Abstract

BV and pomset logic are two logics that both conservatively extend unit-free multiplicative linear logic by a third binary connective, which (i) is non-commutative, (ii) is self-dual, and (iii) lies between the “par” and the “tensor”. It was conjectured early on (more than 20 years ago), that these two logics, that share the same language, that both admit cut elimination, and whose connectives have essentially the same properties, are in fact the same. In this paper we show that this is not the case. We present a formula that is provable in pomset logic but not in BV.

1 Introduction

Pomset logic has been discovered by Christian Retoré [21] through the study of coherence spaces which form a semantics of proofs for linear logic. Retoré observed that next to the two operations $\otimes$ (tensor or multiplicative conjunction) and $\vee$ (par or multiplicative disjunction) there are two other operations $\bowtie$ and $\triangleright$, which are non-commutative, obey $A \bowtie B = B \triangleright A$, and are self-dual, i.e., $\langle A \bowtie B \rangle^\perp = A^\perp \bowtie B^\perp$. From this semantic observation, Retoré derived a proof net syntax together with a correctness criterion and a cut elimination theorem. However, he could not provide a sound and complete cut-free sequent calculus for this logic [20]. Nonetheless, pomset logic has found applications in linguistics, as basis of a new categorial grammar [17], similar to the ones based on the Lambek calculus [16].

System BV was found by Alessio Guglielmi [10] through a syntactic investigation of the connectives of pomset logic and a graph theoretic study of series-parallel orders and cographs. The difficulty of presenting this combination of commutative and non-commutative connectives in the sequent calculus triggered the development of the calculus of structures [11], the first proper deep inference proof formalism. The mixture of commutative and non-commutative connectives in BV immediately found applications in computer science, in particular, Bruscoli [3] established a strict correspondence between the proof-search space of BV and the computations in a fragment of CCS. This work was later extended by quantifiers to capture private names and to establish a correspondence of implication in (first-order) BV and a form of weak bisimulation in the $\pi$-calculus [12, 13].

1 Observe that the order is not inverted, as it is the case with other non-commutative variants of linear logic [29] (see also [9, Section II.9.]).

2 The basic idea of such a rewriting system goes back to Retoré [22] (see also [4]), but not as a proof system admitting cut-elimination.
This leads to the strange situation that we have two logics, pomset logic and BV, which are both conservative extensions of unit-free multiplicative linear logic with mix (MLL₀) [8, 7] with a non-commutative connective ◁ such that $A \otimes B \rightarrow A \triangleright B \rightarrow A \triangleright B$, which both obey a cut elimination result, and which both have found applications that lie outside of pure proof theory.

The only difference between the two logics is that pomset logic naturally extends the proof net correctness criterion of MLL₀ to the new non-commutative connective, but has no deductive proof system, whereas BV naturally extends a deductive system for MLL₀ with the new non-commutative connective, but has no proof nets. This naturally led to the conjecture that both logics ought to be the same [28]. In fact, most researchers working in this area (including the second author of this paper) believed that the two logics comprise the same set of theorems.

In this paper we show that this is not the case. More precisely, we show that the theorems of BV form a proper subset of the theorems of pomset logic. It has already been observed before [22, 28, 26] that every theorem in BV is also a theorem of pomset logic. However, the converse is not true, and we give an example of a formula that is a theorem of pomset logic but not provable in BV.

Organisation of this paper

In the next two sections we give some preliminaries on pomset logic (Section 2) and BV (Section 3). Then, in Section 4 we show that BV is contained in pomset logic. Even though this has been known since more than 20 years [22, 28], there has been no complete proof published so far. The proof we present here is a simplification of the one suggested in [28]. Next, in Section 5, we give our counterexample showing that the converse is not true, i.e., we present a formula that is a theorem of pomset logic but not provable in BV. Finally, in the conclusion (Section 6), we discuss some complexity results and give some intuition on how the counterexample has been found and why it took so long to find it.

2 Preliminaries on Pomset Logic

The formulas of pomset logic and BV are in this paper denoted by capital Latin letters $A, B, C, \ldots$ and are generated from a countable set $V = \{a, b, c, \ldots\}$ of propositional variables and the unit $I$ via the three binary connectives tensor $\otimes$, par $\triangleright$, and seq $\triangleright$, according to the grammar

$$A, B ::= I \mid a \mid a \perp \mid (A \otimes B) \mid [A \triangleright B] \mid \langle A \triangleright B \rangle$$

An atom is either a propositional variable or its dual. For a formula $A$, we define its size $|A|$ to be the number of atom occurrences in $A$. For better readability of large formulas, we use here different kinds of parentheses for the different connectives. In the following, we omit outermost parentheses for better readability. The unit $I$ behaves as unit for all three connectives. We define the relation $\equiv$ on formulas to be the smallest congruence generated by associativity of $\otimes, \triangleright, \triangleright$, commutativity of $\otimes, \triangleright$, and the unit equations:

$$\begin{align*}
A \otimes (B \otimes C) &\equiv (A \otimes B) \otimes C & A \otimes B &\equiv B \otimes A & I \otimes A &\equiv A \\
A \triangleright [B \triangleright C] &\equiv [A \triangleright B] \triangleright C & A \triangleright B &\equiv B \triangleright A & I \triangleright A &\equiv A \\
A \triangleright\langle B \triangleright C \rangle &\equiv \langle A \triangleright B \rangle \triangleright C & I \triangleright A &\equiv A & A \triangleright I &\equiv A \triangleright I
\end{align*}$$

Note that this is redundant and carries no additional meaning. The only purpose is better readability.
The involutive (linear) negation \((\neg)\) is extended from propositional variables to general formulas by taking De Morgan’s laws as its inductive definition, i.e., we define \((\neg a)\) for all propositional variables \(a\), and
\[
1^\perp = I \quad (a \oplus b)^\perp = A^\perp \odot B^\perp \quad (a \boxdot b)^\perp = A^\perp \circ B^\perp
\]
The last equality is what we mean when we say that seq is self-dual. Note that the right-hand side is indeed \(A^\perp \circ B^\perp\) and not \(B^\perp \circ A^\perp\).

We will also need the notion of sequent, which has to be generalized from multisets of formulas to series-parallel orders of formulas.\(^5\) We denote a sequent in pomset logic by capital Greek letters \(\Gamma, \Delta, \ldots\) and they are generated as follows: \(\Gamma, \Delta \::= \emptyset \mid A \mid [\Gamma, \Delta] \mid \langle \Gamma; \Delta \rangle\), where \(\emptyset\) stands for the empty sequent. We consider sequents equal modulo commutativity of \([\cdot, \cdot]\) and associativity of \([\cdot, \cdot, \cdot]\), and the unit-laws for the empty sequent. In the remainder of this paper we will always omit redundant brackets.

The operations \([\cdot, \cdot]\) and \(\langle \cdot, \cdot \rangle\) serve as counterparts on sequents to the connectives \(\oplus\) and \(\circ\) on formulas (just as the sequent \(\vdash A, B, C\) morally means \(A \oplus B \odot C\) in linear logic).

\textbf{Remark 2.1.} Pomset logic is not the only system that features “non-flat” sequents with two distinct connectives. Another famous example is the logic \(\mathbf{BI}\) of bunched implications [19].

In [21], Retoré presents proof nets for pomset logic as \(RB\)-digraphs, that is, directed graphs equipped with perfect matchings, extending his reformulation of MLL\(_0\) proof nets as undirected \(RB\)-graphs [23]. We recall these notions below.

\textbf{Definition 2.2.} A digraph \(G = (V_G, E_G)\) consists of a finite set of vertices \(V_G\) and a set of edges \(E_G \subseteq V_G^2 \setminus \{(u, u) \mid u \in V_G\}\). A digraph \(G\) is labeled if there is a map \(\ell: V_G \rightarrow \mathcal{L}\) assigning each vertex \(v\) of \(V_G\) a label \(\ell(v) \in \mathcal{L}\) in the label set \(\mathcal{L}\). If \(\mathcal{L}\) is the set \(\mathcal{V} \cup V^\perp\) of atoms, we speak of an atom-labeled digraph.

In the remainder of this paper, all digraphs are atom-labelled, and for two digraphs \(G\) and \(H\), we write \(G = H\) iff there is a label-preserving isomorphism between them. Also, we often write \(uv \in E_G\) for \((u, v) \in E_G\), and for a digraph \(G = (V_G, E_G)\), we define the sets \(E_G^0 = \{(u, v) \mid (u, v) \in E_G\}\) and \(E_G^- = \{(u, v) \mid (u, v) \in E_G\} \setminus \{(u, u) \mid u \in V_G\}\), allowing us to treat \((V_G, E_G^0)\) as undirected graph.

\textbf{Definition 2.3.} Let \(G = (V_G, E_G)\) and \(H = (V_H, E_H)\) be disjoint digraphs. We can define the following operations:
\[
\begin{align*}
G \boxdot H &= (V_G \cup V_H, E_G \cup E_H) \\
G \ominus H &= (V_G \cup V_H, E_G \cup E_H \cup \{(u, v) \mid u \in V_G \text{ and } v \in V_H\}) \\
G \odot H &= (V_G \cup V_H, E_G \cup E_H \cup \{(u, v, (v, u) \mid u \in V_G \text{ and } v \in V_H\})
\end{align*}
\]
This allows us to define a mapping \([\cdot]\) from formulas to digraphs as follows:
\[
[\emptyset] = \emptyset \quad [a] = \bullet a \quad [a^\perp] = \bullet a^\perp
\]
\[
[A \oplus B] = [A] \oplus [B] \quad [A \boxdot B] = [A] \boxdot [B] \quad [A \odot B] = [A] \odot [B]
\]
where \(\emptyset\) is the empty graph, and \(\bullet a\) (respectively \(\bullet a^\perp\)) is a single vertex graph whose vertex is labeled by \(a\) (respectively \(a^\perp\)).

\end{footnotesize}

\(^5\) In that respect, pomset logic and \(BV\) are different from other non-commutative variants of linear logic where \(\otimes\) and \(\boxdot\) are non-commutative with \((A \oplus B)^\perp = B^\perp \boxdot A^\perp\) [29, 1].

\(^4\) We follow here mainly the presentation of [24].
Proposition 2.4 ([22]). For all formulas $A$ and $B$, we have $[A] = [B]$ iff $A \equiv B$.

This can be shown by a straightforward induction on the formulas. An immediate consequence of this proposition is that the extension of the mapping $[\cdot]$ to sequents is well-defined, i.e., we have $[\Gamma; \Delta] = [\Gamma] \triangleright [\Delta]$ and $[\Gamma; \Delta] = [\Gamma] \oslash [\Delta]$.

Definition 2.5. Let $G = (V_G, E_G)$ be a digraph and let $V_H \subseteq V_G$. The subdigraph of $G$ induced by $V_H$ is $H = (V_H, E_H)$, where $E_H = \{(u, v) \mid (u, v) \in E_G \text{ and } u \in V_H \text{ and } v \in V_H\}$. In this case we also say that $H$ is an induced subgraph of $G$ and denote that by $H \subseteq G$. If additionally $V_H \subset V_G$ then we write $H \subset G$.

Definition 2.6. An undirected graph is $P_4$-free if it does not contain a $P_4$ (shown on the left below) as induced subgraph, and a directed graph is $N$-free if it does not contain an $N$ (shown on the right below) as induced subgraph.

\[ P_4: \quad \quad N: \]

Definition 2.7. A dicograph is a digraph $G = (V_G, E_G)$, such that
1. the undirected graph $(V_G, E_G^u)$ is $P_4$-free,
2. the directed graph $(V_G, E_G^d)$ is $N$-free, and
3. the relation $E_G$ is weakly transitive:
   - if $(u, v) \in E_G^u$ and $(v, w) \in E_G$ then $(u, w) \in E_G$, and
   - if $(u, v) \in E_G^u$ and $(v, w) \in E_G^d$ then $(u, w) \in E_G$.

Proposition 2.8 ([4]). $G$ is a digraph iff there is a formula $A$ with $G = [A]$.

Definition 2.9. Let $G = (V_G, E_G)$ be a digraph. Then any induced subdigraph of $G$ is also a digraph.

Proposition 2.10. Let $G = (V_G, E_G)$ be a digraph. A perfect matching $B$ of $G$ is a subset of edges such that:
1. any vertex has exactly one outgoing edge in $B$ and exactly one incoming edge in $B$, i.e., for every $u \in V_G$ there is exactly one pair $(v, w) \in V_G \times V_G$ such that $uv \in B$ and $wu \in B$, and
2. for all $u, v \in V_G$, we have that $uv \in B$ iff $vu \in B$.

Item 2 means that $B$ consists of bidirectional edges. In particular, this means that $v = w$ in Item 1. An RB-digraph $G = (V_G, R_G, B_G)$ is a triple where $(V_G, R_G \uplus B_G)$ is a digraph and $B_G$ is a perfect matching in it. Finally, an RB-digraph $G = (V_G, R_G, B_G)$ is an RB-digraph iff $(V_G, R_G)$ is a digraph.\(^6\)

In all figures representing RB-digraphs, we will (following [22]) draw the edges belonging to the matching (the set $B$) \textbf{bold and blue}, and the other edges (the set $R$) \textbf{regular and red}.

Example 2.11. Below we show 7 examples of RB-digraphs. The first 5 are RB-digraphs, the last 2 are not.

\[ \quad \quad \quad \quad \quad \]

\(^6\) Note that the perfect matching $B_G$ is not part of the dicograph. In particular, we allow that two vertices in $V_G$ can be connected by an edge in $R_G$ and in $B_G$.\]
Technically speaking, this not a tree in the graph-theoretical sense, but we use the name as it carries the structure of the formula tree.
If we have a sequent $\Gamma$, then $\mathcal{T}_\text{RB}(\Gamma)$ is obtained from the RB-trees of the formulas in $\Gamma$ which are connected at the roots via the edges corresponding to the series-parallel order of the sequent structure. In order to obtain an RB-digraph, we need to add the $B$-edges corresponding to the linking $\ell$. We denote this RB-digraph, which is in fact an RB-dicograph, by $\tau(\Gamma, \ell)$ and call it the tree-like RB-prenet of $\Gamma$ and $\ell$.

**Definition 2.15.** A relational RB-prenet (resp. tree-like RB-prenet) is correct if it does not contain any chordless $a$-cycle. A correct relational RB-prenet (resp. correct tree-like RB-prenet) is also called a relational RB-net (resp. tree-like RB-net). In both cases we also speak of (pomset logic) proof nets. A sequent $\Gamma$ is provable in pomset logic if there is a linking $\ell$, such that $\rho(\Gamma, \ell)$ or $\tau(\Gamma, \ell)$ is a proof net.

The above definition makes sense because of the following theorem by Retoré:

**Theorem 2.16** ([22, Theorem 7]). For every sequent $\Gamma$ and linking $\ell$, we have that $\rho(\Gamma, \ell)$ is correct if and only if $\tau(\Gamma, \ell)$ is correct.

**Example 2.17.** The three RB-graphs in the middle of (4) are pomset logic proof nets.

### 3 Preliminaries on System BV

In [10] Guglielmi introduces system $\text{BV}$, which is a deductive system for formulas defined in (1). It is defined in the formalism called the calculus of structures, and it works similar to a rewriting system, modulo the equational theory defined in (2).

The inference rules of system $\text{BV}$ are shown in Figure 2. These rules have to be read as rewriting rule schemes, meaning that (i) the variable $a$ can be substituted by any atom, and the variables $A, B, C, D$ can be substituted by any formula, and that (ii) the rules can be applied inside any (positive) context.

A (proof) system is a set of inference rules. We write $s \parallel \delta$, or more concisely $A \vdash^S \delta B$, if there is a derivation from $A$ to $B$ using only rules from the system $S$, and that derivation is named $\delta$. If in that situation $A = \bot$, then we write it as $s \parallel \delta B$ or simply as $\vdash^S B$ and call $\delta$ a proof of $B$. In this case we say that $B$ is provable $S$.

**Example 3.1.** Here are three proofs in $\text{BV}$, corresponding to the three proof nets in the middle of (4):

$$
\begin{align*}
\frac{a}{a \not\in B} & \quad \frac{[A \not\in C] \otimes B \quad (A \otimes B) \not\in C}{A \not\in B} \\
\frac{[A \not\in C] \otimes [B \not\in D]}{(A \otimes B) \not\in (C \otimes D)} & \quad \frac{A}{B} \quad \text{(provided $A \equiv B$)}
\end{align*}
$$

![Figure 2 System BV.](image-url)
An inference rule \( r \) is **derivable** in a system \( \mathcal{S} \) iff for every instance \( \frac{A}{B} \) there is a derivation \( A \vdash_\mathcal{S} B \). An inference rule \( r \) is **admissible** for a system \( \mathcal{S} \) iff for every proof \( \Gamma \vdash_{\mathcal{S} \cup \{r\}} A \) there is a proof \( \Gamma \vdash_\mathcal{S} B \).

**Definition 3.2.** Two system \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are **equivalent** if they prove the same formulas.

To simplify the proofs of our main results, we need a unit-free version of BV. We use here a variant of the one proposed by Kahramanoğulları in [14] in order to reduce the non-determinism in proof search in \( \mathcal{S} \).

The system is called BVu, and its formulas are the same as defined in (1), except that we do not allow any occurrence of the unit \( \mathbb{I} \). This means that we have to restrict the equivalence \( \equiv \) defined in (2) to the unit-free formulas. We define the relation \( \equiv' \) to be the smallest congruence generated by

\[
\begin{align*}
A \circ (B \circ C) & \equiv' (A \circ B) \circ C & A \circ B & \equiv' B \circ A \\
A \circ (B \circ C) & \equiv' (A \circ B) \circ C & A \circ B & \equiv' B \circ A \\
A \circ (B \circ C) & \equiv' (A \circ B) \circ C & A \circ B & \equiv' B \circ A
\end{align*}
\]

(6)

The inference rules for BVu are then shown in Figure 3.\(^8\) Note that the rule \( a_i^\updownarrow \) has no premise. It is an axiom that is used exactly once in a proof which is a derivation without premise (as the unit \( \mathbb{I} \) is not present and cannot take this role).

**Proposition 3.3 ([14]).** The systems BVu and BV are equivalent.

**Proof.** First, if we have a proof \( \Gamma \vdash_{\text{BVu}} A \) then we can simply replace the top instance of \( a_i^\updownarrow \) by \( a_i \) and have a proof of BV. Conversely, assume we have a proof \( \Gamma \vdash_{\text{BV}} B \). Then, in \( \delta \), the unit \( \mathbb{I} \) can occur. Let \( \delta' \) be obtained from \( \delta \) by deleting the unit \( \mathbb{I} \) everywhere (which means that the topmost \( a_i \) is replaced by \( a_i^\updownarrow \)). Then every instance of the rule \( \equiv \) becomes an instance of \( \equiv' \); every instance of \( q_i \) becomes an instance of \( q_2^\downarrow \) or \( q_4^\downarrow \) or \( q_3^\downarrow \) or \( q_4^\downarrow \) or trivial (i.e., premise and conclusion of the rule instance become equal); and similarly for \( s \). However, an instance of \( a_i \) can become an instance of \( a_i^\updownarrow \) or \( a_i^\downarrow \) or \( a_i^\updownarrow \) (which are in BVu), or \( a_i^\updownarrow \) which is shown below.

\[
\begin{align*}
\text{a}_i^\downarrow & B \\
a \circ a \circ B & \equiv A \circ a \circ B
\end{align*}
\]

(7)

This rule is not in BVu, but can be derived with \( \{a_i^\updownarrow, s_2\} \).

---

\(^8\) The rules in the bottom two rows of Figure 3 have already been studied by Retoré in [22], as part of a rewrite system on digraphs to generate theorems of pomset logic.
Remark 3.4. Our version of BVu is slightly different from the one by Kahramanoğullar [14]. In [14] the rule s2 is absent, and instead the rule ai\textsuperscript{1}↓ is shown in (7) is part of the system. It is easy to see that the two variants of BVu are equivalent: first, as we have mentioned above, the rule ai\textsuperscript{1}↓ is derivable in \{ai\textsuperscript{1}, s2\}, and second, the rule s2 is admissible if ai\textsuperscript{1}↓ is present. This can be seen by an easy induction on the size of the derivation. However, note that the same trick does not work for the rule q2↓. This rule cannot be shown admissible, as the formula \(\{a \oplus \{b \oplus c\}\} \equiv \{a \oplus \{b \oplus c\}\} \equiv \{c \oplus \{a \oplus b\}\}\) is not provable in BVu without q2↓.

We will also need a variant of BVu that we call BVu and that is obtained from BVu by restricting rules q2↓ and s2 to cases where neither A nor B has a \(\otimes\) as main connective, i.e., we replace q2↓ and s2 by \(\hat{q}_2\) and \(\hat{s}_2\), respectively:

\[
\begin{align*}
\hat{q}_2↓ \quad & \quad A \otimes B \quad \Rightarrow \quad A \otimes B \\
\hat{s}_2 \quad & \quad A \otimes B \quad \Rightarrow \quad A \otimes B
\end{align*}
\]

and similarly, by restricting the rules q3↓, q3R↓, and s3 to cases where C does not have a \(\otimes\) as main connective, i.e., these three rules are replaced by \(\hat{q}_3↓\), \(\hat{q}_3R↓\), and \(\hat{s}_3\), respectively:

\[
\begin{align*}
\hat{q}_3↓ \quad & \quad [A \otimes C] \otimes B \quad \Rightarrow \quad (A \otimes B) \otimes C \\
\hat{q}_3R↓ \quad & \quad A \otimes [B \otimes C] \quad \Rightarrow \quad (A \otimes B) \otimes C \\
\hat{s}_3 \quad & \quad [A \otimes C] \otimes B \quad \Rightarrow \quad (A \otimes B) \otimes C
\end{align*}
\]

Proposition 3.5. The systems BVu and BVu are equivalent.

Proof. Any derivation in BVu is also a derivation in BVu. Conversely, the rules q2↓ and s2 and s3 are derivable with \(\{\hat{q}_2↓, \hat{q}_3↓, \hat{q}_3R↓, \equiv\}\) and \(\{\hat{s}_2, \hat{s}_3, \equiv\}\) and \(\{\hat{s}_3, \equiv\}\), respectively, as shown below:

\[
\begin{align*}
\hat{q}_2↓ \quad & \quad [A \otimes A''] \otimes [B' \otimes B''] \quad \Rightarrow \quad [A' \otimes A''] \otimes [B' \otimes B''] \\
\hat{q}_3↓ \quad & \quad [A' \otimes A''] \otimes [B' \otimes B''] \quad \Rightarrow \quad [A' \otimes A''] \otimes [B' \otimes B''] \\
\hat{s}_2 \quad & \quad A' \otimes B' \otimes A'' \otimes B'' \quad \Rightarrow \quad A' \otimes B' \otimes A'' \otimes B'' \\
\hat{s}_3 \quad & \quad [A' \otimes A''] \otimes [B' \otimes B''] \quad \Rightarrow \quad [A' \otimes A''] \otimes [B' \otimes B''] \\
\hat{s}_3 \quad & \quad [A' \otimes A''] \otimes [B' \otimes B''] \quad \Rightarrow \quad [A' \otimes A''] \otimes [B' \otimes B'']
\end{align*}
\]

and similarly, the rules q3↓ and q3R↓ are derivable in \(\{\hat{q}_3↓, \equiv\}\) and \(\{\hat{q}_3R↓, \equiv\}\), respectively.
To begin, let $\delta$ be a BV proof of a formula $A$. We denote by $\mathcal{L}(\delta) = \rho(A, \ell(\delta))$ the relational RB-prenet generated from $\delta$ as described in Section 2. Then the main result of this section is the following.

**Theorem 4.1.** For every BV proof $\delta$, the relational RB-prenet $\mathcal{L}(\delta)$ is correct.

**Example 4.2.** The three correct relational RB-prenets in the middle of (4) are obtained from the three BV-proofs in Example 3.1.

In order to prove Theorem 4.1, we first introduce an additional definition.

**Definition 4.3.** A formula is balanced if every propositional variable that occurs in $A$ occurs exactly once positive and exactly once negative. A balanced formula $A$ uniquely determines an axiom linking on $L$, that we denote by $\ell(A)$. Then we write $\mathcal{L}(A)$ for the relational RB-prenet $\rho(A, \ell(A))$, i.e., $\mathcal{L}(A) = (V_A, R_A, B_A)$, where $(V_A, R_A) = [A]$ and $B_A$ is the matching associated to $\ell(A)$.

Conversely, every RB-dicograph uniquely determines a balanced formula, up to renaming of variables and equivalence under $\equiv$. This gives us immediately the following proposition.

**Proposition 4.4.** Let $\delta$ be a proof in BV. Then there is a balanced formula $A$, that is provable in BV and such that $\mathcal{L}(A)$ and $\mathcal{L}(\delta)$ are isomorphic.

**Proof.** Let $B$ be the conclusion of $\delta$. Then $A$ is obtained from $B$ by renaming all variable occurrences such that the result is balanced and the linking is preserved. \hfill $\blacksquare$

**Definition 4.5.** Let $A$ be a formula. A formula $B$ is a pseudo-subformula of $A$, written as $B \sqsubseteq A$, if it is equivalent under $\equiv$ to some $A'$ that can be obtained from $A$ by replacing some atom occurrences in $A$ by $\bot$. If $B \sqsubseteq A$ and $B \neq A$, then we say that $B$ is a proper pseudo-subformula of $A$, and write it as $B \subset A$.

**Example 4.6.** We have that $(a \land b) \land (d \land c) \equiv (b \land ((e \land f) \lor (a \land b)))$ has $(a \land d) \lor (b \land b)$ as pseudo-subformula which is equivalent to $(a \land (\bot \lor \bot)) \lor (b \land (\bot \land \bot))$.

The following proposition explains our choice to denote both pseudo-subformulas and induced subgraphs (Definition 2.5) by $\sqsubseteq$.

**Proposition 4.7.** We have $B \subseteq A$ if and only if $[B] \subseteq [A]$ and $B \subseteq A$ if and only if $[B] \sqsubseteq [A]$.

**Proof.** This follows directly from the definitions of $\sqsubseteq$ and $\sqsubseteq$ and Proposition 2.4. \hfill $\blacksquare$

**Lemma 4.8.** Let $A$ be a balanced formula and $B$ be a balanced pseudo-subformula of $A$. If $A$ is provable in BV, then so is $B$.

**Proof.** Let $\delta$ be the proof of $A$ in BV, and let $\delta'$ be obtained by replacing all atoms that do not occur in $B$ in every line of $\delta$ by $\bot$. Then $\delta'$ is a valid derivation of $B$ in BV. \hfill $\blacksquare$

**Definition 4.9.** A balanced cycle is a balanced formula $H$ such that $\mathcal{L}(H)$ is an $\alpha$-cycle.

**Proposition 4.10.** A formula $H$ is a balanced cycle if and only if there are pairwise distinct atoms $a_1, \ldots, a_n$ for some $n \geq 1$, such that $H \equiv L_1 \lor L_2 \lor \cdots \lor L_n$, where $L_1 = a_n^+ \land a_1$ or $L_1 = a_n^- \land a_1$, and for every $i \in \{2, \ldots, n\}$ we have $L_i = a_{i-1}^+ \land a_i$ or $L_i = a_{i-1}^- \land a_i$.

**Proof.** This follows immediately from the definitions. \hfill $\blacksquare$
Definition 4.11. We say that a balanced formula $A$ contains a cycle if it has a pseudo-subformula $B \subseteq A$ that is a balanced cycle (or, equivalently if $[A]$ contains a chordless $\alpha$-cycle).

We are now ready to state and prove the central lemma to this section.

Lemma 4.12. Let $Q/P$ be an instance of an inference rule in BV. If $P$ is a balanced cycle then $Q$ contains a cycle. If $r \neq \equiv$ then the size of the cycle in $Q$ is strictly smaller than $|P|$.

Proof. By Proposition 4.10 we have that $P \equiv L_1 \varnothing L_2 \varnothing \cdots \varnothing L_n$, where $L_1 = a_n^{-1} \circ a_1$ or $L_1 = a_n^{-1} \circ a_1$, and for every $i \in \{2, \ldots, n\}$ we have $L_i = a_{i-1}^{-1} \circ a_i$ or $L_i = a_{i-1}^{-1} \circ a_i$, with all $a_i$ being pairwise distinct. We proceed by case analysis on the rule $r$. First observe that by Proposition 4.10 the rules $\circ a_i^{-1}$, $a_i^{-1} \circ$, $a_i^{-1}$ cannot be applied to $P$ (seen bottom up), and if $r = \equiv$, then $Q$ trivially contains a cycle, whose size is equal to $|P|$. Now assume $r$ is

- $\circ a_i^{-1}$: Without loss of generality, assume that $A = a_n^{-1}$ and $B = a_1$ and $C = a_i^{-1}$, then
  
  $Q \equiv \langle [a_n^{-1} \circ a_i^{-1}] \circ [a_1 \circ a_i] \circ L_2 \varnothing \cdots \varnothing L_i \varnothing L_{i+1} \varnothing \cdots \varnothing L_n$ 
  which contains the cycle $\langle a_n^{-1} \circ a_i \rangle \varnothing L_{i+1} \varnothing \cdots \varnothing L_n$.

- $a_i^{-1} \circ [A \circ C]$ : Without loss of generality, assume that $A = a_n^{-1}$ and $B = a_1$ and $C = L_i$ for some $i \in \{2, \ldots, n\}$. Then
  
  $Q \equiv \langle [a_n^{-1} \circ L_i] \circ a_1 \circ L_2 \varnothing \cdots \varnothing L_i \varnothing L_{i+1} \varnothing \cdots \varnothing L_n$ 
  which contains the cycle $\langle a_{i-1}^{-1} \circ a_1 \rangle \varnothing L_{i+1} \varnothing \cdots \varnothing L_{i-1}$.

- $a_i^{-1} \circ [A \circ B] C$ : As before, without loss of generality, assume that $A = a_n^{-1}$ and $B = a_1$ and $C = L_i$ for some $i \in \{2, \ldots, n\}$. Then
  
  $Q \equiv \langle [a_n^{-1} \circ L_i] \circ [a_1 \circ L_i] \circ L_2 \varnothing \cdots \varnothing L_i \varnothing L_{i+1} \varnothing \cdots \varnothing L_n$ 
  which contains the cycle $\langle a_n^{-1} \circ a_i \rangle \varnothing L_{i+1} \varnothing \cdots \varnothing L_n$.

- $\circ a_i^{-1}$: We can assume that $A \equiv L_i$ and $B \equiv L_j$ for some $i, j \in \{1, \ldots, n\}$. There are two subcases:
  - $i < j$: Then $Q = \langle L_i \circ L_j \rangle \varnothing L_1 \varnothing \cdots \varnothing L_{i-1} \varnothing L_{i+1} \varnothing \cdots \varnothing L_j \varnothing L_{j+1} \varnothing \cdots \varnothing L_n$ which contains the cycle $L_i \varnothing \cdots \varnothing L_{i-1} \varnothing L_{i+1} \varnothing \cdots \varnothing L_j \varnothing \cdots \varnothing L_n$.
  - $j < i$: Then $Q = \langle L_i \circ L_j \rangle \varnothing L_1 \varnothing \cdots \varnothing L_{j-1} \varnothing L_{j+1} \varnothing \cdots \varnothing L_{i-1} \varnothing L_{i+1} \varnothing \cdots \varnothing L_n$ which contains the cycle $\langle a_{i-1}^{-1} \circ a_j \rangle \varnothing L_{j+1} \varnothing \cdots \varnothing L_{i-1}$.

- $\circ a_i^{-1}$: This case is analogous to the case $\circ a_i^{-1}$ above.

- $a_i^{-1} \circ [A \circ B] C$ : This case is analogous to the case $\circ a_i^{-1}$ above.

In all cases the size of the cycle in $Q$ is strictly smaller than $|Q| = |P|$.

Lemma 4.13. Let $P$ be a balanced formula that contains a cycle. Then $P$ is not provable in BV.
Proof. Let \( H \) be the cycle in \( P \), and let \( n = |H| \) be its size. We proceed by induction on \( n \). Note that \( n \) has to be even. For \( n = 2 \), we have that \( H \equiv a^\perp \triangleleft a \) or \( H \equiv a^\perp \triangleright a \) for some atom \( a \). By way of contradiction, assume \( P \) is provable in \( \text{BV} \). By Lemma 4.8, \( H \) is also provable in \( \text{BV} \), which is impossible. For the inductive case let now \( n > 2 \). As before, we have by Lemma 4.8 that \( H \) is provable in \( \text{BV} \). By Proposition 3.3 and Proposition 3.5, \( H \) is provable in \( \text{BVu} \). Let \( \delta \) be that proof in \( \text{BVu} \). Let now \( Q \) be the premise of the bottommost rule instance \( r \) of \( \delta \) that is not \( a \equiv' \) (i.e., the conclusion of \( r \) is \( H' \equiv' H \) and \( Q \not\equiv' H \)). By Lemma 4.12, \( Q \) contains a cycle whose size is smaller than \( n \). By induction hypothesis \( Q \) is not provable in \( \text{BV} \), and therefore also not provable in \( \text{BVu} \), which is a contradiction to the existence to \( \delta \).

We can now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let \( \delta \) be a proof in \( \text{BV} \). By Proposition 4.4, there is a balanced formula \( P \), such that \( \llbracket P \rrbracket \) is isomorphic to \( \llbracket \delta \rrbracket \), and such that \( P \) is provable in \( \text{BV} \). Now assume, by way of contradiction, that \( \llbracket \delta \rrbracket \) is incorrect. That means that \( \llbracket \delta \rrbracket \) contains a chordless \( \alpha \)-cycle, or equivalently, that \( P \) contains a cycle. By Lemma 4.13, \( P \) is not provable in \( \text{BV} \). Contradiction.

5 \hspace{1em} \textbf{Pomset Logic is not Contained in \( \text{BV} \)}

In this section we present a formula that is provable in pomset logic, i.e., has a correct pomset logic proof net, but that is not provable in \( \text{BV} \). From what has been said in the previous section, it follows that if such a formula exists then there is also a balanced such formula. The formula we discuss in this section is the formula \( Q \) shown below:

\[
Q = ((a \triangleleft b) \otimes (c \triangleleft d)) \triangleright ((e \triangleleft f) \otimes (g \triangleleft h)) \triangleright (a^\perp \triangleleft h^\perp) \triangleright (e^\perp \triangleleft b^\perp) \triangleright (g^\perp \triangleleft d^\perp) \triangleright (c^\perp \triangleleft f^\perp) \tag{10}
\]

or equivalently, the sequent

\[
\Gamma_Q = [(a \triangleleft b) \otimes (c \triangleleft d), (e \triangleleft f) \otimes (g \triangleleft h), a^\perp \triangleleft h^\perp, e^\perp \triangleleft b^\perp, g^\perp \triangleleft d^\perp, c^\perp \triangleleft f^\perp] \tag{11}
\]

Since the formula \( Q \) (resp. the sequent \( \Gamma_Q \)) is balanced, there is a unique axiom linking and therefore a unique relational RB-prenet and a unique tree-like RB-prenet. In Figure 4, we show the tree-like RB prenet for \( \Gamma_Q \), and on the left of Figure 5 we show the relational RB-prenet, which is the same for \( Q \) and \( \Gamma_Q \).

To see that these are provable in pomset logic, we have to show that the RB-prenets do not contain chordless \( \alpha \)-cycles. For this we focus on the tree-like RB-prenet, because in tree-like RB-prenets all \( \alpha \)-paths (and therefore also all \( \alpha \)-cycles) are chordless. Hence, it suffices to show that there are no \( \alpha \)-cycles.

Observe that the \( B \)-edges corresponding to the roots of the formulas in \( \Gamma_Q \) cannot participate in an \( \alpha \)-cycle because they have no adjacent \( R \)-edge at the bottom. We can therefore remove each of these \( B \)-edges, together with the two adjacent \( R \)-edges at the top. The resulting graph is shown on the right of Figure 5.

Another simplification we can do without affecting the \( \alpha \)-cycles in the graph is replacing the two \( B \)-edges labeled \( a \triangleleft b \) and \( c \triangleleft d \), together with the connecting \( R \)-edge by a single \( B \)-edge, and similarly for the two \( B \)-edges \( g \triangleleft h \) and \( e \triangleleft f \). The result is shown on the left of Figure 6.
Finally, observe that there is no \( \alpha \)-cycle that passes through the two \( B \)-edges labeled \( b \) and \( a \). The reason is that the directed \( R \)-edge between them has the opposite direction of the two adjacent \( R \)-edges on the other endpoints of these \( B \)-edges. Thus, we can collapse these two edges (and the adjacent “triangle”) to a single vertex. The same can be done for the pairs \( c/d \) and \( g/h \) and \( e/f \). The result of this operation is shown on the right of Figure 6.

\[ \text{Proposition 5.1.} \quad \text{The formula } Q \text{ and the sequent } \Gamma_Q \text{ shown in Equation (10) and Equation (11) above are provable in pomset logic.} \]

\[ \text{Proof.} \quad \text{In the paragraphs above, we have argued that the tree-like RB-prenet in Figure 4 has an } \alpha \text{-cycle if and only if the RB-digraph on the right of Figure 6 has an } \alpha \text{-cycle. Now it is easy to see that this graph has no } \alpha \text{-cycle. Hence, tree-like RB-prenet for } \Gamma_Q \text{ is correct.} \]

Let us now show that the formula \( Q \) is not provable in \( \text{BV} \). To do so we will show that whenever a \( \text{BV} \) inference has as conclusion \( Q \) then its premise defines an incorrect RB-prenet in pomset logic, and is therefore not provable in pomset logic. Since by Theorem 4.1 all \( \text{BV} \) proofs induce correct pomset proof nets, we can conclude that those premises are not \( \text{BV} \)-provable, therefore there is no way to build a \( \text{BV} \)-proof of \( Q \).

The main difficulty here is to make sure that we do not overlook any case when checking all possible inferences that have \( Q \) as conclusion. Since the unit \( I \) can make these kind of arguments difficult to check, we use here \( \text{BV} \). Now observe that \( Q \) has no subformula of the form \( x \sqtop x^+ \). This means we only have to consider the non-axiom rules of \( \text{BV} \).
Thus, conjugacy preserves provability both in pomset logic (reversing the direction of all automorphisms on the subformulas of $\alpha$ that sends $\beta$ to its “conjugate” $\beta^\top$ defined inductively as follows:

$$x^\top = x \text{ when } x \text{ is an atom \quad } (B \circ C)^\top = C^\top \circ B^\top \text{ for } \circ \in \{\lor, \&\}$

Note that the reversal of the arguments only matters for the non-commutative connective $\&$, and $[Q]^\top$ is the same as $[Q]$, except that all directed $R$-edges have the opposite direction. Thus, conjugacy preserves provability both in pomset logic (reversing the direction of all cycles in the correctness criterion) and in system BV which (the inference rules are closed under conjugacy, with $A^\bot$ and $B^\bot$ being swapped).

We will now go through all the rules of BV and check all possible applications. Using a similar argument as in the proof of Lemma 4.12, we will see that in each case there is a cycle in the resulting premise.

$$\frac{[A \& C] \circ [B \& D]}{(A \& B) \& (C \& D)}$$

Because of the action of the automorphisms $\alpha/\beta$, we can without loss of generality assume that $A = a^\bot$ and $B = b^\bot$. There are three subcases:

- $C = e^\bot$ and $D = f^\bot$. We get the cycle $(e \& h) \equiv (e^\bot \& h^\bot)$ in the premise of the $q_{4\bot}$-application.
- $C = g^\bot$ and $D = d^\bot$. We get the cycle $(a \& d) \equiv (a^\bot \& d^\bot)$ in the premise of the $q_{4\bot}$-application.
- $C = e^\bot$ and $D = f^\bot$. We get the cycle $(b \& c) \equiv (e^\bot \& h^\bot) \equiv (e^\bot \& b^\bot)$ in the premise of the $q_{4\bot}$-application.

$$\frac{[A \& C] \circ B}{(A \& B) \& C}$$

As before, because of the symmetries of $Q$, we only need to consider the case where $A = a^\bot$ and $B = h^\bot$. There are now five subcases of how to match $C$:

- $C = (a \& b) \equiv (c \& d)$. We get the cycle $(e \& h) \equiv (b \& h^\bot) \equiv (e^\bot \& b^\bot)$ in the premise of the $q_{3\bot}$-application.
- $C = (e \& f) \equiv (g \& h)$. We get the cycle $h \& h^\bot$ in the premise of the $q_{3\bot}$-application.
- $C = e^\bot \& b^\bot$. We get the cycle $(e \& h) \equiv (e^\bot \& h^\bot)$ in the premise of the $q_{3\bot}$-application.
\[ C = g^\perp \circ d^\perp. \] We get the cycle \((b \otimes d) \upharpoonright (e \otimes h) \upharpoonright (d^\perp \circ h^\perp) \upharpoonright (e^\perp \circ b^\perp)\) in the premise of the \(\hat{q}_3\) application.

\[ C = c^\perp \circ f^\perp. \] We get the cycle \((f \otimes h) \upharpoonright (f^\perp \circ h^\perp)\) in the premise of the \(\hat{q}_3\) application.

\[ \hat{q}_3 \vdash \frac{A \otimes B}{A \otimes C} : \] Similar to \(\hat{q}_3\), by conjugacy.

\[ \hat{q}_2 \vdash \frac{A \otimes C}{A \otimes B} \] The possible values for the ordered pair \((A, B)\) are all pairs of distinct formulas in the sequent \(\Gamma_Q\) in Equation (11). We first look at the case \(A = \langle a \circ b \rangle \otimes \langle c \circ d \rangle\) and \(B = \langle e \circ f \rangle \otimes \langle g \circ h \rangle\). Here we get the cycle \(\langle d \circ g \rangle \upharpoonright \langle y^\perp \circ d^\perp \rangle\) in the premise. The case \(A = \langle e \circ f \rangle \otimes \langle g \circ h \rangle\) and \(B = \langle a \circ b \rangle \otimes \langle c \circ d \rangle\) is symmetric to the this one via the automorphism \(\beta\). Otherwise, either \(A\) or \(B\) (or both) have the form \(x^\perp \circ y^\perp\). It suffices to treat all the cases \(R = x^\perp \circ y^\perp\). This is because conjugation exchanges the roles of \(A\) and \(B\) in the \(\hat{q}_2\) rule, and \(Q\) is equal to its own conjugate up to the variable renaming performed by \(\gamma\). We may also without loss of generality assume that \(A = a^\perp \circ h^\perp\); as before, this relies on the transitive action of the automorphisms of \(Q\) on the \(x^\perp \circ y^\perp\) that it contains. There are now five cases for \(B\):

\[ B = \langle a \circ b \rangle \otimes \langle c \circ d \rangle. \] We get the cycle \(a^\perp \circ a\) in the premise.

\[ B = \langle e \circ f \rangle \otimes \langle g \circ h \rangle. \] We get the cycle \(h^\perp \circ h\) in the premise.

\[ B = e^\perp \circ b^\perp. \] We get the cycle \((e \circ h) \upharpoonright (h^\perp \circ e^\perp)\) in the premise.

\[ B = g^\perp \circ d^\perp. \] We get the cycle \((a \circ d) \upharpoonright (a^\perp \circ d^\perp)\) in the premise.

\[ B = c^\perp \circ f^\perp. \] We get the cycle \((f \circ h) \upharpoonright (h^\perp \circ f^\perp)\) in the premise.

\[ \hat{q}_1 \vdash \frac{A \otimes C}{A \otimes B} : \] There are two possibilities to match \(A \otimes B\): either with \(\langle a \circ b \rangle \otimes \langle c \circ d \rangle\) or with \(\langle e \circ f \rangle \otimes \langle g \circ h \rangle\). Due to the commutativity of \(\circ\), we have four possibilities to match \(A\) and \(B\). Due to the symmetries discussed above, we only need to consider the case where \(A = a \circ b\) and \(B = c \circ d\). There are now five cases how to match \(C\):

\[ C = \langle e \circ f \rangle \otimes \langle g \circ h \rangle. \] We get the cycle \((f \circ c) \upharpoonright (c^\perp \circ f^\perp)\) in the premise.

\[ C = a^\perp \circ h^\perp. \] We get the cycle \((h^\perp \circ c) \upharpoonright (c^\perp \circ f^\perp) \upharpoonright (f \circ h)\) in the premise.

\[ C = e^\perp \circ b^\perp. \] We get the cycle \((c^\perp \circ d) \upharpoonright (g^\perp \circ d^\perp) \upharpoonright (e \circ g)\) in the premise.

\[ C = g^\perp \circ d^\perp. \] We get the cycle \(d^\perp \circ d\) in the premise.

\[ C = c^\perp \circ f^\perp. \] We get the cycle \(c^\perp \circ c\) in the premise.

\[ \hat{q}_2 \vdash \frac{A \otimes B}{A \otimes B} : \] This case is already subsumed by the case for \(\hat{q}_2\).

In this way, we have completed the proof of the following proposition.

**Proposition 5.2.** The formula \(Q\) shown in Equation (10) is not provable in \(BV\).

**Theorem 5.3.** The theorems of \(BV\) form a proper subset of the theorems of pomset logic.

**Proof.** This follows immediately from Propositions 5.1 and 5.2.

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### 6 Conclusion

Let us end this paper by giving some historical perspective and some explanation how the formula \(Q\) has been found. The main reason that it took more than 20 year to find this (rather simple) formula was that everyone (including the second author) was looking into the wrong direction, trying to prove that \(BV\) and pomset logic are the same. This changed only after the first author (not being aware of the pomset logic vs. \(BV\) problem) observed
that checking pomset logic correctness is \text{coNP}-complete [18]. Since it had been observed before that \text{BV} is \text{NP}-complete [15], this immediately entailed that either \text{NP} = \text{coNP} or \text{BV} \neq \text{pomset logic}.

Unfolding and dissecting the proof of \text{coNP}-completeness of pomset logic correctness led to a relation to classical logic provability. The details of this are subject of ongoing research and would go beyond the scope of this paper. But the outcome let us to the study of linear inferences [5, 6] which are a special case of balanced tautologies [27]. We were looking at linear inferences that are tautologies in classical logic but not provable in linear logic. The simplest such inference is \((A \land D) \lor (B \land C) \Rightarrow [A \lor B] \land [D \lor C]\), which corresponds to the medial rule of system \text{SKS} [2], a formulation of classical logic in the calculus of structures. Its linear version \((A \otimes D) \nabla (B \otimes C) \rightarrow [A \nabla B] \otimes [D \nabla C]\) is, of course, not a theorem of \text{MLL}. This can be immediately seen by inspecting the RB-prenet for the formula \((a \otimes d) \nabla (b \otimes c) \rightarrow [a \nabla b] \otimes [c \nabla d]\), which is shown in Figure 7a, and which contains several (chordless) \(ac\)-cycles. Then, on the right of that “medial RB-prenet”, in Figure 7b, we replace the \(B\)-edges corresponding to the atoms by a pair of \(B\)-edges connected by an (undirected) \(R\)-edge. This does not affect provability, as no \(ac\)-cycles are added or removed. Then, in Figure 7c, we give these new \(R\)-edges a direction. By choosing the “right” direction, we can break all \(ac\)-cycles, which means the result becomes correct with respect to the pomset logic correctness criterion. But the resulting formula (or sequent) remains unprovable in

\begin{itemize}
  \item Figure 7 From the medial of \text{SKS} to our counterexample.
\end{itemize}
BV and Pomset Logic Are Not the Same

To simplify the proof of non-provability in BV, we added further $R$-edges, as shown in Figure 7d, that do not break provability in pomset logic. The result is an intermediate step between the RB-prenets in Figure 4 and Figure 5.

The knowledge that BV and pomset logic are different, leads to four immediate open problems: (i) can we find a proof net correctness criterion for BV, (ii) can we find a deductive proof system for pomset logic that is independent from the prenets, (iii) which of the two logics is better, and (iv) are these two the only ones, or are there more logics having these three connectives and being conservative over $MLL_0$?

References


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9 The sequent system proposed by Slavnov [25], uses labels for encoding the paths in the proof net.


