A 10-Approximation of the $\frac{\pi}{2}$-MST

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Abstract
Bounded-angle spanning trees of points in the plane have received considerable attention in the context of wireless networks with directional antennas. For a point set $P$ in the plane and an angle $\alpha$, an $\alpha$-spanning tree ($\alpha$-ST) is a spanning tree of the complete Euclidean graph on $P$ with the property that all edges incident to each point $p \in P$ lie in a wedge of angle $\alpha$ centered at $p$. The $\alpha$-minimum spanning tree ($\alpha$-MST) problem asks for an $\alpha$-ST of minimum total edge length. The seminal work of Anscher and Katz (ICALP 2014) shows the NP-hardness of the $\alpha$-MST problem for $\alpha = \frac{2\pi}{3}, \pi$ and presents approximation algorithms for $\alpha = \frac{\pi}{2}, \frac{2\pi}{3}, \pi$.

In this paper we study the $\alpha$-MST problem for $\alpha = \frac{\pi}{2}$ which is also known to be NP-hard. We present a 10-approximation algorithm for this problem. This improves the previous best known approximation ratio of 16.

2012 ACM Subject Classification Theory of computation → Computational geometry; Theory of computation → Approximation algorithms analysis

Keywords and phrases Euclidean spanning trees, approximation algorithms, bounded-angle visibility

Digital Object Identifier 10.4230/LIPIcs.STACS.2022.13

Funding Ahmad Biniaz: supported by NSERC.

1 Introduction

Wireless antennas in a wireless network can be modeled by disks in the plane, where the centers of the disks represent locations of antennas and their radii represent transmission ranges of antennas. Two antennas can communicate if they are in each other’s transmission range. In this model antennas are assumed to be omni-directional which can transmit and receive signals in 360 degrees. Replacing omni-directional antennas with directional antennas has received considerable attention in recent years, see for example [1, 3, 6, 8, 9, 10, 11, 13, 14, 21]. Directional antennas can transmit and receive signals only in a circular wedge with some bounded-angle $\alpha$. As noted in [4, 21, 23] such a bounded-angle communication is more secure, requires lower transmission range, and causes less interference. In this model two antennas can communicate if each one is inside the other’s wedge. This model is known as symmetric communication network [4, 5, 23].

The network connectivity is a common problem in designing networks with directional antennas. Aschner and Katz [3] formulated this problem in terms of an $\alpha$-spanning tree ($\alpha$-ST). For a point set $P$ in the plane and an angle $\alpha$, an $\alpha$-ST of $P$ is a spanning tree of the complete Euclidean graph on $P$ such that all edges incident to each point $p \in P$ lie in a wedge of angle $\alpha$ centered at $p$ (see Figure 1). It is known that an $\alpha$-ST always exists when $\alpha \geq \frac{\pi}{2}$ (see e.g. [1, 2, 11]) while it may not exists when $\alpha < \frac{\pi}{2}$, for example if $P$ consists of the three vertices of an equilateral triangle.
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The minimum spanning tree (MST) is the shortest connected network for omni-directional antennas. For directional antennas, the shortest connected network is called the $\alpha$-minimum spanning tree ($\alpha$-MST) which is an $\alpha$-ST of $P$ with minimum total edge length. Although one can compute an MST of $n$ points in the plane optimally in $O(n\log n)$ time, it is not clear how to efficiently compute an $\alpha$-MST. Aschner and Katz [3] proved that the $\alpha$-MST problem is NP-hard for $\alpha = \frac{2\pi}{3}$ and $\alpha = \pi$. They also presented approximation algorithms with ratios $16$, $6$, and $2$ for angles $\alpha = \frac{\pi}{2}$, $\alpha = \frac{2\pi}{3}$, and $\alpha = \pi$, respectively. The approximation ratio $6$ for the $\frac{2\pi}{3}$-MST has been successively improved to $5.34$ [8] and to $4$ [6]. Recently Tran et al. [23] showed that the power assignment problem with directional antennas (described in Section 1.2) of angle $\pi/2$ is NP-hard, by a reduction from the Hamilton path problem on hexagonal grid graphs. A similar reduction can be employed to show that the $\frac{\pi}{2}$-MST problem is also NP-hard.

The above approximation ratios are obtained by considering the weight of the MST as the lower bound (instead of the weight of an optimal $\alpha$-MST). Of these approximation ratios, the ratio $16$ for $\pi/2$ is very interesting because for any $\alpha < \pi/2$ there exists a point set for which the ratio of the weight of any $\alpha$-MST to the weight of any MST is $\Omega(n)$ [5]. In other words, $\alpha = \pi/2$ is the smallest angle for which one can obtain an $\alpha$-ST of weight within some constant factor of the MST weight. However, such a factor cannot be better than $2$ because for points uniformly distributed on a line the weight of $\alpha$-MST could be arbitrary close to $2$ times the weight of MST, for any $\alpha < \pi$ [3, 8].

1.1 Our contributions

We present an algorithm that finds a $\pi/2$-ST of weight at most $10$ times the MST weight (Theorem 6). Thus we obtain a $10$-approximation algorithm for the $\pi/2$-MST problem, improving upon the previous best known ratio of $16$ due to Aschner and Katz [3]. Both our algorithm and that of [3] take linear time after computing an MST.

Towards obtaining the approximation ratio $10$ we extend another interesting result of Aschner et al. [5] which ensures the connectivity of two sets of oriented four points that are separated by a straight line. Our extension (which is given in Theorem 3) relaxes the linear separability constraint. Most of the paper is devoted to proving this theorem.

1.2 Some related problems

There is a relationship between bounded-angle spanning trees and bounded-degree spanning trees which have received a considerable attention [7, 12, 17, 19, 20, 22]. A degree-$k$ ST is a spanning tree in which every vertex has degree at most $k$. It is easily seen that any degree-$k$ ST is an $\alpha$-ST with $\alpha = 2\pi(1 - 1/k)$ because in any degree-$k$ ST all edges that are incident to each vertex lie in some wedge of angle $2\pi(1 - 1/k)$.

The $\alpha$-bottleneck spanning tree ($\alpha$-BST) is a closely related problem in which the goal is to compute an $\alpha$-ST whose longest edge length is minimum. This problem has been studied in the context of designing networks with bounded-range directional antennas, see for
example the results of Aschner et al. [3, 5] for constructing hop-spanners for unit disk graphs, Dobrev et al. [14, 15] and Caragiannis et al. [10] for constructing bounded-degree strongly connected networks, and Carmi et al. [11] for constructing bounded-angle Hamiltonian paths. Another related problem in this context is “power assignment with directional antennas” where the objective is to assign each point \( p \in P \) a wedge of angle \( \alpha \) as well as a range \( r_p \) to obtain a connected symmetric communication network of minimum total power \( \sum_{p \in P} r_p^\beta \) where \( \beta \geq 1 \) is the distance-power gradient [3, 5, 23].

Computing bounded-angle Hamiltonian paths and cycles on points in the plane is another related problem. For paths it is known that any set of points in the plane admits a Hamiltonian path with turning angles at most \( \frac{\pi}{2} \) [11, 18] and this bound on the angle is tight [11, 16]. For cycles no tight bound on the angle is known. Dumitrescu et al. [16] proved that any even-size point set admits a Hamiltonian cycle with angles at most \( \frac{2\pi}{3} \). The most famous conjecture in this context, due to Fekete and Woeginger [18], states that any even-size point set of at least 8 elements admits a Hamiltonian cycle with angles at most \( \frac{\pi}{2} \).

### 1.3 Preliminaries for the algorithm

The following notations are adopted from [8]. Let \( w_p \) be a wedge in the plane having its apex at a point \( p \). We denote the clockwise (right) boundary ray of \( w_p \) by \( \overrightarrow{w_p} \) and its counterclockwise (left) boundary ray by \( \overleftarrow{w_p} \). Let \( w_q \) be another wedge in the plane having its apex at a point \( q \). If \( q \) lies in \( w_p \) then we say that \( p \) sees \( q \) (or \( q \) is visible from \( p \)). We say that \( p \) and \( q \) are mutually visible, denoted by \( p \leftrightarrow q \), if \( p \) sees \( q \) and \( q \) sees \( p \). In Figure 2 \( p \) and \( q \) are mutually visible. Let \( P \) be a set of points in the plane such that some wedge is placed at each point of \( P \). The induced mutual visibility graph of \( P \), denoted by \( G(P) \), is a geometric graph with vertex set \( P \) that has a straight-line edge between two points \( p, q \in P \) if and only if \( p \) and \( q \) are mutually visible. We use the term “orient” to refer to placement of wedges at points. We denote the sum of edge lengths of a geometric graph \( G \) by \( w(G) \).

![Figure 2](image.png) The points \( p \) and \( q \) are mutually visible.

We define the following notations to facilitate the description of our algorithm and its analysis. For two points \( p \) and \( q \) in the plane the slab \( S(p, q) \) is defined as the region between two lines that are perpendicular to the segment \( pq \) at points \( p \) and \( q \) (see Figure 3(a)). We use quadruple to denote a set of four points in the plane. A quadruple \( Q \) is called admissible if it has two points \( p \) and \( q \) such that the other two points lie in \( S(p, q) \) and both on the same side of \( pq \). In this case we refer to \( (p, q) \) as an admissible pair of \( Q \). Notice that a quadruple could have more than one admissible pair. For a quadruple \( Q \) with a fixed admissible pair \( (p, q) \), we define the admissible slab of \( Q \), denoted by \( S(Q) \), to be the same as the slab \( S(p, q) \); see Figure 3(a). The following lemma (though very simple) plays an important role in our algorithm.

**Lemma 1.** Any set \( P \) of five points in the plane contains an admissible quadruple \( Q \) such that all points of \( P \) lie in \( S(Q) \).
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![Figure 3](image)

Figure 3 An admissible quadruple $Q = \{p, q, r, s\}$ with admissible pair $(p, q)$. Illustrations of (a) the slab $S(p, q)$ which is the same as the admissible slab $S(Q)$, (b) the proof of Theorem 2, and (c) the visibility region $V(Q)$ which is the region visible to both $p$ and $q$.

Proof. Let $p$ and $q$ be two points that define a diameter of $P$, i.e., two with maximum distance. Of the remaining three points of $P$ at least two of them, say $r$ and $s$, lie on the same side of $S(p, q)$. Therefore $\{p, q, r, s\}$ is an admissible quadruple which we denote by $Q$. Since $pq$ is a diameter of $P$, all points of $P$ lie in $S(p, q)$ and hence in $S(Q)$.

Our orientation of admissible quadruples in the following theorem is similar to that of Aschner, Katz, and Morgenstern et al. [5] for arbitrary quadruples.

**Theorem 2.** Given an admissible quadruple $Q$, one can place at each point of $Q$ a wedge of angle $\pi/2$ such that the wedges cover the plane and the induced mutual visibility graph of $Q$ is connected.

Proof. Let $Q = \{p, q, r, s\}$. After a suitable relabeling, rotation and reflection assume that $(p, q)$ is an admissible pair of $Q$, the line segment $pq$ is horizontal, $p$ is to the left of $q$, the points $r$ and $s$ lie above $pq$, and $r$ is to the left of $s$ as in Figure 3(b). We place four wedges at points of $Q$ as in Figure 3(b). Formally, we place a wedge $w_p$, at $p$ such that $\overrightarrow{wp}$ passes through $q$, place $w_q$ at $q$ such that $\overrightarrow{wq}$ passes through $p$, place $w_r$ at $r$ such that $q$ lies in $w_r$, and $\overrightarrow{ws}$ is vertical, and place $w_s$ at $s$ such that $p$ lies in $w_s$ and $\overrightarrow{ws}$ is vertical. These four wedges cover the entire plane (if we think of the intersection point of $\overrightarrow{wp}$ and $\overrightarrow{ws}$ as the origin of the coordinate system, then the four wedges cover the four quadrants). Moreover, the induced mutual visibility graph is connected because $p \leftrightarrow q$, $r \leftrightarrow q$, and $p \leftrightarrow s$.

Recall the two points $p$ and $q$ in the proof of Theorem 2 that make $Q$ admissible. Notice that after orientation of Theorem 2 the admissible slab of $Q$ is uniquely defined by $p$ and $q$. We define the visibility region of $Q$, denoted by $V(Q)$, as part of $S(Q)$ that is visible to both $p$ and $q$; see Figure 3(c) for an illustration.

The following theorem, which will be proved in Section 3, plays a crucial role in the correctness of our algorithm. Most of the paper is devoted to proving this theorem.

**Theorem 3.** Let $Q_1$ and $Q_2$ be two admissible quadruples. Assume that wedges of angle $\pi/2$ are placed at points of each of $Q_1$ and $Q_2$ according to the placement in the proof of Theorem 2. Then at least one of the following statements holds:

(i) The induced mutual-visibility graph of $Q_1 \cup Q_2$ is connected.
(ii) At any point $p$ in $S(Q_1) \cup S(Q_2)$ one can place a wedge of angle $\pi/2$ such that $p$ is mutually visible from a point $q_1 \in Q_1$ and from a point $q_2 \in Q_2$. In other words the induced mutual-visibility graph of $Q_1 \cup Q_2 \cup \{p\}$ is connected.
We note that there are admissible quadruples for which statement (i) does not hold, but (ii) holds for them; see for example Figure 12. Theorem 3 extends the following result of Aschner et al. [5] which applies only to quadruples that are separated by a line.

▶ Theorem 4 (Aschner, Katz, and Morgenstern [5], 2013). Let $Q_1$ and $Q_2$ be two quadruples. Assume that wedges of angle $\pi/2$ are placed at points of each of $Q_1$ and $Q_2$ according to the placement in the proof of Theorem 2. If $Q_1$ and $Q_2$ are separated by a straight line, then the induced mutual-visibility graph of $Q_1 \cup Q_2$ is connected.

2 The approximation algorithm

Let $P$ be a set of $n$ points in the plane. In this section we present our algorithm for computing a $\frac{\pi}{2}$-ST of $P$ of weight at most 10 times the weight of the MST of $P$. In Section 2.1 we describe the general framework of the algorithm. In Section 2.2 we provide the details of the algorithm and its analysis.

2.1 A general framework

Our algorithm follows the same framework as previous algorithms [3, 6, 8] which is described below. This framework was first introduced by Aschner and Katz [3].

Start by computing an MST of $P$. From the MST obtain a Hamiltonian path $H$ of weight at most 2 times the weight of MST. It is well-known that such a path can be obtained by doubling the MST edges, computing an Euler tour, and then short-cutting repeated vertices. The constant 2 is tight as Fekete et al. [17] showed that for any fixed $\epsilon > 0$ there exist point sets for which the weight of any Hamiltonian path is at least $2 - \epsilon$ times the weight of MST.

The next step is to partition $H$ into $n/k$ groups each consisting of $k$ consecutive vertices of $H$ for some constant $k$ (assuming $n$ is divisible by $k$). Then orient each group independently in such a way that (I) the vertices in each group are connected, and (II) there is an edge between any pair of consecutive groups. Thus the induced mutual visibility graph on $P$ is connected. Moreover, as the vertices of the groups are connected locally (to the vertices of the same group or a neighboring group), the mutual visibility graph contains a spanning tree whose weight is within some constant factor of the weight of $H$. This constant depends only on $k$.

The original algorithms of Aschner and Katz [3] partition $H$ into groups of size $k = 8$ for $\alpha = \frac{\pi}{6}$ and $k = 3$ for $\alpha = \frac{2\pi}{3}$. The improved algorithms of [8] and [6] (for $\alpha = \frac{2\pi}{3}$) partition $H$ into groups of size $k = 3$ and $k = 2$, respectively.

Our algorithm partitions $H$ into groups of size $k = 5$ for $\alpha = \frac{\pi}{2}$. The most challenging part in our algorithm (and in previous algorithms) is to maintain property (II); the proof of this property often involves detailed case analysis. There is a main difference between our algorithm and previous algorithms [3, 6, 8]. Instead of orienting all five vertices in each group simultaneously, we first select four of them and orient only these selected vertices. The four selected vertices form an admissible quadruple. We refer to the non-selected vertex as a backup. We show that, except for one “special case”, there is always a connection between two oriented admissible quadruples. For the special case we use the backup vertex to make the connection between two quadruples.


### 2.2 Details of our algorithm

In this section we provide details of our algorithm and its analysis. Recall that $P$ is a set of $n$ points in the plane, and that $H$ is a Hamiltonian path on $P$ such that

$$w(H) \leq 2w(\text{MST}).$$

Let $h_1, \ldots, h_{n-1}$ be the sequence of edges of $H$ from one end to another. Partition the edges of $H$ into five sets $H_1 = \{h_1, h_6, \ldots\}$, $H_2 = \{h_2, h_7, \ldots\}$, $H_3 = \{h_3, h_8, \ldots\}$, $H_4 = \{h_4, h_9, \ldots\}$, and $H_5 = \{h_5, h_{10}, \ldots\}$. Let $H_k$ with $k \in \{1, 2, 3, 4, 5\}$ be the edge set with the largest weight. Then

$$w(H_k) \geq \frac{w(H)}{5} \quad \text{and} \quad w(H \setminus H_k) \leq \frac{4w(H)}{5}.$$  

![Illustration of the groups and sub-paths](image)

**Figure 4** Illustration of the groups and sub-paths (dashed edges belong to $H_k$, where $k = 5$).

By removing all edges of $H_k$ from $H$ we obtain a sequence of sub-paths each containing five vertices (except possibly the first and last sub-paths). To simplify our description we assume for now that all sub-paths have five vertices, later in Remark 5 we will take care of the case where the first and last sub-paths have less than five vertices. We refer to the five vertices of each sub-path as a *group*. Let $g_1, g_2, \ldots, g_m$ denote the sequence of the groups that is corresponding to the sequence of sub-paths along $H$ as in Figure 4.

From each group $g_i$ we take an admissible quadruple $Q_i$ (consisting of four vertices) as in the proof of Lemma 1. We denote the remaining vertex of $g_i$ by $b_i$; this is a backup vertex. By Lemma 1, $b_i$ lies in $S(Q_i)$. We orient each admissible quadruple $Q_i$ according to the orientation in the proof of Theorem 2 which ensures the connectivity of the induced mutual visibility graph $G(Q_i)$. Consider any two consecutive oriented quadruples $Q_i$ and $Q_{i+1}$. By Theorem 3 at least one of the following statements holds:

(i) The graph $G(Q_i \cup Q_{i+1})$ is connected, i.e., there is an edge between $Q_i$ and $Q_{i+1}$.

(ii) Any point $p$ in $S(Q_i) \cup S(Q_{i+1})$ can be oriented so that $G(Q_i \cup Q_{i+1} \cup \{p\})$ is connected.

If statement (i) holds then we orient $b_i$ towards a vertex of $Q_i$ that sees $b_i$ (such a vertex exists because the orientation of Theorem 2 covers the entire plane). If (i) does not hold but (ii) holds then we orient $b_i$ in such a way that it connects $Q_i$ and $Q_{i+1}$.

To this end all vertices are oriented except the backup vertex $b_m$ of $g_m$. We orient $b_m$ towards a vertex of $Q_m$ that sees $b_m$. Thus, we obtain a connected induced mutual visibility graph $G(P)$.

Now we obtain a spanning tree $T$ of $G(P)$ as follows: First we take an arbitrary spanning tree $T_i$ from each $G(Q_i)$. Then we connect each pair $T_i$ and $T_{i+1}$ either by a direct edge (if (i) holds) or via a backup vertex (if (ii) holds). Lastly we connect any remaining backup vertex to its corresponding quadruple by an edge. This gives a spanning tree $T$ that we report as the output of our algorithm. Notice that the trees $T_i$ are not necessarily minimum spanning trees of graphs $G(Q_i)$; we will use the triangle inequality to bound the length of $T$. 


Analysis of the approximation ratio. To bound the weight of $T$, we charge the edges of $H$ for the edges of $T$ as follows. By the triangle inequality, the weight of every edge $(p, q)$ of $T$ is at most the weight of the unique path in $H$ between $p$ and $q$. We charge the weight of the edges of this path for the edge $(p, q)$. Every edge of $H_k$ is charged only once and that is for connecting two consecutive trees $T_i$ and $T_{i+1}$ (either directly or via a backup vertex). Every edge of $H \setminus H_k$ (i.e., every edge of each sub-path) is charged at most six times: three times for the three edges of $T_i$, two times for the two edges connecting $T_i$ to $T_{i+1}$ and to $T_{i-1}$, and once for the edge connecting the backup vertex $b_i$ to $T_i$. Therefore
\[
 w(T) \leq w(H_k) + 6w(H \setminus H_k) = w(H) + 5w(H \setminus H_k) \leq w(H) + 5 \cdot \frac{4w(H)}{5} = 5w(H) \leq 10w(MST).
\]

Running-time analysis. After computing an MST in $O(n \log n)$ time, the rest of the algorithm (computing $H$, finding $H_k$, orienting admissible quadruples and backup vertices, and obtaining $T$) takes $O(n)$ time.

Remark 5. Here we handle the case where the first sub-path, denoted by $\delta$, has less than five vertices (the last sub-path will be treated analogously). This case is essentially a simple version of Theorem 3 where fewer points are involved. We will use Theorem 3 to handle this case, however it could also be handled directly but with some case analysis.

We will connect the vertices of $\delta$ to $g_1$ (the first 5-vertex group). Let $Q$ be $g_1$’s admissible quadruple. Since the oriented points in $Q$ cover the entire plane, it might be tempting to orient each point $p$ of $\delta$ towards the point of $Q$ that sees $p$. This approach may not be suitable when $\delta$ has more than one point because to maintain the ratio 10 we should not connect $Q$ to its proceeding group (here to $\delta$) by more than one edge. To remedy this, we use our Theorem 3.

![Figure 5](image-url) ab is the diameter of $\delta$, and $c', d'$ are fake points.

As discussed above, we may assume that $\delta$ has 2, 3, or 4 points. Let $ab$ be a diameter of $\delta$ as in Figure 5. Thus, $\delta$ has points $a$, $b$, and at most two other “real” points. We place a “fake” point $c'$ in $S(a, b)$ and very close to $b$ such that both $c'$ and $b$ lie on the same side of any line through boundary rays of wedges in $Q$. In the same fashion we place a fake point $d'$ very close to $a$, and on the same side of $ab$ as $c'$. Let $Q' = \{a, b, c', d'\}$. Our placement of $c'$ and $d'$ in $S(a, b)$ and on the same side of $ab$ implies that $Q'$ is an admissible quadruple with admissible pair $(a, b)$. We orient $Q'$ according to Theorem 2. By Theorem 3-part (i), a point of $Q'$ and a point of $Q$ are mutually visible (our placement of $c'$ and $d'$ together with Property 1 from the next section imply that part (i) of Theorem 3 holds). If the visibility is through a real point say $b$, then we reflect the orientation of $a$ with respect to $ab$. After reflection, $a$ and $b$ remain mutually visible, and their wedges cover the entire region $S(a, b)$. Then we orient every other real vertex of $\delta$ towards the one of $a$ and $b$ that sees it. If the visibility is through a fake point say $c'$ then the point of $Q$, say $q$, that sees $c'$ also sees $b$ (this is implied by our placement of $c'$). In this case we reflect the orientation of $b$ with respect to $ab$ so that $b$ is mutually visible with $q$, and $a$ and $b$ together see the entire region $S(a, b)$.
Then we orient every other real vertex of \( \delta \) towards the one of \( a \) and \( b \) that sees it. In either case we remove fake points. Therefore the mutual visibility graph on points of \( \delta \) is connected, and it has a connection to a point in \( Q \) via \( a \) or \( b \).

The following theorem summarizes our main result.

**Theorem 6.** For any set of points in the plane and any angle \( \alpha \geq \frac{\pi}{2} \), there is an \( \alpha \)-spanning tree of length at most 10 times the length of the MST. Furthermore, there is an algorithm that finds such an \( \alpha \)-spanning tree in linear time after construction of the MST.

### 3 Proof of Theorem 3

In this section we prove Theorem 3 which says: Let \( Q_1 \) and \( Q_2 \) be two admissible quadruples. Assume that wedges of angle \( \frac{\pi}{2} \) are placed at points of each of \( Q_1 \) and \( Q_2 \) according to the placement in the proof of Theorem 2. Then at least one of the following statements holds

(i) The induced mutual-visibility graph of \( Q_1 \cup Q_2 \) is connected.

(ii) At any point \( p \) in \( S(Q_1) \cup S(Q_2) \) we can place a wedge of angle \( \frac{\pi}{2} \) such that \( p \) is mutually visible from a point \( q_1 \in Q_1 \) and from a point \( q_2 \in Q_2 \). In other words the induced mutual-visibility graph of \( Q_1 \cup Q_2 \cup \{p\} \) is connected.

Our proof is involved. For a better understanding we split our proof into smaller pieces based on the relative position of admissible pairs of \( Q_1 \) and \( Q_2 \). Let \( Q_1 = \{a, b, c, d\} \) and \( Q_2 = \{a', b', c', d'\} \). After a suitable relabeling assume that \((a, b)\) and \((a', b')\) are the admissible pairs of \( Q_1 \) and \( Q_2 \), respectively, that are considered in the orientation of Theorem 2. Also assume that — after the orientation of Theorem 2 — \( c \) looks towards \( a \) while \( d \) looks towards \( b \), and similarly \( c' \) looks towards \( a' \) while \( d' \) looks towards \( b' \) as in Figures 7-13. We use this notation throughout our proof without further mentioning. Up to symmetry we have the following four cases:

A. \( a'b' \) intersects \( ab \).

B. The extension of \( a'b' \) intersects the extension of \( ab \).

C. The extension of \( a'b' \) intersects \( ab \).

D. \( a'b' \) is parallel to \( ab \).

After a suitable rotation we assume that \( ab \) is horizontal and \( a \) is to the left of \( b \). We denote by \( \ell \) the line through \( ab \) and by \( \ell' \) the line through \( a'b' \) as in Figure 7(a). For a point \( x \) we denote by \( \ell_x \) the line through \( x \) that is perpendicular to \( \ell \), and denote by \( \ell'_x \) the line through \( x \) that is perpendicular to \( \ell' \). For a line \( l \) in the plane we use the terms “above” and “below” to refer to the two half planes on the two sides of \( l \). If \( l \) is vertical then “below” refers to the left-side half plane and “above” refers to the right-side half plane. Throughout our proof, we use the following obvious observation about mutual visibility without mentioning it in all occurrences.

**Observation 7.** Assume that wedges \( w_p \) and \( w_q \) of angles \( \frac{\pi}{2} \) are placed at two points \( p \) and \( q \). If the clockwise (resp. counterclockwise) boundary ray of \( w_p \) meets the counterclockwise (resp. clockwise) boundary ray of \( w_q \) at an obtuse or a right angle then \( p \) and \( q \) are mutually visible. See Figure 6.
Some part of our proof (where $Q_1$ and $Q_2$ are separated by a line) could be implied from Theorem 4. However, for the sake of completeness we provide our own proof. We provide the proof of the first cases, $A$ and $B-1$, with more formal details. To simplify our description, we will refer to the clockwise (resp. counterclockwise) boundary ray of the wedge that is placed at a point $p$ by “the clockwise (resp. counterclockwise) ray of $p$”.

![Figure 7](image-url) Illustration of the proof of case $A$.

**A. $a'b'$ intersects $ab$**

We denote by $\alpha$ the intersection angle of $ab$ and $a'b'$ that lies in $V(Q_1) \cap V(Q_2)$. We say that $\alpha$ is defined by the two vertices that lie on this angle. For example in Figure 7(a) the angle $\alpha$ is defined by $a$ and $b'$, depending on the value of $\alpha$ we consider the following two cases.

1. $\alpha \geq \frac{\pi}{2}$. After a suitable relabeling we assume that $\alpha$ is defined by $a$ and $b'$, as in Figure 7(a). In this case the clockwise ray of $a$ and the counterclockwise ray of $b'$ meet at angle $\alpha$, and thus $a$ and $b'$ are mutually visible by Observation 7.

2. $\alpha < \frac{\pi}{2}$. After a suitable relabeling we assume that $\alpha$ is defined by $b$ and $b'$, as in Figure 7(b). If $c'$ is above $\ell$ then the clockwise ray of $a$ and the counterclockwise ray of $c'$ meet at angle $\pi - \alpha$, and thus $c'$ and $a$ are mutually visible by Observation 7. Similarly if $c$ is below $\ell'$ then $c'$ and $a'$ are mutually visible. Assume that $c'$ is below $\ell$ and $c$ is above $\ell'$ as in Figure 7(b). If $d'$ is to the left of $\ell_c$ then the clockwise ray of $d'$ and the counterclockwise ray of $c$ meet at angle $\frac{\pi}{2} + \alpha$, and thus $d'$ and $c$ are mutually visible by Observation 7. Similarly if $d$ is below $\ell'_{c'}$ then $d$ and $c'$ are mutually visible. Assume that $d'$ is to the right of $\ell_{c'}$, and $d$ is above $\ell'_{c'}$. In this setting which is depicted in Figure 7(b), $d$ and $d'$ lie in opposite cones formed by intersection of $\ell_{c}$ and $\ell'_{c'}$, and thus $d$ and $d'$ are mutually visible (observe that the clockwise ray of $d$ and the counterclockwise ray of $d'$ meet at angle $\pi - \alpha$).

**B. The extension of $a'b'$ intersects the extension of $ab$**

Let $\alpha$ be the angle at which the extensions of $ab$ and $a'b'$ meet each other as in Figures 8 and 9. After a suitable reflection and relabeling we assume that $a'b'$ lies below $\ell$, their extensions meet at a point $m$ to the right of $b$, and $a'$ is farther from $m$ than $b'$. Depending on the value of $\alpha$ we consider two cases.

1. $\alpha \geq \frac{\pi}{2}$. Depending on visibility regions of $Q_1$ and $Q_2$ we consider three sub-cases (up to symmetry).

1. $V(Q_1)$ lies below $ab$ and $V(Q_2)$ lies below $a'b'$ as in Figure 8(a). In this case the clockwise ray of $a'$ and the counterclockwise ray of $a$ meet at angle $\alpha$, and hence $a \leftrightarrow a'$ by Observation 7.
2. $V(Q_1)$ lies below $ab$ and $V(Q_2)$ lies above $a'b'$. See Figure 8(b). If $d'$ is below $\ell$ then the clockwise ray of $d'$ and the counterclockwise ray of $a$ meet at angle $\alpha$ and hence $a \leftrightarrow d'$. Assume that $d'$ is above $\ell$. If $d$ is above $\ell_d'$ then the clockwise ray of $d$ and the counterclockwise ray of $d'$ meet at angle $\frac{3\pi}{2} - \alpha$ and thus $d \leftrightarrow d'$. Assume that $d$ is below $\ell_d'$. In this setting which is depicted in Figure 8(b) the clockwise ray of $c'$ and the counterclockwise ray of $d$ meet at angle $\alpha$ and thus $c' \leftrightarrow d$.

3. $V(Q_1)$ lies below $ab$ and $V(Q_2)$ lies above $a'b'$. See Figure 8(c). If $d'$ is above $\ell$ then $a \leftrightarrow d'$. Similarly if $d$ is above $\ell'$ then $a' \leftrightarrow d$. Assume that $d'$ is below $\ell$ and $d$ is below $\ell'$. In this setting which is depicted in Figure 8(c) the clockwise ray of $d'$ and the counterclockwise ray of $d$ meet at angle $\alpha$ and thus $d \leftrightarrow d'$.

2. $\alpha < \frac{\pi}{2}$. Similar to the previous case here we also consider three sub-cases.

1. $V(Q_1)$ lies above $ab$ and $V(Q_2)$ lies above $a'b'$. See Figure 9(a). If $d$ is below $\ell_d'$ then $d$ and $b'$ are mutually visible. If $a'$ is to the left of $\ell_d$ then $a'$ and $c$ are mutually visible. Assume that $d$ is above $\ell_d'$ and $a'$ is to the right of $\ell_d$ as in Figure 9(a). In this setting $d$ and $a'$ are mutually visible.

2. $V(Q_1)$ lies above $ab$ and $V(Q_2)$ lies below $a'b'$. If $d'$ is to the left of $\ell_d$ then $c \leftrightarrow d'$ as in Figure 9(b). Analogously if $d$ is below $\ell_d'$ then $c' \leftrightarrow d$. Therefore assume that $d'$ is to right of $\ell_d$ and $d$ is above $\ell_d'$. In this setting $d \leftrightarrow d'$.

3. $V(Q_1)$ lies below $ab$ and $V(Q_2)$ lies above $a'b'$. See Figure 9(c). Consider $\ell_{a'}$, i.e., the line through $a'$ that is perpendicular to $\ell$. If $b$ is to the right of $\ell_{a'}$ then $d' \leftrightarrow b$. Assume that $b$ is to the left of $\ell_{a'}$ as in Figure 9(c). Now we look at $\ell_{a'}$. If $a$ is above this line then $a \leftrightarrow a'$, otherwise $a \leftrightarrow b'$. (Notice that when $a$ is above $\ell_{a'}$ then $a$ and $b'$ may not be mutually visible, for example when $b'$ is very close to $a'$.)
C. The extension of $a'b'$ intersects $ab$

We denote by $m$ the intersection point of $\ell'$ and $ab$. After a suitable reflection and relabeling, we assume that $a'b'$ lies below $\ell$, $a'$ is farther from $m$ than $b'$, and $\angle \triangle ma \leq \frac{\pi}{2}$, as in Figure 10. Depending on visibility regions of $Q_1$ and $Q_2$ we consider four cases.

1. $V(Q_1)$ lies below $ab$ and $V(Q_2)$ lies below $a'b'$ as in Figure 10(a). In this case $a' \leftrightarrow b$.
2. $V(Q_1)$ lies above $ab$ and $V(Q_2)$ lies above $a'b'$. If $c$ is to the left of $\ell_{c'}$ then so is $d$, as in Figure 10(b). In this case $d$ sees both $a'$ and $b'$, and at least one of $a'$ and $b'$ sees $d$, and thus $d \leftrightarrow a'$ or $d \leftrightarrow b'$. Assume that $c$ is to the right of $\ell_{c'}$. If $c$ is above $\ell'$ then $c \leftrightarrow a'$. Thus, assume that $c$ is below $\ell'$ as in Figure 10(c). Recall that $d'$ is in slab $S(a', b')$. If $d'$ is above the horizontal line through $c$ then $d' \leftrightarrow b$, otherwise $d' \leftrightarrow c$.

3. $V(Q_1)$ lies above $ab$ and $V(Q_2)$ lies below $a'b'$. This case is depicted in Figure 11. If $c$ is below $\ell'$ then $c \leftrightarrow a'$. Assume that $c$ is above $\ell'$. If $d'$ is to the left of $\ell_{c'}$ then $c \leftrightarrow d'$ as in Figure 11(a). Assume that $d'$ is to the right of $\ell_{c'}$ (and hence to the right of $\ell_{d'}$). Now we look at $d$ with respect to $\ell_{d'}$. If $d$ is above $\ell_{d'}$, then $d \leftrightarrow d'$. If $d$ is below $\ell_{d'}$, then it is also below $\ell_{d'}$, and thus $d \leftrightarrow c'$ as in Figure 11(b).

4. $V(Q_1)$ lies below $ab$ and $V(Q_2)$ lies above $a'b'$. This case is depicted in Figure 12. If $d'$ is below $\ell$ then $d' \leftrightarrow b$. Assume that $d'$ is above $\ell$. If $a$ is below $\ell$, then $a \leftrightarrow b'$. Assume that $a$ is above $\ell$, then $a \leftrightarrow d'$. Assume that $c$ is above $\ell_{c'}$. If $c$ is below $\ell_{c'}$ (which is also below $\ell_{c'}$). Notice that $c'$ lies in the slab bounded by $\ell_{c'}$ and $\ell_{c''}$. If $c'$ is to the left
of \( \ell_c \) then \( \ell' \leftrightarrow c \). Assume that \( \ell' \) is to the right of \( \ell_c \). Notice that \( d \) lies in the vertical slab bounded by \( \ell_a \) and \( \ell_c \). Let \( \ell_1 \) be the line through \( \ell' \) parallel to \( \ell_c \). If \( d \) is below \( \ell_1 \) then \( d \leftrightarrow c' \). Assume that \( d \) is above \( \ell_1 \). This configuration is depicted in Figure 12 (the caption of this figure summarizes the constraints). This is the configuration for which statement (i) of the theorem does not hold; for all other configurations statement (i) holds. We will show that statement (ii) holds in the current setting.

First, we extract a property of the current setting which is used in Remark 5. See Figure 12 for a better understanding of this property, and notice that in the current setting the points \( b, c \) lie on different sides of \( \ell'_y \), and the points \( a', d' \) lie on different sides of \( \ell' \).

**Property 1.** If statement (i) in Theorem 3 does not hold then then the points \( b, c \) or the points \( a, d \) of \( Q_1 \) lie on different sides of a line through boundary rays of wedges of \( Q_2 \), and similarly the points \( b', c' \) or the points \( a', d' \) of \( Q_2 \) lie on different sides of a line through boundary rays of wedges of \( Q_1 \).

To verify that statement (ii) holds in the current setting, let \( p \) be any point in the region \( S(Q_1) \cup S(Q_2) \). We show how to place a wedge of angle \( \frac{\pi}{2} \) at \( p \) so that \( p \) is mutually visible from a point in \( Q_1 \) and a point in \( Q_2 \). To simplify our description we partition \( S(Q_1) \cup S(Q_2) \) into eight regions \( R_1, \ldots, R_8 \) as in Figure 13. If \( p \in R_1 \) then we orient \( p \) similar to \( d' \), and thus \( p \leftrightarrow b \) and \( p \leftrightarrow b' \). If \( p \in R_2 \) then we orient \( p \) similar to \( a \), and thus \( p \leftrightarrow c \) and \( p \leftrightarrow b' \). If \( p \in R_3 \) then we orient it similar to \( c' \) so that \( p \leftrightarrow c \) and \( p \leftrightarrow a' \). If \( p \in R_4 \) then we orient it similar to \( b \) so that \( p \leftrightarrow d \) and \( p \leftrightarrow a' \). If \( p \in R_5 \) then we orient it similar to \( b' \), and hence \( p \leftrightarrow d \) and \( p \leftrightarrow d' \). If \( p \in R_6 \) then we orient it similar to \( c \), and thus \( p \leftrightarrow a \) and \( p \leftrightarrow d' \). If \( p \in R_7 \) then we orient it similar to \( a' \) so that \( p \leftrightarrow a \) and \( p \leftrightarrow c' \). Finally if \( p \in R_8 \) then we orient it similar to \( d \), and hence \( p \leftrightarrow b \) and \( p \leftrightarrow c' \). Thus statement (ii) of the theorem holds.
D. $a'b'$ is parallel to $ab$

Assume that $ab$ and $a'b'$ are horizontal, and $ab$ lies above $a'b'$. Consider any horizontal line $h$ between $ab$ and $a'b'$. One pair of points from $Q_1$ (either $(a, b)$ or $(c, d)$) covers the half plane below $h$. Also, one pair of points from $Q_2$ (either $(a', b')$ or $(c', d')$) covers the half plane above $h$. One can simply verify that there is an edge between these two pairs in the induced mutual visibility graph.

This is the end of our proof of Theorem 3.

4 Conclusions

The obvious open problem is to improve our approximation ratio 10 which we think is not the best possible ratio. The use of a Hamiltonian path is a bottleneck towards our analysis as it forces a factor of 2 in the ratio. It might be possible to get better ratios by using the original MST instead of the path. Perhaps the MST may not be the best lower bound either because one may obtain a better ratio by considering the $\frac{\pi}{2}$-MST as a lower bound.
A 10-Approximation of the $\frac{7}{2}$-MST

References


