MSO Undecidability for Hereditary Classes of Unbounded Clique Width

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── Abstract ──────────

Seese's conjecture for finite graphs states that monadic second-order logic (MSO) is undecidable on all graph classes of unbounded clique-width. We show that to establish this it would suffice to show that grids of unbounded size can be interpreted in two families of graph classes: minimal hereditary classes of unbounded clique-width; and antichains of unbounded clique-width under the induced subgraph relation. We explore all the currently known classes of the former category and establish that grids of unbounded size can indeed be interpreted in them.

1 Introduction

The monadic second-order logic (MSO) of graphs has been an object of intensive research for many years now. It is a logic that is highly expressive and yet very well behaved on many interesting classes of graphs. It has enabled the extension of many automata-theoretic and algebraic techniques to the construction of algorithms on graphs (see the comprehensive treatment in [7]). It has become a reference logic against which many others are compared. A key area of investigation is determining on which classes of graphs MSO is algorithmically well-behaved.

The good algorithmic behaviour of MSO on a class $\mathscr{C}$ of graphs is usually taken to mean one of two things: the evaluation (or model-checking) problem for MSO sentences on $\mathscr{C}$ is tractable; or the satisfiability problem of MSO sentences on $\mathscr{C}$ is decidable. Usually, these two are linked. Broadly speaking, the only way we know to show that the MSO theory of a class $\mathscr{C}$ is decidable is to show that $\mathscr{C}$ can be obtained by means of an MSO interpretation from a class of trees, which itself has a decidable theory and this also yields efficient evaluation algorithms for MSO sentences on $\mathscr{C}$. And the only way we know to show that the MSO theory of $\mathscr{C}$ is undecidable is to show that there is an MSO interpretation that yields arbitrarily large grids on $\mathscr{C}$ and this also yields an obstacle to the tractability of MSO evaluation on $\mathscr{C}$.

Seese [20] formalizes the first of these observations into a conjecture: if the MSO theory of a class $\mathscr{C}$ is decidable, there is an MSO interpretation Ψ and a class $\mathscr{T}$ of trees such that Ψ maps $\mathscr{T}$ to $\mathscr{C}$. This remains an open question nearly three decades after it was first posed despite considerable research effort around it. By a theorem of Courcelle and Engelfriet [7], it is known that the classes of graphs obtained by MSO interpretations from trees are exactly those of bounded clique-width. Thus, Seese's conjecture can be understood as saying that any class of graphs of unbounded clique-width has an undecidable MSO theory.

If we similarly formalize the second observation above about grids and combine it with this, we can formulate the following stronger conjecture, which we refer to below as the *strong Seese conjecture*: every class $\mathscr{C}$ of graphs of unbounded clique-width admits an MSO interpretation that defines arbitrarily large grids. Seese's conjecture is often formulated in this stronger form as it seems the only reasonable route to proving it. It can be seen as an interesting analogue of the Robertson-Seymour grid minor theorem to the effect that any class of graphs of unbounded treewidth admits arbitrarily large grids as minors.

In recent years there has been growing interest in clique-width as a measure of the complexity of graphs from a structural and algorithmic point of view, quite separate from questions of logic [10, 4, 18, 11]. In particular, it provides a route for extending algorithmic methods that have had great success on sparse graph classes [17] to more general classes of graphs. A class of graphs may be of bounded clique-width while containing dense graphs – the classic example being the class of cliques.

In the context of the structural study of classes of bounded clique-width, there is particular interest in *hereditary classes*, that is, classes of graphs closed under the operation of taking induced subgraphs. This is because the induced subgraph relation behaves well with respect to clique-width. If a graph H is a subgraph or a minor of a graph G, the clique-width of H can be greater than that of G but if H is an *induced subgraph* of G, then the clique-width of H is no more than that of G. Hence, the hereditary closure of a class $\mathscr{C}$ of bounded clique-width still has bounded clique-width.

The induced subgraph relation is not as well-behaved as the graph minor relation. By the Robertson-Seymour graph minor theorem [19], the graph minor relation is a well-quasi-order. This is not true for the induced subgraph relation. By the same token, the classes of graphs of unbounded treewidth are well understood in that they are precisely the classes which have grid minors of unbounded size, the picture for classes of graphs of unbounded clique-width is somewhat less clear. The relationship between a class having unbounded clique-width and admitting a well-quasi-order of the induced subgraph relation has been the subject of much investigation. It is possible to construct as we see below, infinite descending chains, under inclusion, of hereditary classes of graphs, each of unbounded clique-width.

Lozin [15] identified the first example of a hereditary class $\mathscr{C}$ of graphs of unbounded clique-width that are *minimal* with this property – that is, no hereditary proper subclass of $\mathscr{C}$ has unbounded clique-width. Since then, many other such classes have been constructed. Collins et al. [3] show how to obtain an infinite family of such classes. Their construction has been recently extended by Brignall and Cocks [2] to obtain uncountably many examples. Atminas et al. [1] construct instances of such classes which are characterized by a finite collection of forbidden induced subgraphs. Lozin et al. [16] construct a minimal hereditary class of unbounded clique-width that is well-quasi-ordered under the induced subgraph relation.

This exploration of novel classes of unbounded clique-width also suggests an approach to establishing Seese's conjecture for finite graphs. We establish in Section 3 that a proof of Seese's conjecture would follow from the conjunction of the following two statements: (1) every collection of graphs of unbounded clique-width that forms an infinite anti-chain under the induced subgraph relation interprets arbitrarily large grids; and (2) every minimal hereditary class of unbounded clique-width interprets arbitrarily large grids. This suggests a programme to establish Seese's conjecture by systematically studying antichains and minimal hereditary classes of unbounded clique-width. We do not yet know of a complete classification of minimal hereditary classes of unbounded clique-width, which makes a systematic approach to this programme challenging. Nevertheless, we examine in Sections 4–5 all known classes satisfying

these conditions and show that in all cases we can indeed interpret grids of unbounded size. Thus none of these provides a counterexample to Seese's conjecture. Our construction shows a uniform method of proving that these classes have unbounded clique-width. The proof is often simpler than the *ad hoc* methods by which this was proved for each class in the literature.

It is worth mentioning some significant lines of investigation related to Seese's conjecture. Courcelle [5] shows that proving Seese's conjecture for finite graphs is equivalent to proving the relativized version of the conjecture for particular classes of graphs, two examples being bipartite graphs and split graphs. He further shows the conjecture to be true when relativized to uniformly k-sparse graphs and interval graphs. Another line of work addresses variants of Seese's conjecture obtained by considering logics other than MSO. One such result by Seese [20] shows that guarded second-order logic (GSO) is undecidable on any class of unbounded clique-width. Similarly, Courcelle and Oum [9] show that the extension C_2MSO of MSO obtained by considering modulo 2 counting quantifiers is also undecidable on classes of unbounded clique-width. In all of these cases, the proof goes via interpreting grids in unbounded clique-width classes. There has also been interesting progress looking at Seese's conjecture for structures other than graphs. A significant paper here is by Hliněný and Seese [12] who show the conjecture to be true for matroids representable over any finite field.

2 Preliminaries

The graphs we consider in this paper are simple, undirected and loop-free. For a graph G, we write $V(G)$ for the vertices of G and $E(G)$ for the edges. A graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and for any $x, y \in V(H)$, $\{x, y\} \in E(H)$ if, and only if, $\{x, y\} \in E(G)$. We write $H \subseteq G$ to denote that H is an induced subgraph of G. A class of graphs is said to be *hereditary* if it is closed under induced subgraphs. For any graph class $\mathscr{C}$, we write $\mathscr{C}{\downarrow}$ to denote the hereditary closure of $\mathscr{C}$ – that is, the class of graphs that are induced subgraphs of the graphs in $\mathscr{C}$. We consider monadic second-order logic (MSO) over vocabularies τ containing the binary relation E and finitely many unary relation symbols. A τ-structure can be thought of as an *expansion* of a graph $G = (V, E)$ with unary relations that interpret the unary symbols in τ. Such a structure can be thought of as a vertex-coloured graph. An MSO formula over the vocabulary τ is an expression that is inductively constructed from atomic MSO formulae using the Boolean connectives $\wedge, \vee$, and $\neg$, and existential quantification over vertex variables and set variables. Here an atomic MSO formula is an expression of the form $E(x, y)$ or $Q(x)$ or $X(y)$ or $x = y$ where x, y are vertex variables, the predicates E, Q belong to τ and X is a set variable. A *first order*, or FO, formula is an MSO formula that does not contain any set variable. We often write $\varphi(\bar{x}, \bar{X})$ to denote a formula whose free variables are among $\bar{x}$ and $\bar{X}$, the former being a tuple of vertex variables and the latter a tuple of set variables. Given such a formula, and a graph G along with a tuple $\bar{a}$ of vertices that interprets $\bar{x}$ and a tuple $\bar{A}$ of unary relations that interprets $\bar{X}$, we write $(G, \bar{A}) \models \phi[\bar{a}]$ to denote that the formula ϕ is satisfied in G in this interpretation.

Given a graph G and an MSO formula $\varphi(\bar{x}, \bar{X})$ where the length of $\bar{x}$ is k, we can think of φ as defining a k-ary relation on an expansion of G with an interpretation $\bar{A}$ of $\bar{X}$. Specifically this relation, denoted $\varphi^{(G, \bar{A})}$, is given by $\varphi^{(G, \bar{A})} = \{\bar{a} \mid (G, \bar{A}) \models \varphi[\bar{a}]\}$. Given a sequence $\bar{Z}$ of set variables, an MSO *graph interpretation with parameters* $\bar{Z}$ is a pair $\Psi(\bar{Z}) = (\psi_V(x, \bar{Z}), \psi_E(x, y, \bar{Z}))$ of MSO formulas over the vocabulary $\{E\} \cup \{Z_i \mid Z_i \text{ is an element of } \bar{Z}\}$. Given a graph G together with unary relations $\bar{A}$ interpreting the set variables $\bar{Z}$ in G, the interpretation $\Psi(\bar{Z})$ defines a possibly directed graph $H = \Psi(G, \bar{A})$.

This graph has (i) vertex set $\psi_V^{(G,\bar{A})}$, and (ii) edge set $\psi_E^{(G,\bar{A})}$. In this paper, we are only interested in the case where $\Psi(\bar{Z})$ defines an undirected graph (that is, $\psi_E(x, y, \bar{Z})$ defines an irreflexive and symmetric binary relation). Thus $\Psi(\bar{Z})$ defines a function from the expansions of graphs with $|\bar{Z}|$ unary predicates, to graphs, and therefore in general defines a relation on graphs. Where it causes no confusion, we also refer to the relation defined by an interpretation as an MSO interpretation. If $\bar{Z}$ is emplty, we call the interpretation Ψ *parameterless*, and such a Ψ defines a function from graphs to graphs. An example of a parameterless interpretation is $\Theta = (\theta_V(x), \theta_E(x, y))$ where $\theta_V(x) := (x = x)$ and $\theta_E(x, y) := \neg E(x, y)$; the function it defines maps a graph to its complement. An example of an interpretation with parameters is $\Gamma(Z) = (\gamma_V(x, Z), \gamma_E(x, y, Z))$ where $\gamma_V(x, Z) := Z(x)$ and $\gamma_E(x, y, Z) := E(x, y)$. The function that it defines maps an expansion (G, A) of a graph G to the subgraph of G induced by A; thus the relation on graphs that $\Gamma(Z)$ defines maps a graph to all its induced subgraphs. Given a class $\mathscr{C}$ of graphs and an interpretation Ψ with parameters $\bar{Z}$, we denote by $\Psi(\mathscr{C})$ the class of graphs given by $\Psi(\mathscr{C}) = \{\Psi(G, \bar{A}) \mid G \in \mathscr{C} \text{ and } \bar{A} \text{ is an interpretation of } \bar{Z} \text{ in } G\}$. For example, for the interpretation Γ above and a class $\mathscr{C}$ of graphs, the class $\Gamma(\mathscr{C})$ is exactly the hereditary closure of $\mathscr{C}$. Since they are relations, one can compose interpretations and it is known that the class of MSO interpretations is closed under composition [13]. We call MSO interpretations with parameters simply MSO interpretations for ease of readability, and denote them with the uppercase Greek letters $\Phi, \Gamma, \Psi, \Theta$, etc.

The notion of clique-width is a structural parameter of graphs that was introduced by Courcelle, Engelfriet and Rozenberg in [8] as a generalization of the well-known notion of treewidth. We do not give the definitions of clique-width and treewidth here as we need only specific properties of these for our results that we state below; we point the reader to [7, 17] for more about the notions and results concerning them. We write $\mathrm{cwd}(G)$ and $\mathrm{twd}(G)$ for the clique-width and tree-width of a graph G, respectively. As examples, a clique has clique-width 1, and a cograph has clique-width 2. It is known for any graph G that $\mathrm{cwd}(G) \leq 4 \cdot 2^{\mathrm{twd}(G)-1} + 1$ [10] and for planar G we even have $\mathrm{cwd}(G) \leq 6\mathrm{twd}(G) - 2$ [6]. A class of graphs is said to have *bounded* clique-width if for some number $k \geq 1$, every graph in the class has clique-width at most k. Thus, the class of cliques, the class of cographs and all classes of bounded treewidth have bounded clique-width. A graph class has *unbounded* clique-width if it does not have bounded clique-width. Examples of graph classes of unbounded clique-width include grids, interval graphs, and line graphs [5].

The class of all graphs of clique-width at most k is hereditary since the clique-width of an induced subgraph of G is never more than the clique-width of G. An *antichain* under the induced subgraph relation is a set $\mathcal{A}$ of graphs such that if G and H are distinct graphs in $\mathcal{A}$, then neither of $G \subseteq H$ or $H \subseteq G$ holds. Usually when we say "antichain" without further qualification, we mean an antichain under the induced subgraph relation. A graph class $\mathscr{C}$ is said to be *well-quasi-ordered* (WQO) under induced subgraphs if it does not contain any infinite antichains. For example, the class of all cliques is WQO under induced subgraphs.

The MSO theory of a graph class $\mathscr{C}$ is the class of all MSO sentences that are true in all graphs of $\mathscr{C}$. This theory is decidable if, and only if, the following problem is decidable: given an MSO sentence ϕ decide if ϕ is true in some graph in $\mathscr{C}$. Seese's conjecture states any class whose MSO theory is decidable has bounded clique-width. An $m \times n$ grid is a graph $G = (V, E)$ on $m \cdot n$ vertices with $V = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = \{\{(i, j), (i, j + 1)\} \mid 1 \leq i \leq m, 1 \leq j < n\} \cup \{\{(i, j), (i + 1, j)\} \mid 1 \leq i < m, 1 \leq j \leq n\}$. The grid is *square* if $m = n$. We say a class $\mathscr{C}$ of graphs *interprets grids* via an MSO interpretation Φ, if $\Phi(\mathscr{C})$ contains graphs isomorphic to arbitrarily large square grids. Any class of graphs that contains arbitrarily large grids has undecidable MSO theory [7, Thm. 5.6].

Morover, since MSO decidability is preserved by interpretations [7, Thm. 7.54], any class of graphs that interprets grids via an MSO interpretation has an undecidable MSO theory. The *strong Seese conjecture* is that any class of unbounded clique-width interprets grids via an MSO interpretation. It is known that if the clique-width of a class $\mathscr{C}$ is bounded and Φ is an MSO interpretation, then the clique-width of $\Phi(\mathscr{C})$ is also bounded [7, Cor. 7.38]. A simple observation about classes interpreting grids is the following.

▶ **Proposition 1.** *Suppose $\mathscr{C}$ is a graph class that interprets grids, and $\mathscr{D}$ is a graph class for which there exists an MSO interpretation Ξ such that the hereditary closure of $\Xi(\mathscr{D})$ contains $\mathscr{C}$. Then $\mathscr{D}$ interprets grids as well.*

Specifically, if Θ is the interpretation mapping $\mathscr{C}$ to a class containing arbitrarily large grids, and Γ is the interpretation defined above taking any class to its hereditary closure, then an interpretation Ω such that $\Omega(\mathscr{D})$ contains arbitrarily large square grids, is given by $\Omega = \Theta \circ \Gamma \circ \Xi$ (viewing Θ, Γ and Ξ as functions) where $\circ$ denotes composition.

We say that a class of graphs $\mathscr{C}$ is HUCW if it is hereditary and has unbounded clique-width. An HUCW graph class is said to be *minimal* if it does not contain a proper subclass that is HUCW. For example, bipartite permutation graphs and unit interval graphs are two minimal HUCW graph classes [15]. The existence of countably many minimal HUCW classes is established in [3], and this has been recently extended to uncountably many minimal HUCW classes in [2].

3 Minimal Classes and Well-Quasi-Ordering

In this section we lay out an approach to studying Seese's conjecture that motivates our study of MSO decidability for minimal HUCW classes. The first observation is that, if $\mathscr{C}$ is a counter-example to Seese's conjecture, then so is $\mathscr{C}{\downarrow}$. Recall that a counter-example to Seese's conjecture would be a class $\mathscr{C}$ that has unbounded clique-width and a decidable MSO theory. Clearly if $\mathscr{C}$ has unbounded clique-width, then so does $\mathscr{C}{\downarrow}$. The following proposition is folklore. It follows immediately from the fact that MSO decidability is preserved by interpretations and the existence of the interpretation Γ defined above which takes a class to its hereditary closure.

▶ **Proposition 2.** *If the MSO theory of $\mathscr{C}$ is decidable, then so is the MSO theory of $\mathscr{C}{\downarrow}$.*

Hence, if there is a counter-example to Seese's conjecture, we have one that is a hereditary class of unbounded clique-width, i.e. an HUCW class. In the present section, we establish some basic facts about the HUCW classes that allow us to structure the search for such a counter-example, or indeed the attempt to show that there is none.

The relation of being an induced subgraph is not a well-quasi-order as it admits infinite anti-chains. As an example, let I_n be the graph on $n + 4$ vertices $e_0, e_1, e_2, e_3, c_1, \ldots, c_n$ where for each $i < n$ there is an edge between c_i and c_{i+1}, and in addition we have edges $e_0 - c_1, e_1 - c_1, e_2 - c_n$ and $e_3 - c_n$. In short, there is a path of length n with two additional vertices at each end to mark the ends. Then, it is clear the collection $(I_n)_{n \in \mathbb{N}}$ is an antichain in the induced subgraph order. This particular antichain has bounded clique-width. It is also possible to construct antichains of unbounded clique-width (which therefore must be infinite). An example is obtained by taking the collection of $n \times n$ grids and adding two extra vertices at each corner to form a triangle. In what follows, whenever we refer to an *antichain* we mean one under the induced subgraph relation.

From an antichain of unbounded clique-width, it is possible to construct (as we show below) an infinite descending chain of classes of graphs (under the inclusion relation) all of which are HUCW. Thus, it was a significant discovery to find that there are actually HUCW classes $\mathscr{C}$ that are *minimal*: no proper hereditary subclass of $\mathscr{C}$ has unbounded clique-width. The first such example is due to Lozin [15]. Collins et al. [3] constructed an infinite family of such classes and Lozin et al. [16] give an example that is itself well-quasi-ordered under the induced substructure relation. We examine these in some detail in subsequent sections.

If it were the case that every class that is HUCW contains as a subclass a minimal HUCW class, then showing that every minimal HUCW class interprets grids would suffice to prove Seese's conjecture. Indeed, if $\mathscr{C}$ interprets grids of unbounded size, so does every class that contains $\mathscr{C}$. However, Korpelainen has shown [14] that there are HUCW classes that contain no minimal HUCW subclass. We give a construction of such a class in Section 3.2. This is linked to the existence of antichains of unbounded clique-width. Specifically, we establish the following facts.

1. If $\mathscr{C}$ is a minimal HUCW class, then it cannot contain an antichain of unbounded clique-width (Theorem 6 in Section 3.1).
2. If $\mathscr{C}$ is an HUCW class that contains no minimal class, it must contain an antichain of unbounded clique-width (Theorem 7 in Section 3.1).

From these, the theorem below follows, which suggests a programme for proving Seese's conjecture.

▶ **Theorem 3.** *The strong Seese conjecture holds if, and only if, both of the following are true:*
1. *every antichain of unbounded clique-width interprets grids; and*
2. *every minimal* HUCW *class interprets grids.*

3.1   Antichains and Minimal Classes

We first establish the relationship between the existence of antichains of unbounded clique-width and the minimality of HUCW classes. These are established in Theorems 6 and 7.

We say that a sequence $(\mathscr{C}_i)_{i \in \omega}$ is an infinite *strictly descending* HUCW-*chain* if for each i, $\mathscr{C}_i$ is an HUCW class and $\mathscr{C}_{i+1}$ is a proper subclass of $\mathscr{C}_i$. We say that $\mathscr{C}$ contains an infinite strictly descending HUCW-chain if there is such a chain with $\mathscr{C}_i \subseteq \mathscr{C}$ for all i.

▶ **Lemma 4.** *The following are equivalent:*
1. $\mathscr{C}$ *contains an infinite strictly descending* HUCW-*chain whose intersection is a class of bounded clique-width.*
2. $\mathscr{C}$ *contains an infinite strictly descending* HUCW-*chain whose intersection is empty.*
3. $\mathscr{C}$ *contains an antichain of unbounded clique-width.*

Proof.

3 $\Rightarrow$ 2.   If $\{G_1, G_2, \ldots\}$ is such an antichain, then let $\mathscr{C}_i$ be the hereditary closure of $\{G_i, G_{i+1}, \ldots\}$ for $i \geq 1$. Then $\mathscr{C}_1 \supsetneq \mathscr{C}_2 \supsetneq \ldots$ is an infinite strictly descending HUCW-chain whose intersection is empty.

2 $\Rightarrow$ 1.   Trivial since the empty class has clique-width 0.

1 $\Rightarrow$ 3.   Let $\mathscr{C}_1 \supsetneq \mathscr{C}_2 \supsetneq \ldots$ be such a descending HUCW-chain and $\mathscr{C}_\omega = \bigcap_{i \geq 1} \mathscr{C}_i$. Let $\mathscr{D}_i = \mathscr{C}_i \backslash \mathscr{C}_{i+1}$ for $i \geq 1$. Then for $1 \leq i < j$, we have $\mathscr{D}_i \cap \mathscr{C}_j = \emptyset$; hence $\mathscr{D}_i \cap \mathscr{D}_j = \mathscr{D}_i \cap \mathscr{C}_\omega = \emptyset$. Further, $\mathscr{C}_i = \left(\biguplus_{i \leq k < \omega} \mathscr{D}_k \right) \biguplus \mathscr{C}_\omega$.

▷ **Claim 5.** The following are true:
1. For $1 \leq i < j$, no graph in $\mathscr{D}_i$ is an induced subgraph of a graph in $\mathscr{D}_j$.
2. For $i \geq 1$, for every graph $G \in \mathscr{D}_i$, there exists a number $f(G) > i$ such that for all $j \geq f(G)$, no graph in $\mathscr{C}_j \setminus \mathscr{C}_\omega$ is an induced subgraph of G.

Proof.
1. If $G \subseteq H$ for some $G \in \mathscr{D}_i$ and $H \in \mathscr{D}_j$, then since $\mathscr{D}_j \subseteq \mathscr{C}_j$ and $\mathscr{C}_j$ is hereditary, we would have $G \in \mathscr{C}_j$; but that contradicts the fact that $\mathscr{D}_i \cap \mathscr{C}_j = \emptyset$.
2. Let $H_1, \ldots, H_r$ be an enumeration of the induced subgraphs of G that are not in $\mathscr{C}_\omega$ – clearly r is finite since G is finite. Since $\mathscr{C}_i = \left(\biguplus_{i \leq j < \omega} \mathscr{D}_j \right) \biguplus \mathscr{C}_\omega$, there exist numbers $j_1, \ldots, j_r \in [i, \omega)$ such that $H_i \in \mathscr{D}_{j_i}$ for $i \in \{1, \ldots, r\}$. It then follows by the properties of the $\mathscr{D}_i$'s above that $f(G) = \max\{j_i \mid 1 \leq i \leq r\} + 1$ is indeed as desired. ◁

We now use the above claim to inductively construct an antichain of $\mathscr{C}$ of unbounded clique-width. Let G_0 be a graph in $\mathscr{D}_0$. Assume that we have constructed graphs $G_0, \ldots, G_i$ for $i \geq 0$ such that (i) $G_j \in \mathscr{D}_{l_j}$ and $l_j > l_{j-1}$ for $1 \leq j \leq i$; (ii) $\{G_0, \ldots, G_i\}$ is an antichain; and (iii) the clique-width of G_j is strictly greater than that of G_{j-1} for $1 \leq j \leq i$. Let $k = \max\{f(G_j) \mid 1 \leq j \leq i\} > l_i$ where f is as in Claim 5. Consider the class $\mathscr{C}_k \setminus \mathscr{C}_\omega$ – by Lemma 5, all graphs in this class are incomparable with each of $G_0, \ldots, G_i$ in the induced subgraph order. Further, since $\mathscr{C}_k$ has unbounded clique-width while $\mathscr{C}_\omega$ has bounded clique width, we have that $\mathscr{C}_k \setminus \mathscr{C}_\omega$ has unbounded clique-width, whereby there exists $G_{i+1} \in \mathscr{C}_k \setminus \mathscr{C}_\omega$ such that G_{i+1} has clique-width greater than that of G_i. Let $l_{i+1} \geq k > l_i$ be such that $G_{i+1} \in \mathscr{D}_{l_{i+1}}$. Then we see that G_{i+1} is indeed as desired to complete the induction. ◀

We are now ready to prove the two results linking minimality of HUCW classes and the existence of antichains of unbounded clique-width.

▶ **Theorem 6.** *If $\mathscr{C}$ is a minimal* HUCW *class, then $\mathscr{C}$ does not contain an antichain of unbounded clique-width.*

Proof. If $\mathscr{C}$ contains an antichain of unbounded clique-width, then by Lemma 4, $\mathscr{C}$ contains an infinite strictly descending HUCW-chain, and hence in particular a proper subclass that is HUCW. Hence $\mathscr{C}$ is not minimal. ◀

▶ **Theorem 7.** *If $\mathscr{C}$ is* HUCW *and does not contain a minimal* HUCW *class, then there exists in $\mathscr{C}$ an antichain of unbounded clique-width.*

Proof. We assume without loss of generality that the vertices of the graphs of $\mathscr{C}$ belong to the set $\mathbb{N}$ of natural numbers, so that $\mathscr{C}$ is countable. Suppose that $\mathscr{C}$ does not contain a minimal class. Consider the sequence $(\mathscr{C}_\lambda)_{\lambda \geq 0}$ of classes of structures, for ordinals λ, defined inductively as follows. Let $\mathscr{C}_0 = \mathscr{C}$ and inductively, assume that for all $\nu < \lambda$, the class $\mathscr{C}_\nu$ has been defined and that $\mathscr{C}_\nu \subseteq \mathscr{C}$ for all $\nu < \lambda$. If λ is a limit ordinal, define $\mathscr{C}_\lambda = \bigcap_{\nu < \lambda} \mathscr{C}_\nu$. If λ is a successor ordinal of say λ^-, then define $\mathscr{C}_\lambda$ as follows. If $\mathscr{C}_{\lambda^-}$ is not HUCW, then $\mathscr{C}_\lambda = \mathscr{C}_{\lambda^-}$. Otherwise $\mathscr{C}_{\lambda^-}$ is HUCW and $\mathscr{C}_{\lambda^-} \subseteq \mathscr{C}$; then $\mathscr{C}_{\lambda^-}$ cannot be minimal since by our premise, $\mathscr{C}$ does not contain any minimal HUCW class. Let $\mathscr{C}_\lambda$ be any proper subclass of $\mathscr{C}_{\lambda^-}$ that is HUCW. This completes the construction of the sequence $(\mathscr{C}_\lambda)_{\lambda \geq 0}$.

Consider now the set $\mathcal{P}$ of ordinals defined as $\mathcal{P} = \{\lambda \mid \mathscr{C}_\lambda \text{ is not HUCW}\}$. This set is non-empty – since $\mathscr{C}$ is a class of finite graphs whose vertices are natural numbers, $\mathscr{C}$ is countable and hence $\mathscr{C}_\lambda = \emptyset$ for all uncountable λ. By the definition above, if $\lambda \in \mathcal{P}$, then all ordinals greater than λ are in $\mathcal{P}$ as well. Now since the ordinals are well ordered, $\mathcal{P}$ has a minimum, call it μ^*. We make the following observations about μ^*:

1. μ^* must be a limit ordinal. If it is a successor ordinal of say λ, then $\mathscr{C}_\lambda$ must be HUCW since μ^* is the minimum ordinal in $\mathcal{P}$. But if $\mathscr{C}_\lambda$ is HUCW, then $\mathscr{C}_{\mu^*}$ must be a HUCW class by the inductive definitions above. Therefore, $\mathscr{C}_{\mu^*} = \bigcap_{\nu < \mu^*} \mathscr{C}_\nu$ where $\mathscr{C}_\nu$ is HUCW for all $\nu < \mu^*$.

2. μ^* is countable – this is because $\mathscr{C}$ is countable.

3. $\mathscr{C}_{\mu^*}$ is a hereditary class of bounded clique-width. Let $G \in \mathscr{C}_{\mu^*}$ and $H \subseteq G$. Then by (1) above, $G \in \mathscr{C}_\nu$ for all $\nu < \mu^*$. Since each $\mathscr{C}_\nu$ is hereditary, we have $H \in \mathscr{C}_\nu$ for all $\nu < \mu^*$. Then $H \in \mathscr{C}_{\mu^*}$. So $\mathscr{C}_{\mu^*}$ is hereditary. That $\mathscr{C}_{\mu^*}$ has bounded clique-width now follows from the fact that $\mathscr{C}_{\mu^*}$ is not HUCW.

Now since μ^* is countable, it has cofinality ω so that there exists an increasing function $f : \mathbb{N} \to \mu^*$ (where μ^* is seen as the set of ordinals less than μ^*) such that if $\mathscr{F}_i = \mathscr{C}_{f(i)}$ for $i \in \mathbb{N}$, then $\bigcap_{i \in \mathbb{N}} \mathscr{F}_i = \mathscr{C}_{\mu^*}$. We observe that $\mathscr{F}_1 \supsetneq \mathscr{F}_2 \supsetneq \ldots$ is an infinite strictly descending HUCW-chain in $\mathscr{C}$, whose intersection $\mathscr{C}_{\mu^*}$ is a class of bounded clique-width. It now follows by Lemma 4 that $\mathscr{C}$ contains an antichain of unbounded clique-width. ◄

The converse of Theorem 7 does not hold. That is to say, we can construct an HUCW class that both contains a minimal HUCW class and contains an antichain of unbounded clique-width. Indeed, if $\mathcal{C}_1$ is a minimal HUCW class and $\mathcal{C}_2$ the hereditary closure of an antichain of unbounded clique-width then clearly $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ has this property.

3.2 HUCW Classes which Contain No Minimal Class

Theorem 7 raises the obvious question of whether there exists any class $\mathscr{C}$ which is HUCW but does not contain a minimal HUCW class. The existence of such a class was demonstrated by Korpelainen [14]. Here we give a similar construction which we arrived at independently.

▶ **Theorem 8.** *There is an HUCW class $\mathcal{T}$ that does not contain any minimal HUCW class.*

It suffices to show that if $\mathcal{C}$ is any hereditary subclass of $\mathcal{T}$ of unbounded clique-width, it contains an antichain of unbounded clique-width.

Towards this, let $G_{n,n}$ denote the $n \times n$ grid. Note that, in $G_{n,n}$, every vertex has degree 2, 3 or 4, and there are exactly four vertices (at the corners) of degree 2. For $n \geq 3$, we define T_n as the graph obtained from $G_{n,n}$ by:

1. removing every vertex v of degree 2 and inserting an edge between the two neighbours of v; and

2. replacing every vertex v of degree 4 by four new vertices v_1, v_2, v_3, v_4 that are connected in a 4-cycle so that the four edges incident on v are now each incident on one of the four new vertices.

It is easily seen that T_n is 3-regular, and it is more convenient to work with than grids. The number of vertices in T_n is less than $4n^2$.

Recall that a graph H is a *subdivision* of a graph G if it is obtained from G by replacing every edge by a simple path. For a positive integer t, we write G^t for the t-subdivision of G: the graph obtained from G by replacing each edge of G by a path of length t. We make the following simple observation for later use:

▶ **Lemma 9.** *If H is a subdivision of G and $twd(G) = k$, then $k \leq twd(H) \leq \max(k, 3)$.*

Proof. The lower bound on $\mathrm{twd}(H)$ follows immediately from the fact that G is a minor of H so $\mathrm{twd}(G) \leq \mathrm{twd}(H)$.

Suppose now that (T, β) is a tree decomposition of G of width k. To obtain a tree decomposition of H, consider an edge $\{u, v\}$ of G which is subdivided into a path $u = p_0, \ldots, p_t = v$ in H. As $\{u, v\}$ is an edge of G, there must be a node a of T such that $\{u, v\} \subseteq \beta(t)$. We attach a path $a_1, \ldots, a_t$ of length t to a and let $\beta(a_i) = \{u, v, p_i, p_{i+1}\}$. Doing this for each edge gives us a tree decomposition of H whose width is $\max(k, 3)$. ◄

Define the class $\mathcal{T} = \{H \mid H \subseteq T_n^n \text{ for some } n > 2\}$, i.e. the collection of graphs that are induced subgraphs of the n-subdivision of T_n for some n. We consider the graphs $H \in \mathcal{T}$ where every vertex has degree 2 or 3. We call such graphs *skeleton graphs* and the vertices of degree 3 the *branch vertices*. Note that every graph in $\mathcal{T}$ is an induced subgraph of a skeleton graph.

The next two lemmas establish some useful properties of the graphs in $\mathcal{T}$.

▶ **Lemma 10.** *If $H \in \mathcal{T}$ is a skeleton graph with at most $m > 2$ branch vertices, then $cwd(H) \leq 6m - 2$.*

Proof. Since H has at most m branch vertices, it is the subdivision of some graph G with m vertices. Hence, by Lemma 9, the treewidth of H is at most m. Note further that all graphs in $\mathcal{T}$ are planar and hence H is planar. For any planar graph H, $\mathrm{cwd}(H) \leq 6\mathrm{twd}(H) - 2$ [6, Thm 17], and the result follows. ◄

▶ **Lemma 11.** *If H is a subdivision of T_n for $n > 2$, then the clique-width of H is at least $(n-1)/6$.*

Proof. Since $G_{n-2,n-2}$ is a minor of T_n and $\mathrm{twd}(G_{k,k} = k$ we have that $\mathrm{twd}(T_n) \geq n - 2$. Also, by Lemma 9 we know that $\mathrm{twd}(H) = \mathrm{twd}(T_n)$. Now, for any planar graph G we have $\mathrm{twd}(G) \leq 6\mathrm{cwd}(G) - 1$ by [7, Prop. 2.115]. Since H is planar, the result follows. ◄

Proof of Theorem 8. The class $\mathcal{T}$ is hereditary by definition and has unbounded clique-width by Lemma 11. Thus, it remains to show that for every class $\mathcal{C} \subseteq \mathcal{T}$, if $\mathcal{C}$ has unbounded clique-width, then $\mathcal{C}$ contains an antichain of unbounded clique-width.

So, suppose $\mathcal{C} \subseteq \mathcal{T}$ has unbounded clique-width. For a graph $H \in \mathcal{T}$, write $\mathrm{mn}(H)$ for the length of the shortest path between two branch vertices of H. We define the following sequence of graphs. First, let G_0 be any graph in $\mathcal{C}$ containing at least two branch vertices. Suppose we have defined G_i for $i \geq 0$, and let $t = \max(\mathrm{cwd}(G_i), \mathrm{mn}(G_i))$. We then choose G_{i+1} to be any graph in $\mathcal{C}$ with $\mathrm{cwd}(G_{i+1}) > 24t^2 - 2$.

It is clear that the sequence of graphs $(G_i : i \in \omega)$ is of unbounded clique-width, since $\mathrm{cwd}(G_i) < \mathrm{cwd}(G_{i+1})$ for all i. We now argue that this is also an antichain. For any $i < j$, clearly G_j cannot be an induced subgraph of G_i since $\mathrm{cwd}(G_i) < \mathrm{cwd}(G_j)$, so it remains to show that G_i is not an induced subgraph of G_j. Since $\mathrm{cwd}(G_j) > 24t^2 - 2$, where $t = \max(\mathrm{cwd}(G_i), \mathrm{mn}(G_i))$, it follows by Lemma 10 that G_j has more than $4t^2$ branch vertices. Since T_n^n contains fewer than $4n^2$ branch vertices, it follows that G_j is not an induced subgraph of T_n^n for any $n \leq t$. Hence, $\mathrm{mn}(G_j)$ is at least $t + 1$. However, by the choice of t, $\mathrm{mn}(G_i) \leq t$ and so G_i contains two branch vertices at distance at most t. We conclude that G_i is not an induced subgraph of G_j. ◄

4   Grid-Like Classes

We begin our systematic exploration of all known minimal hereditary classes of unbounded clique-width. Many such classes are defined in terms of a grid-like structure and this is used to show that they have unbounded clique-width. The challenge in these cases is to show how this grid structure can be drawn out through an MSO interpretation. We begin with a collection of minimal HUCW classes (indeed, an uncountable collection of them) defined in terms of certain infinite words and show in Section 4.1 that they interpret grids. This is then extended by reductions in Section 4.2 to a number of other classes.

4.1   Word-defined minimal classes

Our starting point is a construction given by Brignall and Cocks [2] to demonstrate that there are uncountably many minimal HUCW classes, extending a construction by Collins et al. [3] showing the existence of infinitely many such classes. They construct a hereditary class $\mathscr{S}_\alpha$ of graphs for each ω-word $\alpha \in \{0,1,2,3\}^\omega$ and show that as long as α contains infinitely many non-zero letters, the class $\mathscr{S}_\alpha$ has unbounded clique-width. Moreover, for uncountably many distinct such α, $\mathscr{S}_\alpha$ is also minimal. The conditions under which $\mathscr{S}_\alpha$ is minimal need not concern us here. We are able to show that whenever α contains infinitely many non-zero letters $\mathscr{S}_\alpha$ interprets grids via MSO interpretations. In particular, this covers all minimal classes $\mathscr{S}_\alpha$ of unbounded clique-width, including those defined in [3]. Before we proceed to a proof, we give a precise definition of the classes $\mathscr{S}_\alpha$.

The class $\mathscr{S}_\alpha$ is defined as the class of all finite induced subgraphs of a single countably infinite graph $\mathcal{P}_\alpha$. The set of vertices of $\mathcal{P}_\alpha$ is $\{v_{i,j} \mid i,j \in \mathbb{N}\}$. We think of the set as an infinite collection of *columns* $V_j = \{v_{i,j} \mid i \in \mathbb{N}\}$. All edges are between vertices in adjacent columns, i.e. there is no edge between $v_{i,j}$ and $v_{i',j'}$ unless $j' = j + 1$ or $j' = j - 1$. The edges between successive columns are defined by the word α according to the following rules.

1. If $\alpha_j = 0$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i = k$.
2. If $\alpha_j = 1$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i \neq k$ for $i,k \in \mathbb{N}$.
3. If $\alpha_j = 2$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i \leq k$ for $i,k \in \mathbb{N}$.
4. If $\alpha_j = 3$, then $\{v_{i,j}, v_{k,j+1}\} \in E(\mathcal{P}_\alpha)$ if, and only if, $i \geq k$ for $i,k \in \mathbb{N}$.

The class $\mathscr{S}_\alpha$ is now given by $\mathscr{S}_\alpha = \{G \mid G \text{ is a finite induced subgraph of } \mathcal{P}_\alpha\}$. We show the following theorem in this section.

▶ **Theorem 12.** *Let $\alpha \in \{0,1,2,3\}^\omega$ be such that α contains infinitely many non-zero letters. Then there exists an MSO interpretation Θ such that $\Theta(\mathscr{S}_\alpha)$ contains the class of all square grids.*

To prove Theorem 12, we show the existence of an MSO interpretation Ψ such that the hereditary closure of $\Psi(\mathscr{S}_\alpha)$ contains the class of all square grids. Proposition 1 ensures that this indeed suffices. It is clear that graphs in $\mathscr{S}_\alpha$ have a built-in grid-like structure with vertices arranged in rows and columns. The main challenge is to show that a sufficient part of this structure can be made explicit using an MSO interpretation. We give an outline of the construction.

What we show is that we can find in S_α a sequence of graphs G_n for $n \in \mathbb{N}$ within which we can interpret *upper triangular grids*. One can think of an upper triangular grid U_t as the subgraph of the $t \times t$ grid induced by the vertices above the main diagonal, i.e. those vertices in the set $\{(i,j) \mid 1 \leq i,j \leq t\}$ with $i \leq j$. It is clear that U_t has as an induced subgraph an $r \times r$ grid, where $r = \lfloor \frac{t}{2} \rfloor$.

Let $\alpha \in \{0, 1, 2, 3\}^\omega$ be an ω-word containing infinitely many non-zero letters. We write α_i for the ith letter of α. Let $p < \omega$ be the least value such that $\alpha_p \neq 0$. Fix $n \geq 1$ and let l be the length of the shortest contiguous subsequence of α starting at α_p that contains exactly $2n + 2$ elements which are not 0. We write $\beta_0 \cdots \beta_{l-1}$ for this sequence, so $\beta_0 = \alpha_p$.

Recall that the vertices of P_α are $\{v_{i,j} \mid i, j \in \mathbb{N}\}$, and we write V_j for the set $\{v_{i,j} \mid i \in \mathbb{N}\}$. We define the graph G_n to be the subgraph of P_α induced by the set $C = \bigcup\limits_{i=0}^{i=l-1} C_i$ where $C_i \subseteq V_{p+i}$ is defined as follows for $0 \leq i < l$.

1. $C_0 = \{v_{0,p}, v_{1,p}, v_{2,p}, v_{3,p}\}$; and
2. $C_{i+1} = \{v_{0,p+i+1}, v_{1,p+i+1}, \ldots, v_{t-1,p+i+1}\}$ where $t = |C_i|$ if $\beta_{i+1} = 0$ and $t = |C_i| + 1$ otherwise.
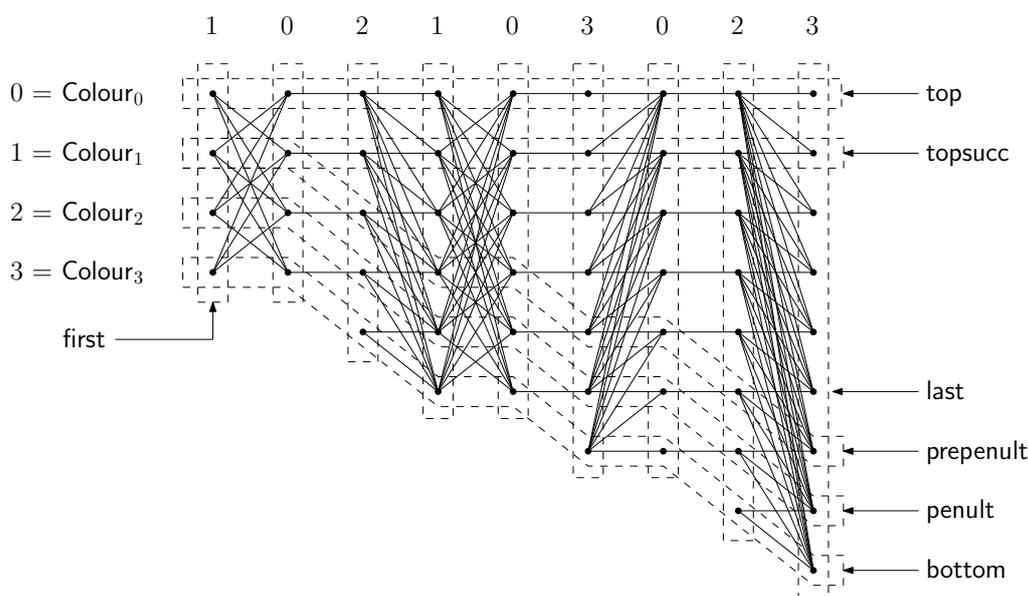
Figure 1 The graph H_2 for $\alpha = (102103023)^\omega$. The unlabeled graph underlying H_2 is G_2.

It is clear that $G_n \in \mathscr{S}_\alpha$. We show that we can interpret upper triangular grids in this class of graphs. The key challenge in defining the required interpretation is to define the two binary relations: one that relates vertices that are in the same column and the other that relates vertices that are in the same row. In constructing the interpretation we make use of a number of set parameters to obtain a labeled version H_n of G_n as illustrated in Figure 1. In particular, H_n uses unary predicates for the vertices corresponding to the possible values of β_i, for the first and last column, the top, bottom and penultimate rows, and the rows immediately succeeding and preceding the top and penultimate rows respectively. The "diagonal" nature of the bottom row is vital to allowing us to define when two vertices are in successive columns, which we need in order to define the two relations of being in the same row and in the same column.

4.2 Composing Interpretations

We now consider the classes of graphs shown to be minimal HUCW in [15, 1], and prove that these interpret grids using Theorem 12 above. Specifically, we show that for each class $\mathscr{C}$ among them, there is some $\alpha \in \{0, 1, 2, 3\}^\omega$ and an MSO interpretation Ξ such that the hereditary closure of $\Xi(\mathscr{C})$ contains $\mathscr{S}_\alpha$; then $\mathscr{C}$ interprets grids by Proposition 1.

▶ **Theorem 13.** *The following minimal* HUCW *classes of graphs interpret grids:*
1. *Bichain graphs*
2. *Split permutation graphs*
3. *Bipartite permutation graphs*
4. *Unit interval graphs*

▶ Remark 14. We mention that Theorem 13(4) follows from the results of Courcelle in [5]. It is shown in [5] that Seese's conjecture holds for the class of interval graphs. More specifically, it can be inferred from the results in [5] that any unbounded clique-width subclass of interval graphs admits MSO interpretability of grids. It follows, in particular, that this is true of the unit interval graphs. We therefore show parts (1)–(3) of Theorem 13 to complete its proof.

Bichain graphs

We need some terminology to talk about these graphs. Given a graph G, a sequence $v_1, \ldots, v_k$ of vertices of G is said to be a *chain* if $N(v_i) \subseteq N(v_j)$ whenever $i \leq j$, where $N(v) := \{u \mid E(u, v)\}$ denotes the neighbourhood of v. A bipartite graph $(A \cup B, E)$ is called a *k-chain graph* if each of the two parts A and B can be further partitioned into at most k chains. A *bichain graph* is a 2-chain graph.

We now describe the bichain graph Z_n as defined in [1]. This graph is *n-universal* in that all bichain graphs on at most n vertices are induced subgraphs of Z_n. The graph has vertex set $\{z_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ (which can thus be seen as an $n \times n$ grid of points), and $\{z_{i,j}, z_{i',j'}\}$ is an edge if, and only if, one of the following holds: (i) j is odd, $j' = j + 1$ and $i < i'$; (ii) j is even, $j' = j + 1$ and $i' \leq i$; or (iii) j is even, j' is odd and $j' \geq j + 3$. Since bichain graphs are hereditary, it follows that the class Bichain of all bichain graphs is exactly the hereditary closure of the class $\{Z_n \mid n \geq 1\}$.

Again, the grid structure is implicit in the graph Z_n. What we show is that when Z_n is expanded with unary relations for the bottom row $\{z_{n,j} \mid 1 \leq j \leq n\}$ and the last column $\{z_{i,n} \mid 1 \leq i \leq n\}$, we can construct an FO interpretation to a class that contains the class $\mathscr{S}_\alpha$ for $\alpha = (23)^\omega$ in its hereditary closure.

Split permutation graphs

Recall that a *split graph* is a graph G whose vertex set can be partitioned into two sets C and I such that C induces a clique in G and I is an independent set in G. A *permutation graph* is a graph whose vertices represent the domain of a permutation, and each of whose edges determines an inversion in the permutation. Following [1], we use the following characterization of split permutation graphs.

▶ **Proposition 15** ([1, Prop. 2.3]). *Let G be a split graph given together with a partition of its vertex set into a clique C and an independent set I. Let H be the bipartite graph obtained from G by deleting the edges of C. Then G is a split permutation graph if, and only if, H is a bichain graph.*

Let G be a split permutation graph with (C, I) being a partition of its vertex set into a clique C and an independent set I. Let G^* be the expansion of G with a unary predicate P which is interpreted as the set C. Consider the FO interpretation Ψ which removes from G^* all edges inside P. It is easy to see that $\Psi(G^*)$ is a bichain graph by Proposition 15.

Let Ψ be the FO interpretation as described above and SP be the class of split permutation graphs. Then $\Psi(\mathsf{SP})$, and hence its hereditary closure, contains the class Bichain. We are then done by Theorem 13(1) and Proposition 1.

Bipartite permutation graphs

These graphs are graphs that are bipartite as well as being permutation graphs. For our purposes, the following characterization is useful. Consider the graph P_n on vertex set $\{v_{i,j} \mid 1 \le i, j \le n\}$ where the only edges are between $v_{i,j}$ and $v_{i+1,j'}$ for $j' \le j$. Then, the class of bipartite permutation graphs is exactly the hereditary closure of the class $\{P_n \mid n \ge 1\}$ [15]. Now, it is easily seen that this class is exactly the class $\mathscr{S}_\alpha$ as described in Section 4.1, for $\alpha = 2^\omega$, and this has been observed in [3]. Thus, Theorem 13(3) follows from Theorem 12.

5 Power Graphs
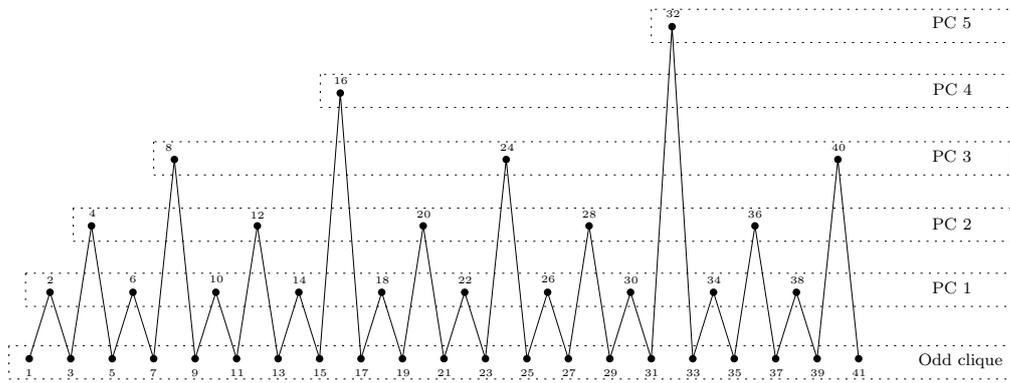
D_{41}

■ **Figure 2** The power graph D_{41} with "PC i" denoting the power clique corresponding to i.

In this section, we consider the class of *power graphs* as defined in [16] in the context of well-quasi-ordering and clique-width. Most of the classes that we have seen so far can be shown to not be well-quasi ordered under the induced subgraph relation. In particular, all word-defined classes, unit interval graphs and bipartite permutation graphs can be seen to contain the antichain $\{I_n \mid n \ge 1\}$ described after Proposition 2. We do not know whether bichain graphs and split permutation graphs are well-quasi ordered, though it has been shown that their expansion with two labels is not a well-quasi ordered class [1]. In contrast, power graphs constitute a class of graphs that is HUCW, that *is* well-quasi ordered [16] and, as we show, is a minimal HUCW class. It was introduced precisely to demonstrate an HUCW class that is well-quasi ordered. Minimality follows from arguments contained in [16], but was not observed there. We now define the class of power graphs. We show that they are minimal and then in the remainder of the section show that they admit interpretability of grids.

For $n \ge 1$, we define the graph D_n as follows. The vertex set of D_n is $[n] = \{1, \dots, n\}$. For each $i < n$, there is an edge between i and $i+1$ – we call these *path edges*. Furthermore, there is an edge between i and j if the largest power of 2 that divides i is the same as the largest power of 2 that divides j – we call these *clique edges*. To understand this terminology, note that we can see D_n as consisting of a simple path of length n, along with, for each k such that $2^k \le n$, a clique on all vertices $j = 2^k \cdot (2r+1)$ for some $r \ge 0$ – we call this the *power clique* corresponding to k. In particular, taking $k = 0$, there is a clique formed by all the odd elements, which we call the *odd clique*. Observe that the path edges, which are the

only edges with endpoints in different power cliques always have one end point in the odd clique, and one outside it. The class of power graphs, denoted Power-graphs, is now defined as the hereditary closure of the class $\{D_n \mid n \geq 1\}$.

5.1   Minimality of Power Graphs

▶ **Proposition 16.** *The class* Power-graphs *is a minimal hereditary class of unbounded clique-width.*

That Power-graphs is a hereditary class of unbounded clique-width has already been shown in [16]. Thus, we only need to show that no proper subclass has this property.

Given a graph $G \in$ Power-graphs which is a subgraph of D_n, define an *interval* in G to be a set $S \subseteq [n]$ of vertices of G such that if $i, j \in S$ with $i < j$ and k is a vertex of G with $i < k < j$ then $k \in S$. We call a subgraph of G induced by an interval a *factor* of G. We now recall the following two results proved in [16].

▶ **Lemma 17** (Lemma 11, [16]). *Let G be a graph in* Power-graphs. *Then there exists an integer $t = t(G)$ such that for any $n \geq t$, every factor of D_n of length at least t contains G as an induced subgraph.*

▶ **Theorem 18** (Theorem 2, [16]). *Let G be a graph in* Power-graphs *such that the length of the longest factor in G is t. Then the clique-width of G is at most $2(\log t + 4)$.*

Proof of Proposition 16. Consider a proper hereditary subclass $\mathscr{S}$ of Power-graphs; then $\mathscr{S}$ excludes a graph $G \in$ Power-graphs. Let $t = t(G)$ be as given by Lemma 17. Let $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$ where $\mathscr{S}_1 = \mathscr{S} \cap \{D_n \mid n < t\}\!\downarrow$ and $\mathscr{S}_2 = \mathscr{S} \cap \{D_n \mid n \geq t\}\!\downarrow$. Observe that $\mathscr{S}_1$ has finitely many graphs up to isomorphism.

We show that for each $X \in \mathscr{S}_2$, every factor of X has length $< t$. For otherwise X has a factor Y of length $\geq t$ and there is $p \geq 1$ such that $X \subseteq D_p$ and so Y is also a factor of D_p. Hence by Lemma 17, we have G is an induced subgraph of Y, whereby it is also an induced subgraph of X. Since $\mathscr{S}$ is hereditary, $G \in \mathscr{S}$ which is a contradiction.

By Theorem 18, every $X \in \mathscr{S}_2$ has clique-width $\leq k = 2(\log t + 4)$. Then $\mathscr{S}_2$ has bounded clique-width, and hence so does $\mathscr{S}$ since $\mathscr{S}_1$ is finite.   ◀

5.2   Interpreting grids in Power Graphs

We now establish the main result of this section, showing that power graphs do not provide a counter-example to Seese's conjecture.

▶ **Theorem 19.** *There exists an* MSO *interpretation Θ such that $\Theta($Power-graphs$)$ contains all square grids.*

We show Theorem 19 by showing that there exists an MSO interpretation Φ such that the hereditary closure of $\Phi($Power-graphs$)$ contains all bipartite permutation graphs. We are then done by Theorem 13 and Proposition 1. Indeed, it suffices to show that we can interpret grids in a subset of Power-graphs and we do this for the set $\{D_n \mid n \in \mathbb{N} \text{ even and } n > 9\}$.

We first show that there exists an FO formula $\mathsf{odd}(x)$ such that if x is a number in D_n with $n \geq 9$, then $\mathsf{odd}(x)$ is true if, and only if, x is an odd number. The formula asserts that there exist three elements y, z, w which together with x form a clique except that there is no edge $z - w$. It is easy to see that for $n \geq 9$, all odd numbers in D_n satisfy $\mathsf{odd}(x)$. If x is odd with $x < n - 3$, this is witnessed by $y = x + 2$, $w = x + 4$ and $z = x + 1$, otherwise by $y = x - 2$, $w = x - 4$ and $z = x - 1$.

To show that the even numbers of D_n do not satisfy $\mathsf{odd}(x)$, first observe that in any power clique other than the odd clique, since the numbers in the clique are of the form $2^k \cdot (2r+1)$ for fixed k, the difference between any two numbers in the clique is at least 2^{k+1}, which is at least 4 since $k \geq 1$. Suppose now that x is an even number in D_n and x, y, z form a 3-clique. We argue that any w that is adjacent to both x and y must also be adjacent to z showing that $\mathsf{odd}(x)$ is not satisfied. Consider the two cases:

- The edge between x and y is a clique edge. Then $|x - y| \geq 4$. If z is in a different power clique, then $|x - z| = 1$ and $|z - y| = 1$, whereby $|x - y| \leq 2$ – a contradiction. Thus z is in the same power clique as x and y. By the same argument, w is the same power clique as x and y, so there is a clique edge $z - w$.
- The edge between x and y is a path edge and so $|x - y| = 1$. Then the edges from z to x and y cannot both be path edges, as you cannot have a triangle of such edges. So, one of them is a clique edge. If z is in the same power clique as x, then $|x - z| \geq 4$ and $|y - z| = 1$, which is a contradiction, so z must be in the same power clique as y. By the same argument, w is in the same clique as y, so there is a clique edge $z - w$.

With the formula $\mathsf{odd}(x)$, we can distinguish path edges from clique edges. Indeed, an edge is a path edge if, and only if, it has exactly one end point that is odd. In D_n, the path edges form a simple path of length $n - 1$ and, if n is even, then only one of the two end points satisfies $\mathsf{odd}(x)$. This allows us to give this simple path an orientation: for each path edge $(x, x+1)$ we can identify the direction $x \to x + 1$. The transitive closure of this relation (which is definable in MSO), gives us a definition of the natural linear order on D_n.

Once we have defined a linear order $\leq$ on D_n, this induces a linear order on the power cliques: namely, a clique C is below C' if the $\leq$-minimal element of C is less than the $\leq$-minimal element of C'. Indeed, we can also define a successor relation on cliques from this. From these, we define a relation that relates a pair x and y precisely if y occurs after x in the linear order $\leq$ and occurs in the power clique that is successor to the power clique containing x. It is easy to see that the graph induced by this relation contains arbitrarily large bipartite permutation graphs P_k as defined on page 13.

6 Conclusion

The study of monadic second-order logic on graphs has attracted great attention in recent years. An important aspect of work on this logic is to identify classes of graphs on which MSO is well behaved. Seese's conjecture is an important focus of this classification effort. In its stronger form it offers a dichotomy: any class of graphs is either interpretable in trees and therefore has bounded clique-width and is well-behaved *or* it interprets arbitrarily large grids and its MSO theory is then undecidable.

We show that Seese's conjecture could be established by considering two kinds of graph classes: minimal hereditary classes of unbounded clique-width and antichains of unbounded clique-width. Showing that all such classes interpret unbounded grids would suffice. While we do not have a complete taxonomy of such classes, we investigated all the ones known and showed that none of them provides a counter-example to Seese's conjecture.

One could weaken the strong conjecture by requiring only that the classes of unbounded clique-width admit MSO *transductions* of grids, rather than interpretations (see [7] for a discussion of transductions). This would still suffice to establish Seese's conjecture. In all the cases we consider, however, we establish the stronger form, i.e. an interpretation of grids.

It is also worth pointing out that for many of the classes we consider, the original proofs that they have unbounded clique-width require sophisticated bespoke arguments. The interpretation of grids in the classes also provides a uniform method of proving that they have unbounded clique-width.

As a final remark, it is worth noting that there are standard graph operations which allow us to construct new minimal HUCW graph classes from the ones we have. For example, taking the graph complement of all graphs in a class $\mathscr{C}$ yields a class that is also minimal HUCW if $\mathscr{C}$ is. Since this operation is itself an MSO interpretation, the results about interpreting arbitrarily large grids apply to the resulting classes as well.

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