication gate of C computing

$$x = \prod_{j=1}^k x_j,$$

we replace that gate with an AND gate placed in parallel with $p-1$ $\text{AND} \circ \text{MOD}_{p-1}$ circuits, implementing the conditions

$$b_0(x) = \bigwedge_{j=1}^k b_0(x_j),$$

and for $i \neq 0$,

$$b_i(x) = G_i \wedge \bigwedge_{j=1}^k b_0(x_j).$$

where G_i is a MOD_{p-1} gate such that

$$G_i = 1 \iff (p - \log_g(i)) + \sum_{j=1}^k \sum_{i'=2}^{p-1} b_{i'}(x_j) \cdot \log_g(i') \equiv 0 \pmod{p-1}.$$

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To see why this works, we observe that the inner sum computes the discrete logarithm of x_j (for the same reason that the inner sum in the addition case computes x_j). Since $x = \prod x_j$, we have (for non-zero x) $\log_g x = \sum \log_g x_j$, so the outer sum computes the discrete logarithm of x . Now $(\log_g x) + (p - \log_g i) \equiv 0 \pmod{p-1}$ precisely when $x = i$.

Finally, we take the output wire of C' to be the negation of the b_0 wire from the output gate of C . ◀

We note that as a special case, an arithmetic circuit over \mathbb{F}_2 can be viewed directly as an $\text{AC}^0[2]$ circuit (with the same size and depth), since an element of \mathbb{F}_2 is simply a bit, addition in \mathbb{F}_2 is MOD_2 , and multiplication is AND. Additionally, when $p-1$ is not square-free, we can improve the modulus in the circuit above.

► **Lemma 21.** *Let p be prime, and let C be an arithmetic circuit over \mathbb{F}_p of size s and depth $2d$ (with alternating layers of $+$ and \times gates, with $+$ at the output) on n inputs, such that for every $\mathbf{x} \in \{0,1\}^n$, $C(\mathbf{x}) \in \{0,1\}$. Then C is equivalent to an $\text{AC}^0[pm]$ circuit of size $O((sp)^{p-1})$ and depth $3d$, where m is the product of the distinct prime factors of $p-1$.*

Proof. We start with the circuit C' given by Lemma 20. We now use Theorem 11 (Lucas) and the Chinese Remainder Theorem to simulate each MOD_{p-1} gate of fan-in f using an $\text{AND} \circ \text{MOD}_m \circ \text{AND}$, as follows. We observe that

$$\sum_{i=1}^f y_i \equiv 0 \pmod{p-1} \iff \bigwedge_q \bigwedge_k \left[\sum_S \prod_{i \in S} y_i \equiv 0 \pmod{q} \right], \quad (2)$$

where the outermost AND ranges over all primes q dividing $p-1$, the inner AND ranges over all $k \geq 0$ such that q^k (strictly) divides $p-1$, and the summation ranges over all subsets $S \subseteq [f]$ of size q^k .

In particular, letting $p-1 = q_1^{k_1} \cdots q_t^{k_t}$, $\sum_i y_i$ is divisible by $p-1$ if and only if it is divisible by $q_i^{k_i}$ for all i , by the Chinese Remainder Theorem. Theorem 11 says we can check that $\sum_i y_i$ is divisible by $q_i^{k_i}$, by checking that $\left(\sum_{q_i^j} y_i\right)$ is divisible by q , for all $j = 0, \dots, k_i - 1$. Equation (2) is checking precisely these conditions.

For each q, k , the condition in square brackets is a homogeneous polynomial of degree at most $p-1$, and the degree of the polynomial is different for each choice of q and k . Therefore the total number of monomials over all such polynomials is at most

$$\sum_{i=1}^{p-1} \binom{f}{i} \leq f^{p-1}.$$

Each monomial can be thought of an AND gate in the natural way. The two outer ANDs (over q and k) can be collapsed into a single AND of fan-in at most $\log(p-1)$. Recall that C' from Lemma 20 is of the form

$$(\text{AND} \circ \text{MOD}_{p-1} \circ \text{MOD}_p)^d.$$

Therefore the bottom layer of AND gates (the monomials above) have MOD_p gates as inputs. Applying 10, each of these ANDs can be replaced by a sum (mod m) of fan-in $O(p^{p-1})$ which is Boolean-valued. These sums can be absorbed into the layer of MOD_m gates. Since the MOD_{p-1} gates in C' can only be inputs to AND gates, the top layer of AND gates in our circuit for (2) can be absorbed into the AND gates for which they are inputs. Our final circuit has the form

$$(\text{AND} \circ \text{MOD}_m \circ \text{MOD}_p)^d.$$

Note the above construction increases the size to at most $O((sp)^{p-1})$. ◀

Putting these results together, we obtain the following:

► **Theorem 22.** *Let d be a multiple of 6, let n be a natural number, and let $\alpha \in (0, 1]$. Set*

$$s := \left\lceil \frac{3\alpha \log_2 n}{7} \right\rceil, \quad t := \left\lceil \frac{2\alpha \log_3 n}{7} \right\rceil, \quad u := \left\lceil \frac{2\alpha \log_7 n}{7} \right\rceil,$$

and $m := 2^s 3^t 7^u$. Then there is an $\text{AC}^0[42]$ circuit of depth $d+1$ and size $2^{\tilde{O}(n^{6\alpha/7d})}$ computing the MOD_m function on n inputs, where the output gate is an AND gate.

Proof. Applying Lemma 13, we construct:

- arithmetic circuits C_1, C_2, \dots, C_{2^s} over \mathbb{F}_2 of depth d , where C_i computes the i -th elementary symmetric polynomial modulo 2,
 - arithmetic circuits D_1, D_3, \dots, D_{3^t} over \mathbb{F}_3 of depth $2d/3$, where D_i computes the i -th elementary symmetric polynomial modulo 3, and
 - arithmetic circuits E_1, E_7, \dots, E_{7^u} over \mathbb{F}_7 of depth $2d/3$, where E_i computes the i -th elementary symmetric polynomial modulo 7,
- all of which have size $n^{\mathcal{O}(n^{6\alpha/7d})}$, given our parameters.

We convert each of the D_i and E_i into $\text{AC}^0[42]$ circuits D'_i and E'_i using Lemma 20, and as previously observed, the C_i are already $\text{AC}^0[2]$ circuits.

Finally, from Lemma 12 and the Chinese Remainder Theorem, all of the $C_i(\mathbf{x})$, $D'_i(\mathbf{x})$, and $E'_i(\mathbf{x})$ output 1 if and only if $|\mathbf{x}|_1 \equiv 0 \pmod m$. Our final circuit for MOD_m is obtained by taking the AND of all of these circuits. ◀

We are now ready to prove Theorem 19.

Proof. Let $d \equiv 2 \pmod 6$, let f be a symmetric function on n inputs, and let g be its companion function; that is, for every \mathbf{x} , $f(\mathbf{x}) = g(|\mathbf{x}|_1)$. We begin with the same opening move as Oliveira, Santhanam, and Srinivasan [32], observing that

$$f(\mathbf{x}) = \bigvee_{i \in g^{-1}(1)} \bigwedge_{j \neq i} D_{i,j},$$

where $D_{i,j}(\mathbf{x}) = 1$ if $|\mathbf{x}|_1 = i$ and $D_{i,j}(\mathbf{x}) = 0$ if $|\mathbf{x}|_1 = j$ (and has otherwise arbitrary behavior). Thus it suffices to construct circuits $C_{i,j}$ computing functions consistent with $D_{i,j}$.

When $|i - j| \geq n^{7/13}$, Lemma 14 gives an AC^0 circuit $C_{i,j}$ of depth $d - 1$ and size $\exp(\tilde{O}(n^{6/(13(d-2))}))$ computing $D_{i,j}$.

When $|i - j| \leq n^{7/13}$, we observe that a circuit for MOD_m suffices, with $m > n^{7/13}$. We take $\alpha = 7/13$ in Theorem 22. Then we have a circuit $C'_{i,j}$ of depth $d - 1$ and size $\exp(\tilde{O}(n^{6/(13(d-2))}))$ computing the MOD_m function on $2n$ inputs, where $m > n^{7/13}$. We now take $C_{i,j}(\mathbf{x}) = C'_{i,j}(\mathbf{x}1^{m-i}0^{n-m+i})$. Finally, we set

$$C = \bigvee_{i \in g^{-1}(1)} \bigwedge_{j \neq i} C_{i,j}.$$

We can collapse the output AND gates of all of the $C_{i,j}$ into the second layer AND gates, so C has depth d and size $\exp(\tilde{O}(n^{6/(13(d-2))}))$, as desired. ◀

More generally, for certain m which are the product of r primes, we can improve the results of Theorem 19. Recall from the introduction that we defined a product m of primes q_1, \dots, q_r to be **good** if every prime factor of $\phi(m)$ divides m , and we noted that the primorial $m = p_r\#$, the product of the first r primes, is good.

► **Reminder of Theorem 4.** *Let m be a good product of r primes. For every symmetric function f on n inputs and every depth $d \geq 4$ congruent to 1 modulo 3, there exists an $\text{AC}^0[m]$ circuit of depth d and size $\exp(\tilde{O}(n^{3/((r+3)(d-1)-3)}))$ computing f .*

Proof. Let

$$m = \prod_{a=1}^r p_a$$

be a good product of r primes. For each $a \in [r]$, let $s_a = \lceil \alpha \log_{p_a} n \rceil$ for some α to be defined later. By Lemma 13, there are arithmetic circuits $C_{a,b}$ over \mathbb{F}_{p_a} of depth $(2/3)(d-1)$ and size $n^{O(n^{3\alpha/(d-1)})}$ computing the p_a^b -th elementary symmetric polynomial in $2n$ inputs over \mathbb{F}_{p_a} . By Lemma 21, we can convert these into $\text{AC}^0[m]$ circuits $C'_{a,b}$ of depth $d-1$ and size $2^{\tilde{O}(n^{3\alpha/(d-1)})}$. When $|i-j| \leq n^{\alpha r}$, $i \not\equiv j \pmod{p_a^{s_a}}$ for at least one a by the Chinese Remainder Theorem, so we can construct a circuit $E_{i,j}$ computing $D_{i,j}$ by taking

$$E_{i,j}(\mathbf{x}) = -C'_{a,b}(\mathbf{x}, 0^{p_a^{s_a}-j} 1^{n+j-p_a^{s_a}})$$

for some pair (a, b) . When $|i-j| \geq n^{\alpha r}$, we use Lemma 14 to get a circuit $E_{i,j}$ of depth $d-1$ and size $\exp(\tilde{O}(n^{(1-\alpha r)/(d-2)}))$ computing $D_{i,j}$. All of the $E_{i,j}$ have AND gates as output gates, so we take $\alpha = \frac{d-1}{(r+3)(d-1)-3}$ to balance the sizes of the two circuit constructions and complete the proof as per Theorem 19. ◀

It is worth noting that when $2 \mid m$, we can improve this construction slightly. When $2 \mid m$ (and $6 \mid d-1$), the $(r+3)(d-1)-3$ in the denominator of the double exponent instead becomes $(r + \frac{7}{2})(d-1) - 3$.

5 Conclusion

We believe our work demonstrates that CC^0 circuits are not as weak as conventional wisdom anticipates, even at depth three. We hope that researchers seriously consider (possibly refuting) the $\text{SYM} \circ \text{AND}$ hypothesis, as it stands in the way of obtaining significantly smaller CC^0 and ACC^0 circuits for symmetric functions.

A natural next step would be to explore how much further our constructions can be pushed beyond symmetric functions. Our Theorem 8 already demonstrates that TC^0 circuits with linearly many gates and linear fan-in can be non-trivially simulated with CC^0 circuits in subexponential size. Another question is whether NC^1 circuits or Boolean formulas can be simulated similarly. For another example, it is well-known that time t and space s computations can be simulated with depth-3 AC^0 circuits of size $2^{O(\sqrt{t \cdot s})}$; this follows from efficient simulations in the polynomial hierarchy of space-bounded computation [30]. Could the size of this construction be improved, using MOD_m gates? If such an improved circuit could be constructed in a uniform way, it would likely imply new time-space lower bounds for decision problems in PP or the counting hierarchy [4]. However, even a non-uniform construction would be very interesting.

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A Proof of Theorem 8

Let us recall the $\text{SYM} \circ \text{AND}$ Hypothesis and its consequence stated in the introduction.

► **Reminder of Hypothesis 7.** *There are constants $c, k > 1$ such that for sufficiently large n , there is a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ computable by TC^0 circuits of depth c with at most $\tilde{O}(n)$ gates where each gate has fan-in $\tilde{O}(n)$, such that f does not have an $\exp(O(n^{1/k}))$ size $\text{SYM} \circ \text{AND}$ circuit.*

► **Reminder of Theorem 8.** *Assuming the SYM ◦ AND Hypothesis (Hypothesis 7), there is a fixed $\alpha > 0$ such that for every m and d , every depth- d $\text{ACC}^0[m]$ circuit computing the MAJORITY function on n inputs requires size at least $\exp(n^{\frac{\alpha}{rd}})$ for sufficiently large n , where r is the number of distinct prime factors of m .*

We prove the contrapositive. We start with the negation of the theorem's conclusion:

Suppose for every $\alpha > 0$, there is some modulus m which is a product of r primes, along with some depth d , such that MAJORITY can be computed by a depth- d $\text{ACC}^0[m]$ circuit of size $\exp(O(n^{\frac{\alpha}{rd}}))$.

Assuming the above, we will refute the SYM ◦ AND Hypothesis: we will show for all $c, k > 1$ and every function f computable by the appropriate depth- c TC^0 circuits, f has an $\exp(O(n^{1/k}))$ size SYM ◦ AND circuit.

Let $c, k > 1$ be arbitrary. Let C be a TC^0 circuit C with depth c and $\tilde{O}(n)$ gates each of fan-in at most $\tilde{O}(n)$. Suppose we substitute each MAJORITY gate of C with a copy of the assumed $\text{ACC}^0[m]$ circuit. We obtain a $\text{ACC}^0[m]$ circuit C' of depth at most $c \cdot d$ and of size $\exp(\tilde{O}(n^{\frac{\alpha}{rd}}))$ such that C' is equivalent to C .

Chen and Papakonstantinou [18] prove that for every depth- d' size- s circuit D over AND, OR, and MOD_m gates, where m is the product of r distinct primes, D is equivalent to a SYM ◦ AND circuit D' of size at most

$$S'(s, m, r, d') = 2^{(m \log s)^{10rd'}}.$$

Applying their reduction to our C' , we obtain a SYM ◦ AND circuit C'' of size $\exp(\tilde{O}(n^{10\alpha c}))$ that is equivalent to our original circuit C . For all $\alpha < 1/(10ck)$, we obtain a SYM ◦ AND circuit equivalent to C with size $\exp(O(n^{1/k}))$.