# Covering Many (Or Few) Edges with $k$ Vertices in Sparse Graphs


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## Abstract

We study the following two fixed-cardinality optimization problems (a maximization and a minimization variant). For a fixed $\alpha$ between zero and one we are given a graph and two numbers $k \in \mathbb{N}$ and $t \in \mathbb{Q}$. The task is to find a vertex subset $S$ of exactly $k$ vertices that has value at least (resp. at most for minimization) $t$. Here, the value of a vertex set computes as $\alpha$ times the number of edges with exactly one endpoint in $S$ plus $1-\alpha$ times the number of edges with both endpoints in $S$. These two problems generalize many prominent graph problems, such as DENSEST $k$-SUBGRAPH, SPARSEST $k$-SUBGRAPH, PARTIAL VERTEX COVER, and MAX $(k,n-k)$-CUT.

In this work, we complete the picture of their parameterized complexity on several types of sparse graphs that are described by structural parameters. In particular, we provide kernelization algorithms and kernel lower bounds for these problems. A somewhat surprising consequence of our kernelizations is that PARTIAL VERTEX COVER and MAX $(k, n-k)$-CUT not only behave in the same way but that the kernels for both problems can be obtained by the same algorithms.



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## 1 Introduction

Fixed-cardinality optimization problems are a well-studied class of graph problems where one seeks for a given graph $G$, a vertex set $S$ of size $k$ such that $S$ optimizes some objective function $\mathrm{val}_G(S)$ [6, 8, 7, 26]. Prominent examples of these problems are DENSEST $k$-SUBGRAPH [5, 15, 26], SPARSEST $k$-SUBGRAPH [18, 19, 33], PARTIAL VERTEX COVER [1, 17, 21], and MAX $(k,n-k)$-CUT [7, 31, 32].

A common thread in these examples is that all these problems are formulated in terms of the number of edges that have one or two endpoints in $S$: In the decision version of DENSEST $k$-SUBGRAPH we require that there are at least $t$ edges with both endpoints in $S$; CLIQUE is the special case where $t = \binom{k}{2}$. Conversely, in SPARSEST $k$-SUBGRAPH we require that at

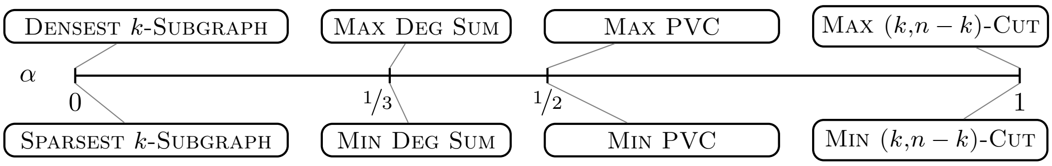


**Figure 1** Problem definition cheat sheet.

most $t$ edges have both endpoints in $S$ and INDEPENDENT SET is the special case with $t = 0$. In PARTIAL VERTEX COVER we require that at least $t$ edges have at least one endpoint in $S$. Finally, in MAX $(k,n-k)$-CUT we require that at least $t$ edges have exactly one endpoint in $S$.

We study the following general problem first defined by Bonnet et al. [3] that contains all of the above problems as special case.[1]

MAX $\alpha$-FIXED CARDINALITY GRAPH PARTITIONING (MAX $\alpha$-FCGP)
**Input:** A graph $G$, $k \in \mathbb{N}$, and $t \in \mathbb{Q}$.
**Question:** Is there a set $S$ of exactly $k$ vertices such that

$$\mathrm{val}(S) \coloneqq (1-\alpha) \cdot m(S) + \alpha \cdot m(S, V(G) \setminus S) \geq t \text{ ?}$$

Here, $\alpha \in [0,1]$, and $m(S)$ denotes the number of edges with two endpoints in $S$ and $m(S, V(G) \setminus S)$ denotes the number of edges with exactly one endpoint in $S$. Naturally, one may also consider the minimization problem, denoted as MIN $\alpha$-FIXED CARDINALITY GRAPH PARTITIONING (MIN $\alpha$-FCGP), where we are looking for a set $S$ such that $\mathrm{val}(S) \leq t$.

The value of $\alpha$ describes how strongly edges with exactly one endpoint in $S$ influence the value of $S$ relative to edges with two endpoints in $S$. For $\alpha = 1/3$, edges with two endpoints in $S$ count twice as much as edges with one endpoint in $S$ and, thus, every vertex contributes exactly its degree to the value of $S$. Hence, in this case, we simply want to find a vertex set with a largest or smallest degree sum.

More importantly, MAX $\alpha$-FCGP and MIN $\alpha$-FCGP contain all of the above-mentioned problems as special cases (see Figure 1). For example, PARTIAL VERTEX COVER (MAXPVC) is MAX $\alpha$-FCGP with $\alpha = 1/2$ as all edges with at least one endpoint in $S$ count the same. MAX $(k, n-k)$-CUT is MAX $\alpha$-FCGP with $\alpha = 1$ since edges with both endpoints in $S$ are ignored. SPARSEST $k$-SUBGRAPH is MIN $\alpha$-FCGP with $\alpha = 0$ as only the edges with both endpoints in $S$ count. Consequently, there exist values of $\alpha$ such that MAX $\alpha$-FCGP and MIN $\alpha$-FCGP are NP-hard and W[1]-hard on general graphs with respect to the natural parameter $k$ [7, 9, 12, 17]. This hardness makes it interesting to study these problems on input graphs with special structure and Bonnet et al. [3] and Shachnai and Zehavi [32] studied this problem on bounded-degree graphs.

We continue this line of research and give a complete picture of the parameterized complexity of MIN $\alpha$-FCGP and MAX $\alpha$-FCGP on several types of sparse graphs that are described by structural parameters. In particular, we provide kernelization algorithms and kernel lower bounds for these problems, see Figure 2 for an overview.

[1] On the face of it, the definition of Bonnet et al. [3] seems to be more general as it has separate weight parameters for the internal and outgoing edges. It can be reduced to our formulation by adapting the value of $t$ and thus our results also hold for this formulation.

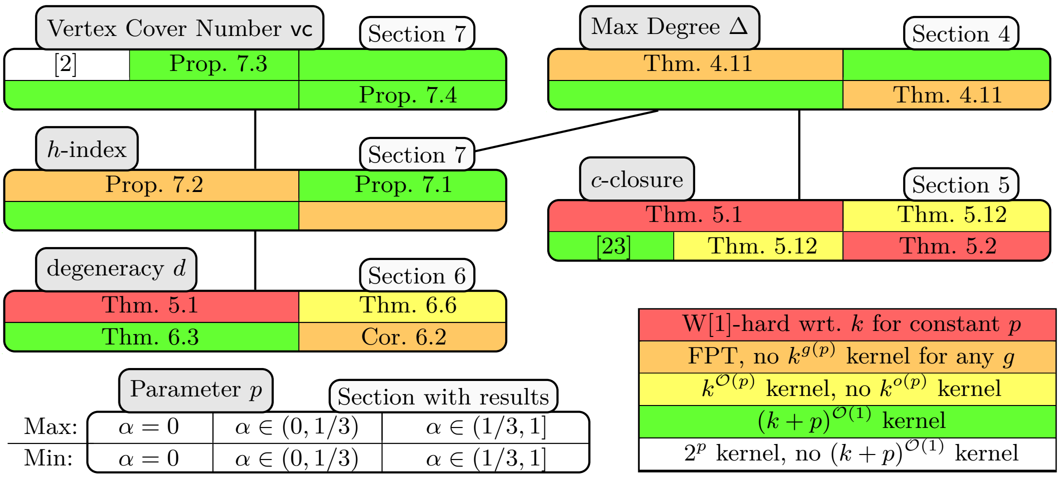


**Figure 2** Overview over our results. Each box displays the parameterized results (see also bottom right) with respect to $k$ and the corresponding parameter $p$ for all variants (maximization, minimization, and all $\alpha \in [0, 1]$, see bottom left). Note that the split of the boxes is not proportional to the corresponding values of $\alpha$. See Section 2 (paragraph "Graph parameter definitions.") for the definitions of the parameters. A line from a box for parameter $p$ to a box *above* for parameter $p'$ implies that $p \in \mathcal{O}(p')$ on all graphs. Thus, hardness results hold also for all parameters below and tractability results for all parameters above.

**Known results.** MaxPVC can be solved in $\mathcal{O}^*((\Delta + 1)^k)$ time where $\Delta$ is the maximum degree of the input graph [30]. For the degeneracy $d$, Amini et al. [1] gave an $\mathcal{O}^*((dk)^k)$-time algorithm which was recently improved to an algorithm with running time $\mathcal{O}^*(2^{\mathcal{O}(dk)})$ [28]. Bonnet et al. [3] showed that in $\mathcal{O}^*(\Delta^k)$ time one can solve Max $\alpha$-FCGP for all $\alpha > 1/3$ and Min $\alpha$-FCGP for all $\alpha < 1/3$. Bonnet et al. [3] call these two problem cases *degrading*. This name reflects the fact that in Max $\alpha$-FCGP with $\alpha > 1/3$, adding a vertex $v$ to a set $S$ increases the value at least as much as adding $v$ to a superset of $S$.[2] This is because here one edge with both endpoints in $S$ is less valuable than two edges each with one endpoint in $S$. In Min $\alpha$-FCGP this effect is reversed since we aim to minimize val. The other problem cases are called *non-degrading*. For non-degrading problems, Bonnet et al. [3] achieved a running time of $\mathcal{O}^*((\Delta k)^{\mathcal{O}(k)})$ and asked whether they can also be solved in $\mathcal{O}^*(\Delta^{\mathcal{O}(k)})$ time. This question was answered positively by Shachnai and Zehavi [32], who showed that Max $\alpha$-FCGP and Min $\alpha$-FCGP can be solved in $\mathcal{O}^*(4^{k+o(k)}\Delta^k)$ time.

Kernelization has been studied only for special cases. Max $(k,n-k)$-Cut admits a polynomial problem kernel when parameterized by $t$ [31]. This also gives a polynomial kernel for $k + \Delta$ since instances with $t > \Delta k$ are trivial no-instances. It is also known that Sparsest $k$-Subgraph admits a kernel with $\gamma k^2$ vertices [23]. Here, $\gamma$ is a parameter bounded by $\max(c, d + 1)$ [16]. In contrast, Densest-$k$ Subgraph is unlikely to admit a polynomial problem kernel when parameterized by $\Delta + k$ since Clique is a special case.

More broadly, for graph problems that are W[1]-hard for the standard parameter solution $k$ size, the study of kernelization on sparse input graphs has received much attention in recent years [10, 11, 25, 29].

[2] Note that this matches the definition of submodularity.

Independent of our work, a polynomial *compression* for MaxPVC (the special case of Max $\alpha$-FCGP with $\alpha = 1/2$) of size $(dk)^{\mathcal{O}(d)}$ was recently discovered by Panolan and Yaghoubizade [28].

**Our results.** We provide a complete picture of the parameterized complexity of Max $\alpha$-FCGP and Min $\alpha$-FCGP for all $\alpha$ with respect to the combination of $k$ and five parameters describing the graph structure: the maximum degree $\Delta$ of $G$, the $h$-index of $G$, the degeneracy of $G$, the $c$-closure of $G$, and the vertex cover number vc of $G$. With the exception of the $c$-closure, all parameters are sparseness measures. The $c$-closure, first described by Fox et al. [16], measures how strongly a graph adheres to the triadic closure principle. Informally, the closure of a graph is small whenever all vertices with many common neighbors are also neighbors of each other. For a formal definition of all parameters refer to Section 2.

Our results are summarized by Figure 2. On a very general level, our main finding suggests that the degrading problems are much more amenable to FPT algorithms and kernelizations than their non-degrading counterparts. No such difference is observed when considering the running time of FPT algorithms for the parameter $k + \Delta$ but it becomes striking in the context of kernelization and when using secondary parameters that are smaller than $\Delta$. Given the importance of the distinction between the degrading and non-degrading cases, we distinguish these subcases of Max $\alpha$-FCGP and Min $\alpha$-FCGP by name (Degrading Max $\alpha$-FCGP, Non-Degrading Max $\alpha$-FCGP, Degrading Min $\alpha$-FCGP, Non-Degrading Min $\alpha$-FCGP).

On a technical level, by introducing an annotated version of the problem that keeps track of removed vertices, we separate and unify arguments that deal with vertices identified as (not) being part of a solution. In particular, we show that by introducing vertex weights (called counter) we can deal with vertices whose contribution is substantially below or above the average contribution that is necessary to reach the threshold $t$. More precisely, if the contribution of a vertex $v$ is much above $t/k$, then we can add $v$ to the solution and if it is much below $t/k$, then we can remove $v$. As a consequence, we can show that the weights can be bounded in the maximum degree of the annotated instance. This gives the kernels for the parameter $k + \Delta$.

The main step in the more sophisticated kernelizations for the degeneracy $d$ and the $c$-closure is now to decrease the maximum degree of the instance as this allows us to use the kernel for $k + \Delta$. To decrease the maximum degree, for these parameters, we make use of Ramsey bounds. More precisely, the Ramsey bounds help to find a large independent set $I$ such that all vertices outside of $I$ have only a bounded number of neighbors in $I$. This then allows to prove by pigeonholing the following for the vertex $v$ of $I$ with the currently worst contribution to the objective function: No matter what the optimal solution selects outside of $I$, there is always some vertex of $I \setminus \{v\}$ that gives at least as good a contribution to the final solution as $v$. For the parameter $c$, we also need an additional pigeonhole argument excluding large cliques in order to apply the Ramsey bound. For the parameter $d$, we establish a new constructive Ramsey bound for $K_{i,j}$-free graphs that may be of independent interest.

We remark that when we describe the kernel size for $\alpha > 0$ (for instance, Proposition 4.2), the factor $1/\alpha$ is hidden in the $\mathcal{O}$ notation. We would like to emphasize, however, that the exponents in the kernel size such as $\mathcal{O}(c)$ and $\mathcal{O}(d)$ do not depend on $1/\alpha$. On the other hand, the lower bounds such as Theorem 4.11 hold indeed for all $\alpha$ in the range corresponding to the case.

We believe that this general approach could be useful for other parameterizations that are not considered in this work. A somewhat surprising consequence of our kernelizations is that Partial Vertex Cover and Max $(k, n-k)$-Cut not only behave in the same way but that the kernels for both problems can also be obtained by the same algorithms.

Due to lack of space, several proofs (marked with $(\star)$) are deferred to the full version [22].

## 2 Preliminaries

For $q \in \mathbb{N}$, we write $[q]$ to denote the set $\{1, 2, \ldots, q\}$. For a graph $G$, we denote its *vertex set* by $V(G)$ and its *edge set* by $E(G)$. Let $X, Y \subseteq V(G)$ be vertex subsets. We use $G[X]$ to denote the *subgraph induced* by $X$. We let $G - X$ denote the graph obtained by *removing* the vertices in $X$. We denote by $N_G(X) \coloneqq \{y \in V(G) \setminus X \mid xy \in E(G), x \in X\}$ the *open neighborhood* and by $N_G[X] \coloneqq N_G(X) \cup X$ the *closed neighborhood* of $X$. By $E_G(X, Y) \coloneqq \{xy \in E(G) \mid x \in X, y \in Y\}$ we denote the set of edges *between* $X$ and $Y$. As a shorthand, we set $E_G(X) \coloneqq E_G(X, X)$. Furthermore, we denote by $m_G(X, Y) \coloneqq |E_G(X, Y)|$ and $m_G(X) \coloneqq |E_G(X)|$ the *sizes* of these edge sets. For all these notations, when $X$ is a singleton $\{x\}$ we may write $x$ instead of $\{x\}$. Let $v \in V(G)$. We denote the *degree* of $v$ by $\deg_G(v)$. We drop the subscript $\cdot_G$ when it is clear from context.

**Graph parameter definitions.** For more information on parameterized complexity, we refer to the standard monographs [9, 12]. We denote the size of a smallest vertex cover (a set of vertices that covers all edges) of a graph $G$ by $\mathsf{vc}_G$. The maximum and minimum degree of $G$ are $\Delta_G \coloneqq \max_{v \in V(G)} \deg_G(v)$ and $\delta_G \coloneqq \min_{v \in V(G)} \deg_G(v)$, respectively. The degeneracy of $G$ is $d_G \coloneqq \max_{S \subseteq V(G)} \delta_{G[S]}$. The *$h_G$-index* of a graph $G$ is the largest integer $h$ such that $G$ has at least $h$ vertices of degree at least $h$ [14]. We say that $G$ is $c$-closed for $c \coloneqq \max(\{0\} \cup \{|N_G(u) \cap N_G(v)| \mid uv \notin E(G)\}) + 1$ [16].

**Ramsey numbers.** Ramsey's theorem states that for every $p, q \in \mathbb{N}$, there exists an integer $R(p, q)$ such that any graph on at least $R(p, q)$ vertices contains either a clique of size $p$ or an independent set of size $q$. The numbers $R(p, q)$ are referred to as *Ramsey numbers*. Although the precise values of Ramsey numbers are not known, some upper bounds have been proven. For instance, it holds that $R(p, q) \leq \binom{p+q-2}{p-1}$ (see e.g. [20]). The proof for this upper bound is constructive. More precisely, given a graph $G$ on at least $\binom{p+q-2}{p-1}$ vertices, we can find in time $n^{\mathcal{O}(1)}$ either a clique of size $p$ or an independent set of size $q$.

## 3 A Data Reduction Framework via Annotation

In this section, we introduce an annotated variant which allows easier handling for kernelization by giving more options for encoding information in the instances. Moreover, to avoid repeating certain basic arguments, we provide general data reduction rules and statements used in the subsequent sections. Finally, we describe how to reduce from the annotated to the non-annotated problem variants in polynomial time.

In the annotated problem variant, we have additionally as input a (possibly empty) partial solution $T \subseteq V(G)$ and $\mathsf{counter}\colon V \to \mathbb{N}$ which encodes for each vertex, the number of neighbors in the original graph that are guaranteed to be not in a solution. For a set $S \subseteq V(G)$, we set

- $\mathsf{counter}(S) \coloneqq \sum_{v \in S} \mathsf{counter}(v)$ and
- $\mathrm{val}_G(S) \coloneqq \alpha(m(S, V(G) \setminus S) + \mathsf{counter}(S)) + (1 - \alpha)m(S)$.

For $v \in S$ we set $\deg^{+c}(v) \coloneqq \deg(v) + \mathsf{counter}(v)$.

Annotated Max $\alpha$-FCGP
**Input:** A graph $G$, $T \subseteq V(G)$, $\mathsf{counter}\colon V(G) \to \mathbb{N}$, $k \in \mathbb{N}$, and $t \in \mathbb{Q}$.
**Question:** Is there a vertex set $S$, $T \subseteq S \subseteq V(G)$, of size $k$ with $\mathrm{val}_G(S) \geq t$ (Max) or $\mathrm{val}_G(S) \leq t$ (Min), respectively?

For a partial solution $T \subseteq V(G)$, we define the *contribution* of a vertex $v$. Note that our definition slightly differs from that of Bonnet et al. [3]:

$$\begin{aligned} \mathrm{cont}(v,T) &\coloneqq \alpha \cdot (|N(v) \setminus T| + \mathsf{counter}(v)) + (1-2\alpha)|N(v) \cap T| \\ &= \alpha \deg^{+c}(v) + (1-3\alpha)|N(v) \cap T| \end{aligned}$$

This definition is chosen so that the value $\mathrm{val}(S)$ of a vertex set $S$ computes as follows.

▶ **Lemma 3.1** ($\star$). *Let $G$ be a graph and $S \coloneqq \{v_1, \dots, v_n\} \subseteq V(G)$ a vertex set. Then, it holds that* $\mathrm{val}(S) = \sum_{i \in [|S|]} \mathrm{cont}(v_i, \{v_1, \dots, v_{i-1}\})$.

**Main reduction rules.** Annotations are helpful for data reductions in the following way: If we identify a vertex $v$ that is (or is not) in a solution, then, we can simplify the instance as follows using the annotations.

▶ **Reduction Rule 3.2** (Inclusion Rule). If there is a solution $S$ with $v \in S \setminus T$, then add $v$ to $T$. If there is a vertex $v \in T$ with $\mathsf{counter}(v) > 0$, then reduce $t$ by $\alpha \cdot \mathsf{counter}(v)$ and set $\mathsf{counter}(v) := 0$.

▶ **Reduction Rule 3.3** (Exclusion Rule). If there is a solution $S$ with $v \notin S$, then for each $u \in N(v)$ increase $\mathsf{counter}(u)$ by one and remove $v$ from $G$.

The reduction rules themselves are simple. The difficulty lies in identifying vertices that are included in or excluded from some solution. In the respective arguments, we use the subsequently discussed notion of better vertices.

**Better vertices.** The following notion captures a situation that frequently appears in our arguments and allows for simple exchange arguments (see following lemma).

▶ **Definition 3.4.** *A vertex $v \in V(G)$ is* better *than $u \in V(G)$ with respect to a vertex set $T \subseteq V(G)$ if $\mathrm{cont}(v,T) \geq \mathrm{cont}(u,T)$ for the maximization variant (if $\mathrm{cont}(v,T) \leq \mathrm{cont}(u,T)$ for the minimization variant).*

*A vertex $v \in V(G)$ is* strictly better *than $u \in V(G)$ if for all $T \subseteq V(G)$ of size at most $k$ we have $\mathrm{cont}(v,T) \geq \mathrm{cont}(u,T)$ for the maximization variant ($\mathrm{cont}(v,T) \leq \mathrm{cont}(u,T)$ for the minimization variant).*

When we simply say that $v$ is better than $u$, we mean that $v$ is better than $u$ with respect to the empty set. The following lemma immediately follows from Lemma 3.1.

▶ **Lemma 3.5.** *Let $S$ be a solution of an instance of $\alpha$-FCGP. Suppose that there are two vertices $v \in S$ and $v' \notin S$ such that $v'$ is better than $v$ with respect to $S \setminus \{v\}$ or $v'$ is strictly better than $v$. Then, $S' \coloneqq (S \setminus \{v\}) \cup \{v'\}$ is also a solution.*

**Proof.** We give a proof for the maximization variant; the minimization variant follows analogously. By Lemma 3.1, we have $\mathrm{val}(S') = \mathrm{val}(S \setminus \{v\}) + \mathrm{cont}(v', S \setminus \{v\}) \geq \mathrm{val}(S \setminus \{v\}) + \mathrm{cont}(v, S \setminus \{v\}) = \mathrm{val}(S)$. ◀

Observe that the contribution of any vertex $v$ differs from $\alpha \deg^{+c}(v)$ by at most $|(1-3\alpha)k|$. This observation allows us to identify some strictly better vertices in the following. This is helpful when we wish to apply the second part of Lemma 3.5 on strictly better vertices.

▶ **Lemma 3.6** ($\star$). *Let $u, v \in V(G)$. Vertex $v$ is strictly better than $u$ if*
- *(Maximization:)* $\alpha \deg^{+c}(u) \leq \alpha \deg^{+c}(v) - |(1-3\alpha)k|$.
- *(Minimization:)* $\alpha \deg^{+c}(u) \geq \alpha \deg^{+c}(v) + |(1-3\alpha)k|$.

**Reduction to the non-annotated problem.** The following two lemmas (for maximization and minimization variant respectively) remove annotations and generate an equivalent instance of $\alpha$-FCGP. We remark that the resulting instance size depends on $\Gamma := \max_{v \in V(G)} \mathsf{counter}(v) + 1$. We obtain an upper bound on $\Gamma$ in terms of $k + \Delta$ in the next section.

▶ **Lemma 3.7.** *Given an instance $\mathcal{I} := (G, T, \mathsf{counter}, k, t)$ of* Annotated Max $\alpha$-FCGP *with $\alpha \in (0, 1]$, we can compute an equivalent instance $\mathcal{I}'$ of* Max $\alpha$-FCGP *of size $\mathcal{O}((\Delta + \Gamma) \cdot |V(G)| + k \cdot |T|)$ in polynomial time.*

**Proof.** We may assume that $G$ has at least $k$ vertices (otherwise the lemma holds for a trivial No-instance $\mathcal{I}'$). We construct an equivalent instance $\mathcal{I}' := (G', k, t')$ of Max $\alpha$-FCGP. The graph $G'$ is obtained from $G$ as follows:
1. Add $\mathsf{counter}(v) + \lfloor 1/\alpha \rfloor$ degree-one neighbors to every vertex $v \in V(G)$.
2. Additionally, add $\ell := \Delta + \Gamma + |1/\alpha - 3| \cdot k + \lfloor 1/\alpha \rfloor$ degree-one neighbors to every vertex $v \in T$.

By $L_v$ we denote the set of degree-one vertices added to vertex $v \in V(G)$ and by $L := \bigcup_{v \in V(G)} L_v$ we denote the set of all newly added leaf vertices. To conclude the construction of $\mathcal{I}'$, we set $t' := t + \alpha(\ell \cdot |T| + \lfloor 1/\alpha \rfloor \cdot k)$. Since $G$ has at most $\Delta \cdot |V(G)|$ edges and we add at most $\mathcal{O}((\Gamma + 1) \cdot |V(G)| + (\Delta + k) \cdot |T|)$ edges, we see that $G'$ has at most $\mathcal{O}((\Delta + \Gamma) \cdot |V(G)| + k \cdot |T|)$ edges.

Next, we prove the equivalence between $\mathcal{I}$ and $\mathcal{I}'$. For a solution $S$ of $\mathcal{I}$, its value in $G'$ is increased by $\alpha \cdot \lfloor 1/\alpha \rfloor$ for every vertex in $S$ and additionally, by $\alpha \cdot \ell$ for every vertex in $T$, amounting to $t + \alpha(\ell \cdot |T| + \lfloor 1/\alpha \rfloor \cdot k)$.

Conversely, consider a solution $S'$ of $\mathcal{I}$. First, we show that there is a solution containing all vertices of $T$ and no leaf vertex of $L$ using Lemma 3.5. Suppose that for some vertex $v \in V(G)$, one of its degree-one neighbors $v' \in L_v$ is in $S'$ but not $v$ itself. We then have $\mathrm{cont}(v', S' \setminus \{v\}) = \alpha$ and $\mathrm{cont}(v, S' \setminus \{v\}) \geq \alpha$, implying that $(S' \setminus \{v'\}) \cup \{v\}$ is also a solution by Lemma 3.5. Thus, in the following we can assume that $S' \cap L_v = \emptyset$ for every vertex $v \in V(G) \setminus S'$. If there is a vertex $v' \in S' \cap L_v$ for some $v \in V(G)$, then by the assumption that $|V(G)| \geq k$, the pigeonhole principle gives us a vertex $w \in V(G) \setminus S'$ with $S' \cap L_w = \emptyset$. Since $|L_w| \geq \mathsf{counter}(w) + \lfloor 1/\alpha \rfloor \geq \lfloor 1/\alpha \rfloor$, we have $\mathrm{cont}(w, S' \setminus \{v'\}) \geq \alpha \cdot \lfloor 1/\alpha \rfloor \geq \alpha(1/\alpha - 1) = 1 - \alpha$. We thus have $\mathrm{cont}(v', S \setminus \{v'\}) = 1 - \alpha \leq \mathrm{cont}(w, S' \setminus \{v'\})$. Hence, $(S' \setminus \{v'\}) \cup \{w\}$ is a solution, again by Lemma 3.5. Thus, in the following, we can assume that $S' \cap L = \emptyset$.

So we may assume that $S'$ consists only of vertices in $G$. Suppose that some vertex $v \in T$ is not in $S'$. For any vertex $v' \in S' \setminus T$, we have $\deg(v') \leq \deg_G(v) + \mathsf{counter}(v) + \lfloor 1/\alpha \rfloor \leq \Delta + \Gamma + \lfloor 1/\alpha \rfloor$. So we have $\deg(v') \geq \Delta + \Gamma + |1/\alpha| \cdot k + \lfloor 1/\alpha \rfloor \geq \deg(v) + |1/\alpha - 3| \cdot k$. Applying Lemma 3.6 with $\mathsf{counter}(v) = \mathsf{counter}(v') = 0$, we obtain that $v$ is strictly better than $v'$. Now, it follows from Lemma 3.5 that $\mathcal{I}'$ has a solution $S'$ such that $T \subseteq S' \subseteq V(G')$. Hence, $S'$ is also a solution for $\mathcal{I}$. ◀

▶ **Lemma 3.8** ($\star$). *Given an instance $\mathcal{I} := (G, T, \mathsf{counter}, k, t)$ of* Annotated Min $\alpha$-FCGP *for $\alpha \in (0, 1]$, we can compute an equivalent instance $\mathcal{I}'$ of* Min $\alpha$-FCGP *of size $\mathcal{O}((\Delta + \Gamma)^3 \cdot |V(G)|)$ in polynomial time.*

# 4 Parameterization By Maximum Degree

## 4.1 Polynomial Kernels in Degrading Cases

Recall that in the degrading cases we have $\alpha \in (1/3, 1]$ for maximization and $\alpha \in [0, 1/3)$ for minimization. Furthermore, recall that for two vertices $u$ and $v$, $v$ is said to be better than $u$ if $\text{cont}(v, T) \geq \text{cont}(u, T)$ (vice versa for the minimization variant).

For the annotated version we define $\Delta_{\overline{T}} := \max_{v \in V(G) \setminus T} \deg(v)$. Clearly, $\Delta_{\overline{T}} \leq \Delta$.

▶ **Reduction Rule 4.1** ($\star$)**.** Let $\mathcal{I}$ be an instance of Annotated Degrading $\alpha$-FCGP. If there are more than $\Delta_{\overline{T}} k + 1$ vertices that are better than $v$ with respect to $T$, then apply the Exclusion Rule (Reduction Rule 3.3) to $v$.

Next, we show that the exhaustive application of Reduction Rule 4.1 yields a polynomial kernel for Degrading $\alpha$-FCGP.

▶ **Proposition 4.2.** Degrading $\alpha$-FCGP *has a kernel of size*
- $\mathcal{O}(\Delta^2 k)$ *for maximization and* $\alpha \in (1/3, 1]$*, and*
- $\mathcal{O}(\Delta^4 k)$ *for minimization and* $\alpha \in (0, 1/3)$*.*

**Proof.** Given an instance of Degrading $\alpha$-FCGP, we transform it into an equivalent instance of Annotated Degrading $\alpha$-FCGP and apply Reduction Rule 4.1 exhaustively. Observe that $|V(G)| \leq \Delta_{\overline{T}} k + 1 \leq \Delta k + 1$. Moreover, we have $T = \emptyset$ and $\Gamma \leq \Delta$ since each neighbor of a vertex can increase its counter by at most one. By Lemma 3.7 (maximization) resp. Lemma 3.8 (minimization), we obtain an equivalent instance of $\alpha$-FCGP of size $\mathcal{O}(\Delta^2 k)$ (maximization) resp. $\mathcal{O}(\Delta^4 k)$ (minimization). ◀

Note that Proposition 4.2 does not cover the case $\alpha = 0$ for minimization. Note that this problem is referred to as Sparsest $k$-Subgraph. We remark that this is complemented by Proposition 6.4, in which we provide a kernel of size $\mathcal{O}(d^2 k)$. Since $d \leq \Delta$, this implies also a kernel of size $\mathcal{O}(\Delta^2 k)$.

Proposition 4.2 basically shows that given an instance of Degrading $\alpha$-FCGP, we can find in polynomial time an equivalent instance of Degrading $\alpha$-FCGP of size $\mathcal{O}(\Delta + k)^{O(1)}$. In the following, we will show that an equivalent instance of Degrading $\alpha$-FCGP that has size $(\Delta + k)^{\mathcal{O}(1)}$ can be constructed even if an instance of Annotated Degrading $\alpha$-FCGP is given (see Proposition 4.10). Proposition 4.10 plays an important role in kernelizations in subsequent sections. Essentially, the task of kernelization boils down to bounding the maximum degree $\Delta$ by Proposition 4.10.

As shown in the proof of Proposition 4.2, the instance size becomes polynomial in $k + \Delta$ by exhaustively applying Reduction Rule 4.1. Recall that in Section 3, we presented a polynomial-time procedure to remove annotations with an additional polynomial factor in $\Delta + \Gamma$ on the instance size, where $\Gamma$ denotes the maximum counter. To prove Proposition 4.10, it remains to bound $\Gamma$ for Annotated Degrading $\alpha$-FCGP.

**Bounding the largest counter $\Gamma$.** Throughout the section, let $k' := k - |T|$ and $t' := t - \text{val}(T)$. First, we identify vertices which are contained in a solution, if one exists.

▶ **Definition 4.3.** *Let $\mathcal{I}$ be a Yes-instance of* Annotated Degrading $\alpha$-FCGP*. A vertex $v \in V(G) \setminus T$ is called* satisfactory *if*
- *(Maximization:)* $\text{cont}(v, T) \geq t'/k' + (3\alpha - 1)(k - 1)$ *and* $\alpha \in (1/3, 1]$*.*
- *(Minimization:)* $\text{cont}(v, T) \leq t'/k' - (1 - 3\alpha)(k - 1)$ *and* $\alpha \in (0, 1/3)$*.*

▶ **Reduction Rule 4.4** ($\star$)**.** Let $\mathcal{I}$ be an instance of Annotated Degrading $\alpha$-FCGP with $\alpha > 0$ and let $v \in V(G) \setminus T$ be a satisfactory vertex. Apply the Inclusion Rule (Reduction Rule 3.2) on vertex $v$.

We henceforth assume that Reduction Rule 4.4 is exhaustively applied on every satisfactory vertex. Next, we identify vertices which are not contained in any solution.

▶ **Definition 4.5.** *Let $\mathcal{I}$ be a Yes-instance of* Annotated Degrading $\alpha$-FCGP*. A vertex $v \in V(G) \setminus T$ is called* needless *if*

- *(Maximization:)* $\mathrm{cont}(v, T) \leq t'/k' - (3\alpha - 1)(k - 1)^2$ *for maximization and* $\alpha \in (1/3, 1]$.
- *(Minimization:)* $\mathrm{cont}(v, T) \geq t'/k' + (1 - 3\alpha)(k - 1)^2$ *for minimization and* $\alpha \in (0, 1/3)$.

▶ **Reduction Rule 4.6** ($\star$)**.** Let $\mathcal{I}$ be an instance of Annotated Degrading $\alpha$-FCGP with $\alpha > 0$ and let $v \in V(G) \setminus T$ be a needless vertex. Apply the Exclusion Rule (Reduction Rule 3.3) on vertex $v$.

We henceforth assume that Reduction Rule 4.6 is applied on every needless vertex. The following reduction rule decreases the counter of each vertex in $V(G) \setminus T$. After this rule is exhaustively applied, we may assume that $\mathsf{counter}(v) = 0$ for at least one vertex $v \in V(G) \setminus T$. Recall that we already have $\mathsf{counter}(v) = 0$ for every vertex in $T$.

▶ **Reduction Rule 4.7.** If $\mathsf{counter}(v) > 0$ for every vertex $v \in V(G) \setminus T$, then decrease $\mathsf{counter}(v)$ by 1 for every vertex $v \in V(G) \setminus T$ and decrease $t$ by $\alpha k'$.

Next, we show that after the exhaustive application of Reduction Rule 4.7 the counter of each vertex is bounded polynomially in terms of $\Delta$ and $k$.

▶ **Lemma 4.8.** *Let $\mathcal{I}$ be a Yes-instance of* Annotated Degrading $\alpha$-FCGP *with $\alpha > 0$. We have $\mathsf{counter}(v) \in \mathcal{O}(\Delta + k^2)$ for every vertex $v \in V(G) \setminus T$.*

**Proof.** First and foremost, observe that there exists at least one vertex $u \in V(G) \setminus T$ with $\mathsf{counter}(u) = 0$, since otherwise Reduction Rule 4.7 is still applicable. Since Reduction Rule 3.3 is applied to each needless vertex, we conclude that every vertex in $V(G) \setminus T$ has contribution at least $t'/k' - (3\alpha - 1)(k - 1)^2$ for maximization. Furthermore, since Reduction Rule 3.2 is applied to each satisfactory vertex, we conclude that every vertex $v \in V(G) \setminus T$ has contribution at least $t'/k' - (1 - 3\alpha)(k - 1)$ for minimization. In particular, we have

$$\mathrm{cont}(u, T) \geq t'/k' - (3\alpha - 1)(k - 1)^2 \text{ for maximization, and}$$
$$\mathrm{cont}(u, T) \geq t'/k' - (1 - 3\alpha)(k - 1) \text{ for minimization.}$$

Since also $\mathrm{cont}(u, T) = \alpha \cdot \deg(u) + (1 - 3\alpha)|N(u) \cap T|$ we obtain that

$$t'/k' \leq \alpha \cdot \deg(u) + (1 - 3\alpha)|N(u) \cap T| + (3\alpha - 1)(k - 1)^2 \text{ for maximization, and} \tag{1}$$
$$t'/k' \geq \alpha \cdot \deg(u) + (1 - 3\alpha)[(k - 1) + |N(u) \cap T|] \text{ for minimization.} \tag{2}$$

Moreover, since Reduction Rule 3.2 is applied to each satisfactory vertex, we conclude that every vertex $v \in V(G) \setminus T$ has contribution at most $t'/k' + (3\alpha - 1)(k - 1)$ for maximization. Furthermore, since Reduction Rule 3.3 is applied to each needless vertex, we conclude that every vertex in $V(G) \setminus T$ has contribution at most $t'/k' + (1 - 3\alpha)(k - 1)^2$ for minimization. This implies that in particular

$$\alpha \cdot \mathsf{counter}(v) \leq t'/k' + (3\alpha - 1)(k - 1) \text{ for maximization, and.} \tag{3}$$
$$\alpha \cdot \mathsf{counter}(v) \leq t'/k' + (1 - 3\alpha)(k - 1)^2 \text{ for minimization.} \tag{4}$$

For maximization and $\alpha \in (1/3, 1]$ it then follows from Equations (1) and (3) that

$$\mathsf{counter}(v) \leq \deg(u) + \frac{3\alpha - 1}{\alpha}[k(k-1) - |N(v) \cap T|] \in \mathcal{O}(\Delta + k^2).$$

For minimization and $\alpha \in (0, 1/3)$ it then follows from Equations (2) and (4) that

$$\mathsf{counter}(v) \leq \deg(u) + \frac{1 - 3\alpha}{\alpha}[k(k-1) + |N(v) \cap T|] \in \mathcal{O}(\Delta + k^2) \text{ for minimization.}$$

This concludes the proof. ◀

**Putting everything together.** We use the following proposition in our kernels for Degrading $\alpha$-FCGP. Therefore, we first transform the instance into an equivalent instance of Annotated Degrading $\alpha$-FCGP. Second, we apply our reduction rules. Third, we reduce back to the unannotated version.

▶ **Proposition 4.9** (⋆). *Let $\alpha > 0$. Given an instance $(G, T, \mathsf{counter}, k, t)$ of*

- Annotated Degrading Max $\alpha$-FCGP, *we can compute in polynomial time an equivalent* Degrading Max $\alpha$-FCGP *instance of size $\mathcal{O}(|V(G)|^2 + |V(G)|k^2) \subseteq \mathcal{O}(|V(G)|^3)$, and*
- Annotated Degrading Min $\alpha$-FCGP, *we can compute in polynomial time an equivalent* Degrading Min $\alpha$-FCGP *instance of size $\mathcal{O}(|V(G)| \cdot (|V(G)| + k^2)^3) \subseteq \mathcal{O}(|V(G)|^7)$.*

▶ **Proposition 4.10.** *Given an instance $(G, T, \mathsf{counter}, k, t)$ of* Annotated Degrading $\alpha$-FCGP *with $\alpha > 0$, we can compute in polynomial time an equivalent* Degrading $\alpha$-FCGP *instance of size $\mathcal{O}((\Delta + k)^{\mathcal{O}(1)})$.*

**Proof.** Follows from Lemmas 3.7–3.8, and Proposition 4.2. ◀

## 4.2 No Polynomial Kernels in Non-Degrading Cases

Note that if $\alpha = 0$, then Max $\alpha$-FCGP corresponds to Densest $k$-Subgraph. It is already known that Densest $k$-Subgraph does not admit a polynomial kernel for $\Delta + k$ [26]. We strengthen and generalize this result: First, we observe that Densest $k$-Subgraph does not admit a polynomial kernel for $k$, even when $\Delta = 3$. Second, we extend this negative result to Non-Degrading Max $\alpha$-FCGP and Non-Degrading Min $\alpha$-FCGP when $\Delta$ is a constant.

▶ **Theorem 4.11** (⋆). *Unless* coNP $\subseteq$ NP/poly,

1. Non-Degrading Max $\alpha$-FCGP *on subcubic graphs does not admit a polynomial kernel for $k$, and*
2. Non-Degrading Min $\alpha$-FCGP *on graphs with constant maximum degree does not admit a polynomial kernel for $k$.*

# 5 Parameterization by $c$-closure

**Hardness for the non-degrading case.** We start with showing that the Non-Degrading case is intractable.

▶ **Theorem 5.1** (⋆). Non-Degrading Max $\alpha$-FCGP *remains W[1]-hard with respect to the solution size $k$ even on 2-closed and 2-degenerate graphs.*

▶ **Theorem 5.2** (⋆). Non-Degrading Min $\alpha$-FCGP *remains W[1]-hard with respect to the solution size $k$ even on 2-closed graphs.*

**A $k^{\mathcal{O}(c)}$-size kernel for the degrading case.** In contrast to the non-degrading case, we develop a kernel of size $k^{\mathcal{O}(c)}$ for the degrading case. To this end, we apply a series of reduction rules to obtain an upper bound of $k^{\mathcal{O}(c)}$ on the maximum degree. Then, the kernel of size $k^{\mathcal{O}(c)}$ follows from Proposition 4.10. In order to upper-bound the maximum degree, we rely on a polynomial Ramsey bound for $c$-closed graphs [24].

▶ **Lemma 5.3** ([24, Lemma 3.1]). *Any $c$-closed graph $G$ on at least $R_c(a,b) := (c-1)\cdot\binom{b-1}{2} + (a-1)(b-1)+1$ vertices contains a clique of size $a$ or an independent set of size $b$. Moreover, a clique of size $a$ or an independent set of size $b$ can be found in polynomial time.*

Using a similar approach as Reduction Rule 4.1 (but exploiting the $c$-closure instead of the maximum degree) yields the following.

▶ **Reduction Rule 5.4.** Let $\mathcal{I}$ be an instance of Annotated Degrading $\alpha$-FCGP. Let $v \in V(G)$ be some vertex and let $X_v \subseteq N(v)$ be the set of vertices better than $v$. If $|X_v| > (c-1)k$, then apply the Exclusion Rule (Reduction Rule 3.3) to $v$.

▶ **Lemma 5.5.** *Reduction Rule 5.4 is correct.*

**Proof.** We provide a proof for the maximization version; the minimization version follows analogously. Let $S$ be a solution. Assume that $v \in S$ (we are done otherwise). We show that there is a vertex $v' \neq v$ such that $S' := (S \setminus \{v\}) \cup \{v'\}$ constitutes a solution. By Lemma 3.5, it suffices to show that $\mathrm{cont}(v', S \setminus \{v\}) \geq \mathrm{cont}(v, S \setminus \{v\})$. Let $S'_v := S \setminus N[v]$. Each vertex in $S'_v$ is, by definition, nonadjacent to $v$, and hence it shares at most $c-1$ neighbors in common with $v$. This implies $|X_v \setminus N(S'_v)| \geq |X_v| - (c-1)\cdot|S'_v| > 0$ as $X_v \subseteq N(v)$. Thus, there exists a vertex $v' \in X_v \setminus N(S'_v)$, that is, $N(v') \cap S'_v = \emptyset$. Then, we have $N(v') \cap (S \setminus \{v\}) \subseteq S \cap N(v)$ and thus $|N(v') \cap (S \setminus \{v\})| \leq |N(v) \cap (S \setminus \{v\})|$. Moreover, we have $\alpha \deg^{+c}(v') \geq \alpha \deg^{+c}(v)$ (recall that $v'$ is better than $v$). Since $\alpha \in (1/3, 1]$, it follows that

$$\begin{aligned}
&\mathrm{cont}(v', S \setminus \{v\}) = \alpha \deg^{+c}(v') + (1-3\alpha)|N(v') \cap (S \setminus \{v\})| \\
&\geq \alpha \deg^{+c}(v) + (1-3\alpha)|N(v) \cap (S \setminus \{v\})| = \mathrm{cont}(v, S \setminus \{v\}).
\end{aligned}$$

This concludes the proof. ◀

Note that if there is a clique of size $(c-1)k+1$, then Reduction Rule 5.4 applies to one of the vertices with the smallest contribution. Thus, applying Reduction Rule 5.4 exhaustively removes all cliques of size $(c-1)k+1$.

▶ **Lemma 5.6.** *Let $v \in V(G)$ be a vertex such that $\deg(v) \geq R_c((c-1)k+1, (k+1)k^{c-2})$. Then, we can find in polynomial time a set $X$ of $i \in [c-1]$ vertices and an independent set $I$ with the following properties:*

- **(i)** *The set $X$ contains $v$.*
- **(ii)** *The set $I \subseteq \bigcap_{x \in X} N(x)$ is an independent set of size at least $(k+1)k^{c-i}$.*
- **(iii)** *For every vertex $u \in V(G) \setminus X$, it holds that $|N(u) \cap I| \leq (k+1)k^{c-i-1}$.*

**Proof.** Since there is no clique of size $(c-1)k+1$, there is an independent set $I_v$ of size $(k+1)k^{c-2}$ in $N(v)$ by Lemma 5.3 (which can be found in polynomial time). Let $X$ be an inclusion-wise maximal set of $i$ vertices including $v$ such that $|\bigcap_{x \in X} N(x) \cap I_v| > (k+1)k^{c-i}$. One of such sets can be found by the following polynomial-time algorithm: We start with $X := \{v\}$ and $i := 1$. We will maintain the invariant that $|X| = i$. If there exists a vertex $v' \in V(G) \setminus X$ with $|N(v') \cap \bigcap_{x \in X} N(x) \cap I_v| > (k+1)k^{c-i-1}$, then we add $v'$ to $X$ and increase $i$ by 1. We keep doing so until there remains no such vertex $v'$.

We show that this algorithm terminates for $i = |X| \le c - 1$. Assume to the contrary that the algorithm continues for $i = c - 1$. We then have that $|N(v') \cap \bigcap_{x\in X} N(x) \cap I_v| > (k+1)k^{c-i-1} = k + 1 \ge 2$ for some vertex $v' \in V(G) \setminus X$. Since $I_v$ is an independent set, the set $N(v') \cap \bigcap_{x\in X} N(x) \cap I_v$ contains two nonadjacent vertices. Note, however, that these two vertices have at least $|X \cup \{v\}| = c$ common neighbors, contradicting the $c$-closure of $G$.

Finally, we show that a set $X$ found by this algorithm and $I := \bigcap_{x\in X} N(x) \cap I_v$ satisfy the three properties of the lemma. We have $|\bigcap_{x\in X} N(x) \cap I_v| = |N(v') \cap \bigcap_{x\in X\setminus\{v\}} N(x) \cap I_v| > (k+1)k^{c-(i-1)-1} = (k+1)k^{c-i}$, where $v'$ is the last vertex added to $X$. Moreover, since $X$ is inclusion-wise maximal, we have $|N(u) \cap I| = |N(u) \cap \bigcap_{x\in X} N(x) \cap I_v| \le (k+1)k^{c-i-1}$ for every vertex $u \in V(G) \setminus X$. ◀

▶ **Reduction Rule 5.7.** Let $\mathcal{I}$ be an instance of Annotated Degrading $\alpha$-FCGP. Let $X, I$ be as specified in Lemma 5.6 and let $v \in I$ be a vertex such that every other vertex in $I$ is better than $v$. If $k \ge 2$, then apply the Exclusion Rule (Reduction Rule 3.3) to $v$.

▶ **Lemma 5.8.** *Reduction Rule 5.7 is correct.*

**Proof.** Again, we show the proof for the maximization variant; the minimization variant follows analogously. For the sake of contradiction, assume that every solution $S$ contains $v$. By Lemma 5.6, every vertex $u \in V(G) \setminus X$ has at most $(k+1)^{c-i}$ neighbors in $I$. Moreover, since $I$ is an independent set, we have $|I \cap N[v']| = 1$ for every vertex $v' \in I$ (including $v$). For $S' := S \setminus X$, we have

$$\begin{aligned}|I \setminus N[S']| &\ge |I| - |I \cap N[v]| - |I \cap N[S' \setminus \{v\}]| \\ &\ge (k+1)k^{c-i} - (k-1)(k+1)k^{c-i-1} - 1 = k^{c-i} + k^{c-i-1} - 1 > 0.\end{aligned}$$

Let $v'$ be an arbitrary vertex in $I \setminus N[S']$. We show that $\text{cont}(v', S \setminus \{v\}) \ge \text{cont}(v, S \setminus \{v\})$. By Lemma 3.5, this would imply that $(S \setminus \{v\}) \cup \{v'\}$ is a solution not containing $v$. Since $v$ and $v'$ are both adjacent to all vertices of $X$ and $\alpha \in (1/3, 1]$, we have $|N(v) \cap (S \setminus \{v\})| > |X \cap (S \setminus \{v\})|$. We thus have

$$\begin{aligned}\text{cont}(v', S \setminus \{v\}) &= \alpha \deg^{+c}(v') + (1 - 3\alpha)|X \cap (S \setminus \{v\})| \\ &\ge \alpha \deg^{+c}(v) + (1 - 3\alpha)|X \cap (S \setminus \{v\})| \\ &\ge \alpha \deg^{+c}(v) + (1 - 3\alpha)|N(v) \cap (S \setminus \{v\})| = \text{cont}(v, S \setminus \{v\}).\end{aligned}$$

Here, the first inequality follows from the fact that $v'$ is better than $v$. ◀

By applying these reduction rules, we can ensure that $\Delta \le R_c((c-1)k+1, (k+1)k^{c-2}) \in k^{\mathcal{O}(c)}$. Proposition 4.10 leads to the following:

▶ **Proposition 5.9.** Degrading $\alpha$-FCGP *has a kernel of size* $k^{\mathcal{O}(c)}$.

**Matching lower bounds.** Next, we show that the kernels provided in Theorem 5.9 cannot be improved under standard assumptions.

▶ **Proposition 5.10 (⋆).** Degrading Max $\alpha$-FCGP *has no kernel of size* $\mathcal{O}(k^{c-3-\epsilon})$ *unless* coNP ⊆ NP/poly.

▶ **Proposition 5.11 (⋆).** *For each* $\alpha \in (0, 1/3)$*,* Min $\alpha$-FCGP *does not admit a kernel of size* $\mathcal{O}(k^{c-3-\epsilon})$ *unless* coNP ⊆ NP/poly.

Note that Min $\alpha$-FCGP for $\alpha = 0$ is equivalent to Sparsest $k$-Subgraph which admits a kernel of size $\mathcal{O}(c^2k^3)$ [23].

Now, Propositions 5.9–5.11 imply the following.

▶ **Theorem 5.12.** Degrading $\alpha$-FCGP *admits a kernel of size* $k^{\mathcal{O}(c)}$*. For* $\alpha > 0$*,* Degrading $\alpha$-FCGP *does not admit a kernel of size* $k^{o(c)}$ *unless* coNP $\subseteq$ NP/poly*.*

## 6 Parameterization by Degeneracy

**Minimization variant.** We start with the minimization variant which turns out to be easier than the maximization variant. This is most likely because of the following bound on $t$.

▶ **Lemma 6.1** ($\star$)**.** *In non-trivial instances* $(G, k, t)$ *of* Min $\alpha$-FCGP *we have* $t \le dk$*.*

Shachnai and Zehavi [32] showed that Min $\alpha$-FCGP with $\alpha \in (0, 1]$ admits an FPT-algorithm with respect to $k + t$. Hence, we obtain the following.

▶ **Corollary 6.2.** Min $\alpha$-FCGP *for* $\alpha > 0$ *is FPT parameterized by* $d + k$*.*

Naturally, we may now ask whether this FPT result can be strengthened to a polynomial kernel. As shown by Theorem 4.11, the non-degrading case of Min $\alpha$-FCGP does not admit a polynomial kernel even on graphs with constant maximum degree which implies constant degeneracy. In contrast, the degrading has a kernel whose size is polynomial in $d + k$.

▶ **Theorem 6.3** ($\star$)**.** Degrading Min $\alpha$-FCGP *admits a kernel of size* $(d + k)^{\mathcal{O}(1)}$*.*

▶ **Proposition 6.4** ($\star$)**.** Sparsest $k$-Subgraph *admits a kernel with* $\mathcal{O}(dk)$ *vertices and of size* $\mathcal{O}(d^2k)$*.*

**Maximization variant.** Recall, that MaxPVC is the special case of Max $\alpha$-FCGP with $\alpha = 1/2$. Amini et al. [1] showed that MaxPVC can be solved in $\mathcal{O}^*((dk)^k)$ time. Adapting this algorithm leads to an FPT-algorithm for $\alpha$-FCGP with respect to $d + k$ for $\alpha \neq 0$:

▶ **Proposition 6.5** ($\star$)**.** Degrading $\alpha$-FCGP *can be solved in* $\mathcal{O}^*((dk)^k)$ *time for* $\alpha \neq 0$*.*

The rest of this section is devoted to the proof of the next theorem.

▶ **Theorem 6.6.** Degrading Max $\alpha$-FCGP *admits a kernel of size* $k^{\mathcal{O}(d)}$ *but, unless* coNP $\subseteq$ NP/poly*, no kernel of size* $\mathcal{O}(k^{d-2-\epsilon})$*.*

In particular, this implies that MaxPVC admits a kernel of size $k^{\mathcal{O}(d)}$. We remark that a compression of size $(dk)^{\mathcal{O}(d)}$ was obtained independently by Panolan and Yaghoubizade [28].

**A kernel for biclique-free graphs in the degrading case.** We next develop a kernel of size $k^{\mathcal{O}(d)}$. In fact, our algorithm works for *biclique-free graphs* – graphs that do not have a biclique $K_{a,b}$ as a subgraph for $a \le b \in \mathbb{N}$. Note that a $d$-degenerate graph has no $K_{d+1,d+1}$ as a subgraph, since otherwise every vertex in $K_{d+1,d+1}$ has at least degree $d + 1$.

Note that a clique of size $a + b$ contains $K_{a,b}$ as a subgraph. So given a graph $G$ with no occurrence of $K_{a,b}$ on at least $\binom{a+b+k-2}{k-1} \in k^{\mathcal{O}(a+b)}$ vertices, one can find an independent set of size $k$ in polynomial time (see Section 2). We show that this upper bound on the number of vertices can be improved: the sum $a + b$ in the exponent can be replaced by $\min\{a, b\}$.

▶ **Lemma 6.7.** *For* $a \le b \in \mathbb{N}$*, let* $G$ *be a graph that contains no* $K_{a,b}$ *as a subgraph. If* $G$ *has at least* $R(k)$ *vertices, then we can find in polynomial time an independent set of size* $k$*, where* $R(k) \in (a + b)^{\mathcal{O}(a)} \cdot k^a$*.*

**Proof.** We first show that if $G$ has at least $k + b\binom{k}{a} + \sum_{\ell\in[a-1]} R(a+b,\ell+1)\binom{k}{\ell}$ vertices, then it contains an independent set of size $k$. We give an algorithm to find an independent set of size $k$ in polynomial time later. Let $I$ be a maximum independent set in $G$. We assume for contradiction that $|I| < k$. We prove that there are at most $t\binom{k}{a}$ vertices that have at least $a$ neighbors in $I$ and that there are at most $\sum_{\ell\in[a-1]} R(a+b,\ell+1)$ vertices that have at most $a-1$ neighbors in $I$.

For each subset $X \subseteq I$ of size exactly $a$, note that there are at most $b$ vertices $v$ such that $N(v) \supseteq X$, since otherwise there is a $K_{a,b}$ in $G$. It follows that the number of vertices with at least $a$ neighbors in $I$ is at most $t\binom{|I|}{a} \le b\binom{k}{a}$. Consider a set $X \subseteq I$ of size $\ell \in [a-1]$. Let $V_X \coloneqq \{v \in V(G)\setminus I \mid N(v)\cap I = X\}$. Then, there is no independent set $I'$ of size $\ell+1$ in $V_X$, since otherwise $(I\setminus X)\cup I'$ is an independent set of size at least $|I|+1$, contradicting the fact that $I$ is an independent set of maximum size. Moreover, there is no clique of size $a+b$ in $V_X$. Thus, $|V_X| < R(a+b,\ell+1)$. The number of vertices with at most $a-1$ neighbors in $I$ is then at most $\sum_{X\subseteq I,|X|=\ell\in[a-1]} R(a+b,\ell+1)\binom{|I|}{\ell} \le \sum_{\ell\in[a-1]} R(a+b,\ell+1)\binom{k}{\ell}$.

We turn the argument above into a polynomial-time algorithm as follows. Suppose that we have an independent set $I'$ of size smaller than $k$. As discussed above, there are at most $b\cdot\binom{k}{a}$ vertices that have at least $s$ neighbors in $I'$. Hence, there is a vertex set $X \subseteq I'$ of size $\ell$ such that $|V_X| > R(a+b,\ell+1)$. Note that $X$ can be found in polynomial time, for instance, by counting the number of vertices $v'$ such that $N(v')\cap I' = N(v)\cap I'$ for each vertex $v \in V(G)$. We can then find an independent set $I''$ of size $\ell+1$ in $X$ (this can be done in polynomial time as discussed in Section 2). This way, we end up with an independent set $(I'\setminus X)\cup I''$ of size at least $|I'|+1$. Note that this procedure of finding an independent set of greater size is repeated at most $k$ times, and thus the overall running time is polynomial. ◀

We remark that for fixed $a \le b \in \mathbb{N}$, Lemma 6.7 gives us an $\mathcal{O}(n^{1-1/a})$-approximation algorithm for Independent Set that runs in $n^\ell$ time with a constant $\ell$ not depending on $a$ or $b$. An $\mathcal{O}(n^{1-1/a})$-approximation algorithm is known on graphs where $K_{a,b}$ is excluded as an *induced subgraph* [4, 13]. However, these algorithms have running time $n^{\Omega(a)}$.

We then apply Lemma 6.7 to obtain a lemma analogous to Lemma 5.6.

▶ **Lemma 6.8.** *Let $v \in V(G)$ be a vertex such that $\deg(v) \ge R(bk^{a-1})$. Then, we can find in polynomial time a set $X$ of $i \in [a-1]$ vertices and an independent set $I$ with the following properties:*

**(i)** *The set $X$ contains $v$.*
**(ii)** *The set $I \subseteq \bigcap_{x\in X} N(x)$ is an independent set of size at least $bk^{a-i}+1$.*
**(iii)** *For every vertex $u \in V(G)\setminus X$, it holds that $|N(u)\cap I| \le bk^{a-i-1}$.*

**Proof.** By Lemma 6.7, there is an independent set $I_v$ of size $bk^{b-1}$ in $N(v)$ (which can be found in polynomial time). Let $X$ be an inclusion-wise maximal set of $i$ vertices including $v$ with $|\bigcap_{x\in X} N(x)\cap I_v| > bk^{a-i}$. Such a set can be found by the following polynomial-time algorithm: We start with $X = \{v\}$ and $i = 1$. We will maintain the invariant that $|X| = i$. If there exists a vertex $v' \in V(G)\setminus X$ with $|N(v')\cap\bigcap_{x\in X} N(x)\cap I_v| > bk^{a-i-1}$, then we add $v'$ to $X$ and increase $i$ by 1. We keep doing so until there remains no such vertex $v'$.

We show that this algorithm terminates for $i = |X| \le a-1$. Assume to the contrary that the algorithm continues for $i = a-1$. We then have that $|N(v')\cap\bigcap_{x\in X} N(x)\cap I_v| > bk^{a-i-1}$ for some vertex $v' \in V(G)\setminus X$. It follows that the set $X\cup\{v'\}$ (which is of size $a$) has more than $b$ common neighbors, contradicting the fact that $G$ has no $K_{a,b}$ as a subgraph.

Finally, we show that a set $X$ found by this algorithm and $I := \bigcap_{x\in X} N(x) \cap I_v$ satisfy the three properties of the lemma. We have $|I| = |\bigcap_{x\in X} N(x) \cap I_v| = |N(v') \cap \bigcap_{x\in X\setminus\{v'\}} N(x) \cap I_v| > bk^{a-(i-1)-1} = bk^{a-i}$, where $v'$ is the last vertex added to $X$. Moreover, since $X$ is inclusion-wise maximal, we have $|N(u) \cap I| = |N(u) \cap \bigcap_{x\in X} N(x) \cap I_v| \le bk^{a-i-1}$ for every vertex $u \in V(G) \setminus X$. ◀

▶ **Reduction Rule 6.9.** Let $\mathcal{I}$ be an instance of Annotated Degrading $\alpha$-FCGP. Let $X, I$ be as specified in Lemma 6.8 and let $v \in I$ be a vertex such that every other vertex in $I$ is better than $v$. Then, apply the Exclusion Rule (Reduction Rule 3.3) to $v$.

▶ **Lemma 6.10.** *Reduction Rule 6.9 is correct.*

**Proof.** We show the proof for the maximization variant; the minimization variant follows analogously. For the sake of contradiction, assume that every solution $S$ contains $v$. By Lemma 6.8, every vertex $u \in V(G) \setminus X$ has at most $bk^{a-i}$ neighbors in $I$. Moreover, since $I$ is an independent set, we have $|I \cap N[v']| = 1$ for every vertex $v' \in I$ (including $v$). For $S' := S \setminus X$, we have

$$\begin{aligned}|I \setminus N[S']| &\ge |I| - |I \cap N[v]| - |I \cap N[S' \setminus \{v\}]| \\ &\ge (bk^{a-i} + 1) - (k-1)bk^{a-i-1} - 1 = bk^{a-i-1} > 0.\end{aligned}$$

Let $v'$ be an arbitrary vertex in $I \setminus N[S']$. We show that $\operatorname{cont}(v', S \setminus \{v\}) \ge \operatorname{cont}(v, S \setminus \{v\})$. By Lemma 3.5, this would imply that $(S \setminus \{v\}) \cup \{v'\}$ is a solution not containing $v$. Since $v$ and $v'$ are both adjacent to all vertices of $X$ and $\alpha \in (1/3, 1]$, we have

$$\begin{aligned}\operatorname{cont}(v', S \setminus \{v\}) &= \alpha \deg^{+c}(v') + (1-3\alpha)|X \cap (S \setminus \{v\})| \\ &\ge \alpha \deg^{+c}(v) + (1-3\alpha)|X \cap (S \setminus \{v\})| \\ &\ge \alpha \deg^{+c}(v) + (1-3\alpha)|N(v) \cap (S \setminus \{v\})| = \operatorname{cont}(v, S \setminus \{v\}).\end{aligned}$$

Here, the first inequality follows from the fact that $v'$ is better than $v$. ◀

By applying Reduction Rule 6.9 exhaustively, we end up with an instance with maximum degree $\Delta \le R(bk^{a-1})$. The following proposition then follows from Proposition 4.10 using the bound in Lemma 6.7:

▶ **Proposition 6.11.** *For any $a \le b \in \mathbb{N}$,* Degrading $\alpha$-FCGP *on graphs that do not contain $K_{a,b}$ as a subgraph has a kernel of size $(R(bk^{a-1}) + k)^{\mathcal{O}(1)} \in b^{\mathcal{O}(a)} k^{\mathcal{O}(a^2)}$.*

Note that a $d$-degenerate graph contains no $K_{d+1,d+1}$ as a subgraph. We obtain the following result using the folklore fact that any $d$-degenerate graph on at least $(d+1)k$ vertices has an independent set of size $k$.

▶ **Lemma 6.12** (⋆)**.** Degrading $\alpha$-FCGP *admits a kernel of size $k^{\mathcal{O}(d)}$.*

Now, we show that significant improvement in Lemma 6.12 is unlikely. This, together with Lemma 6.12, implies Theorem 6.6.

▶ **Proposition 6.13** (⋆)**.** Degrading Max $\alpha$-FCGP *admits no kernel of size $\mathcal{O}(k^{d-2-\epsilon})$ unless* coNP $\subseteq$ NP/poly.

## 7 Parameterization by vc and $h$-index

To complete the picture of the parameterized complexity landscape, we consider two parameters that are larger than the degeneracy of $G$: the $h$-index of $G$ and the vertex cover number of $G$. We start with the maximization variant and the $h$-index. As Non-Degrading Max $\alpha$-FCGP does not admit a polynomial kernel with respect to $k$ even if $\Delta$ is constant (see Thm. 4.11), the same holds for the $h$-index. The degrading case admits a polynomial kernel.

▶ **Proposition 7.1** (⋆). Degrading Max $\alpha$-FCGP *admits a kernel of size* $\mathcal{O}(h^2k^2 + k^4)$.

We complement this with showing fixed-parameter tractability for $k + h$.

▶ **Proposition 7.2** (⋆). Non-Degrading Max $\alpha$-FCGP *is fixed-parameter tractable with respect to* $k + h$.

For the larger parameter vertex cover number vc, we achieve a kernel for all $\alpha > 0$.

▶ **Proposition 7.3** (⋆). *If* $\alpha \neq 0$, *then* Max $\alpha$-FCGP *admits a kernel of size* $\mathcal{O}(\mathsf{vc}^4 + \mathsf{vc} \cdot k^3)$.

For $\alpha = 0$, Max $\alpha$-FCGP corresponds to Densest $k$-Subgraph and Clique is one of its special cases ($t = \binom{k}{2}$). Since Clique does not admit a polynomial kernel with respect to vc [2] (and any clique is of size at most $\mathsf{vc} + 1$), Densest $k$-Subgraph does not admit a polynomial kernel. However, Densest $k$-Subgraph can be solved by a straightforward algorithm in $\mathcal{O}^*(2^{\mathsf{vc}})$ time. Thus, Densest $k$-Subgraph admits a kernel of size $\mathcal{O}(2^{\mathsf{vc}})$.

**Minimization variant.** Note that Degrading Min $\alpha$-FCGP has a polynomial kernel with respect to $d + k$ (see Theorem 6.3) and, thus, also with respect to $h + k$ and $\mathsf{vc} + k$. As Non-Degrading Min $\alpha$-FCGP does not admit a polynomial kernel with respect to $k$ even if $\Delta$ is constant (see Thm. 4.11), the same holds for the $h$-index. It remains to consider Non-Degrading Min $\alpha$-FCGP parameterized by $\mathsf{vc} + k$.

▶ **Proposition 7.4** (⋆). Min $\alpha$-FCGP *admits a kernel of size* $\mathcal{O}(\mathsf{vc}^8 + \mathsf{vc} \cdot k^7)$ *for* $\alpha > 0$ *and of size* $\mathcal{O}(\mathsf{vc}^2 + \mathsf{vc} \cdot k)$ *for* $\alpha = 0$.

## 8 Conclusion

We provided a systematic parameterized complexity analysis for $\alpha$-FCGP (see Figure 2). Although we settled the existence of polynomial kernels with respect to various parameters combined with the solution size $k$, several open questions remain. First, our polynomial kernels are not optimized and thus the polynomials are of high degree. Looking for smaller kernels is thus an obvious first task. Second, can our positive results for $c$-closure and degeneracy be extended to the smaller parameter weak closure [16]? Furthermore, while we looked at parameters that are small in sparse graphs, can similar results be achieved for dense graphs as considered e. g. by Lochet et al. [27]? Finally, we believe that an experimental verification of our data reduction rules would demonstrate their practical usefulness.

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