Characterizing Omega-Regularity Through Finite-Memory Determinacy of Games on Infinite Graphs

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Abstract

We consider zero-sum games on infinite graphs, with objectives specified as sets of infinite words over some alphabet of *colors*. A well-studied class of objectives is the one of *ω-regular objectives*, due to its relation to many natural problems in theoretical computer science. We focus on the strategy complexity question: given an objective, how much memory does each player require to play as well as possible? A classical result is that finite-memory strategies suffice for both players when the objective is ω-regular. We show a reciprocal of that statement: when both players can play optimally with a *chromatic* finite-memory structure (i.e., whose updates can only observe colors) in all infinite game graphs, then the objective must be ω-regular. This provides a game-theoretic characterization of ω-regular objectives, and this characterization can help in obtaining memory bounds. Moreover, a by-product of our characterization is a new *one-to-two-player lift*: to show that chromatic finite-memory structures suffice to play optimally in two-player games on infinite graphs, it suffices to show it in the simpler case of one-player games on infinite graphs. We illustrate our results with the family of discounted-sum objectives, for which ω-regularity depends on the value of some parameters.

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1 Introduction

Games on graphs and synthesis. We study *zero-sum turn-based games on infinite graphs*. In such games, two players, $\mathcal{P}_1$ and $\mathcal{P}_2$, interact for an infinite duration on a graph, called an *arena*, whose state space is partitioned into states controlled by $\mathcal{P}_1$ and states controlled by $\mathcal{P}_2$. The game starts in some state of the arena, and the player controlling the current state may choose the next state following an edge of the arena. Moves of the players in the game are prescribed by their *strategy*, which can use information about the past of the play. Edges

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of the arena are labeled with a (possibly infinite) alphabet of *colors*, and the interaction of the players in the arena generates an *infinite word* over this alphabet of colors. These infinite words can be used to specify the players' objectives: a *winning condition* is a set of infinite words, and $\mathcal{P}_1$ wins a game on a graph if the infinite word generated by its interaction with $\mathcal{P}_2$ on the arena belongs to this winning condition – otherwise, $\mathcal{P}_2$ wins.

This game-theoretic model has applications to the *reactive synthesis* problem [4]: a system (modeled as $\mathcal{P}_1$) wants to guarantee some specification (the winning condition) against an uncontrollable environment (modeled as $\mathcal{P}_2$). Finding a *winning strategy* in the game for $\mathcal{P}_1$ corresponds to building a controller for the system that achieves the specification against all possible behaviors of the environment.

Strategy complexity. We are interested in the *strategy complexity* question: given a winning condition, how *complex* must winning strategies be, and how *simple* can they be? We are interested in establishing the sufficient and necessary amount of memory to play *optimally*. We consider in this work that an *optimal strategy* in an arena must be winning from any state from which winning is possible (a property sometimes called *uniformity* in the literature). The amount of memory relates to how much information about the past is needed to play in an optimal way. With regard to reactive synthesis, this has an impact in practice on the resources required for an optimal controller.

Three classes of strategies are often distinguished, depending on the number of states of memory they use: memoryless, finite-memory, and infinite-memory strategies. A notable subclass of finite-memory strategies is the class of strategies that can be implemented with finite-memory structures that only observe the sequences of colors (and not the sequences of states nor edges). Such memory structures are called *chromatic* [30]. By contrast, finite-memory structures that have access to the states and edges of arenas are called *general*. Chromatic memory structures are syntactically less powerful and may require more states than general ones [11], but they have the benefit that they can be defined independently of arenas.

We seek to characterize the winning conditions for which chromatic-finite-memory strategies suffice to play optimally against arbitrarily complex strategies, for both players, in all finite and infinite arenas. We call this property *chromatic-finite-memory determinacy*. This property generalizes *memoryless determinacy*, which describes winning conditions for which memoryless strategies suffice to play optimally for both players in all arenas. Our work follows a line of research [6, 8] giving various characterizations of chromatic-finite-memory determinacy for games on *finite* arenas (see Remark 2 for more details).

ω-regular languages. A class of winning conditions commonly arising as natural specifications for reactive systems (it encompasses, e.g., linear temporal logic specifications [38]) consists of the *ω-regular languages*. They are, among other characterizations, the languages of infinite words that can be described by a *finite parity automaton* [36]. It is known that all ω-regular languages are chromatic-finite-memory determined, which is due to the facts that an ω-regular language is expressible with a parity automaton, and that *parity conditions* admit memoryless optimal strategies [27, 42]. Multiple works study the strategy complexity of ω-regular languages, giving, e.g., precise general memory requirements for all Muller conditions [18] or a characterization of the chromatic memory requirements of Muller conditions [11, Theorem 28].

A result in the other direction is given by Colcombet and Niwiński [17]: they showed that if a *prefix-independent* winning condition is memoryless-determined in infinite arenas, then this winning condition must be a parity condition. As parity conditions are memoryless-determined, this provides an elegant characterization of parity conditions from a strategic perspective, under prefix-independence assumption.

Congruence. A well-known tool to study a language L of finite (resp. infinite) words is its *right congruence relation* $\sim_L$: for two finite words w_1 and w_2, we write $w_1 \sim_L w_2$ if for all finite (resp. infinite) words w, $w_1 w \in L$ if and only if $w_2 w \in L$. There is a natural deterministic (potentially infinite) automaton recognizing the equivalence classes of the right congruence, called the *minimal-state automaton of* $\sim_L$ [41, 35].

The relation between a regular language of *finite* words and its right congruence is given by the Myhill-Nerode theorem [37], which provides a natural bijection between the states of the minimal deterministic automaton recognizing a regular language and the equivalence classes of its right congruence relation. Consequences of this theorem are that a language is regular if and only if its right congruence has finitely many equivalence classes, and a regular language can be recognized by the minimal-state automaton of its right congruence.

For the theory of languages of *infinite* words, the situation is not so simple: ω-regular languages have a right congruence with finitely many equivalence classes, but having finitely many equivalence classes does not guarantee ω-regularity (for example, a language is *prefix-independent* if and only if its right congruence has exactly one equivalence class, but this does not imply ω-regularity). Moreover, ω-regular languages cannot necessarily be recognized by adding a natural acceptance condition (parity, Rabin, Muller. . .) to the minimal-state automaton of their right congruence [1]. There has been multiple works about the links between a language of infinite words and the minimal-state automaton of its right congruence; one relevant question is to understand when a language can be recognized by this minimal-state automaton [41, 35, 1].

Contributions. We characterize the ω-regularity of a language of infinite words W through the strategy complexity of the zero-sum turn-based games on infinite graphs with winning condition W: the ω-regular languages are *exactly* the chromatic-finite-memory determined languages (seen as winning conditions) (Theorem 9). As discussed earlier, it is well-known that ω-regular languages admit chromatic-finite-memory optimal strategies [36, 42, 11] – our results yield the other implication. This therefore provides a characterization of ω-regular languages through a game-theoretic and strategic lens.

Our technical arguments consist in providing a precise connection between the representation of W as a parity automaton and a chromatic memory structure sufficient to play optimally. If strategies based on a chromatic finite-memory structure are sufficient to play optimally for both players, then W is recognized by a parity automaton built on top of the direct product of the *minimal-state automaton of the right congruence* and this *chromatic memory structure* (Theorem 8). This result generalizes the work from Colcombet and Niwiński [17] in two ways: by relaxing the prefix-independence assumption about the winning condition, and by generalizing the class of strategies considered from memoryless to chromatic-finite-memory strategies. We recover their result as a special case.

Moreover, we actually show that chromatic-finite-memory determinacy in *one-player* games of both players is sufficient to show ω-regularity of a language. As ω-regular languages are chromatic-finite-memory determined in two-player games, we can reduce the problem of chromatic-finite-memory determinacy of a winning condition in *two-player* games to the easier chromatic-finite-memory determinacy in *one-player* games (Theorem 10). Such a *one-to-two-player lift* holds in multiple classes of zero-sum games, such as deterministic games on finite arenas [23, 6, 31] and stochastic games on finite arenas [24, 8]. The proofs for finite arenas all rely on an *edge-induction technique* (also used in other works about strategy complexity in finite arenas [28, 21, 13]) that appears unfit to deal with infinite arenas. Although not mentioned by Colcombet and Niwiński, it was already noticed [30] that for prefix-independent winning conditions in games on infinite graphs, a one-to-two-player lift for *memoryless* determinacy follows from [17].

Related works. We have already mentioned [18, 42, 17, 29, 11] for fundamental results on the memory requirements of ω-regular conditions, [23, 24, 6, 8] for characterizations of "low" memory requirements in finite (deterministic and stochastic) arenas, and [41, 35, 1] for links between an ω-regular language and the minimal-state automaton of its right congruence.

One stance of our work is that our assumptions about strategy complexity affect *both* players. Another intriguing question is to understand when the memory requirements of only *one* player are finite. In finite arenas, results in this direction are sufficient conditions for the existence of memoryless optimal strategies for one player [28, 3], and a procedure to compute the chromatic memory requirements of prefix-independent ω-regular conditions [29, 30].

Other articles study the strategy complexity of (non-necessarily ω-regular) winning conditions in infinite arenas; see, e.g., [20, 25, 16]. In such non-ω-regular examples, as can be expected given our main result, at least one player needs infinite memory to play optimally, or the arena model is different from ours (e.g., only allowing finite branching – we discuss such differences in more depth after Theorem 8). A particularly interesting example w.r.t. our results is considered by Chatterjee and Fijalkow [15]. They study the strategy complexity of *finitary Büchi and parity conditions*, and show that $\mathcal{P}_1$ has chromatic-finite-memory optimal strategies for finitary Büchi and finitary parity. However, for these (non-ω-regular) winning conditions, $\mathcal{P}_2$ needs infinite memory. This example illustrates that our main result would not hold if we just focused on the strategy complexity of one player.

We mention works on finite-memory determinacy in different contexts: finite arenas [34], non-zero-sum games [33], countable one-player stochastic games [26], concurrent games [32, 7].

Structure. We fix definitions in Section 2. Our main results are discussed in Section 3. We apply our results to discounted-sum and mean-payoff winning conditions in Section 4. Due to a lack of space, we only sketch some technical details; the complete proofs as well as additional examples and remarks are found in the full version of the article [9].

2 Preliminaries

Let C be an arbitrary non-empty set of *colors*. Given a set A, we write A^* for the set of finite sequences of elements of A and A^ω for the set of infinite sequences of elements of A.

Arenas. We consider two players $\mathcal{P}_1$ and $\mathcal{P}_2$. An arena is a tuple $\mathcal{A} = (S, S_1, S_2, E)$ such that $S = S_1 \uplus S_2$ (disjoint union) is a non-empty set of *states* (of any cardinality) and $E \subseteq S \times C \times S$ is a set of *edges*. States in S_1 are controlled by $\mathcal{P}_1$ and states in S_2 are controlled by $\mathcal{P}_2$. We allow arenas with infinite branching. Given $e \in E$, we denote by in, col, and out the projections to its first, second, and third component, respectively (i.e., $e = (\mathsf{in}(e), \mathsf{col}(e), \mathsf{out}(e))$). We assume arenas to be *non-blocking*: for all $s \in S$, there exists $e \in E$ such that $\mathsf{in}(e) = s$.

Let $\mathcal{A} = (S, S_1, S_2, E)$ be an arena with $s \in S$. We denote by $\mathsf{Plays}(\mathcal{A}, s)$ the set of *plays of* $\mathcal{A}$ *from* s, that is, infinite sequences of edges $\rho = e_1e_2 \ldots \in E^\omega$ such that $\mathsf{in}(e_1) = s$ and for all $i \geq 1$, $\mathsf{out}(e_i) = \mathsf{in}(e_{i+1})$. For $\rho = e_1e_2 \ldots \in \mathsf{Plays}(\mathcal{A}, s)$, we write $\mathsf{col}^\omega(\rho)$ for the infinite sequence $\mathsf{col}(e_1)\mathsf{col}(e_2)\ldots \in C^\omega$. We denote by $\mathsf{Hists}(\mathcal{A}, s)$ the set of *histories of* $\mathcal{A}$ *from* s, which are all finite prefixes of plays of $\mathcal{A}$ from s. We write $\mathsf{Plays}(\mathcal{A})$ and $\mathsf{Hists}(\mathcal{A})$ for the sets of all plays of $\mathcal{A}$ and all histories of $\mathcal{A}$ (from any state), respectively. If $h = e_1 \ldots e_k$ is a history of $\mathcal{A}$, we define $\mathsf{in}(h) = \mathsf{in}(e_1)$ and $\mathsf{out}(h) = \mathsf{out}(e_k)$. For convenience, for every $s \in S$, we also consider the *empty history* λ_s from s, and we set $\mathsf{in}(\lambda_s) = \mathsf{out}(\lambda_s) = s$. For $i \in \{1, 2\}$, we denote by $\mathsf{Hists}_i(\mathcal{A})$ the set of histories h such that $\mathsf{out}(h) \in S_i$. An arena $\mathcal{A} = (S, S_1, S_2, E)$ is a *one-player arena of* $\mathcal{P}_1$ (resp. *of* $\mathcal{P}_2$) if $S_2 = \emptyset$ (resp. $S_1 = \emptyset$).

Skeletons. A *skeleton* is a tuple $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ such that M is a finite set of *states*, $m_{\mathsf{init}} \in M$ is an *initial state*, and $\alpha_{\mathsf{upd}} \colon M \times C \to M$ is an *update function.* We denote by α^*_{upd} the natural extension of α_{upd} to finite sequences of colors. We always assume that all states of skeletons are reachable from their initial state. We define the trivial skeleton $\mathcal{M}_{\mathsf{triv}}$ as the only skeleton with a single state. Although we require skeletons to have finitely many states, we allow them to have infinitely many transitions (which happens when C is infinite).

We say that a non-empty sequence $\pi = (m_1, c_1) \ldots (m_k, c_k) \in (M \times C)^+$ is a *path of* $\mathcal{M}$ *(from* m_1 *to* $\alpha_{\mathsf{upd}}(m_k, c_k)$*)* if for all $i \in \{1, \ldots, k-1\}$, $\alpha_{\mathsf{upd}}(m_i, c_i) = m_{i+1}$. For convenience, we also consider every element $(m, \bot)$ for $m \in M$ and $\bot \notin C$ to be an *empty path of* $\mathcal{M}$ *(from* m *to* m*)*. A non-empty path of $\mathcal{M}$ from m to m' is a *cycle of* $\mathcal{M}$ *(on* m*)* if $m = m'$. Cycles of $\mathcal{M}$ are usually denoted by letter γ. For $\pi = (m_1, c_1) \ldots (m_k, c_k)$ a path of $\mathcal{M}$, we define $\mathsf{col}^*(\pi)$ to be the sequence $c_1 \ldots c_k \in C^*$. For an infinite sequence $(m_1, c_1)(m_2, c_2) \ldots \in (M \times C)^\omega$, we also write $\mathsf{col}^\omega((m_1, c_1)(m_2, c_2) \ldots)$ for the infinite sequence $c_1c_2 \ldots \in C^\omega$.

For $m, m' \in M$, we write $\Pi_{m,m'}$ for the set of paths of $\mathcal{M}$ from m to m', Γ_m for the set of cycles of $\mathcal{M}$ on m, and $\Gamma_{\mathcal{M}}$ for the set of all cycles of $\mathcal{M}$ (on any skeleton state). When considering sets of paths or cycles of $\mathcal{M}$, we add a c in front of the set to denote the projections of the corresponding paths or cycles to colors (e.g., $\mathsf{c}\Gamma_{\mathcal{M}} = \{\mathsf{col}^*(\gamma) \in C^+ \mid \gamma \in \Gamma_{\mathcal{M}}\}$).

For $w = c_1c_2 \ldots \in C^\omega$, we define $\mathsf{skel}(w)$ as the infinite sequence $(m_1, c_1)(m_2, c_2) \ldots \in (M \times C)^\omega$ that w induces in the skeleton ($m_1 = m_{\mathsf{init}}$ and for all $i \geq 1$, $\alpha_{\mathsf{upd}}(m_i, c_i) = m_{i+1}$).

Let $\mathcal{M}_1 = (M_1, m^1_{\mathsf{init}}, \alpha^1_{\mathsf{upd}})$ and $\mathcal{M}_2 = (M_2, m^2_{\mathsf{init}}, \alpha^2_{\mathsf{upd}})$ be two skeletons. Their *(direct) product* $\mathcal{M}_1 \otimes \mathcal{M}_2$ is the skeleton $(M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ where $M = M_1 \times M_2$, $m_{\mathsf{init}} = (m^1_{\mathsf{init}}, m^2_{\mathsf{init}})$, and, for all $m_1 \in M_1$, $m_2 \in M_2$, $c \in C$, $\alpha_{\mathsf{upd}}((m_1, m_2), c) = (\alpha^1_{\mathsf{upd}}(m_1, c), \alpha^2_{\mathsf{upd}}(m_2, c))$.

Strategies. Let $\mathcal{A} = (S, S_1, S_2, E)$ be an arena and $i \in \{1, 2\}$. A *strategy of* $\mathcal{P}_i$ *on* $\mathcal{A}$ is a function $\sigma_i \colon \mathsf{Hists}_i(\mathcal{A}) \to E$ such that for all $h \in \mathsf{Hists}_i(\mathcal{A})$, $\mathsf{out}(h) = \mathsf{in}(\sigma_i(h))$. We denote by $\Sigma_i(\mathcal{A})$ the set of strategies of $\mathcal{P}_i$ on $\mathcal{A}$. Given a strategy σ_i of $\mathcal{P}_i$, we say that a play ρ is *consistent with* σ_i if for all finite prefixes $h = e_1 \ldots e_i$ of ρ such that $\mathsf{out}(h) \in S_i$, $\sigma_i(h) = e_{i+1}$. For $s \in S$, we denote by $\mathsf{Plays}(\mathcal{A}, s, \sigma_i)$ the set of plays from s that are consistent with σ_i.

For $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ a skeleton, a strategy $\sigma_i \in \Sigma_i(\mathcal{A})$ is *based on (memory)* $\mathcal{M}$ if there exists a function $\alpha_{\mathsf{nxt}} \colon S \times M \to E$ such that for all $s \in S_i$, $\sigma_i(\lambda_s) = \alpha_{\mathsf{nxt}}(s, m_{\mathsf{init}})$, and for all non-empty paths $h \in \mathsf{Hists}_i(\mathcal{A})$, $\sigma_i(h) = \alpha_{\mathsf{nxt}}(\mathsf{out}(h), \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, \mathsf{col}^*(h)))$. A strategy is *memoryless* if it is based on $\mathcal{M}_{\mathsf{triv}}$.

▶ Remark 1. Our memory model is *chromatic* [30], i.e., it observes the sequences of colors and not the sequences of edges of arenas, since the argument of the update function of a skeleton is in $M \times C$. It was recently shown that the amount of memory states required to play optimally for a winning condition using chromatic skeletons may be strictly larger than using *general* memory structures (i.e., using memory structures observing edges) [11, Proposition 32]. The example provided is a Muller condition (hence an ω-regular condition), in which both kinds of memory requirements are still finite. A result in this direction is also provided by Le Roux [32] for games on *finite* arenas: it shows that in many games, a strategy using general finite memory can be swapped for a (larger) chromatic finite memory.

For games on infinite arenas, which we consider in this article, we do not know whether there exists a winning condition with *finite* general memory requirements, but *infinite* chromatic memory requirements. Our results focus on chromatic memory requirements. ⌟

Winning conditions. A *(winning) condition* is a set $W \subseteq C^\omega$. When a condition W is clear in the context, we say that an infinite word $w \in C^\omega$ is winning if $w \in W$, and losing if not. For a condition W and a word $w \in C^*$, we write $w^{-1}W = \{w' \in C^\omega \mid ww' \in W\}$ for the set of *winning continuations of* w. We write $\overline{W}$ for the complement $C^\omega \setminus W$ of a condition W.

A *game* is a tuple $\mathcal{G} = (\mathcal{A}, W)$ where $\mathcal{A}$ is an arena and W is a winning condition.

Optimality and determinacy. Let $\mathcal{G} = (\mathcal{A} = (S, S_1, S_2, E), W)$ be a game, and $s \in S$. We say that $\sigma_1 \in \Sigma_1(\mathcal{A})$ *is winning from* s if $\mathsf{col}^\omega(\mathsf{Plays}(\mathcal{A}, s, \sigma_1)) \subseteq W$, and we say that $\sigma_2 \in \Sigma_2(\mathcal{A})$ *is winning from* s if $\mathsf{col}^\omega(\mathsf{Plays}(\mathcal{A}, s, \sigma_2)) \subseteq \overline{W}$.

A strategy of $\mathcal{P}_i$ is *optimal in* $(\mathcal{A}, W)$ if it is winning from all the states from which $\mathcal{P}_i$ has a winning strategy. We often write *optimal in* $\mathcal{A}$ if condition W is clear from the context. We stress that this notion of optimality requires a *single* strategy to be winning from *all* the winning states (a property sometimes called *uniformity*).

A winning condition W is *determined* if for all games $\mathcal{G} = (\mathcal{A} = (S, S_1, S_2, E), W)$, for all $s \in S$, either $\mathcal{P}_1$ or $\mathcal{P}_2$ has a winning strategy from s. Let $\mathcal{M}$ be a skeleton. We say that a winning condition W is $\mathcal{M}$*-determined* if (*i*) W is determined and (*ii*) in all arenas $\mathcal{A}$, both players have an optimal strategy based on $\mathcal{M}$. A winning condition W is *one-player* $\mathcal{M}$*-determined* if in all one-player arenas $\mathcal{A}$ of $\mathcal{P}_1$, $\mathcal{P}_1$ has an optimal strategy based on $\mathcal{M}$ *and* in all one-player arenas $\mathcal{A}$ of $\mathcal{P}_2$, $\mathcal{P}_2$ has an optimal strategy based on $\mathcal{M}$. A winning condition W is (one-player) *memoryless-determined* if it is (one-player) $\mathcal{M}_{\mathsf{triv}}$-determined. A winning condition W is *(one-player) chromatic-finite-memory determined* if there exists a skeleton $\mathcal{M}$ such that it is (one-player) $\mathcal{M}$-determined.

▶ Remark 2. It might seem surprising that for chromatic-finite-memory determinacy, we require the existence of a *single* skeleton that suffices to play optimally in *all* arenas, rather than the seemingly weaker existence, for each arena, of a finite skeleton (which may depend on the arena) that suffices to play optimally. In infinite arenas, it turns out that these notions are equivalent (proof in [9]).

▶ Lemma 3. *Let* $W \subseteq C^\omega$ *be a winning condition. The following are equivalent:*
1. *for all arenas* $\mathcal{A}$*, there exists a skeleton* $\mathcal{M}^\mathcal{A}$ *such that both players have an optimal strategy based on* $\mathcal{M}^\mathcal{A}$ *in* $\mathcal{A}$*;*
2. W *is chromatic-finite-memory determined.*

When restricted to finite arenas, we do not have an equivalence between these two notions (hence the distinction between finite-memory determinacy and *arena-independent* finite-memory determinacy [6, 8]). Our proof of Lemma 3 exploits that an infinite "union" of arenas is still an arena, which is not true when restricted to finite arenas. ⌟

ω-regular languages. We define a *parity automaton* as a pair $(\mathcal{M}, p)$ where $\mathcal{M}$ is a skeleton and $p\colon M \times C \to \{0, \ldots, n\}$; function p assigns *priorities* to every transition of $\mathcal{M}$. This definition implies that we consider deterministic and complete parity automata (i.e., in every state, reading a color leads to exactly one state). Following [12], if $\mathcal{M}$ is a skeleton, we say that a parity automaton $(\mathcal{M}', p)$ is *defined on top of* $\mathcal{M}$ if $\mathcal{M}' = \mathcal{M}$.

A parity automaton $(\mathcal{M}, p)$ defines a language $L_{(\mathcal{M},p)}$ of all the infinite words $w \in C^\omega$ such that, for $\mathsf{skel}(w) = (m_1, c_1)(m_2, c_2)\ldots$, $\limsup_{i\geq 1} p(m_i, c_i)$ is even. We say that $W \subseteq C^\omega$ is *recognized by* $(\mathcal{M}, p)$ if $W = L_{(\mathcal{M},p)}$. A language of infinite words is *ω-regular* if it is recognized by a parity automaton. We emphasize that we consider *transition-based* parity conditions: we assign priorities to transitions (and not states) of $\mathcal{M}$. For more information on links between state-based and transition-based acceptance conditions, we refer to [11].

Right congruence. For $\sim$ an equivalence relation, we call the *index of* $\sim$ the number of equivalence classes of $\sim$. We denote by $[a]_\sim$ the equivalence class of an element a for $\sim$.

Let W be a winning condition. We define the *right congruence* $\sim_W \subseteq C^* \times C^*$ *of* W as $w_1 \sim_W w_2$ if $w_1^{-1}W = w_2^{-1}W$ (meaning that w_1 and w_2 have the same winning continuations). Relation $\sim_W$ is an equivalence relation. When W is clear from the context, we write $\sim$ for $\sim_W$. We denote by ε the empty word. When $\sim$ has finite index, we can associate a natural skeleton $\mathcal{M}_\sim = (M_\sim, m^\sim_{\mathsf{init}}, \alpha^\sim_{\mathsf{upd}})$ to $\sim$ such that $M_\sim$ is the set of equivalence classes of $\sim$, $m^\sim_{\mathsf{init}} = [\varepsilon]_\sim$, and $\alpha^\sim_{\mathsf{upd}}([w]_\sim, c) = [wc]_\sim$. This transition function is well-defined since it follows from the definition of $\sim$ that if $w_1 \sim w_2$, then for all $c \in C$, $w_1c \sim w_2c$. Hence, the choice of representatives for the equivalence classes does not have an impact in this definition. We call skeleton $\mathcal{M}_\sim$ the *minimal-state automaton of* $\sim$ [41, 35].

3 Concepts and characterization

We define two concepts at the core of our characterization, one of them dealing with *prefixes* and the other one dealing with *cycles*. Let $W \subseteq C^\omega$ be a winning condition and $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ be a skeleton.

Prefix-independence. Let $\sim$ be the right congruence of W.

▶ **Definition 4.** *Condition W is $\mathcal{M}$-*prefix-independent *if for all $m \in M$, for all $w_1, w_2 \in$ $\mathsf{c}\Pi_{m_{\mathsf{init}},m}$, $w_1 \sim w_2$.*

In other words, W is $\mathcal{M}$-prefix-independent if finite words reaching the same state of $\mathcal{M}$ from its initial state have the same winning continuations. The classical notion of *prefix-independence* is equivalent to $\mathcal{M}_{\mathsf{triv}}$-prefix-independence (as all finite words have the exact same set of winning continuations, which is W). If $\sim$ has finite index, W is in particular $\mathcal{M}_\sim$-prefix-independent: indeed, two finite words reach the same state of $\mathcal{M}_\sim$ (if and) only if they are equivalent for $\sim$. Any skeleton $\mathcal{M}$ such that W is $\mathcal{M}$-prefix-independent must have at least one state for each equivalence class of $\sim$, but multiple states may partition the same equivalence class.

Cycle-consistency. For $w \in C^*$, we define

$$\Gamma^{\mathsf{win},w}_{\mathcal{M}} = \{\gamma \in \Gamma_m \mid m = \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, w) \text{ and } (\mathsf{col}^*(\gamma))^\omega \in w^{-1}W\}$$

as the cycles on the skeleton state reached by w in $\mathcal{M}$ that induce winning words when repeated infinitely many times after w. We define

$$\Gamma^{\mathsf{lose},w}_{\mathcal{M}} = \{\gamma \in \Gamma_m \mid m = \alpha^*_{\mathsf{upd}}(m_{\mathsf{init}}, w) \text{ and } (\mathsf{col}^*(\gamma))^\omega \in w^{-1}\overline{W}\}$$

as their losing counterparts. We emphasize that cycles are allowed to go through the same edge multiple times.

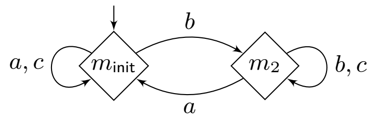

Figure 1 Skeleton $\mathcal{M}$ such that $W = \mathsf{Büchi}(a) \cap \mathsf{Büchi}(b)$ is $\mathcal{M}$-cycle-consistent (Example 6). In figures, we use rhombuses (resp. circles, squares) to depict skeleton states (resp. arena states controlled by $\mathcal{P}_1$, arena states controlled by $\mathcal{P}_2$).

▶ **Definition 5.** *Condition W is $\mathcal{M}$-*cycle-consistent *if for all $w \in C^*$, $(\mathsf{c}\Gamma_{\mathcal{M}}^{\mathsf{win},w})^\omega \subseteq w^{-1}W$ and $(\mathsf{c}\Gamma_{\mathcal{M}}^{\mathsf{lose},w})^\omega \subseteq w^{-1}\overline{W}$.*

What this says is that after any finite word, if we concatenate infinitely many winning (resp. losing) cycles on the skeleton state reached by that word, then it only produces winning (resp. losing) infinite words.

▶ **Example 6.** For $c' \in C$, let $\mathsf{Büchi}(c')$ be the set of infinite words on C that see color c' infinitely often. Let $C = \{a, b, c\}$. Condition $W = \mathsf{Büchi}(a) \cap \mathsf{Büchi}(b)$ is $\mathcal{M}_{\mathsf{triv}}$-prefix-independent, but not $\mathcal{M}_{\mathsf{triv}}$-cycle-consistent: for any $w \in C^*$, a and b are both in $\mathsf{c}\Gamma_{\mathcal{M}_{\mathsf{triv}}}^{\mathsf{lose},w}$ (as wa^ω and wb^ω are losing), but word $w(ab)^\omega$ is winning. However, W is $\mathcal{M}$-cycle-consistent for the skeleton $\mathcal{M}$ with two states m_{init} and m_2 represented in Figure 1. For finite words reaching m_{init}, the losing cycles only see a and c, and combining infinitely many of them gives an infinite word without b, which is a losing continuation of any finite word. The winning cycles are the ones that go to m_2 and then go back to m_{init}, as they must see both a and b; combining infinitely many of them guarantees a winning continuation after any finite word. A similar reasoning applies to state m_2. Notice that W is also $\mathcal{M}$-prefix-independent. With regard to memory requirements, condition W is not $\mathcal{M}_{\mathsf{triv}}$-determined but is $\mathcal{M}$-determined. ⌟

Both $\mathcal{M}$-prefix-independence and $\mathcal{M}$-cycle-consistency hold symmetrically for a winning condition and its complement, and are stable by product with an arbitrary skeleton (as products generate even smaller sets of prefixes and cycles to consider).

▶ **Lemma 7.** *Let $W \subseteq C^\omega$ be a winning condition and $\mathcal{M}$ be a skeleton. Then, W is $\mathcal{M}$-prefix-independent (resp. $\mathcal{M}$-cycle-consistent) if and only if $\overline{W}$ is $\mathcal{M}$-prefix-independent (resp. $\mathcal{M}$-cycle-consistent). If W is $\mathcal{M}$-prefix-independent (resp. $\mathcal{M}$-cycle-consistent), then for all skeletons $\mathcal{M}'$, W is $(\mathcal{M} \otimes \mathcal{M}')$-prefix-independent (resp. $(\mathcal{M} \otimes \mathcal{M}')$-cycle-consistent).*

Moreover, an ω-regular language recognized by a parity automaton $(\mathcal{M}, p)$ is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent.

Main results. We state our main technical tool. We recall that *one-player $\mathcal{M}$-determinacy* of a winning condition W is both about one-player arenas of $\mathcal{P}_1$ (trying to achieve a word in W) *and* of $\mathcal{P}_2$ (trying to achieve a word in $\overline{W}$).

▶ **Theorem 8.** *Let $W \subseteq C^\omega$ be a winning condition and $\sim$ be its right congruence.*

1. *If there exists a skeleton $\mathcal{M}$ such that W is one-player $\mathcal{M}$-determined, then $\sim$ has finite index (in particular, W is $\mathcal{M}_\sim$-prefix-independent) and W is $\mathcal{M}$-cycle-consistent.*
2. *If there exists a skeleton $\mathcal{M}$ such that W is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent, then W is ω-regular and can be recognized by a deterministic parity automaton defined on top of $\mathcal{M}$.*

Technical sketch. We prove the first and second items of this theorem in [9, Sections 4 and 5]. We comment briefly on our proof technique for each item.

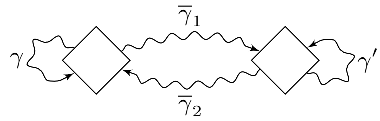

Figure 2 Comparing cycles γ and γ' using intermediate cycle $\overline{\gamma} = \overline{\gamma}_1\overline{\gamma}_2$. Squiggly arrows indicate a sequence of transitions. Cycles γ and $\gamma\overline{\gamma}_1\overline{\gamma}_2$ are winning, and cycles γ' and $\gamma'\overline{\gamma}_2\overline{\gamma}_1$ are losing.

1. For the first item, we assume that W is one-player $\mathcal{M}$-determined for a skeleton $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$. We define a preorder $\preceq$ on C^* such that $w_1 \preceq w_2$ if $w_1^{-1}W \subseteq w_2^{-1}W$. Notice that the right congruence $\sim$ of W is equal to $\preceq \cap \succeq$. By exhibiting well-chosen one-player arenas, using the $\mathcal{M}$-determinacy assumption, we can show that for each $m \in M$, in the set $\mathsf{c}\Pi_{m_{\mathsf{init}},m}$, relation $\preceq$ is total and there is no infinite increasing nor decreasing sequence (for $\preceq$). This shows that $\sim$ has finite index on each $\mathsf{c}\Pi_{m_{\mathsf{init}},m}$; as M is finite and $C^* = \bigcup_{m\in M} \mathsf{c}\Pi_{m_{\mathsf{init}},m}$, relation $\sim$ has finite index on C^*. The proof of $\mathcal{M}$-cycle-consistency is more direct: if a player had an interest in mixing multiple losing cycles of $\mathcal{M}$ to make them into a winning play, we could find a (possibly infinite) one-player arena of that player in which strategies based on $\mathcal{M}$ would not suffice to play optimally.

2. For the second item, we assume that W is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent for a skeleton $\mathcal{M}$. Our technical lemmas focus on *cycles* of $\mathcal{M}$, how they relate to each other, and what happens when we combine them. Our main tool is to define a partial preorder on cycles, which will help assign priorities to transitions of $\mathcal{M}$ – the aim being to define a parity condition on top of $\mathcal{M}$ that recognizes W. As we consider $\mathcal{M}$-prefix-independence along with $\mathcal{M}$-cycle-consistency, for m a state of $\mathcal{M}$, each cycle in Γ_m has a well-defined accepting status: it generates either a winning or a losing infinite word when repeated infinitely often after any finite word in $\mathsf{c}\Pi_{m_{\mathsf{init}},m}$.

Intuitively, for some state m of $\mathcal{M}$, for γ a winning cycle on m and γ' a losing cycle on m, we can look at which cycle *dominates* the other, that is, whether the combined cycle $\gamma\gamma'$ is winning, in which case γ dominates γ', or losing, in which case γ' dominates γ ($\gamma\gamma'$ and $\gamma'\gamma$ necessarily have the same accepting status). This shows how to compare cycles with different accepting statuses that start on the same skeleton state. This notion and some properties about this notion generalize part of the proof technique of [17], in which colors rather than cycles are compared.

We can extend this idea to some pairs of a winning cycle γ and a losing cycle γ' that have no state in common: our criterion to compare two such cycles is that there is a cycle $\overline{\gamma}$ connecting them such that $\overline{\gamma}$ is not "powerful enough" to alter the values of each cycle separately, that is, such that $\gamma\overline{\gamma}$ is winning and $\gamma'\overline{\gamma}$ is losing. To know which cycle dominates the other, we look at the accepting value of the cycle $\gamma\overline{\gamma}_1\gamma'\overline{\gamma}_2$, for some adequate break of $\overline{\gamma}$ into two paths $\overline{\gamma}_1$ and $\overline{\gamma}_2$. We illustrate the situation in Figure 2. If $\gamma\overline{\gamma}_1\gamma'\overline{\gamma}_2$ is winning, then γ dominates γ', and if it is losing, then γ' dominates γ.

This defines a partial preorder on cycles of $\mathcal{M}$. We show that there is no infinite decreasing nor increasing sequence for this preorder, and after defining a related equivalence relation, that there are finitely many equivalence classes of cycles. We can assign finitely many priorities to these cycles in a way consistent with the partial preorder, and then transfer these priorities to *transitions* of $\mathcal{M}$, as a function $p\colon M \times C \to \{0, \dots, n\}$. We conclude by showing that W is recognized by parity automaton $(\mathcal{M}, p)$. ◀

We state two consequences of Theorem 8: a strategic characterization of ω-regular languages, and a novel one-to-two-player-lift.

▶ **Theorem 9** (Characterization). *Let $W \subseteq C^\omega$ be a language of infinite words. Language W is ω-regular if and only if it is chromatic-finite-memory determined (in infinite arenas).*

Proof. One implication is well-known [36, 42]: if W is ω-regular, then it can be recognized by a deterministic parity automaton whose skeleton we can use as a memory that suffices to play optimally for both players, in arenas of any cardinality. The other direction is given by Theorem 8: if W is chromatic-finite-memory determined, then there exists in particular a skeleton $\mathcal{M}$ such that W is one-player $\mathcal{M}$-determined, so $\sim$ has finite index and W is $\mathcal{M}$-cycle-consistent. In particular, by Lemma 7, W is $(\mathcal{M}_\sim \otimes \mathcal{M})$-prefix-independent and $(\mathcal{M}_\sim \otimes \mathcal{M})$-cycle-consistent, so W is ω-regular and can be recognized by a deterministic parity automaton defined on top of $\mathcal{M}_\sim \otimes \mathcal{M}$. ◀

▶ **Theorem 10** (One-to-two-player lift). *Let $W \subseteq C^\omega$ be a winning condition. Language W is **one-player** chromatic-finite-memory determined if and only if it is chromatic-finite-memory determined.*

Proof. The implication from two-player to one-player arenas is trivial. The other implication is given by Theorem 8: if W is one-player $\mathcal{M}$-determined, then $\sim$ has finite index and W is $\mathcal{M}$-cycle-consistent. Again by Lemma 7 and Theorem 8, as W can be recognized by a parity automaton defined on top of $\mathcal{M}_\sim \otimes \mathcal{M}$, W is determined and strategies based on $\mathcal{M}_\sim \otimes \mathcal{M}$ suffice to play optimally in all two-player arenas. ◀

We discuss two specific situations in which we can easily derive interesting consequences using our results: the prefix-independent case, and the case where the minimal-state automaton suffices to play optimally.

Prefix-independent case. If a condition W is prefix-independent (i.e., $\sim$ has index 1 and $\mathcal{M}_\sim = \mathcal{M}_{\mathsf{triv}}$), and skeleton $\mathcal{M}$ suffices to play optimally in one-player games, then W is recognized by a parity automaton defined on top of $\mathcal{M}_{\mathsf{triv}} \otimes \mathcal{M}$, which is isomorphic to $\mathcal{M}$. This implies that the exact same memory can be used by both players to play optimally in two-player arenas, with no increase in memory. Note that we do not know in general whether this product is necessary to go from one-player to two-player arenas, but the question is automatically solved for prefix-independent conditions.

If, moreover, $\mathcal{M} = \mathcal{M}_{\mathsf{triv}}$ (i.e., memoryless strategies suffice to play optimally in one-player arenas), we recover exactly the result from Colcombet and Niwiński [17]: W can be recognized by a parity automaton defined on top of $\mathcal{M}_{\mathsf{triv}}$, so we can directly assign a priority to each color with a function $p\colon C \to \{0, \ldots, n\}$ such that an infinite word $w = c_1c_2\ldots \in C^\omega$ is in W if and only if $\limsup_{i\geq 1} p(c_i)$ is even.

$\mathcal{M}_\sim$-determined case. An interesting property of some ω-regular languages is that they can be recognized by defining an acceptance condition on top of the minimal-state automaton of their right congruence [35], which is a useful property for the learning of languages [1]. Here, Theorem 8 shows that W can be recognized by defining a transition-based parity acceptance condition on top of the minimal-state automaton $\mathcal{M}_\sim$ if and only if W is $\mathcal{M}_\sim$-determined. The transition-based parity acceptance condition was not considered in the cited results [35, 1].

▶ **Corollary 11.** *Let $W \subseteq C^\omega$ be an ω-regular language and $\mathcal{M}_\sim$ be the minimal-state automaton of its right congruence. The following are equivalent:*
1. *W is recognized by defining a transition-based parity acceptance condition on top of $\mathcal{M}_\sim$;*
2. *W is $\mathcal{M}_\sim$-determined;*
3. *W is $\mathcal{M}_\sim$-cycle-consistent.*

Proof. Implication 1. $\implies$ 2. follows from the memoryless determinacy of parity games [42]. Implication 2. $\implies$ 3. follows from the first item of Theorem 8. Implication 3. $\implies$ 1. follows from the second item of Theorem 8: by definition, W is $\mathcal{M}_\sim$-prefix-independent; if it is also $\mathcal{M}_\sim$-cycle-consistent, then W is recognized by a parity automaton defined on top of $\mathcal{M}_\sim$. ◀

Classes of arenas. We discuss the sensitivity of Theorem 8 w.r.t. our model of arenas.

There are multiple conditions that are chromatic-finite-memory determined if we only consider *finite arenas* (finitely many states and edges) and not infinite arenas. A few examples are discounted-sum games [40], mean-payoff games [19], total-payoff games [22], one-counter games [10] which are all memoryless-determined in finite arenas but which require infinite memory to play optimally in some infinite arenas (we discuss some of these in Section 4). In particular, Theorem 9 tells us that the derived winning conditions are not ω-regular.

Strangely, the fact that our arenas have colors on *edges* and not on *states* is crucial for the result. Indeed, there exists a winning condition (a generalization of a parity condition with infinitely many priorities [25]) that is memoryless-determined in state-labeled infinite arenas, but not in edge-labeled infinite arenas (as we consider here). This particularity was already discussed [17], and it was also shown that the same condition is memoryless-determined in edge-labeled arenas with finite branching. Therefore, the fact that we allow *infinite branching* in our arenas is also necessary for Theorem 9. Another example of a winning condition with finite memory requirements in finitely branching arenas for one player but infinite memory requirements in infinitely branching arenas is presented in [16, Section 4].

4 Applications

We provide applications of our results to discounted-sum and mean-payoff conditions.

4.1 Discounted sum

We apply our results to a *discounted-sum* condition in order to illustrate our notions. A specificity of this example is that its ω-regularity depends on some parameters – we use our results to characterize the parameters for which it is ω-regular or, equivalently (Theorem 9), chromatic-finite-memory determined. The ω-regularity of discounted-sum conditions has also been studied in [14, 2] with different techniques and goals.

Let $C \subseteq \mathbb{Q}$ be non-empty and bounded. For $\lambda \in (0,1) \cap \mathbb{Q}$, we define the *discounted-sum function* $\mathsf{DS}_\lambda \colon C^\omega \to \mathbb{R}$ such that for $w = c_1c_2\ldots \in C^\omega$, $\mathsf{DS}_\lambda(w) = \sum_{i=1}^{\infty} \lambda^{i-1} \cdot c_i$. This function is always well-defined for a bounded C, and takes values in $[\frac{\inf C}{1-\lambda}, \frac{\sup C}{1-\lambda}]$.

We define the winning condition $\mathsf{DS}_\lambda^{\geq 0} = \{w \in C^\omega \mid \mathsf{DS}_\lambda(w) \geq 0\}$ as the set of infinite words whose discounted sum is non-negative, and let $\sim$ be its right congruence. We will analyze cycle-consistency and prefix-independence of $\mathsf{DS}_\lambda^{\geq 0}$ to conclude under which conditions (on C and λ) it is chromatic-finite-memory determined (or equivalently, ω-regular by Theorem 9). First, we discuss a few properties of the discounted-sum function.

Basic properties. We extend function DS_λ to finite words in a natural way: for $w \in C^*$, we define $\mathsf{DS}_\lambda(w) = \mathsf{DS}_\lambda(w0^\omega)$. For $w \in C^*$, we define $|w|$ as the length of w (so $w \in C^{|w|}$). First, we notice that for $w \in C^*$ and $w' \in C^\omega$, we have $\mathsf{DS}_\lambda(ww') = \mathsf{DS}_\lambda(w) + \lambda^{|w|}\mathsf{DS}_\lambda(w')$. Therefore, $ww' \in \mathsf{DS}_\lambda^{\geq 0}$ if and only if $\frac{\mathsf{DS}_\lambda(w)}{\lambda^{|w|}} \geq -\mathsf{DS}_\lambda(w')$. This provides a characterization of the winning continuations of a finite word $w \in C^*$ by comparing their discounted sum to the value $\frac{\mathsf{DS}_\lambda(w)}{\lambda^{|w|}}$.

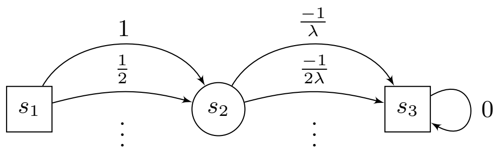

Figure 3 Arena with infinitely many edges in which $\mathcal{P}_1$ needs infinite memory to win for condition $\mathsf{DS}_\lambda^{\geq 0}$ from s_1 for any $\lambda \in (0,1) \cap \mathbb{Q}$, with $C = [-k, k] \cap \mathbb{Q}$ for k sufficiently large.

This leads us to define the *gap* of a finite word $w \in C^*$, following ideas in [5], as

$$\mathsf{gap}(w) = \begin{cases} \top & \text{if } \frac{\mathsf{DS}_\lambda(w)}{\lambda^{|w|}} \geq -\frac{\inf C}{1-\lambda}, \\ \bot & \text{if } \frac{\mathsf{DS}_\lambda(w)}{\lambda^{|w|}} < -\frac{\sup C}{1-\lambda}, \\ \frac{\mathsf{DS}_\lambda(w)}{\lambda^{|w|}} & \text{otherwise.} \end{cases}$$

Intuitively, the gap of a finite word $w \in C^*$ represents how far it is from going back to 0: if $w' \in C^\omega$ is such that $\mathsf{DS}_\lambda(w') = -\mathsf{gap}(w)$, then $\mathsf{DS}_\lambda(ww') = 0$. We can see that for all words $w \in C^*$, if $\mathsf{gap}(w) = \top$, then all continuations are winning (i.e., $w^{-1}W = C^\omega$) as it is not possible to find an infinite word with a discounted sum less than $\frac{\inf C}{1-\lambda}$. Similarly, if $\mathsf{gap}(w) = \bot$, then all continuations are losing (i.e., $w^{-1}W = \emptyset$).

Cycle-consistency. We have that $\mathsf{DS}_\lambda^{\geq 0}$ is $\mathcal{M}_{\mathsf{triv}}$-cycle-consistent (proof in [9, Section 6]).

▶ **Proposition 12.** *For all bounded $C \subseteq \mathbb{Q}$, $\lambda \in (0,1) \cap \mathbb{Q}$, winning condition $\mathsf{DS}_\lambda^{\geq 0}$ is $\mathcal{M}_{\mathsf{triv}}$-cycle-consistent.*

Prefix-independence. If $C = [-k, k] \cap \mathbb{Q}$ for some $k \in \mathbb{N} \setminus \{0\}$, winning condition $\mathsf{DS}_\lambda^{\geq 0}$ is not $\mathcal{M}$-prefix-independent for any $\mathcal{M}$, as $\sim$ has infinite index. Indeed, we have for instance that elements in $\{\frac{1}{i} \in C^* \mid i \geq 1\}$ are all in different equivalence classes of $\sim$. We can see how to use this to exhibit an arena in which $\mathcal{P}_1$ can win but needs infinite memory to do so in Figure 3.

For finite $C \subseteq \mathbb{Z}$, the picture is more complicated; for $C = [-k, k] \cap \mathbb{Z}$ for some $k \in \mathbb{N}$, we characterize when $\mathsf{DS}_\lambda^{\geq 0}$ is $\mathcal{M}$-prefix-independent for some finite skeleton $\mathcal{M}$. We give an intuition of the two situations in which that happens: (i) if C is too small, then the first non-zero color seen determines the outcome of the game, as it is not possible to compensate this color to change the sign of the discounted sum; (ii) if $\lambda = \frac{1}{n}$ for some integer $n \geq 1$, then the gap function actually takes only finitely many values, which is not the case for a different λ.

▶ **Proposition 13.** *Let $\lambda \in (0,1) \cap \mathbb{Q}$, $k \in \mathbb{N}$, and $C = [-k, k] \cap \mathbb{Z}$. Then, the right congruence $\sim$ of $\mathsf{DS}_\lambda^{\geq 0}$ has finite index if and only if $k < \frac{1}{\lambda} - 1$ or λ is equal to $\frac{1}{n}$ for some integer $n \geq 1$.*

Proof (sketch). Full proof in [9, Section 6]. The key property is to show that gaps characterize equivalence classes of prefixes: for $w_1, w_2 \in C^*$, $w_1 \sim w_2$ if and only if $\mathsf{gap}(w_1) = \mathsf{gap}(w_2)$. Once this is proven, it is left to determine the number of different gap values in each situation, which corresponds to the index of $\sim$. We illustrate one situation in which the index is finite by depicting the minimal-state automaton of $\sim$ for $\lambda = \frac{1}{2}$ and $k = 2 \geq \frac{1}{\lambda} - 1$ in Figure 4. ◀

Connecting Propositions 12 and 13, here is the characterization we obtain using Theorem 8.

▶ **Corollary 14.** *Let $\lambda \in (0,1) \cap \mathbb{Q}$, $k \in \mathbb{N}$, and $C = [-k, k] \cap \mathbb{Z}$. Condition $\mathsf{DS}_\lambda^{\geq 0}$ is chromatic-finite-memory determined (or equivalently, ω-regular) if and only if $k < \frac{1}{\lambda} - 1$ or λ is equal to $\frac{1}{n}$ for some integer $n \geq 1$.*

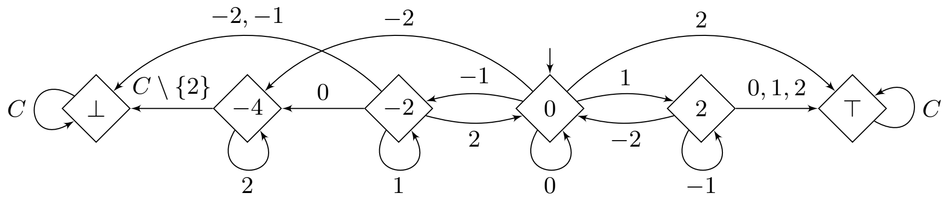

Figure 4 Minimal-state automaton of $\sim$ for $\lambda = \frac{1}{2}$ and $C = \{-2, -1, 0, 1, 2\}$. The value in a state is the gap value characterizing the equivalence class of $\sim$. Here, $\frac{\sup C}{1-\lambda} = 4$ and $\frac{\inf C}{1-\lambda} = -4$. The asymmetry around 0 comes from the ≥ 0 in the definition of the condition: when state -4 is reached, there is exactly one winning continuation (2^ω), but a state with gap value 4 would only have winning continuations (hence, it is part of state $\top$). Notice that we can define a parity condition on top of this automaton that recognizes $\mathsf{DS}_\lambda^{\geq 0}$: an infinite word is winning as long as it does not reach $\bot$.

4.2 Mean payoff

Let $C \subseteq \mathbb{Q}$ be non-empty. We define the *mean-payoff function* $\mathsf{MP}\colon C^\omega \to \mathbb{R} \cup \{-\infty, \infty\}$ such that for $w = c_1c_2\ldots \in C^\omega$, $\mathsf{MP}(w) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n c_i$. We define the winning condition $\mathsf{MP}^{\geq 0} = \{w \in C^\omega \mid \mathsf{MP}(w) \geq 0\}$ as the set of infinite words whose mean payoff is non-negative. This condition is $\mathcal{M}_{\mathsf{triv}}$-prefix-independent for any set of colors. However, it is known that infinite-memory strategies may be required to play optimally in some infinite arenas [39, Section 8.10]; the example provided uses infinitely many colors. Here, we show that chromatic-finite-memory strategies do not suffice to play optimally, even for $C = \{-1, 1\}$. Let us analyze cycle-consistency of $\mathsf{MP}^{\geq 0}$. If we consider, for $n \in \mathbb{N}$,

$$w_n = \underbrace{1\ldots1}_{n \text{ times}}\underbrace{-1\ldots-1}_{n+1 \text{ times}},$$

we have that $(w_n)^\omega$ is losing for all $n \in \mathbb{N}$, but the infinite word $w_0w_1w_2\ldots$ has a mean payoff of 0 and is thus winning. This shows directly that $\mathsf{MP}^{\geq 0}$ is not $\mathcal{M}_{\mathsf{triv}}$-cycle-consistent. The argument can be adapted to show that $\mathsf{MP}^{\geq 0}$ is not $\mathcal{M}$-cycle-consistent for any skeleton $\mathcal{M}$ (see [9, Section 6]).

5 Conclusion

We proved an equivalence between chromatic-finite-memory determinacy of a winning condition in games on infinite graphs and ω-regularity of the corresponding language of infinite words, generalizing a result by Colcombet and Niwiński [17]. A "strategic" consequence is that chromatic-finite-memory determinacy in one-player games of both players implies the seemingly stronger chromatic-finite-memory determinacy in zero-sum games. A "language-theoretic" consequence is a link between the representation of ω-regular languages by parity automata and the memory structures used to play optimally in zero-sum games, using as a tool the minimal-state automata classifying the equivalence classes of the right congruence.

For future work, one possible improvement over our result is to deduce tighter chromatic memory requirements in two-player games compared to one-player games. Our proof technique gives as an upper bound on the two-player memory requirements a product between the minimal-state automaton and a sufficient skeleton for one-player arenas, but smaller skeletons often suffice. We do not know whether the product with the minimal-state automaton is necessary in general in order to play optimally in two-player arenas (although it is necessary in Theorem 8 to describe W using a parity automaton). This behavior contrasts with the case of finite arenas, in which it is known that a skeleton sufficient for both players in finite one-player arenas also suffices in finite two-player arenas [6, 8]. More generally, it would be interesting to characterize precisely the (chromatic) memory requirements of ω-regular winning conditions, extending work on the subclass of Muller conditions [18, 11].

References

1 Dana Angluin and Dana Fisman. Regular ω-languages with an informative right congruence. *Inf. Comput.*, 278:104598, 2021. doi:10.1016/j.ic.2020.104598.

2 Suguman Bansal, Swarat Chaudhuri, and Moshe Y. Vardi. Comparator automata in quantitative verification. *CoRR*, abs/1812.06569, 2018. arXiv:1812.06569.

3 Alessandro Bianco, Marco Faella, Fabio Mogavero, and Aniello Murano. Exploring the boundary of half-positionality. *Ann. Math. Artif. Intell.*, 62(1-2):55–77, 2011. doi:10.1007/s10472-011-9250-1.

4 Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph games and reactive synthesis. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, editors, *Handbook of Model Checking*, pages 921–962. Springer, 2018. doi:10.1007/978-3-319-10575-8_27.

5 Udi Boker, Thomas A. Henzinger, and Jan Otop. The target discounted-sum problem. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015*, pages 750–761. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.74.

6 Patricia Bouyer, Stéphane Le Roux, Youssouf Oualhadj, Mickael Randour, and Pierre Vandenhove. Games where you can play optimally with arena-independent finite memory. In Igor Konnov and Laura Kovács, editors, *31st International Conference on Concurrency Theory, CONCUR 2020, September 1-4, 2020, Vienna, Austria (Virtual Conference)*, volume 171 of *LIPIcs*, pages 24:1–24:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.CONCUR.2020.24.

7 Patricia Bouyer, Stéphane Le Roux, and Nathan Thomasset. Finite-memory strategies in two-player infinite games. *CoRR*, abs/2107.09945, 2021. arXiv:2107.09945.

8 Patricia Bouyer, Youssouf Oualhadj, Mickael Randour, and Pierre Vandenhove. Arena-independent finite-memory determinacy in stochastic games. In Serge Haddad and Daniele Varacca, editors, *32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference*, volume 203 of *LIPIcs*, pages 26:1–26:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.CONCUR.2021.26.

9 Patricia Bouyer, Mickael Randour, and Pierre Vandenhove. Characterizing omega-regularity through finite-memory determinacy of games on infinite graphs. *CoRR*, abs/2110.01276, 2021. arXiv:2110.01276.

10 Tomás Brázdil, Václav Brozek, and Kousha Etessami. One-counter stochastic games. In Kamal Lodaya and Meena Mahajan, editors, *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010, December 15-18, 2010, Chennai, India*, volume 8 of *LIPIcs*, pages 108–119. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010. doi:10.4230/LIPIcs.FSTTCS.2010.108.

11 Antonio Casares. On the minimisation of transition-based Rabin automata and the chromatic memory requirements of Muller conditions. *CoRR*, abs/2105.12009, 2021. arXiv:2105.12009.

12 Antonio Casares, Thomas Colcombet, and Nathanaël Fijalkow. Optimal transformations of games and automata using Muller conditions. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, volume 198 of *LIPIcs*, pages 123:1–123:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ICALP.2021.123.

13 Krishnendu Chatterjee and Laurent Doyen. Perfect-information stochastic games with generalized mean-payoff objectives. In Martin Grohe, Eric Koskinen, and Natarajan Shankar, editors, *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16, New York, NY, USA, July 5-8, 2016*, pages 247–256. ACM, 2016. doi:10.1145/2933575.2934513.

14 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Expressiveness and closure properties for quantitative languages. In *Proceedings of the 24th Annual IEEE Symposium on Logic in Computer Science, LICS 2009, 11-14 August 2009, Los Angeles, CA, USA*, pages 199–208. IEEE Computer Society, 2009. doi:10.1109/LICS.2009.16.

15 Krishnendu Chatterjee and Nathanaël Fijalkow. Infinite-state games with finitary conditions. In Simona Ronchi Della Rocca, editor, *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy*, volume 23 of *LIPIcs*, pages 181–196. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013. doi:10.4230/LIPIcs.CSL.2013.181.

16 Thomas Colcombet, Nathanaël Fijalkow, and Florian Horn. Playing safe. In Venkatesh Raman and S. P. Suresh, editors, *34th International Conference on Foundation of Software Technology and Theoretical Computer Science, FSTTCS 2014, December 15-17, 2014, New Delhi, India*, volume 29 of *LIPIcs*, pages 379–390. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014. doi:10.4230/LIPIcs.FSTTCS.2014.379.

17 Thomas Colcombet and Damian Niwiński. On the positional determinacy of edge-labeled games. *Theor. Comput. Sci.*, 352(1-3):190–196, 2006. doi:10.1016/j.tcs.2005.10.046.

18 Stefan Dziembowski, Marcin Jurdzinski, and Igor Walukiewicz. How much memory is needed to win infinite games? In *Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science, Warsaw, Poland, June 29 - July 2, 1997*, pages 99–110. IEEE Computer Society, 1997. doi:10.1109/LICS.1997.614939.

19 Andrzej Ehrenfeucht and Jan Mycielski. Positional strategies for mean payoff games. *Int. Journal of Game Theory*, 8(2):109–113, 1979. doi:10.1007/BF01768705.

20 Hugo Gimbert. Parity and exploration games on infinite graphs. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, 18th International Workshop, CSL 2004, 13th Annual Conference of the EACSL, Karpacz, Poland, September 20-24, 2004, Proceedings*, volume 3210 of *Lecture Notes in Computer Science*, pages 56–70. Springer, 2004. doi:10.1007/978-3-540-30124-0_8.

21 Hugo Gimbert and Edon Kelmendi. Submixing and shift-invariant stochastic games. *CoRR*, abs/1401.6575, 2014. arXiv:1401.6575.

22 Hugo Gimbert and Wieslaw Zielonka. When can you play positionally? In Jirí Fiala, Václav Koubek, and Jan Kratochvíl, editors, *Mathematical Foundations of Computer Science 2004, 29th International Symposium, MFCS 2004, Prague, Czech Republic, August 22-27, 2004, Proceedings*, volume 3153 of *Lecture Notes in Computer Science*, pages 686–697. Springer, 2004. doi:10.1007/978-3-540-28629-5_53.

23 Hugo Gimbert and Wieslaw Zielonka. Games where you can play optimally without any memory. In Martín Abadi and Luca de Alfaro, editors, *CONCUR 2005 - Concurrency Theory, 16th International Conference, CONCUR 2005, San Francisco, CA, USA, August 23-26, 2005, Proceedings*, volume 3653 of *Lecture Notes in Computer Science*, pages 428–442. Springer, 2005. doi:10.1007/11539452_33.

24 Hugo Gimbert and Wieslaw Zielonka. Pure and stationary optimal strategies in perfect-information stochastic games with global preferences. Unpublished, 2009. URL: https://hal.archives-ouvertes.fr/hal-00438359.

25 Erich Grädel and Igor Walukiewicz. Positional determinacy of games with infinitely many priorities. *Log. Methods Comput. Sci.*, 2(4), 2006. doi:10.2168/LMCS-2(4:6)2006.

26 Stefan Kiefer, Richard Mayr, Mahsa Shirmohammadi, Patrick Totzke, and Dominik Wojtczak. How to play in infinite MDPs (invited talk). In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 3:1–3:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ICALP.2020.3.

27 Nils Klarlund. Progress measures, immediate determinacy, and a subset construction for tree automata. *Ann. Pure Appl. Log.*, 69(2-3):243–268, 1994. doi:10.1016/0168-0072(94)90086-8.

28 Eryk Kopczyński. Half-positional determinacy of infinite games. In Michele Bugliesi, Bart Preneel, Vladimiro Sassone, and Ingo Wegener, editors, *Automata, Languages and Programming, 33rd International Colloquium, ICALP 2006, Venice, Italy, July 10-14, 2006, Proceedings, Part II*, volume 4052 of *Lecture Notes in Computer Science*, pages 336–347. Springer, 2006. doi:10.1007/11787006_29.

29 Eryk Kopczyński. Omega-regular half-positional winning conditions. In Jacques Duparc and Thomas A. Henzinger, editors, *Computer Science Logic, 21st International Workshop, CSL 2007, 16th Annual Conference of the EACSL, Lausanne, Switzerland, September 11-15, 2007, Proceedings*, volume 4646 of *Lecture Notes in Computer Science*, pages 41–53. Springer, 2007. doi:10.1007/978-3-540-74915-8_7.

30 Eryk Kopczyński. *Half-positional Determinacy of Infinite Games*. PhD thesis, Warsaw University, 2008.

31 Alexander Kozachinskiy. One-to-two-player lifting for mildly growing memory. *CoRR*, abs/2104.13888, 2021. arXiv:2104.13888.

32 Stéphane Le Roux. Time-aware uniformization of winning strategies. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, *Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings*, volume 12098 of *Lecture Notes in Computer Science*, pages 193–204. Springer, 2020. doi:10.1007/978-3-030-51466-2_17.

33 Stéphane Le Roux and Arno Pauly. Extending finite-memory determinacy to multi-player games. *Inf. Comput.*, 261(Part):676–694, 2018. doi:10.1016/j.ic.2018.02.024.

34 Stéphane Le Roux, Arno Pauly, and Mickael Randour. Extending finite-memory determinacy by Boolean combination of winning conditions. In Sumit Ganguly and Paritosh K. Pandya, editors, *38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2018, December 11-13, 2018, Ahmedabad, India*, volume 122 of *LIPIcs*, pages 38:1–38:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.FSTTCS.2018.38.

35 Oded Maler and Ludwig Staiger. On syntactic congruences for omega-languages. *Theor. Comput. Sci.*, 183(1):93–112, 1997. doi:10.1016/S0304-3975(96)00312-X.

36 Andrzej Wlodzimierz Mostowski. Regular expressions for infinite trees and a standard form of automata. In Andrzej Skowron, editor, *Computation Theory - Fifth Symposium, Zaborów, Poland, December 3-8, 1984, Proceedings*, volume 208 of *Lecture Notes in Computer Science*, pages 157–168. Springer, 1984. doi:10.1007/3-540-16066-3_15.

37 A. Nerode. Linear automaton transformations. *Proceedings of the American Mathematical Society*, 9(4):541–544, 1958. doi:10.2307/2033204.

38 Amir Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science, Providence, Rhode Island, USA, 31 October - 1 November 1977*, pages 46–57. IEEE Computer Society, 1977. doi:10.1109/SFCS.1977.32.

39 Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley Series in Probability and Statistics. Wiley, 1994. doi:10.1002/9780470316887.

40 L. S. Shapley. Stochastic games. *Proceedings of the National Academy of Sciences*, 39(10):1095–1100, 1953. doi:10.1073/pnas.39.10.1095.

41 Ludwig Staiger. Finite-state omega-languages. *J. Comput. Syst. Sci.*, 27(3):434–448, 1983. doi:10.1016/0022-0000(83)90051-X.

42 Wieslaw Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theor. Comput. Sci.*, 200(1-2):135–183, 1998. doi:10.1016/S0304-3975(98)00009-7.