# Constructing the Space of Valuations of a Quasi-Polish Space as a Space of Ideals

**Matthew de Brecht** ✉
Kyoto University, Japan

## Abstract

We construct the space of valuations on a quasi-Polish space in terms of the characterization of quasi-Polish spaces as spaces of ideals of a countable transitive relation. Our construction is closely related to domain theoretical work on the probabilistic powerdomain, and helps illustrate the connections between domain theory and quasi-Polish spaces. Our approach is consistent with previous work on computable measures, and can be formalized within weak formal systems, such as subsystems of second order arithmetic.

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## 1 Introduction

Quasi-Polish spaces [2] are a class of well-behaved countably based sober spaces that includes Polish spaces, $\omega$-continuous domains, and countably based spectral spaces. They can be interpreted via Stone-duality as the spaces of models of countably axiomatized propositional geometric theories [12, 1]. In [7] another characterization of quasi-Polish spaces was presented that is a natural generalization of the notion of an *abstract basis* for $\omega$-continuous domains [8]. In this paper we use this latter characterization to extend domain theoretical work on probabilistic powerdomains to the study of valuations on quasi-Polish spaces.

Valuations are a substitute for Borel measures which are used in the denotational semantics of probabilistic programming languages [14] and in computable approaches to measure theory, probability theory, and randomness [19, 13, 18]. See R. Heckmann's excellent paper [11] for more on the theory of valuations, spaces of valuations, and integration[1]. Every valuation on a quasi-Polish space can be extended to a Borel measure [5], and this extension is unique if the valuation is locally finite [3]. Conversely, it is easy to see that the restriction of a Borel measure to the open sets is a valuation. Thus, in particular, there is a bijection between probabilistic valuations and probabilistic Borel measures on quasi-Polish spaces.

The main result in this paper is a construction of the space of valuations on a quasi-Polish space as a space of ideals of a transitive relation on a countable set (Theorem 13). Our construction is closely related to domain theoretical work on the probabilistic powerdomain (see [14] and [8, Section IV-9]). Along with the constructions of the upper and lower powerspaces of quasi-Polish spaces as spaces of ideals given in [4], our results demonstrate how some domain theoretic results generalize well to quasi-Polish spaces (see also [6] for more on the upper and lower powerspaces of quasi-Polish spaces).

[1] The valuations in this note correspond to the *Scott-continuous valuations* in [11].


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An immediate corollary of our construction is that the space of valuations on a quasi-Polish space is again a quasi-Polish space, although this already follows from well-known results. A locale theoretic proof easily follows from S. Vickers' geometricity result in [20, Proposition 5] by using R. Heckmann's characterization of quasi-Polish spaces as countably presented locales [12]. A proof based on quasi-metrics, at least for the case of subprobabilistic valuations, follows from J. Goubault-Larrecq's work on continuous Yoneda-complete quasi-metric spaces in [9, Section 11] and his characterization of quasi-Polish spaces in [10, Theorem 8.18]. Independently, the first proof we found (which we presented at the Domains XII conference in August 2015) was largely based on M. Schröder's work in [19] on the space of (probabilistic) measures within the cartesian closed category $\mathsf{QCB}_0$. That proof starts with the observation that the $\mathsf{QCB}_0$ exponential $\mathbb{S}^{\mathbb{S}^X}$ is quasi-Polish whenever $X$ is[2], then uses the cartesian closed structure of $\mathsf{QCB}_0$ to show that $Y^{\mathbb{S}^X}$ is quasi-Polish whenever $X$ and $Y$ are, and finally observes that M. Schröder's construction of the space of valuations on $X$ can be obtained as the equalizer of the continuous functions $\ell, r\colon \overline{\mathbb{R}}_+^{\mathbb{S}^X} \to \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+^{\mathbb{S}^X \times \mathbb{S}^X}$ defined as:

$$\begin{aligned} \ell(\nu) &= \big\langle \nu(\emptyset), \lambda\langle U,V\rangle.\nu(U)+\nu(V)\big\rangle, \text{ and} \\ r(\nu) &= \big\langle 0, \lambda\langle U,V\rangle.\nu(U\cup V)+\nu(U\cap V)\big\rangle. \end{aligned}$$

It follows that the space of valuations is quasi-Polish because the space of extended reals $\overline{\mathbb{R}}_+$ is quasi-Polish and the category of quasi-Polish spaces is closed under countable limits.

A nice characteristic of the construction we give in this paper is that it can be formalized within relatively weak formal systems. For example, our approach is related to C. Mummert's formalization of general topology within subsystems of second order arithmetic [15, 16, 17][3].

## 2 Main result

We let $\overline{\mathbb{R}}_+$ denote the positive extended reals (i.e., $[0,\infty]$) with the Scott-topology induced by the usual order. Given a topological space $X$, we let $\mathbf{O}(X)$ denote the lattice of open subsets of $X$ with the Scott-topology.

▶ **Definition 1** (Valuations). *Let $X$ be a topological space. A* valuation *on $X$ is a continuous function $\nu\colon \mathbf{O}(X) \to \overline{\mathbb{R}}_+$ satisfying:*

1. $\nu(\emptyset) = 0$, *and* (strictness)
2. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$. (modularity)

*The* space of valuations *on $X$ is the set $\mathbf{V}(X)$ of all valuations on $X$ with the* weak topology, *which is generated by subbasic opens of the form*

$$\langle U, q\rangle := \{\nu \in \mathbf{V}(X) \mid \nu(U) > q\}$$

*with $U \in \mathbf{O}(X)$ and $q \in \overline{\mathbb{R}}_+ \setminus \{\infty\}$.*

In this paper we will only consider the whole space of valuations $\mathbf{V}(X)$, but it is straightforward to modify our results for the subspaces of $\mathbf{V}(X)$ consisting of *probabilistic valuations* (i.e., valuations satisfying $\nu(X) = 1$) and *sub-probabilistic valuations* (i.e., valuations satisfying $\nu(X) \leq 1$).

[2] See [6] for a proof. The $\mathbb{S}$ here is the Sierpinski space, and the space $\mathbf{O}(\mathbf{O}(X))$ defined in [6] is homeomorphic to the $\mathsf{QCB}_0$ exponential object $\mathbb{S}^{\mathbb{S}^X}$ when $X$ is quasi-Polish.

[3] Note that C. Mummert's *MF-spaces* are in general $\mathbf{\Pi}^1_1$-complete spaces, whereas quasi-Polish spaces correspond to the $\mathbf{\Pi}^0_2$-level of the Borel hierarchy. This explains why $\mathbf{\Pi}^1_1 - \mathsf{CA}_0$ is required to prove MF-spaces are closed under $G_\delta$-subsets, whereas our construction of $\mathbf{\Pi}^0_2$-subspaces of quasi-Polish spaces in Theorem 3 of [4] can be done within $\mathsf{ACA}_0$.

Quasi-Polish spaces were introduced in [2]. In this paper we will define them using the following equivalent characterization from [7] (see also [4]).

▶ **Definition 2.** *Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an* ideal *(with respect to $\prec$) if and only if:*

1. $I \neq \emptyset$, ($I$ is non-empty)
2. $(\forall a \in I)(\forall b \in \mathbb{N})\,(b \prec a \Rightarrow b \in I)$, ($I$ is a lower set)
3. $(\forall a, b \in I)(\exists c \in I)\,(a \prec c\,\&\,b \prec c)$. ($I$ is directed)

*The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$. A space is* quasi-Polish *if and only if it is homeomorphic to $\mathbf{I}(\prec)$ for some transitive relation $\prec$ on $\mathbb{N}$.*

We often apply the above definition to other countable sets with the implicit assumption that it has been suitably encoded as a subset of $\mathbb{N}$.

Fix a transitive relation $\prec$ on $\mathbb{N}$ for the rest of this section. Let $\mathcal{B}$ be the (countable) set of all partial functions $r :\subseteq \mathbb{N} \to \mathbb{Q}_{>0}$ such that $dom(r)$ is finite, where $\mathbb{Q}_{>0}$ is the set of rational numbers strictly larger than zero.

▶ **Definition 3.** *Define the transitive relation $\prec_V$ on $\mathcal{B}$ as $r \prec_V s$ if and only if*

$$\sum_{b\in F} r(b) < \sum_{c\in\uparrow F\cap dom(s)} s(c)$$

*for every non-empty $F \subseteq dom(r)$, where $\uparrow F = \{c \in \mathbb{N} \mid (\exists b \in F)\, b \prec c\}$.*

Transitivity of $\prec_V$ follows from the transitivity of $\prec$. Note that if $dom(r) = \emptyset$ then $r \prec_V s$ for every $s \in \mathcal{B}$. We will sometimes use the fact that if $r \prec_V s$ and $b \in dom(r)$ then there is $c \in dom(s)$ with $b \prec c$.

▶ **Definition 4.** *Define $f_V : \mathbf{V}(\mathbf{I}(\prec)) \to \mathbf{I}(\prec_V)$ and $g_V : \mathbf{I}(\prec_V) \to \mathbf{V}(\mathbf{I}(\prec))$ as*

$$f_V(\nu) = \left\{ r \in \mathcal{B} \;\middle|\; \sum_{b\in F} r(b) < \nu\Big(\bigcup_{b\in F}[b]_\prec\Big) \text{ for every non-empty } F \subseteq dom(r) \right\},$$

$$g_V(I) = \lambda U. \bigvee \left\{ \sum_{b\in dom(r)} r(b) \;\middle|\; r \in I \text{ and } \bigcup_{b\in dom(r)} [b]_\prec \subseteq U \right\}.$$

We next prove a few lemmas which will be used to show that $f_V$ and $g_V$ are continuous inverses of each other.

▶ **Lemma 5.** *If $I \in \mathbf{I}(\prec_V)$, $r \in I$, and $A \subseteq dom(r)$, then $r|_A \in I$, where $r|_A$ is the partial function obtained by restricting the domain of $r$ to $A$.*

**Proof.** Since $I$ is directed there is $s \in I$ with $r \prec_V s$. Then clearly $r|_A \prec_V s$ hence $r|_A \in I$ because $I$ is a lower set. ◀

▶ **Definition 6.** *Define the transitive binary relation $\prec_U$ on $\mathcal{P}_{\text{fin}}(\mathbb{N})$ (the set of finite subsets of $\mathbb{N}$) as $F \prec_U G$ if and only if $(\forall n \in G)\,(\exists m \in F)\, m \prec n$.*

We write $\mathbf{K}(X)$ for the space of saturated compact subsets of $X$ (see [6]).

▶ **Lemma 7** (Lemma 9 & Theorem 10 of [4])**.** *Given $J \in \mathbf{I}(\prec_U)$, the set*

$$g_U(J) = \{I \in \mathbf{I}(\prec) \mid (\forall F \in J)(\exists m \in I)\, m \in F\}$$

*is in $\mathbf{K}(\mathbf{I}(\prec))$. Furthermore, for any $S \subseteq \mathbb{N}$, $g_U(J) \subseteq \bigcup_{b\in S}[b]_\prec$ if and only if there is finite $F \subseteq S$ with $F \in J$.*

▶ **Lemma 8.** *If $I \in \mathbf{I}(\prec_V)$ and $r \in I$, then there exists $s \in I$ with $r \prec_V s$ and $dom(r) \prec_U dom(s)$.*

**Proof.** Choose any $t \in I$ with $r \prec_V t$. Let $s$ be the restriction of $t$ to have $dom(s) = \{c \in dom(t) \mid (\exists b \in dom(r))\, b \prec c\}$. Clearly $r \prec_V s$ and $dom(r) \prec_U dom(s)$, and Lemma 5 implies $s \in I$. ◀

▶ **Lemma 9.** *Assume $I \in \mathbf{I}(\prec_V)$ and $r \in I$. Then there exists $K \in \mathbf{K}(\mathbf{I}(\prec))$ such that*

- *$K \subseteq \bigcup_{b \in dom(r)} [b]_\prec$, and*
- *For any finite $F \subseteq \mathbb{N}$, if $K \subseteq \bigcup_{b \in F} [b]_\prec$, then there is $s \in I$ with $r \prec_V s$ and $F \prec_U dom(s)$ and $K \subseteq \bigcup_{c \in dom(s)} [c]_\prec \subseteq \bigcup_{b \in F} [b]_\prec$.*

**Proof.** Fix $I \in \mathbf{I}(\prec_V)$ and $r \in I$. Using Lemma 8, we can find a $\prec_V$-ascending sequence $(r_i)_{i \in \mathbb{N}}$ in $I$ with $r = r_0$ and $dom(r_i) \prec_U dom(r_{i+1})$ for each $i \in \mathbb{N}$. Then $J = \{F \in \mathcal{P}_{\text{fin}}(\mathbb{N}) \mid (\exists i \in \mathbb{N})\, F \prec_U dom(r_i)\}$ is in $\mathbf{I}(\prec_U)$, hence $K = g_U(J) \in \mathbf{K}(\mathbf{I}(\prec))$ and $K \subseteq \bigcup_{b \in dom(r)} [b]_\prec$ by Lemma 7 and the fact that $dom(r) \in J$. Assume $F \subseteq \mathbb{N}$ is finite and $K \subseteq \bigcup_{b \in F} [b]_\prec$. Then $F \in J$ by Lemma 7, hence $F \prec_U dom(r_i)$ for some $i \in \mathbb{N}$. Since $\prec_U$ is transitive, we can assume without loss of generality that $i > 0$. Setting $s = r_i$, we have $s \in I$ and $r \prec_V s$ and $F \prec_U dom(s)$, and since $dom(s) \in J$ it follows from Lemma 7 that $K \subseteq \bigcup_{c \in dom(s)} [c]_\prec$. The claim $\bigcup_{c \in dom(s)} [c]_\prec \subseteq \bigcup_{b \in F} [b]_\prec$ follows from $F \prec_U dom(s)$. ◀

▶ **Lemma 10.** *Let $D \subseteq \mathbb{N}$ be finite, and let $\mathcal{P}_+(D)$ be the set of non-empty subsets of $D$. Define*

$$U_G = \bigcap_{b \in G} [b]_\prec$$
$$V_G = U_G \cap \bigcup_{b \in D \setminus G} [b]_\prec$$

*for each $G \in \mathcal{P}_+(\mathcal{D})$. Let $P \subseteq \mathcal{P}_+(D)$ be an upper set (i.e., if $F \in P$ and $F \subseteq G \subseteq D$ then $G \in P$). If $\nu \in \mathbf{V}(\mathbf{I}(\prec))$ and $\nu(U_G) < \infty$ for each $G \in P$, then*

$$\sum_{G \in P} (\nu(U_G) - \nu(V_G)) = \nu\left(\bigcup_{G \in P} U_G\right).$$

**Proof.** The proof is by induction on the size of $P$. It is trivial when $P = \emptyset$, so assume $P$ is a non-empty upper set and that the lemma holds for all upper sets of size strictly less than $P$. If $F$ is any minimal element of $P$, then

$$\begin{aligned} V_F &= \bigcup_{b \in D \setminus F} U_{F \cup \{b\}} \\ &= \bigcup_{G \in P \setminus \{F\}} U_{F \cup G} \\ &= U_F \cap \bigcup_{G \in P \setminus \{F\}} U_G, \end{aligned}$$

so the induction hypothesis and modularity yields

$$\sum_{G\in P}(\nu(U_G)-\nu(V_G)) = \nu(U_F)-\nu(V_F)+\sum_{G\in P\setminus\{F\}}(\nu(U_G)-\nu(V_G))$$
$$= \nu(U_F)-\nu\left(U_F\cap\bigcup_{G\in P\setminus\{F\}}U_G\right)+\nu\left(\bigcup_{G\in P\setminus\{F\}}U_G\right)$$
$$= \nu\left(\bigcup_{G\in P}U_G\right). \qquad ◀$$

▶ **Lemma 11.** *$f_V$ is well-defined and continuous.*

**Proof.** We first show that $f_V(\nu)\in \mathbf{I}(\prec_V)$ for each $\nu\in\mathbf{V}(\mathbf{I}(\prec))$.

1. *($f_V(\nu)$ is non-empty).* The partial function with empty domain is in $f_V(\nu)$.
2. *($f_V(\nu)$ is a lower set).* Assume $r\prec_V s\in f_V(\nu)$. Let $F\subseteq dom(r)$ be non-empty, and define $G=\uparrow F\cap dom(s)$. Since $b\prec c$ implies $[c]_\prec\subseteq[b]_\prec$ it follows that $\bigcup_{c\in G}[c]_\prec\subseteq\bigcup_{b\in F}[b]_\prec$. Then
$$\sum_{b\in F}r(b)<\sum_{c\in G}s(c) \qquad \text{(because } r\prec_V s)$$
$$<\nu(\bigcup_{c\in G}[c]_\prec) \qquad \text{(because } s\in f_V(\nu))$$
$$\le\nu(\bigcup_{b\in F}[b]_\prec) \qquad \text{(because } \nu \text{ is monotonic)},$$
hence $r\in f_V(\nu)$.
3. *($f_V(\nu)$ is directed).* Our proof is related to the series of lemmas leading up to Theorem IV-9.16 in [8]. Assume $r_0,r_1\in f_V(\nu)$. For each $i\in\{0,1\}$ and non-empty $F\subseteq dom(r_i)$ fix some real number $\beta_F^i$ satisfying
$$\sum_{b\in F}r_i(b)<\beta_F^i<\nu\left(\bigcup_{b\in F}[b]_\prec\right),$$
and set
$$\beta=\min\left\{\frac{\beta_F^i-\sum_{b\in F}r_i(b)}{\sum_{b\in F}r_i(b)}\,\middle|\, i\in\{0,1\}\ \&\ \emptyset\neq F\subseteq dom(r_i)\right\}.$$
Then $\alpha=1/\left(1+\beta/2\right)$ satisfies $0<\alpha<1$ and is such that
$$\sum_{b\in F}r_i(b)<\alpha\nu\left(\bigcup_{b\in F}[b]_\prec\right)$$
for each $i\in\{0,1\}$ and non-empty $F\subseteq dom(r_i)$ (see Lemma IV-9.11 (iii) of [8]). Set $M=1+\sum_{b\in dom(r_0)}r_0(b)+\sum_{b\in dom(r_1)}r_1(b)$, and $D=dom(r_0)\cup dom(r_1)$. Let $U_G$ and $V_G$ be defined as in Lemma 10 for each non-empty $G\subseteq D$.
We define a finite set $h(G)\subseteq\mathbb{N}$ and a function $s_G\colon h(G)\to\mathbb{Q}_>$ for each non-empty $G\subseteq D$ as follows. If $\nu(U_G)=\nu(V_G)$ then let $h(G)=\emptyset$ and let $s_G$ be the empty function. Otherwise, the set
$$C=\{c\in\mathbb{N}\mid(\forall b\in D)\,[b\prec c\iff b\in G]\}$$

is non-empty because $\nu(U_G) > \nu(V_G)$ implies there is some ideal containing $G$ which is not in $V_G$. If there is some $c \in C$ with $\nu([c]_\prec) = \infty$, then set $h(G) = \{c\}$ and define $s_G : h(G) \to \mathbb{Q}_>$ as $s_G(c) = M$. If no such $c \in C$ exists, then let $(c_i)_{i\in\mathbb{N}}$ be an enumeration of $C$ and define

$$p_i = \nu([c_i]_\prec) - \nu\left([c_i]_\prec \cap \left(\bigcup_{k<i}[c_k]_\prec \cup V_G\right)\right).$$

Using modularity and a simple inductive argument, we have

$$\begin{aligned}\sum_{i\leq n} p_i &= \nu(\bigcup_{i\leq n}[c_i]_\prec) - \nu\left(\bigcup_{i\leq n}[c_i]_\prec \cap V_G\right)\\ &= \nu(\bigcup_{i\leq n}[c_i]_\prec \cup V_G) - \nu(V_G)\end{aligned}$$

for each $n \in \mathbb{N}$. Since $U_G = \bigcup_{i\in\mathbb{N}}[c_i]_\prec \cup V_G$ and $\nu$ is Scott-continuous, there is $n_0 \in \mathbb{N}$ with

$$\left(\frac{1+\alpha}{2}\right)\sum_{i\leq n_0} p_i \geq \alpha(\nu(U_G) - \nu(V_G))$$

if $\nu(U_G) < \infty$, and

$$\left(\frac{1+\alpha}{2}\right)\sum_{i\leq n_0} p_i \geq M$$

if $\nu(U_G) = \infty$. Define

$$h(G) = \{c_i \mid i \leq n_0 \,\&\, p_i > 0\}$$

and define $s_G : h(G) \to \mathbb{Q}_>$ so that $s_G(c_i)$ is a positive rational satisfying

$$\left(\frac{1+\alpha}{2}\right) p_i \leq s_G(c_i) < p_i.$$

Since $h(G) \cap h(G') \neq \emptyset$ implies $G = G'$, there is $s \in \mathcal{B}$ with

$$dom(s) = \bigcup\{h(G) \mid G \subseteq D\}$$

satisfying $s(c) = s_G(c)$ for the unique $G \subseteq D$ with $c \in h(G)$. From the construction of $s$, if $F \subseteq h(G)$ is non-empty then

$$\sum_{c\in F} s(c) < \nu(\bigcup_{c\in F}[c]_\prec) - \nu(\bigcup_{c\in F}[c]_\prec \cap V_G). \tag{1}$$

Furthermore, if $h(G) \neq \emptyset$, then $\nu(U_G) < \infty$ implies

$$\alpha\left(\nu(U_G) - \nu(V_G)\right) \leq \sum_{c\in h(G)} s(c), \tag{2}$$

and $\nu(U_G) = \infty$ implies

$$M \leq \sum_{c\in h(G)} s(c). \tag{3}$$

To show $s \in f_V(\nu)$, we must prove $\sum_{c\in F} s(c) < \nu(\bigcup_{c\in F}[c]_\prec)$ for each non-empty $F \subseteq dom(s)$. This clearly holds when $F = \{c\}$ is a singleton. Next, assume it holds for all sets of size less than or equal to $n$, and let $F$ be a set of size $n+1$. We can assume $\nu(\bigcup_{c\in F}[c]_\prec) < \infty$, since otherwise the claim is trivial. Let $G \subseteq D$ be a set of minimal size satisfying $F \cap h(G) \neq \emptyset$. This implies that either $F \setminus h(G)$ is empty or else it satisfies the induction hypothesis. Furthermore, for any $c \in F \setminus h(G)$ there is $G' \subseteq D$ with $c \in h(G')$, and since the minimality of $G$ implies $G' \not\subseteq G$, there is $b \in G' \setminus G$ with $b \prec c$, which implies $U_G \cap [c]_\prec \subseteq V_G$. Therefore,

$$\begin{aligned}
\sum_{c\in F} s(c) &= \sum_{c\in F\cap h(G)} s_G(c) + \sum_{c\in F\setminus h(G)} s(c) \\
&< \nu(\bigcup_{c\in F\cap h(G)} [c]_\prec) - \nu(\bigcup_{c\in F\cap h(G)} [c]_\prec \cap V_G) + \nu(\bigcup_{c\in F\setminus h(G)} [c]_\prec) \\
&\text{(by (1) and the induction hypothesis)} \\
&\leq \nu(\bigcup_{c\in F\cap h(G)} [c]_\prec) - \nu(\bigcup_{c\in F\cap h(G)} [c]_\prec \cap \bigcup_{c\in F\setminus h(G)} [c]_\prec) \\
&+ \nu(\bigcup_{c\in F\setminus h(G)} [c]_\prec) \\
&\text{(because } U_G \cap [c]_\prec \subseteq V_G \text{ for each } c \in F \setminus h(G)) \\
&= \nu(\bigcup_{c\in F} [c]_\prec),
\end{aligned}$$

which proves $s \in f_V(\nu)$.
Finally, we must show $r_0 \prec_V s$ and $r_1 \prec_V s$. Fix $i \in \{0,1\}$ and non-empty $F \subseteq dom(r_i)$. Set $P = \{G \subseteq D \mid G \cap F \neq \emptyset\}$ and note that $\uparrow F \cap dom(s) = \bigcup_{G\in P} h(G)$. If $\nu(U_G) < \infty$ for each $G \in P$, then using (2) and the fact that $G \neq G'$ implies $h(G) \cap h(G') = \emptyset$, we have

$$\begin{aligned}
\sum_{c\in\uparrow F\cap dom(s)} s(c) &\geq \sum_{G\in P} \alpha(\nu(U_G) - \nu(V_G)) \\
&= \alpha\nu\left(\bigcup_{G\in P} U_G\right) \qquad \text{(by Lemma 10)} \\
&= \alpha\nu\left(\bigcup_{b\in F} [b]_\prec\right) \\
&> \sum_{b\in F} r_i(b).
\end{aligned}$$

Otherwise, there is $G \in P$ with $\nu(U_G) = \infty$, so (3) implies

$$\sum_{c\in\uparrow F\cap dom(s)} s(c) \geq M > \sum_{b\in F} r_i(b).$$

This completes the proof that $f_V(\nu)$ is directed.
It only remains to show that $f_V$ is continuous. Fix $r \in \mathcal{B}$. For each $F \subseteq dom(r)$ define $W_F = \bigcup_{b\in F}[b]_\prec$ and $q_F = \sum_{b\in F} r(b)$, and set $D = \{F \subseteq dom(r) \mid F \neq \emptyset\}$. Then $f_V(\nu) \in [r]_{\prec_V}$ if and only if

$$\nu \in \bigcap_{F\in D} \langle W_F, q_F\rangle,$$

hence $f_V$ is continuous. ◀

▶ **Lemma 12.** *$g_V$ is well-defined and continuous.*

**Proof.** We first show that $\nu = g_V(I)$ is a valuation for each $I \in \mathbf{I}(\prec_V)$.

1. $\nu(\emptyset) = 0$: Assume $U \in \mathbf{O}(\mathbf{I}(\prec))$ and $\nu(U) > 0$. Then there is $r_0 \in I$ and $b_0 \in dom(r_0)$ such that $[b_0]_\prec \subseteq U$ and $0 < r_0(b_0)$. Since $I$ is directed, there is an infinite sequence $r_0 \prec_V r_1 \prec_V \cdots$ in $I$. Since $b_0 \in dom(r_0)$ and $r_0 \prec_V r_1$, there is $b_1 \in dom(r_1)$ with $b_0 \prec b_1$. Similarly, there must be $b_2 \in dom(r_2)$ with $b_1 \prec b_2$. This yields an infinite sequence $b_0 \prec b_1 \prec \cdots$, hence $\{c \in \mathbb{N} \mid (\exists i \in \mathbb{N})\, c \prec b_i\}$ is an element of $[b_0]_\prec \subseteq U$. Therefore, $U \neq \emptyset$.
2. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$: We first show $\nu(U) + \nu(V) \leq \nu(U \cup V) + \nu(U \cap V)$. Let $r, s \in I$ be such that $(\forall b \in dom(r))\, [b]_\prec \subseteq U$ and $(\forall b \in dom(s))\, [b]_\prec \subseteq V$. Set
$$p_r = \sum_{b \in dom(r)} r(b), \qquad p_s = \sum_{b \in dom(s)} s(b).$$
Let $t \in I$ be a $\prec_V$-upper bound of $r$ and $s$. Let
$$D_r = \{c \in dom(t) \mid (\exists b \in dom(r))\, b \prec c\},$$
$$D_s = \{c \in dom(t) \mid (\exists b \in dom(s))\, b \prec c\}.$$
Note that $c \in D_r \cap D_s$ implies $[c]_\prec \subseteq U \cap V$. Set
$$q_0 = \sum_{c \in D_r \setminus D_s} t(c), \qquad q_1 = \sum_{c \in D_s \setminus D_r} t(c), \qquad q_2 = \sum_{c \in D_r \cap D_s} t(c).$$
Then $r \prec_V t$ implies $p_r \leq q_0 + q_2$ and $s \prec_V t$ implies $p_s \leq q_1 + q_2$. Furthermore, using the fact $t \in I$, Lemma 5, and the definition of $\nu$, we obtain $\nu(U \cup V) \geq q_0 + q_1 + q_2$ and $\nu(U \cap V) \geq q_2$, hence $p_r + p_s \leq \nu(U \cup V) + \nu(U \cap V)$. It follows that $\nu(U) + \nu(V) \leq \nu(U \cup V) + \nu(U \cap V)$.
Next we show $\nu(U \cup V) + \nu(U \cap V) \leq \nu(U) + \nu(V)$. Let $r, s \in I$ be such that $(\forall b \in dom(r))\, [b]_\prec \subseteq U \cup V$ and $(\forall b \in dom(s))\, [b]_\prec \subseteq U \cap V$. Let $K \subseteq \bigcup_{b \in dom(r)} [b]_\prec$ be as in Lemma 9. Since $K$ is compact and $K \subseteq U \cup V$, there exists a finite set $F \subseteq \mathbb{N}$ with $K \subseteq \bigcup_{b \in F} [b]_\prec$ and such that each $b \in F$ satisfies $[b]_\prec \subseteq U$ or $[b]_\prec \subseteq V$. Apply Lemma 9 to get $t \in I$ with $r \prec_V t$ and $F \prec_U dom(t)$ and $K \subseteq \bigcup_{c \in dom(t)} [c]_\prec \subseteq \bigcup_{b \in dom(r)} [b]_\prec$. Next let $u \in I$ be a $\prec_V$-upper bound of $t$ and $s$. By restricting the domain of $u$ if necessary, we can assume that $(dom(t) \cup dom(s)) \prec_U dom(u)$, hence every $c \in dom(u)$ satisfies $[c]_\prec \subseteq U$ or $[c]_\prec \subseteq V$. Let $u_0$ be the restriction of $u$ to have domain $dom(u_0) = \{b \in dom(u) \mid [b]_\prec \subseteq U\}$, and let $u_1$ be the restriction of $u$ to have domain $dom(u_1) = \{b \in dom(u) \mid [b]_\prec \subseteq V\}$. Note that $u_0$ and $u_1$ are both in $I$ by Lemma 5, and that $dom(u) = dom(u_0) \cup dom(u_1)$. Then using the fact that $r \prec_V u$ and $s \prec_V u$, we have
$$\begin{aligned}\sum_{b \in dom(r)} r(b) + \sum_{b \in dom(s)} s(b) &\leq \sum_{c \in dom(u)} u(c) + \sum_{c \in dom(u_0) \cap dom(u_1)} u(c) \\ &= \sum_{c \in dom(u_0)} u_0(c) + \sum_{c \in dom(u_1)} u_1(c) \\ &\leq \nu(U) + \nu(V).\end{aligned}$$
Therefore, $\nu(U \cup V) + \nu(U \cap V) \leq \nu(U) + \nu(V)$.
3. $\nu$ is a continuous function: Assume $U \in \mathbf{O}(\mathbf{I}(\prec))$ and $q \in \mathbb{Q}_{>0}$ and $\nu(U) > q$. Since $\mathbf{I}(\prec)$ is consonant (see [6]), it suffices to find $K \in \mathbf{K}(\mathbf{I}(\prec))$ such that $K \subseteq U$ and $\nu(W) > q$ whenever $W$ is an open set containing $K$. By definition of $g_V(I)$, there must be $r \in I$

such that $(\forall b \in dom(r))\,[b]_\prec \subseteq U$ and $\sum_{b\in dom(r)} r(b) > q$. Now let $K \in \mathbf{K}(\mathbf{I}(\prec))$ be as in Lemma 9. Then $K \subseteq U$, and if $K \subseteq W$ then there is $s \in I$ with $r \prec_V s$ and $K \subseteq \bigcup_{c\in dom(s)}[c]_\prec \subseteq W$, hence $q < \sum_{c\in dom(s)} s(c) \leq \nu(W)$.

It only remains to show that $g_V$ is continuous. Assume $g_V(I) \in \langle U, q\rangle$. Then there is $r \in I$ satisfying $(\forall b \in dom(r))\,[b]_\prec \subseteq U$ and $q < \sum_{b\in dom(r)} r(b)$. Then $I \in [r]_{\prec_V} \subseteq g_V^{-1}(\langle U, q\rangle)$, hence $g_V$ is continuous. ◀

▶ **Theorem 13.** *$\mathbf{V}(\mathbf{I}(\prec))$ and $\mathbf{I}(\prec_V)$ are homeomorphic (via $f_V$ and $g_V$).*

**Proof.** It only remains to show that $f_V$ and $g_V$ are inverses of each other.

To show that $g_V \circ f_V$ is the identity function, it suffices to show that $g_V(f_V(\nu)) \in \langle U, q\rangle$ if and only if $\nu \in \langle U, q\rangle$ for each $\nu \in \mathbf{V}(\mathbf{I}(\prec))$ and each subbasic open $\langle U, q\rangle$. If $g_V(f_V(\nu)) \in \langle U, q\rangle$, then there must be $r \in f_V(\nu)$ with $q < \sum_{b\in dom(r)} r(b)$ and $\bigcup_{b\in dom(r)}[b]_\prec \subseteq U$. This implies that $dom(r) \neq \emptyset$, and using the definition of $f_V$ we obtain $q < \sum_{b\in dom(r)} r(b) < \nu(\bigcup_{b\in dom(r)}[b]_\prec) \leq \nu(U)$, hence $\nu \in \langle U, q\rangle$. Conversely, if $\nu \in \langle U, q\rangle$ then since $\nu$ is continuous there exist $b_0, \ldots, b_n \in \mathbb{N}$ such that $\bigcup_{i\leq n}[b_i]_\prec \subseteq U$ and $q < \nu(\bigcup_{i\leq n}[b_i]_\prec)$. If $\nu([b_i]_\prec) = \infty$ for some $i \leq n$, then the partial function $r$ defined as $dom(r) = \{b_i\}$ and $r(b_i) = q + 1$ is in $f_V(\nu)$, which implies $g_V(f_V(\nu)) \in \langle U, q\rangle$. Otherwise $\nu([b_i]_\prec) < \infty$ for each $i \leq n$, so define

$$m_i = \nu([b_i]_\prec) - \nu\big([b_i]_\prec \cap \bigcup_{j<i}[b_j]_\prec\big).$$

Note that the modularity of $\nu$ implies $m_i = \nu(\bigcup_{j\leq i}[b_j]_\prec) - \nu(\bigcup_{j<i}[b_j]_\prec)$, hence a simple inductive argument yields $\sum_{i\leq n} m_i = \nu(\bigcup_{i\leq n}[b_i]_\prec)$, which is strictly larger than $q$. Let $G = \{i \mid m_i > 0\}$. Then there exists $r \in \mathcal{B}$ with $dom(r) = \{b_i \mid i \in G\}$ and $(\forall i \in G)\, r(b_i) < m_i$ and $q < \sum_{b\in dom(r)} r(b)$. If $F \subseteq G$ is non-empty, then

$$\sum_{i\in F} r(b_i) < \sum_{i\in F} m_i = \sum_{i\in F}\left(\nu([b_i]_\prec) - \nu\big([b_i]_\prec \cap \bigcup_{j<i}[b_j]_\prec\big)\right)$$
$$\leq \sum_{i\in F}\left(\nu([b_i]_\prec) - \nu\big([b_i]_\prec \cap \bigcup_{\substack{j<i\\ j\in F}}[b_j]_\prec\big)\right) = \nu\left(\bigcup_{i\in F}[b_i]_\prec\right).$$

Thus, $r \in f_V(\nu)$ and $q < \sum_{b\in dom(r)} r(b)$, hence $g_V(f_V(\nu)) \in \langle U, q\rangle$.

Next we show that $f_V(g_V(I)) = I$ for each $I \in \mathbf{I}(\prec_V)$. By unwinding the definitions of $f_V$ and $g_V$, we have $r \in f_V(g_V(I))$ if and only if for every non-empty $F \subseteq dom(r)$ there is $s \in I$ such that $\bigcup_{c\in dom(s)}[c]_\prec \subseteq \bigcup_{b\in F}[b]_\prec$ and $\sum_{b\in F} r(b) < \sum_{c\in dom(s)} s(c)$. Thus, given any $r \in I$, by Lemma 8 there is $s \in I$ with $r \prec_V s$ and $dom(r) \prec_U dom(s)$, hence $\bigcup_{c\in dom(s)}[c]_\prec \subseteq \bigcup_{b\in F}[b]_\prec$ and $\sum_{b\in F} r(b) < \sum_{c\in dom(s)} s(c)$, which implies $r \in f_V(g_V(I))$. Therefore, $I \subseteq f_V(g_V(I))$.

To prove $f_V(g_V(I)) \subseteq I$, fix any $r \in f_V(g_V(I))$. Then for every non-empty $F \subseteq dom(r)$ there is $s_F \in I$ such that $\bigcup_{c\in dom(s_F)}[c]_\prec \subseteq \bigcup_{b\in F}[b]_\prec$ and $\sum_{b\in F} r(b) < \sum_{c\in dom(s_F)} s_F(c)$. Using Lemma 9, we can assume that $F \prec_U dom(s_F)$. Let $s \in I$ be a $\prec_V$-upper bound of all of the $s_F$. Then for any non-empty $F \subseteq dom(r)$, we have

$$\sum_{b\in F} r(b) < \sum_{c\in\uparrow F\cap dom(s_F)} s_F(c) \qquad \text{(by choice of } s_F)$$
$$< \sum_{c\in\uparrow F\cap dom(s)} s(c) \qquad \text{(because } s_F \prec_V s \text{ and } \prec \text{ is transitive)}.$$

Therefore $r \prec_V s$, hence $r \in I$ because $I$ is a lower-set. It follows that $f_V(g_V(I)) \subseteq I$, which completes the proof that $f_V(g_V(I)) = I$. ◀

We remark that the homeomorphisms $f_V$ and $g_V$ are computable in the sense of TTE [21] when $\prec$ is computably enumerable, and therefore our approach is consistent with previous work on computable measures in [19, 13, 18]. The computability of $f_V$ is obvious. For $g_V$, note that for any $U \in \mathbf{O}(\mathbf{I}(\prec))$ and any $A \subseteq \mathbb{N}$ satisfying $U = \bigcup_{a \in A}[a]_\prec$, Lemma 9 implies

$$g_V(I)(U) = \bigvee \left\{ \sum_{c \in dom(s)} s(c) \;\middle|\; s \in I \,\&\, (\forall c \in dom(s))(\exists a \in A)\, a \prec c \right\},$$

which shows that $g_V$ is computable.

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